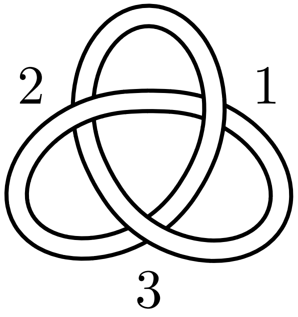
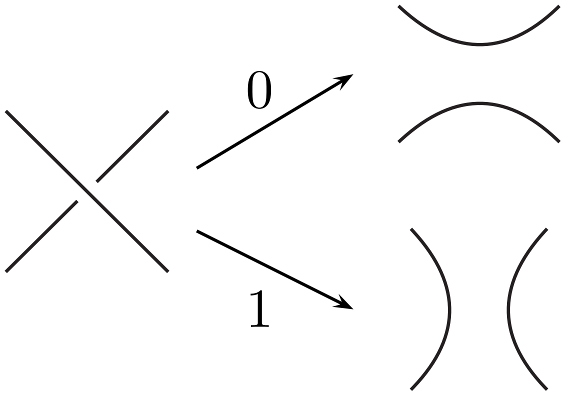
Coloured poset homology

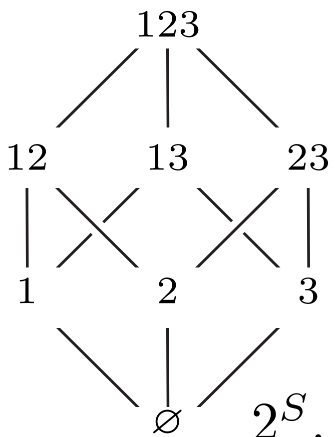
Brent Everitt (York) –joint with Paul Turner (Fribourg)

arXiv:0808.1686
0711.0103

Example: Jones polynomial





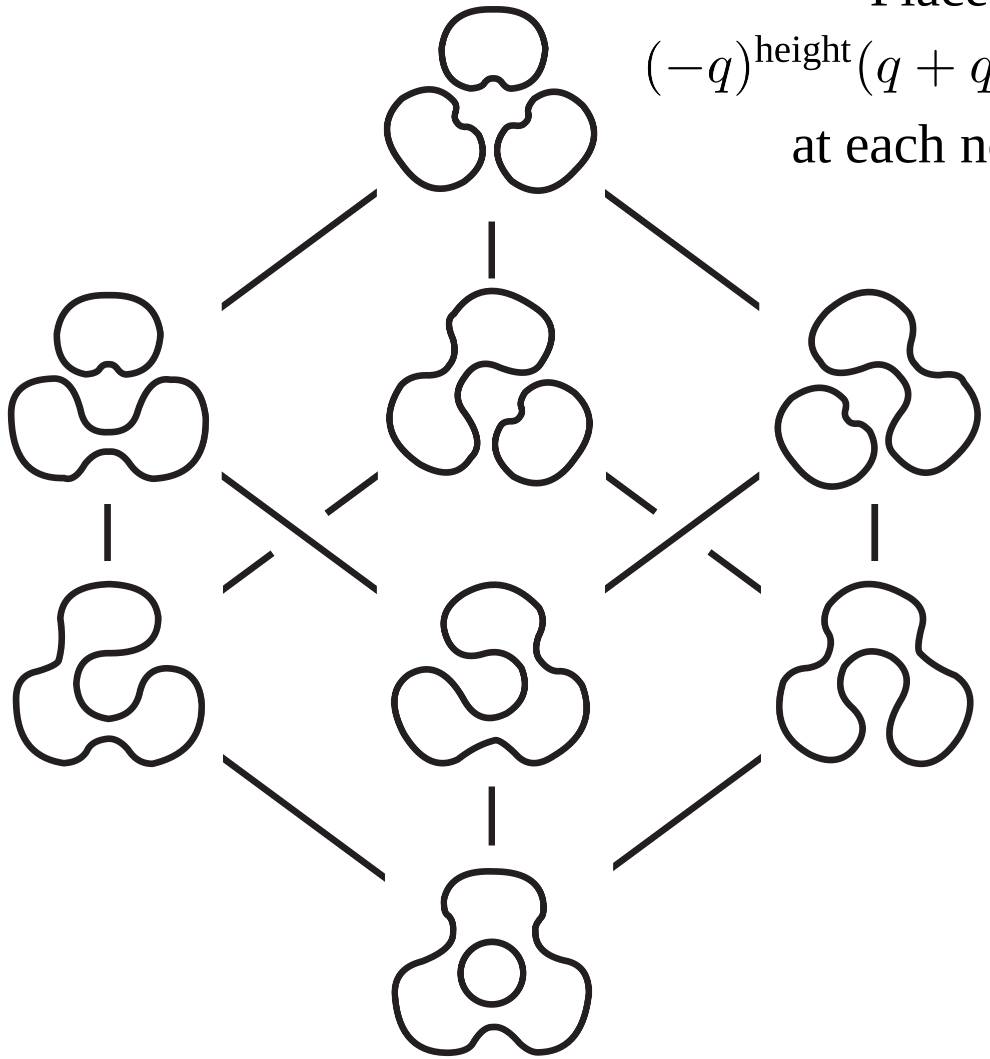


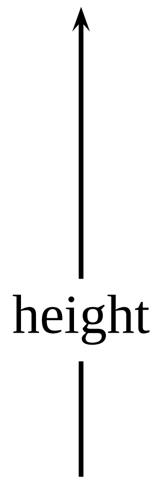
$2^S, S = \{1, 2, 3\}$

Place

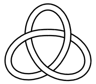
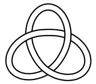
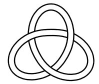
$(-q)^{\text{height}}(q + q^{-1})^{\#\text{circles}}$

