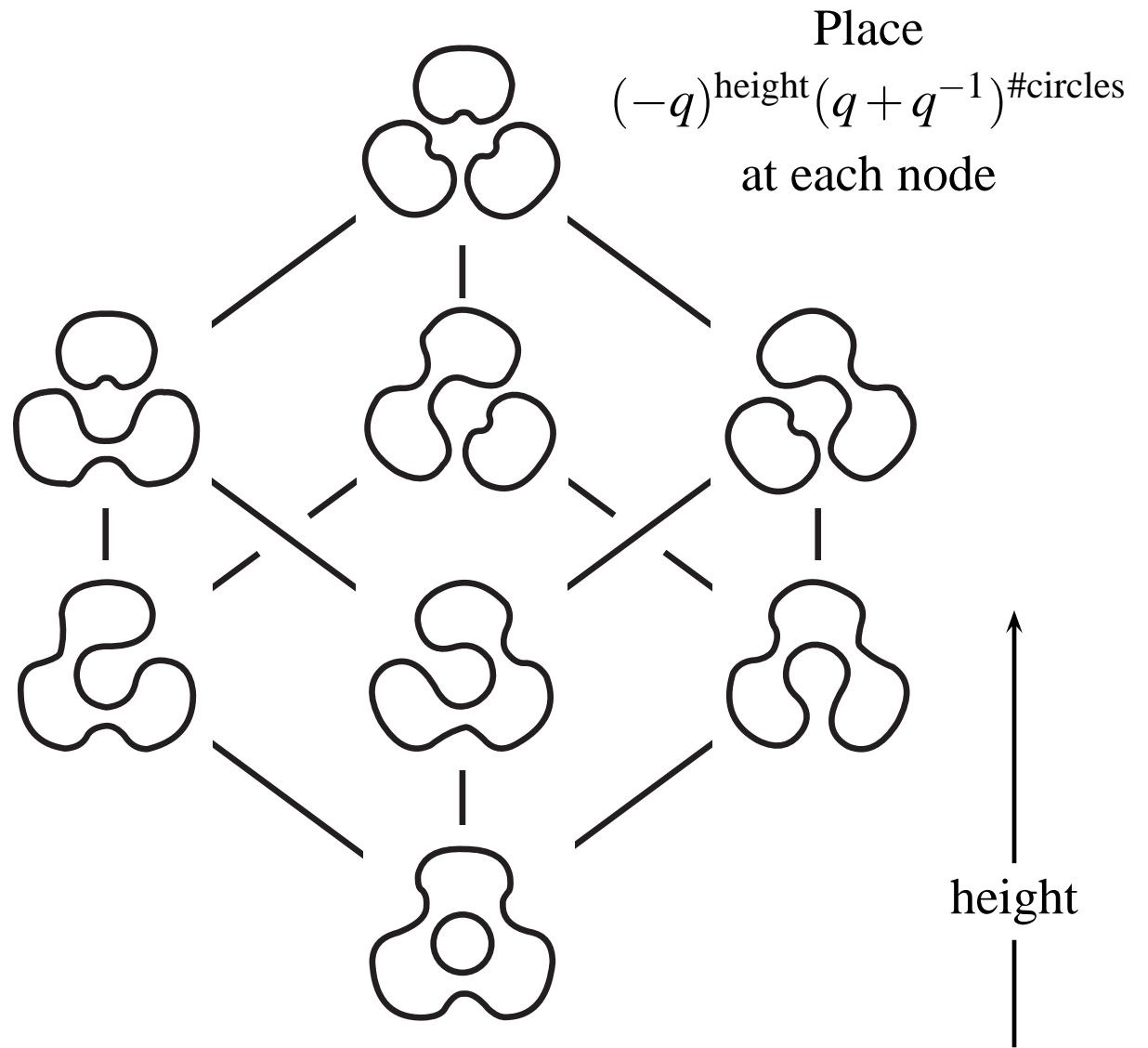
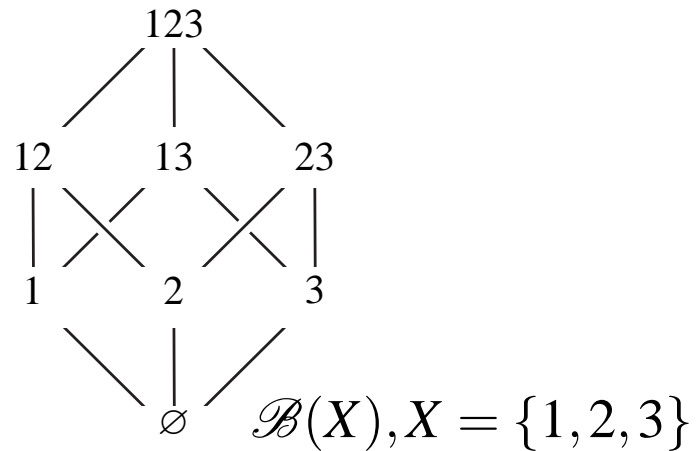
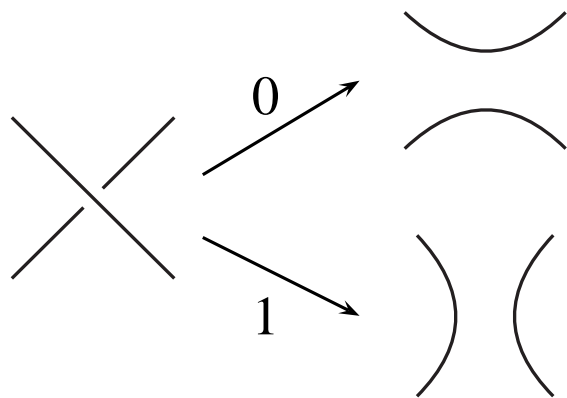
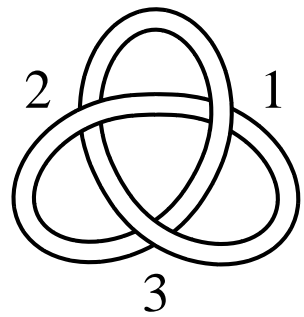


