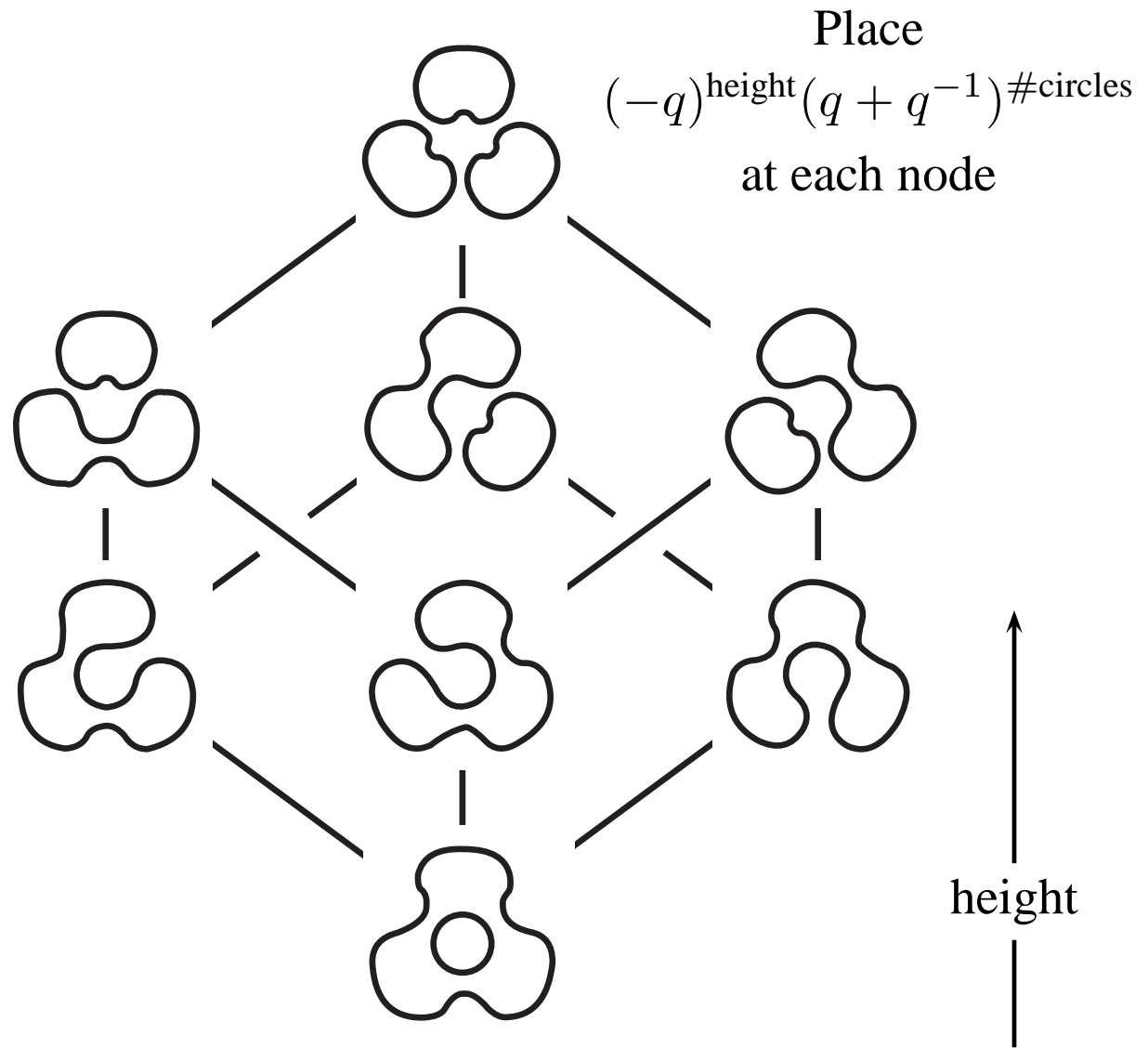
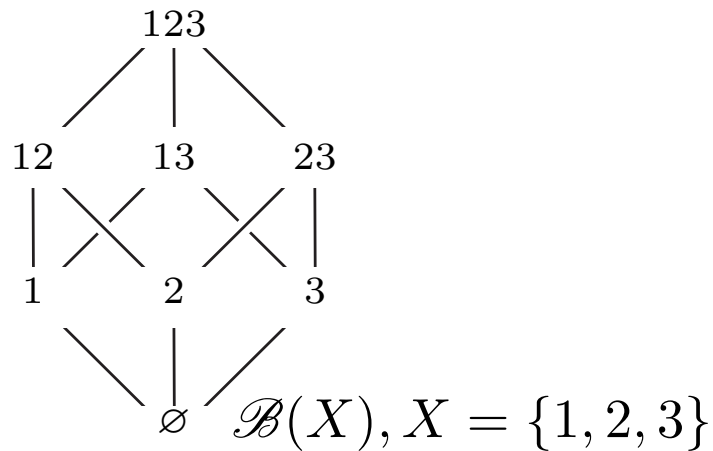
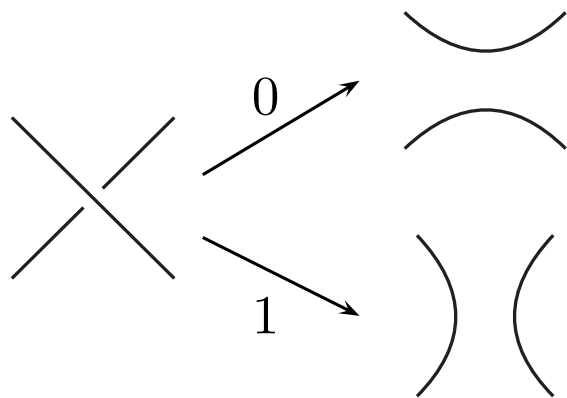
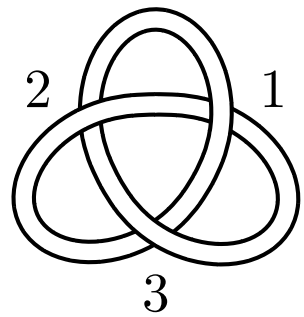