at each node

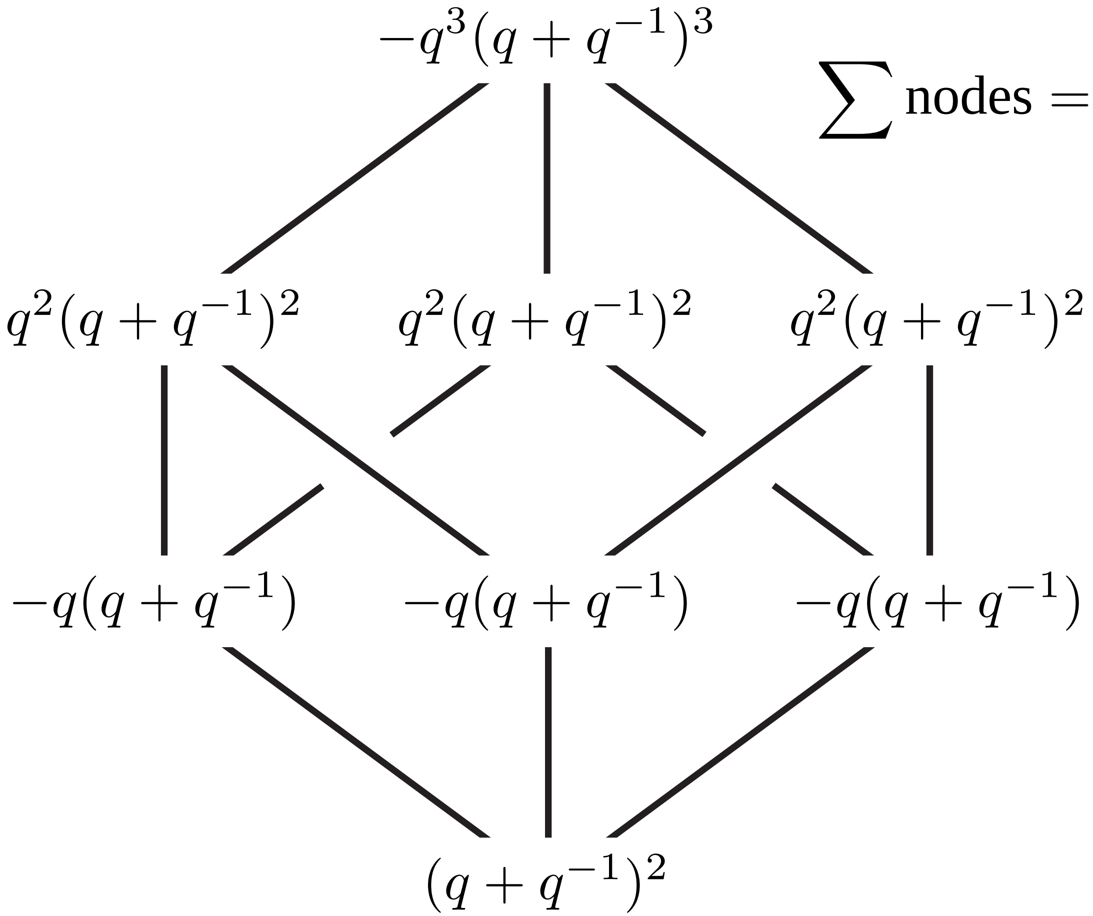


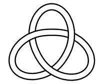


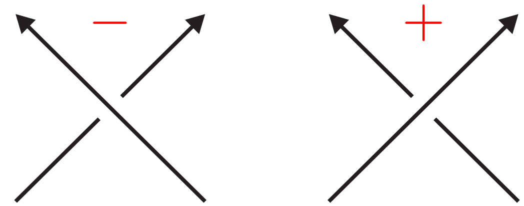
Example: Jones polynomial

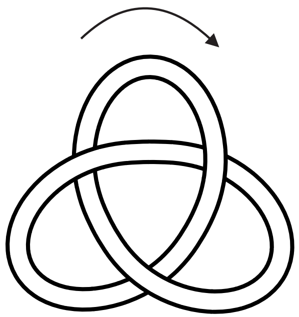
$J\Big($  $\Big) = \dfrac{1}{(q+q^{-1})}\widehat{J}\Big($  $\Big) \longleftarrow (-1)^{n_-}q^{n_+-2n_-}\Big\langle$  $\Big\rangle$

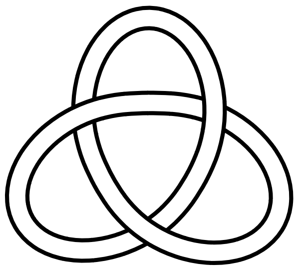
(Jones) (unnormalized Jones) (Kauffman bracket)



$\sum$ nodes $= \Big\langle$  $\Big\rangle = -q^6 + q^2 + 1 + q^{-2}$







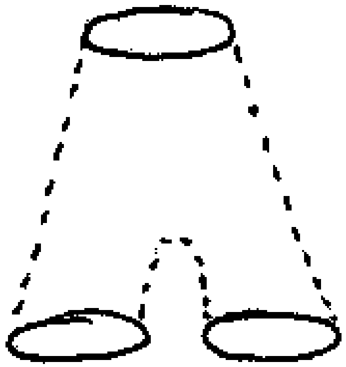
Jones →

$f \in \mathbb{Z}[q, q^{-1}]$

link L_2

↑

link L_1



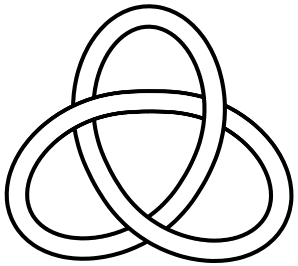
functor →

$\bigoplus_{i,j} M_{ij}$

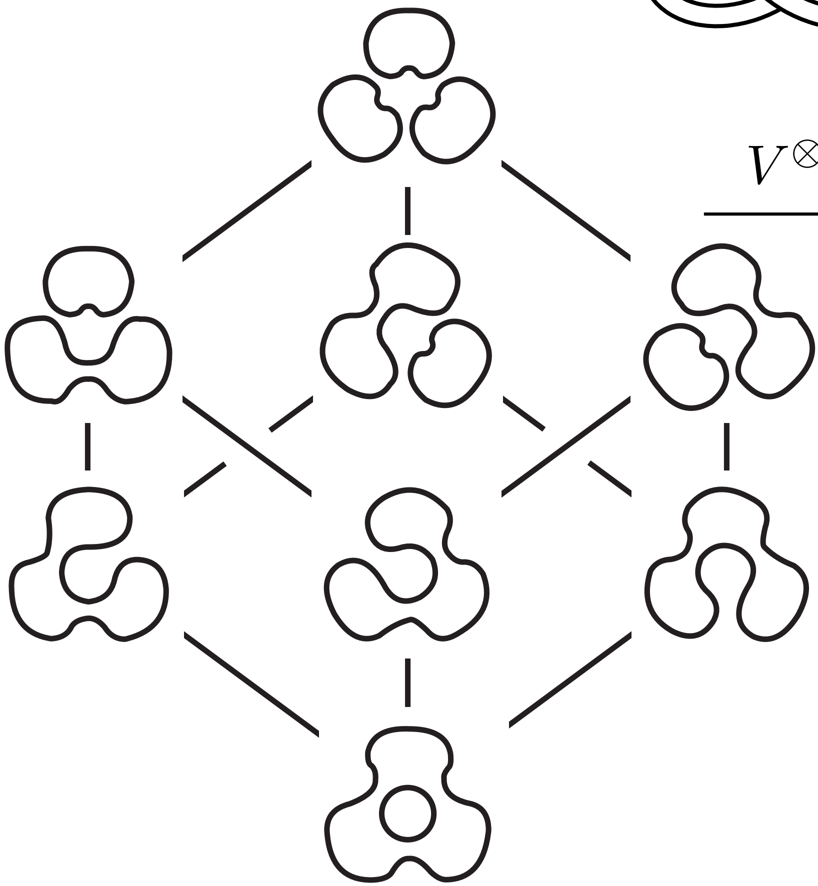
↑ (bi-graded spaces)

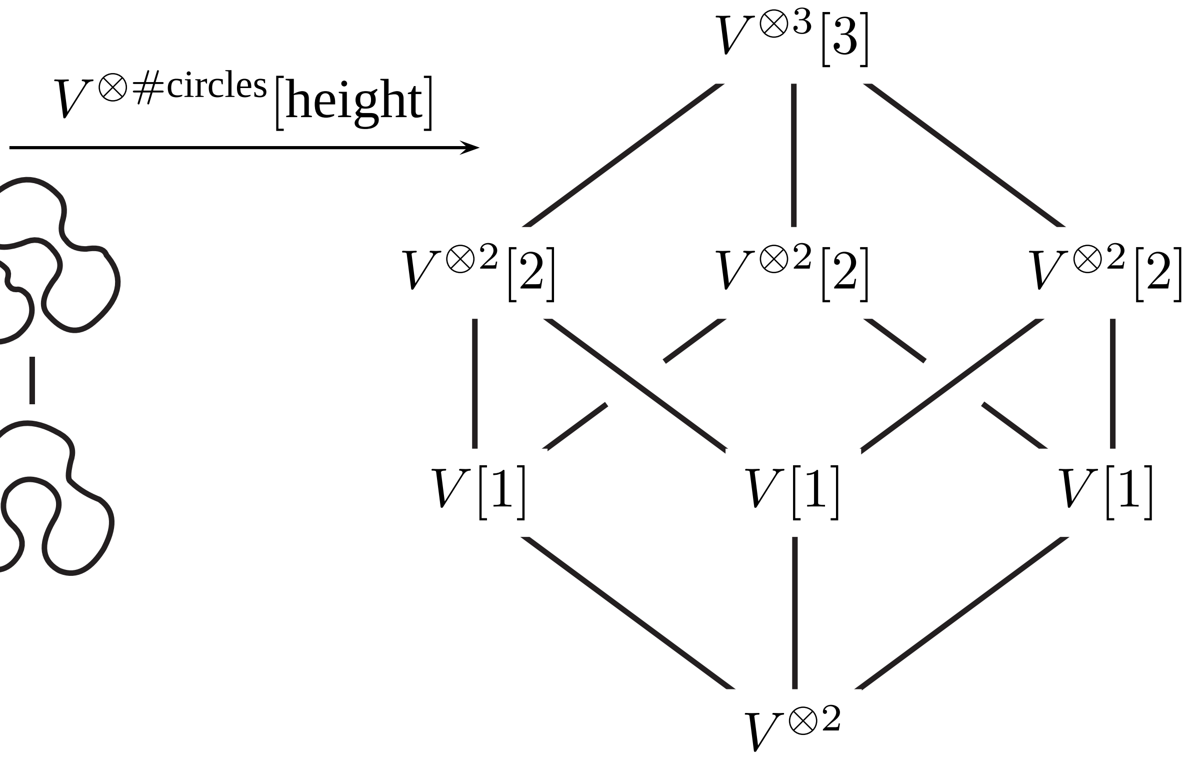
$\bigoplus_{i,j} N_{ij}$

$$\chi(\bigoplus_{i,j} M_{ij}) = \sum_i (-1)^i \text{qdim}\left(\bigoplus_j M_{ij}\right) = \sum_i (-1)^i \sum_j (\dim M_{ij}) q^j = \text{“Jones”}$$