Coloured poset homology

Brent Everitt (York) –joint with **Paul Turner** (Fribourg)

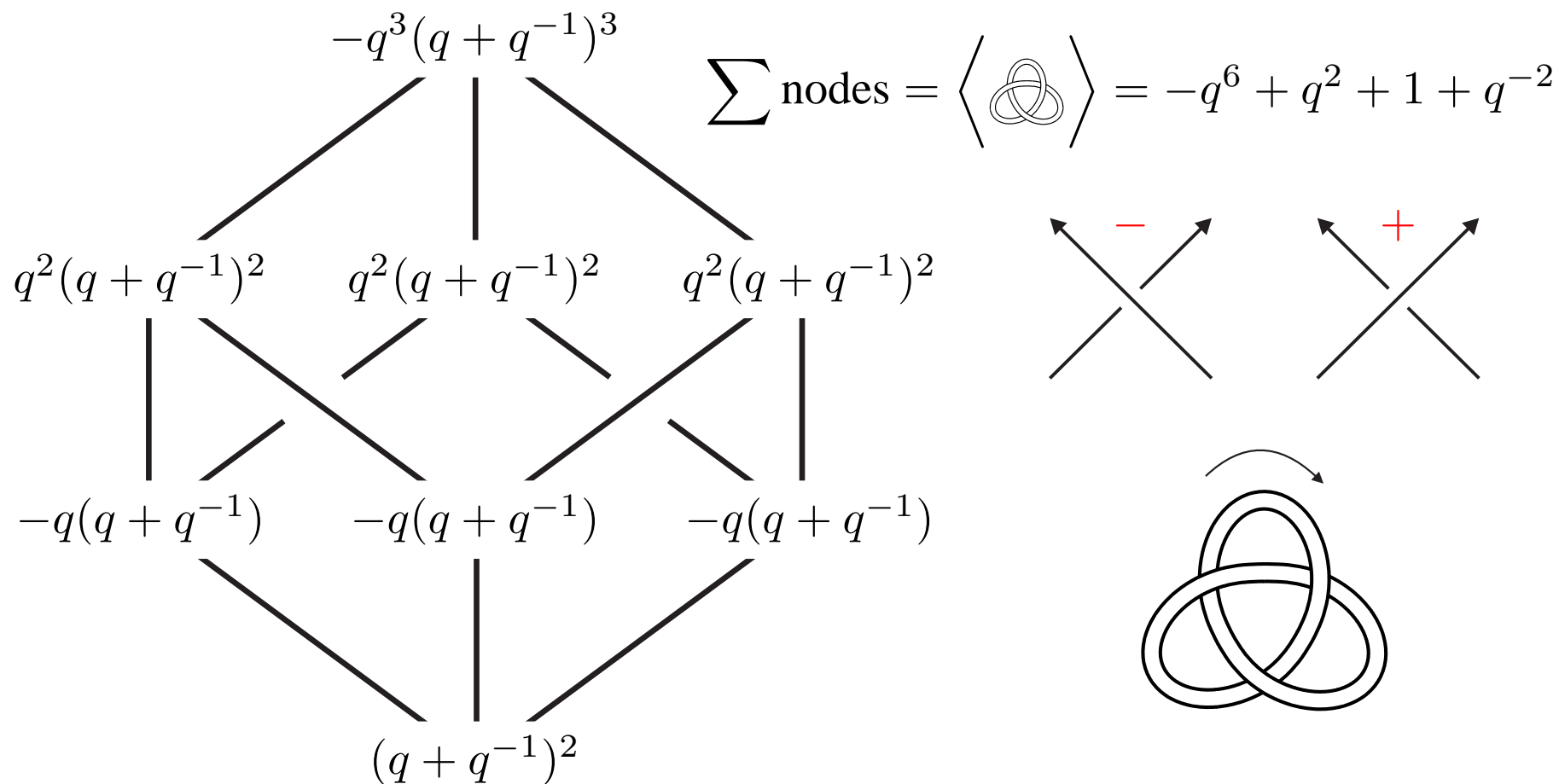
arXiv:0808.1686

arXiv:0711.0103 (in *J. Algebra*)



$$J\left(\text{trefoil}\right) = \frac{1}{(q + q^{-1})} \hat{J}\left(\text{trefoil}\right) \longleftarrow (-1)^{n_-} q^{n_+ - 2n_-} \left\langle \text{trefoil} \right\rangle$$

(Jones)
(unnormalized Jones)
(Kauffman bracket)



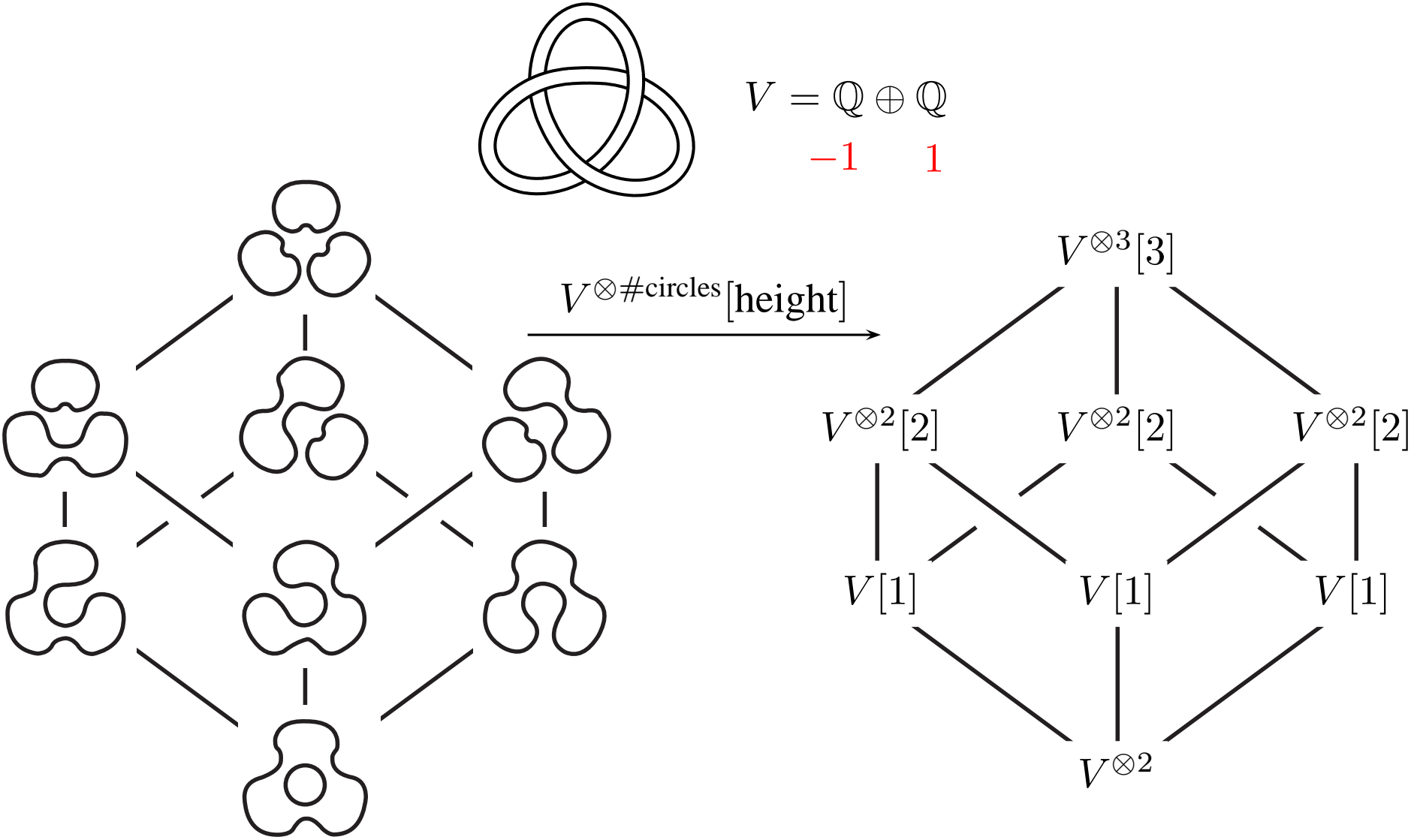
$$\mathbb{N} = \{0, 1, 2, \dots\} \begin{array}{c} \xrightarrow{\text{categorify}} \\ \xleftarrow{\text{de-categorify (dim)}} \end{array} \text{Spaces} \quad \begin{array}{l} \text{Obj: } k\text{-spaces } V \\ \text{Mor: } k\text{-linear maps} \end{array}$$

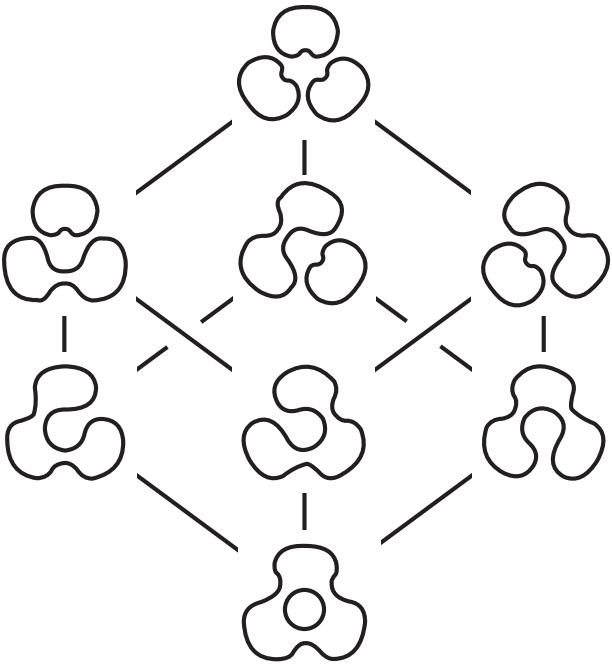
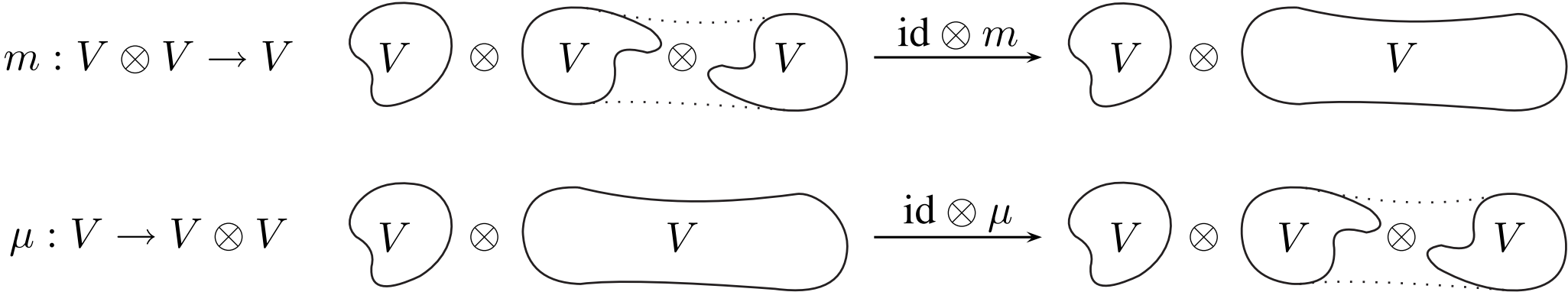
$$\mathbb{N}[q, q^{-1}] = \left\{ \sum_{\mathbb{Z}} n_i q^i \right\} \begin{array}{c} \xrightarrow{\text{categorify}} \\ \xleftarrow{\text{de-categorify (qdim)}} \end{array} \text{Graded Spaces}$$