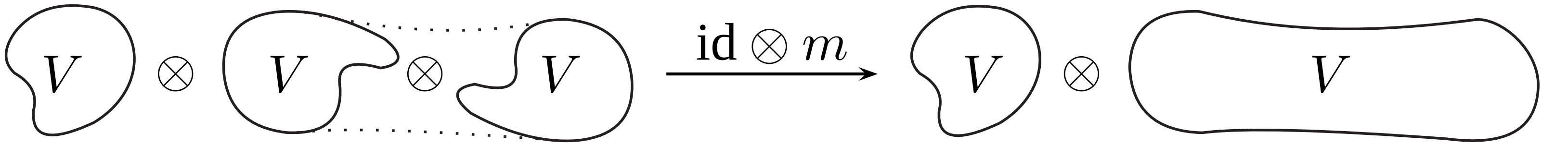
$$V = \mathbb{Q} \oplus \mathbb{Q}$$
$$\quad -1 \quad\; 1$$



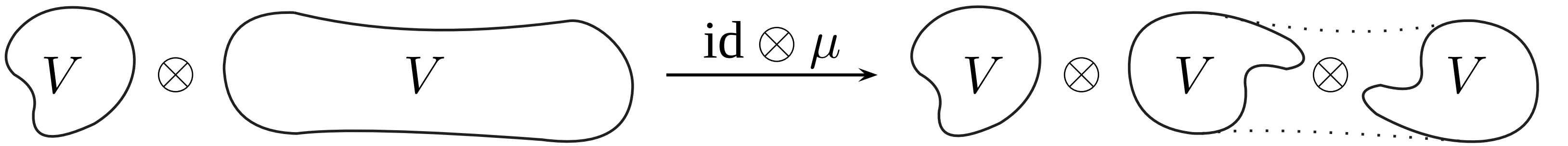


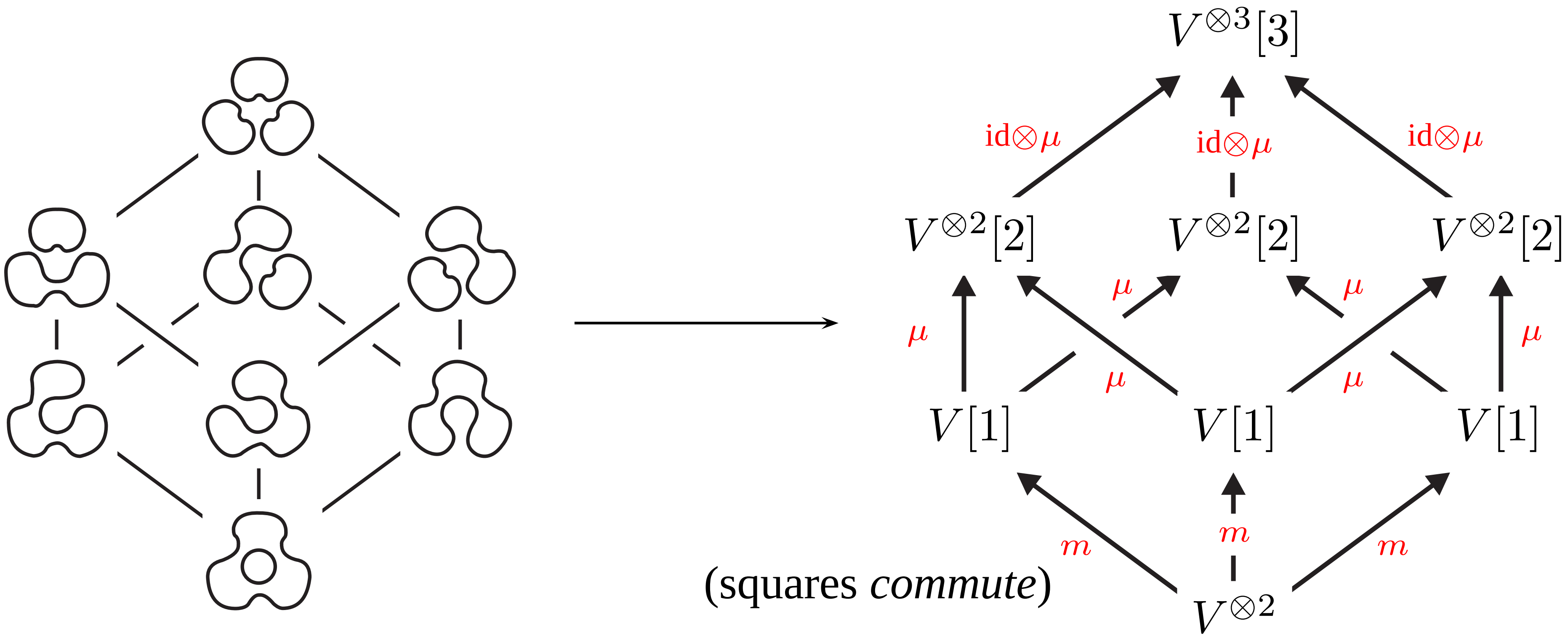
Example: Khovanov complex

$m : V \otimes V \to V$

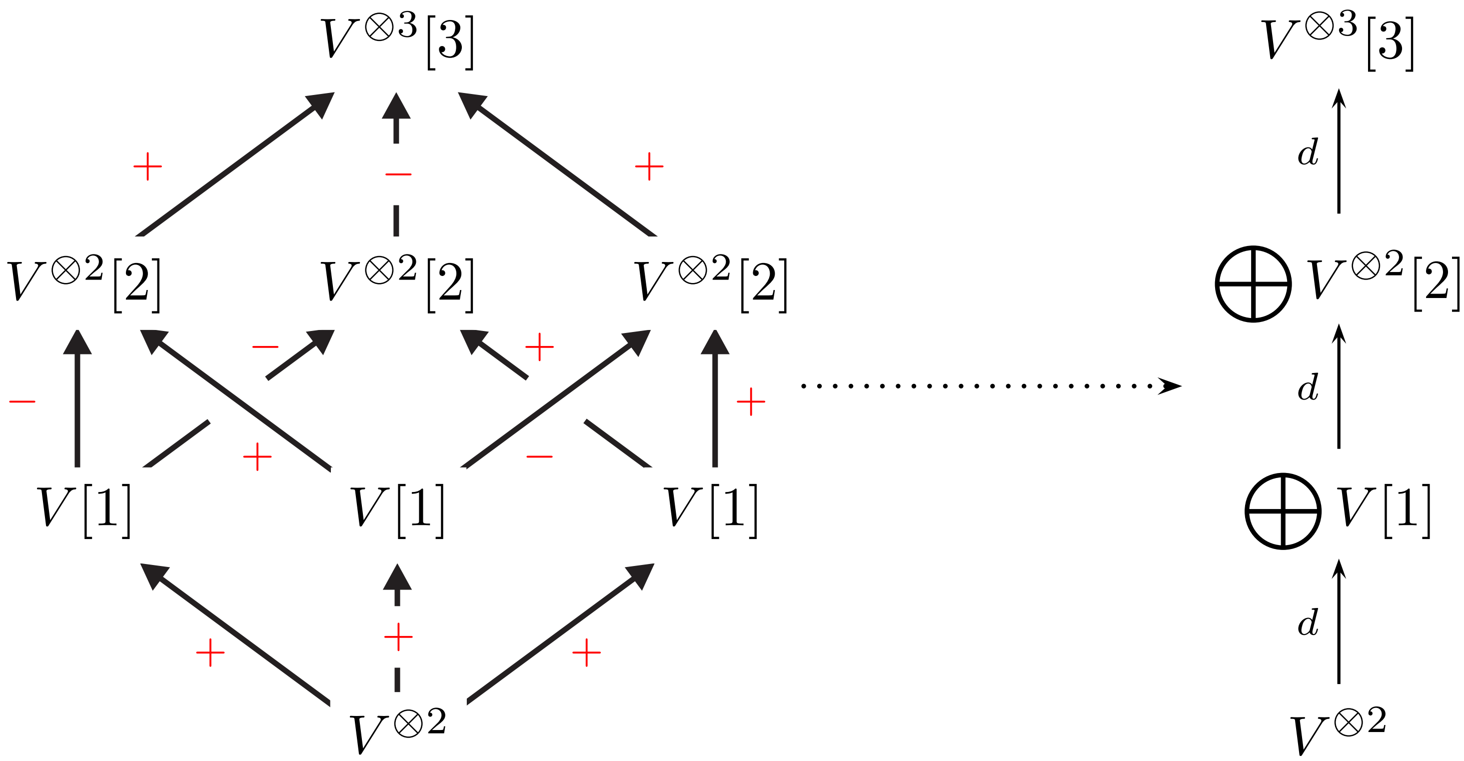


$\mu : V \to V \otimes V$





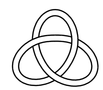
(squares *commute*)

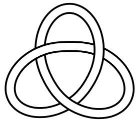


$\mathcal{K}_*$

Khovanov (chain) complex

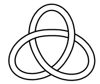
add $\pm$'s to edge maps so squares *anticommute*

Khovanov homology $KH_*\Big($  $, \mathbb{Q}\Big) = H_*(\mathcal{K}_*)$



	6	4	2	0	-2	$q\dim$
KH_0	$\mathbb{Q}$					q^6
KH_1			$\mathbb{Q}$			q^2
KH_2						0