$$\text{Obj: } V = \cdots \begin{array}{|c|c|c|c|} \hline & V_{-1} & V_0 & V_1 \\ \hline \end{array} \cdots$$

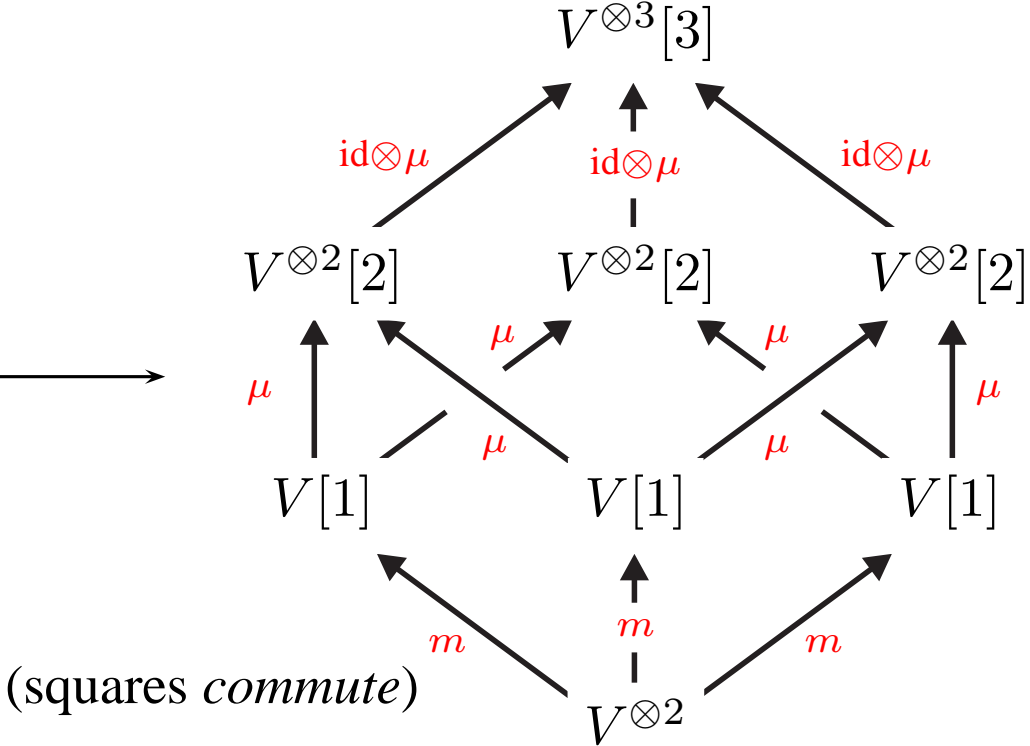
Mor: graded maps

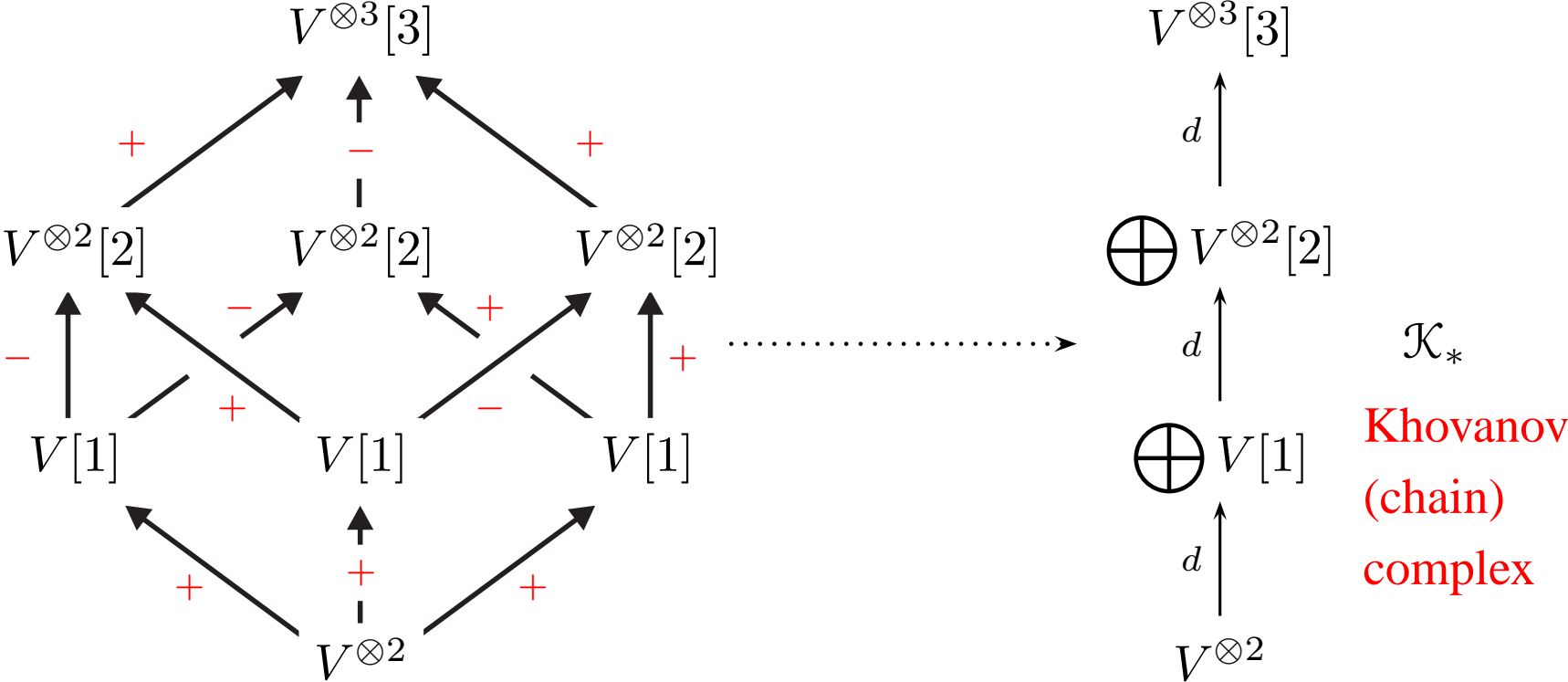
$$q\text{dim } V = \sum \dim V_i q^i$$





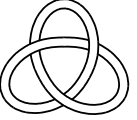
→





add \pm 's to edge maps so squares *anticommute*

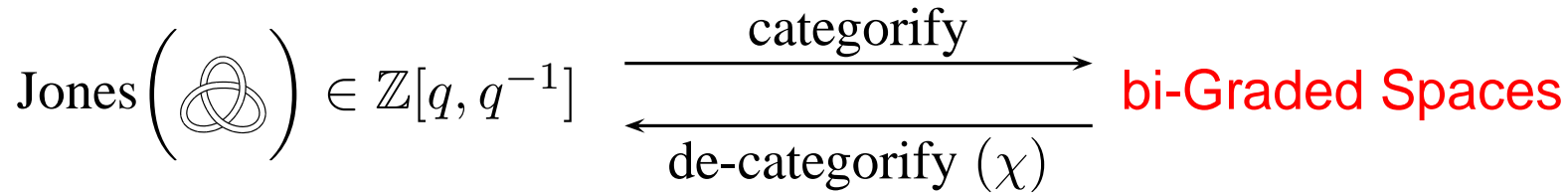
Khovanov homology $KH_* \left(\text{trefoil}, \mathbb{Q} \right) = H_*(\mathcal{K}_*)$

	6	4	2	0	-2	$q\dim$
KH_0	\mathbb{Q}					q^6
KH_1			\mathbb{Q}			q^2
KH_2						0
KH_3				\mathbb{Q}	\mathbb{Q}	$1 + q^{-2}$

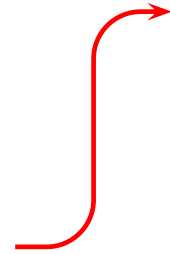
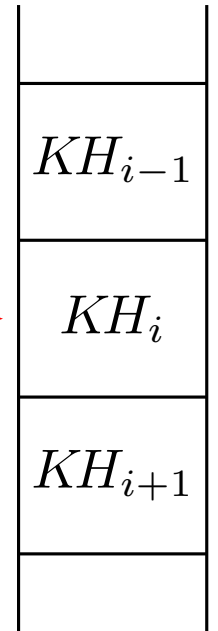
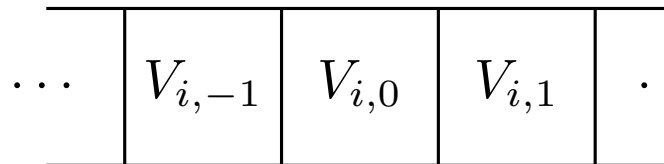
Euler characteristic $\chi(\mathcal{K}_*)$