KH_3				$\mathbb{Q}$	$\mathbb{Q}$	$1 + q^{-2}$

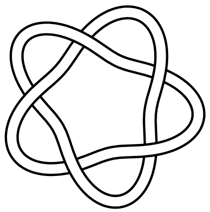
Euler characteristic $\chi(\mathcal{K}_*)$

$= \sum(-1)^i q\dim KH_i\Big($  $, \mathbb{Q}\Big)$

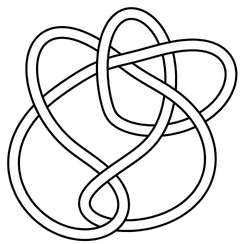
$= q^6 - q^2 - 1 - q^{-2}$

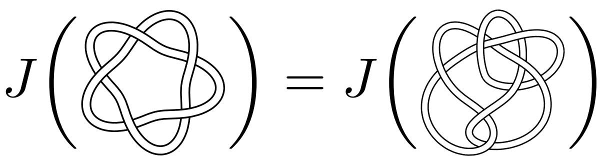
- $\chi(\mathcal{K}_*)$ "=" Jones
- KH_* an invariant (after a bit of nudging)

$\mathbb{Q}$						
		$\mathbb{Q}$				
		$\mathbb{Q}$				
				$\mathbb{Q}$		
					$\mathbb{Q}$	$\mathbb{Q}$



$\mathbb{Q}$							
		$\mathbb{Q}$					
		$\mathbb{Q}$					
			$\mathbb{Q}$	$\mathbb{Q}$			
			$\mathbb{Q}$		$\mathbb{Q}$		
					$\mathbb{Q}\oplus\mathbb{Q}$		
							$\mathbb{Q}$
						$\mathbb{Q}$	$\mathbb{Q}$