$$= \sum (-1)^i q\dim KH_i \left(\text{trefoil}, \mathbb{Q} \right)$$

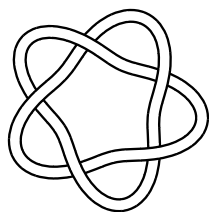
$$= q^6 - q^2 - 1 - q^{-2}$$



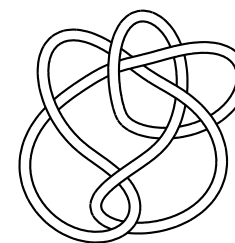
$$\chi(V) = \sum (-1)^i q\dim KH_i$$



Q						
		Q				
		Q				
				Q		
					Q	Q



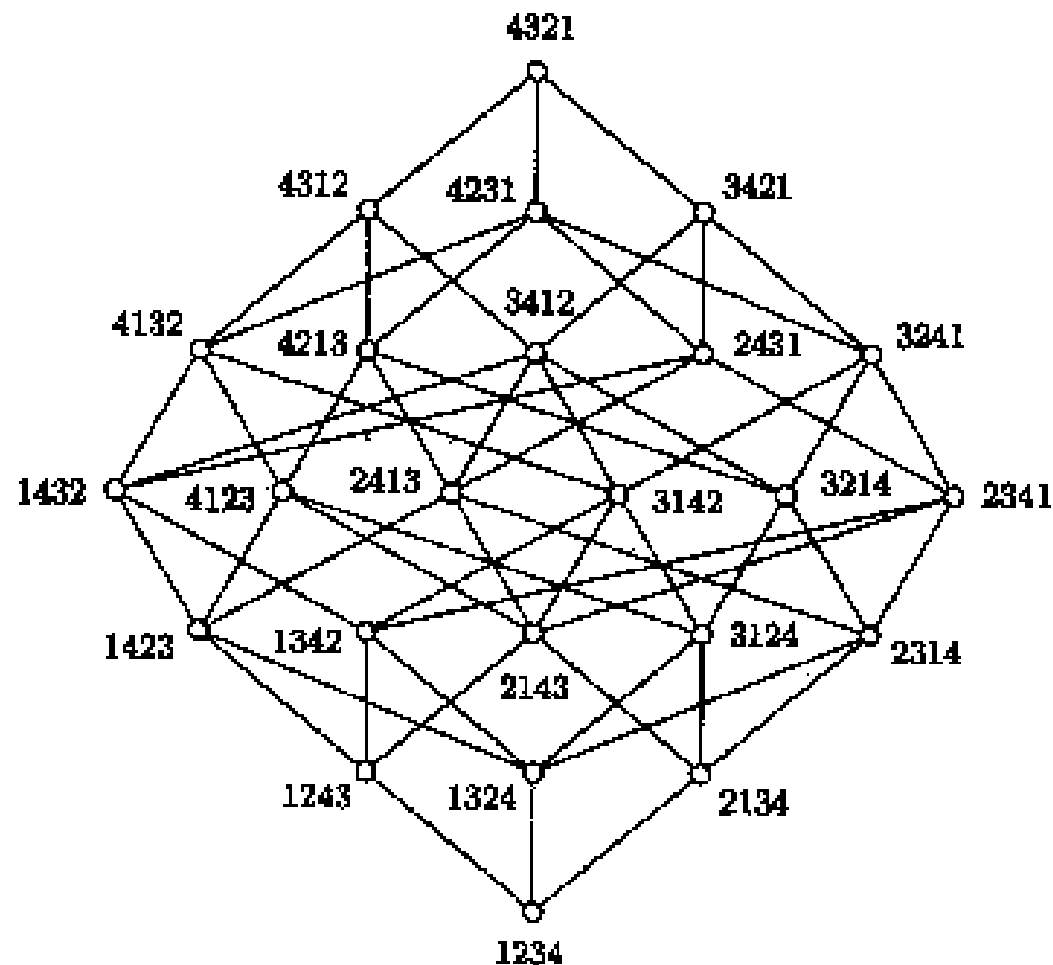
Q							
		Q					
		Q					
			Q	Q			
			Q		Q		
					Q+Q		
							Q
						Q	Q



$$\text{Jones} \left(\text{trefoil} \right) = \text{Jones} \left(\text{trefoil} \right)$$

Eg:

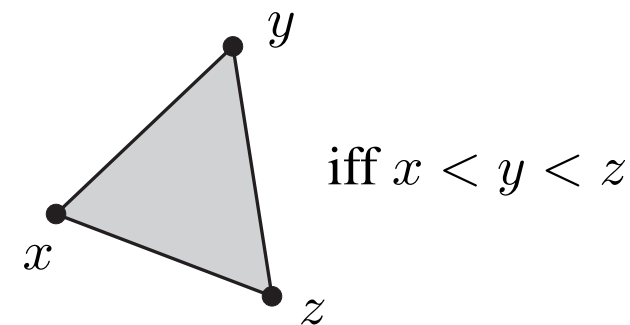
- Boolean lattice $\mathcal{B}(X)$ on a set X .
- Coxeter group with the Bruhat-Chevalley order, Eg: $\mathfrak{S}_4 \longrightarrow$
- cell poset of a CW-complex.
- intersection lattice of a hyperplane arrangement.



- poset $P \longrightarrow |P|$ order (simplicial) complex.

- **poset homology** = simplicial homology of $|P|$

ie: $H_*(P, R) := H_*(|P|, R) =$ homology of chain complex



$$C_n(P, R) = \bigoplus_{x_0 < \dots < x_n} R$$

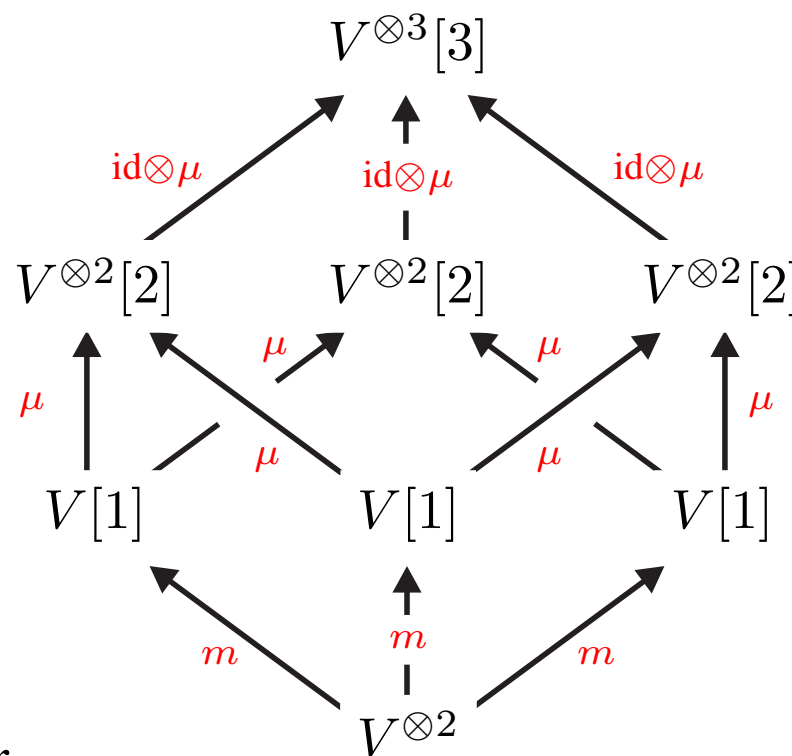
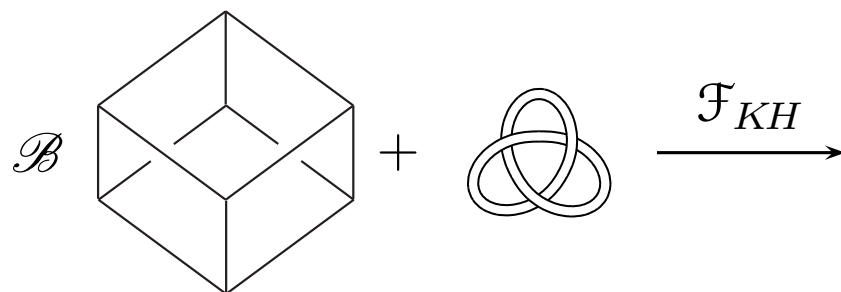
with differential $d : C_n(P, R) \rightarrow C_{n-1}(P, R)$

$$\lambda \cdot (x_0 < \dots < x_n) \xrightarrow{d} \sum_{j=0}^n (-1)^j \lambda \cdot (x_0 < \dots < \widehat{x}_j < \dots < x_n)$$

- Eg: [Folkman] P finite geometric lattice

$$\widetilde{H}_n(P \setminus \{0, 1\}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{|\mu(0,1)|} & n = \text{rk}P - 2, \\ 0 & \text{otherwise.} \end{cases}$$

- Eg: “Khovanov colouring”:



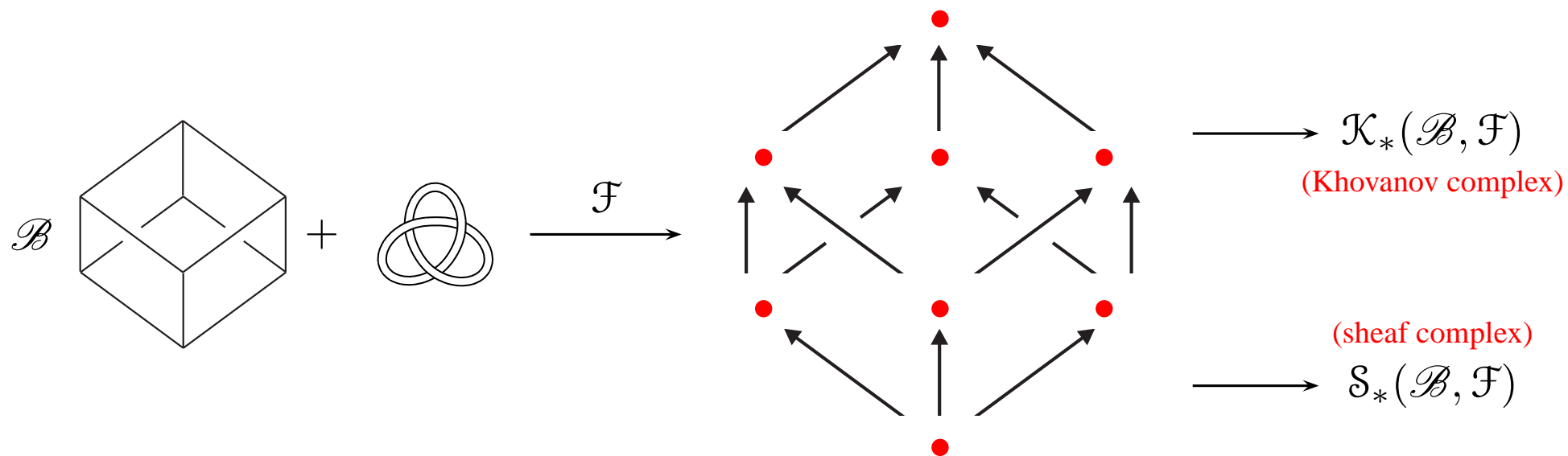
- in general: $P \xrightarrow{\mathcal{F}} R\text{-mod}$ (covariant) functor
(= pre-sheaf of modules over P)

- $P \xrightarrow{\mathcal{F}} R\text{-mod}$ coloured poset/sheaf
- **coloured poset** or **sheaf homology** $\mathcal{H}_*(P, \mathcal{F}) =$ homology of chain complex

$$\mathcal{S}_n(P, \mathcal{F}) = \bigoplus_{x_0 < \cdots < x_n} \mathcal{F}(x_0)$$

with differential $d : \mathcal{S}_n(P, \mathcal{F}) \rightarrow \mathcal{S}_{n-1}(P, \mathcal{F})$

$$\begin{aligned} \lambda \cdot (x_0 < \cdots < x_n) &\xrightarrow{d} \mathcal{F}(x_0 < x_1)(\lambda) \cdot (\hat{x}_0 < x_1 < \cdots < x_n) \\ &+ \sum_{j=1}^n (-1)^j \lambda \cdot (x_0 < \cdots < \hat{x}_j < \cdots < x_n) \end{aligned}$$



- **Theorem:** \mathcal{B} Boolean and $\mathcal{B} \xrightarrow{\mathcal{F}} R\text{-mod}$ a sheaf, then