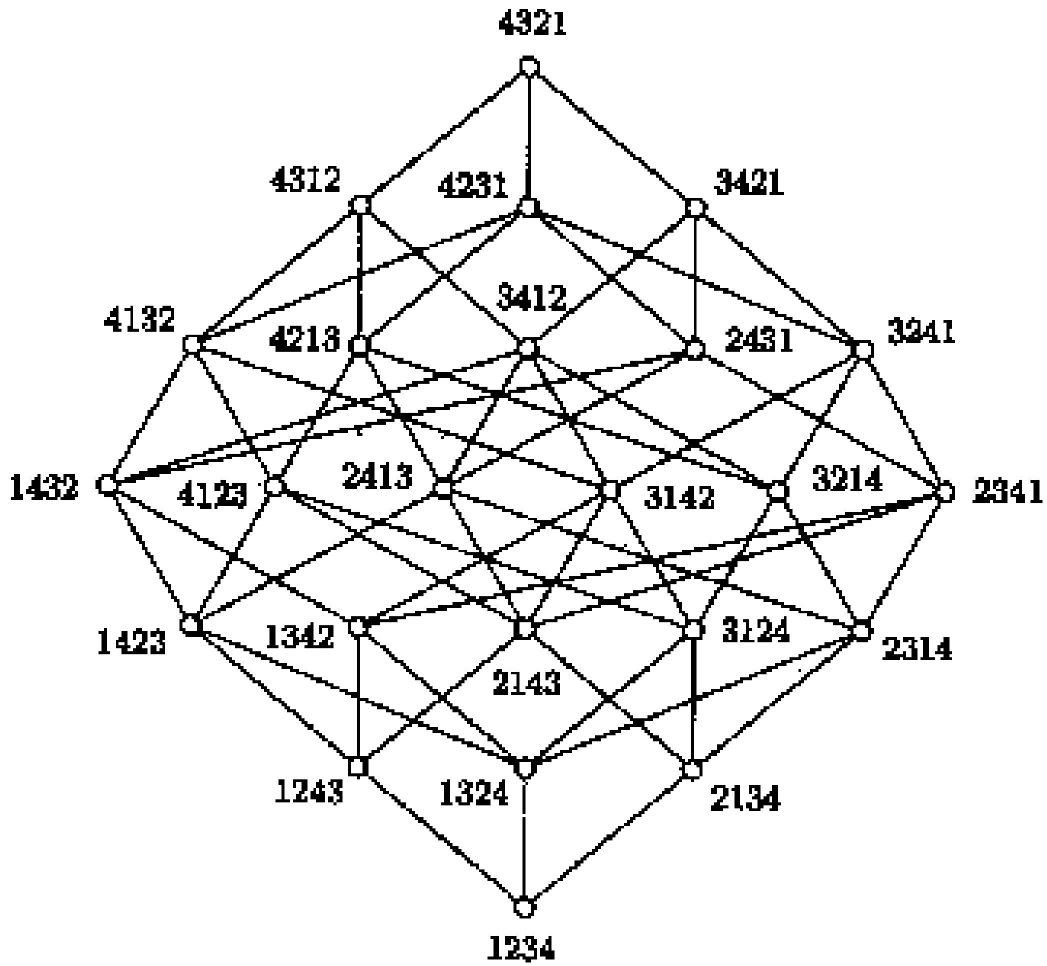
- Alexander polynomial: Heegaard-Floer homology (Ozsváth and Szabó)
- HOMFLY polynomial: Khovanov-Rozansky homology
- chromatic polynomial: graph homology (Helme-Guizon and Rong)

- Boolean lattice $\mathbb{B}(X)$ on a set X.

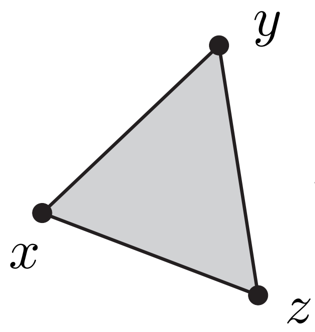
- Coxeter group with the Bruhat order.

- "quiver" poset of a directed graph.

- intersection lattice of a hyperplane arrangement.



- poset $P \longrightarrow |P|$ order (simplicial) complex.



iff $x < y < z$

- poset homology = simplicial homology of $|P|$

ie: $H_*(P, R) := H_*(|P|, R)$ = homology of chain complex

$$C_n(P, R) = \bigoplus_{x_0 < \cdots < x_n} R$$

with differential $d : C_n(P, R) \to C_{n-1}(P, R)$

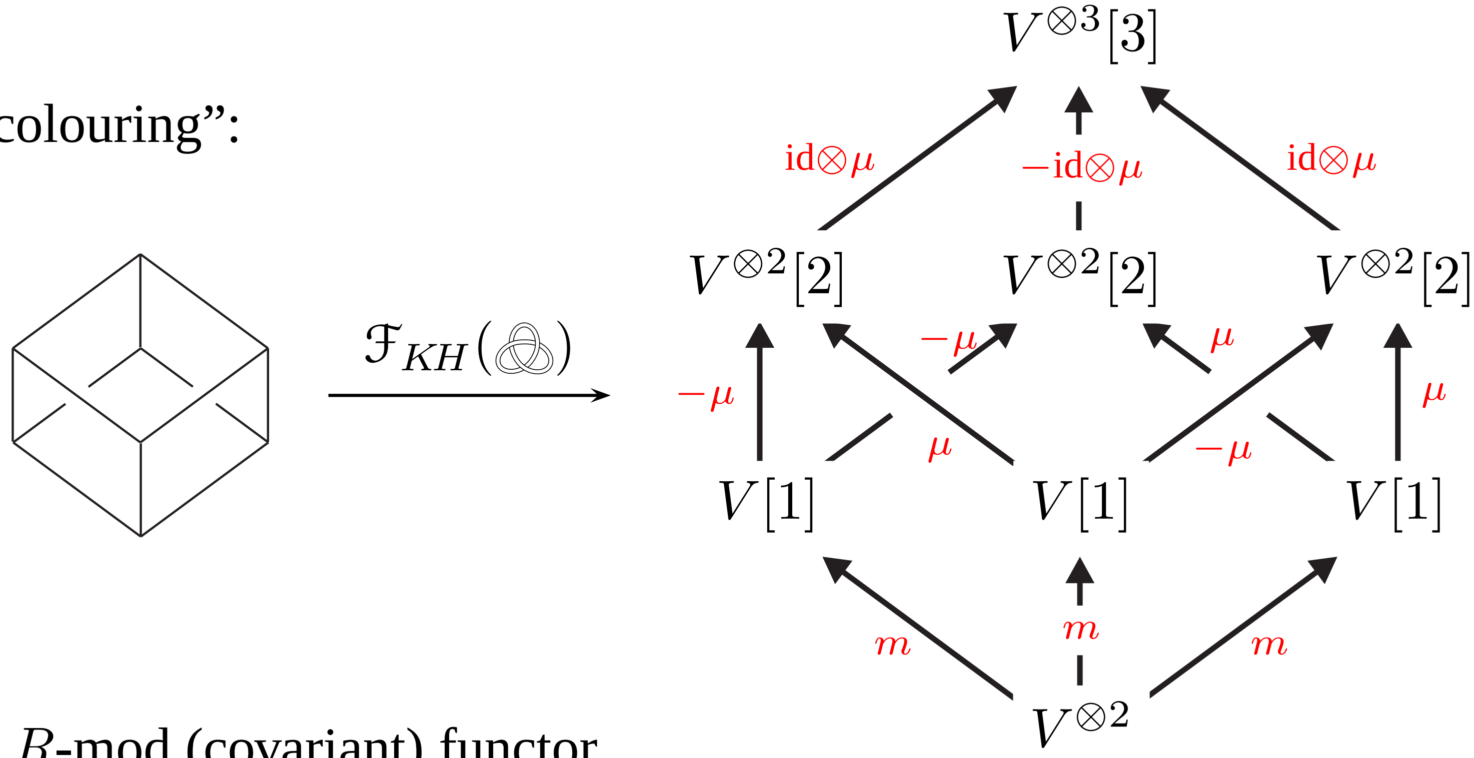
$$\lambda \cdot (x_0 < \cdots < x_n) \stackrel{d}{\mapsto} \sum_{j=0}^{n} (-1)^j \lambda \cdot (x_0 < \cdots < \widehat{x}_j < \cdots < x_n)$$

- Eg: [Folkman] P finite geometric lattice

$$\widetilde{H}_n(P \setminus \{0, 1\}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{|\mu(0,1)|} & n = \text{rk}P - 2, \\ 0 & \text{otherwise.} \end{cases}$$

• Eg: "Khovanov colouring":

$\mathbb{B}$



• in general: $P \xrightarrow{\mathfrak{F}} R$-mod (covariant) functor

(= pre-sheaf of modules over P)

- $P \xrightarrow{\mathcal{F}} R$-mod coloured poset/sheaf

- coloured poset or sheaf homology $\mathcal{H}_*(P,\mathcal{F})$ = homology of chain complex

$$\mathcal{S}_n(P,\mathcal{F}) = \bigoplus_{x_0<\cdots<x_n} \mathcal{F}(x_0)$$

with differential $d : \mathcal{S}_n(P,\mathcal{F}) \to \mathcal{S}_{n-1}(P,\mathcal{F})$

$$\lambda \cdot (x_0 < \cdots < x_n) \overset{d}{\mapsto} \mathcal{F}(x_0 < x_1)(\lambda) \cdot (\widehat{x}_0 < x_1 < \cdots < x_n) + \sum_{j=1}^{n} (-1)^j \lambda \cdot (x_0 < \cdots < \widehat{x}_j < \cdots < x_n)$$

- **Theorem** [E-Turner]: $\mathbb{B}$ Boolean and $\mathbb{B} \xrightarrow{\mathcal{F}} R$-mod a sheaf/colouring, then

$$KH_*(\mathbb{B},\mathcal{F}) \cong \widetilde{\mathcal{H}}_{*-1}(\mathbb{B} \setminus 1, \mathcal{F})$$

• X regular CW-complex $\longrightarrow P = S(X) =$ (closed) cells under reverse $\subseteq$ $S(X) \stackrel{\mathcal{F}}{\rightarrow} R$-mod sheaf/colouring.

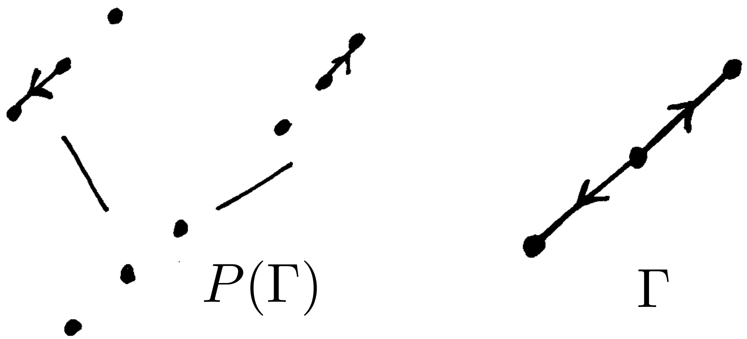
• cellular homology of X (with coefficients in sheaf $\mathcal{F}$) $H_*^{cell}(X, \mathcal{F}) =$ homology of chain complex

$$\mathcal{K}_n(X, \mathcal{F}) = \bigoplus_{n\text{-cells } \sigma} \mathcal{F}(\sigma) \qquad \lambda \cdot \sigma \stackrel{d}{\mapsto} \sum_{(n-1)\text{-cells } \tau} [\sigma : \tau]\mathcal{F}(\sigma < \tau)(\lambda) \cdot \tau$$

(*cf*: Khovanov homology . . .)