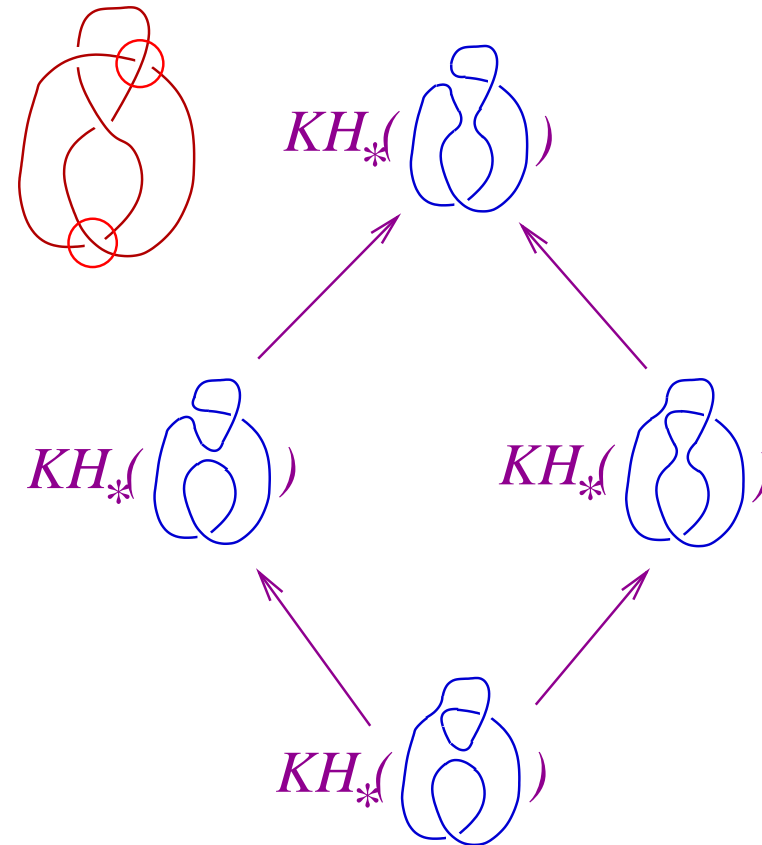
$$KH_*(\mathcal{B}, \mathcal{F}) \cong \tilde{\mathcal{H}}_{*-1}(\mathcal{B} \setminus 1, \mathcal{F})$$

- **Theorem:** P cellular poset and $P \xrightarrow{\mathcal{F}} R\text{-mod}$ a sheaf, then

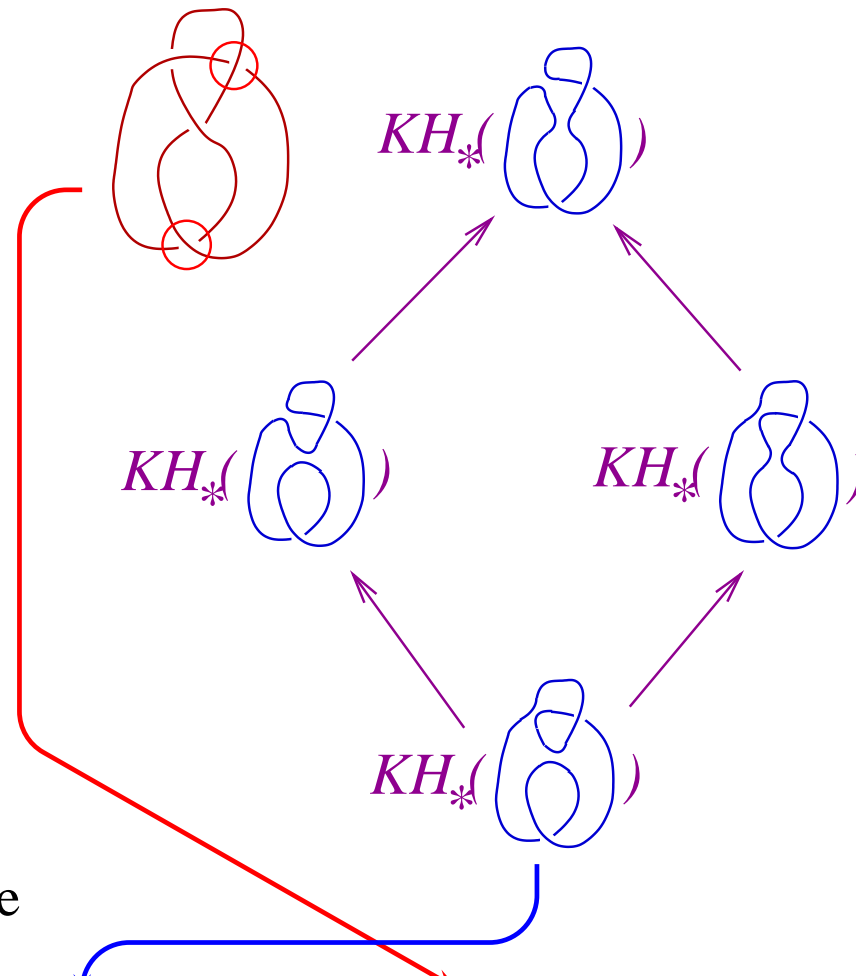
$$H_*^{\text{cell}}(P, \mathcal{F}) \cong \mathcal{H}_*(P, \mathcal{F})$$

(Eg: Cohen-Macaulay posets, cell posets of regular CW-complexes, . . .)

- Take an N -crossing link diagram D and fix k crossings.
- Resolve each of the remaining crossings as usual.
- Put the resulting 2^{N-k} diagrams on a Boolean lattice \mathcal{B} .
- Define a sheaf on \mathcal{B} by taking $KH_*(-)$.



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- Define a sheaf on \mathcal{B} by taking $KH_*(-)$.



Theorem: There is a spectral sequence

$$E_{p,q}^2 = KH_p(\mathcal{B}, KH_q) \implies KH_{p+q}(\quad)$$