• **Theorem** [E-Turner]: $P = S(X)$, X regular CW-complex, $P \stackrel{\mathcal{F}}{\rightarrow} R$-mod sheaf/colouring,

$$H_*^{cell}(P, \mathcal{F}) \cong \mathcal{H}_*(P, \mathcal{F})$$



- A = associative R-algebra.
- $P(\Gamma)$ = quiver poset of directed graph Γ.
- $P(\Gamma) \xrightarrow{\mathcal{F}_A}$

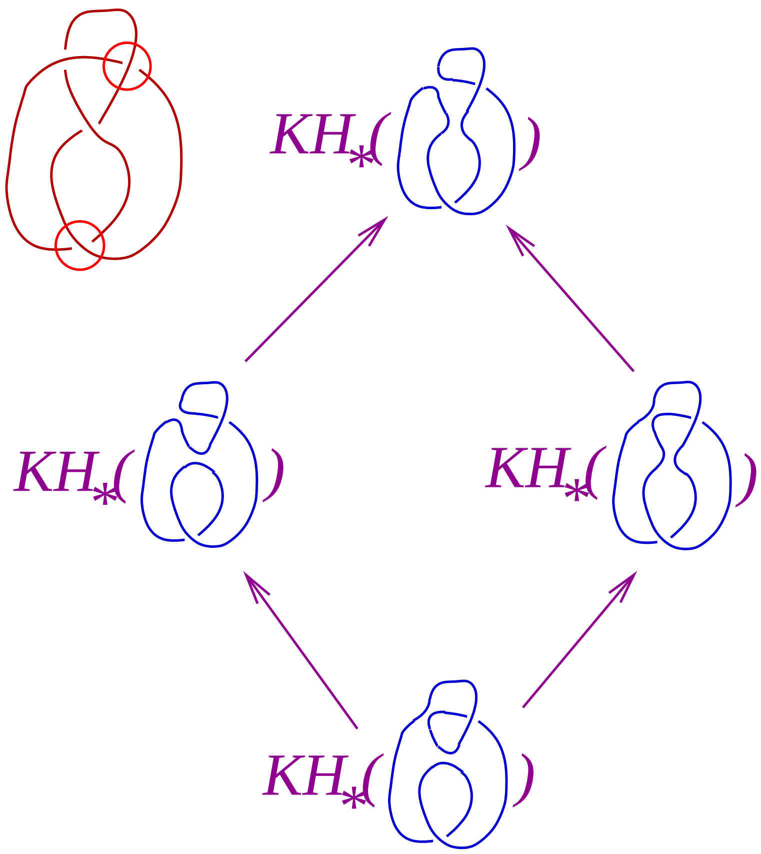
$$\begin{array}{ccc} A \otimes A & & A \otimes A \\ & \nwarrow \quad \nearrow & \\ & A \otimes A \otimes A & \end{array}$$

- **Corollary** [Turner-Wagner]:

$$\mathcal{H}_i(P(n\text{-gon}), \mathcal{F}_A) \cong HH_i(A, R), \ (i < n)$$

($HH_*(A, R)$ = Hochschild homology)

- Take an N-crossing link diagram D and fix k crossings.
- Resolve each of the remaining crossings as usual.
- Put the resulting 2^{N-k} diagrams on a Boolean lattice $\mathbb{B}$.
- Define a sheaf on $\mathbb{B}$ by taking $KH_*(-)$.



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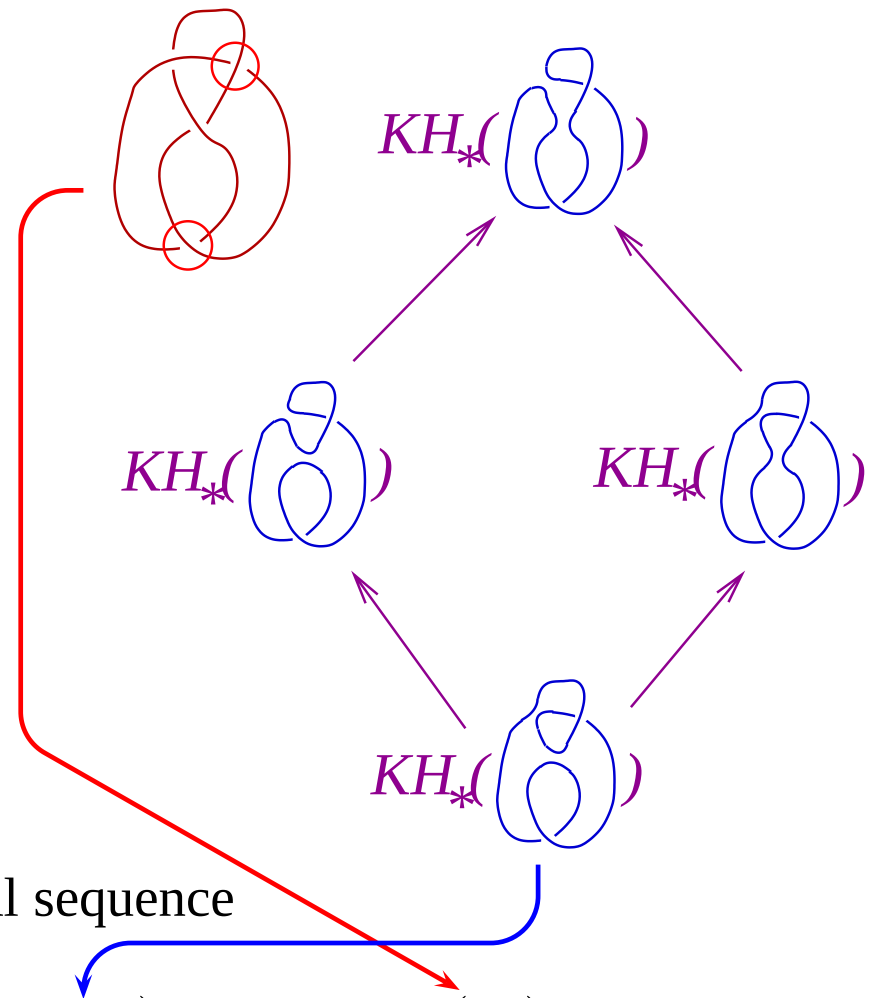

Theorem [E-Turner] there is a spectral sequence

$$E^2_{p,q} = KH_p(\mathbb{B}, KH_q) \Longrightarrow KH_{p+q}(\quad)$$