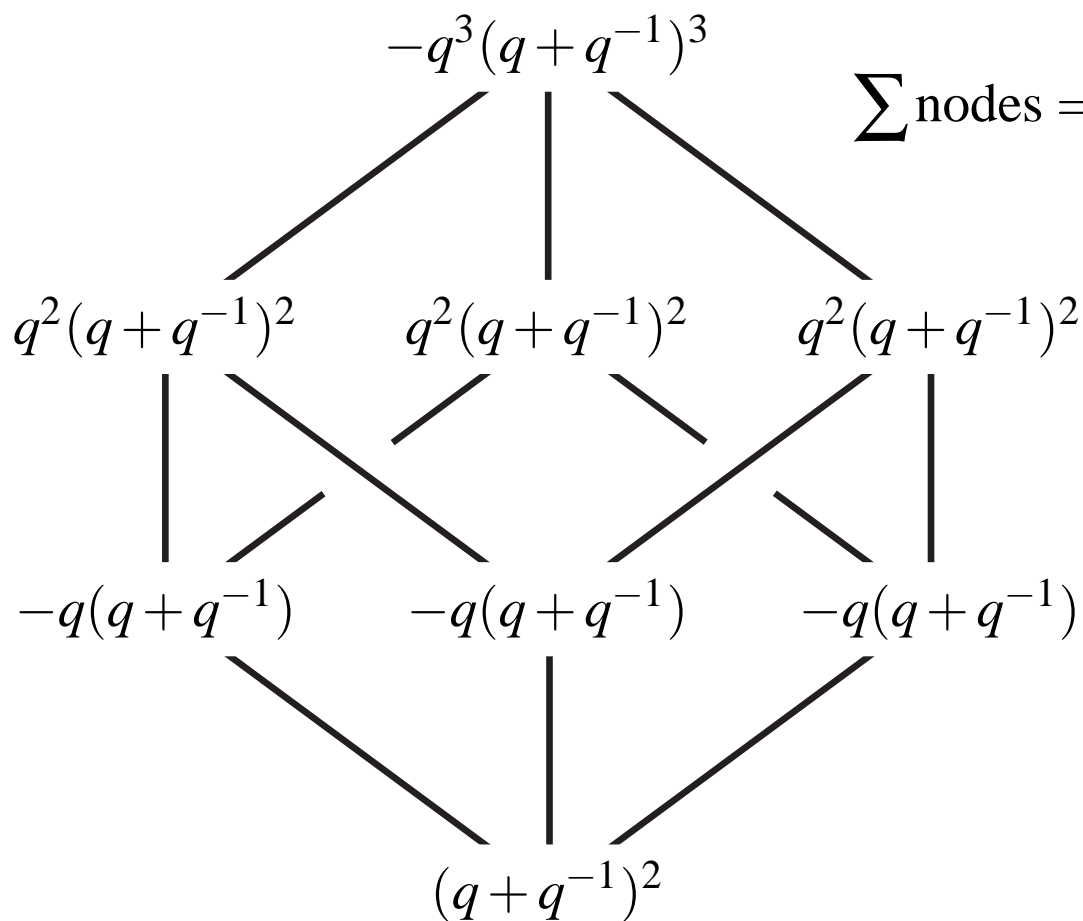
Knots, posets and sheaves

Brent Everitt (York) –joint with Paul Turner (Fribourg)

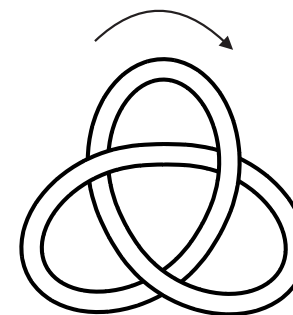
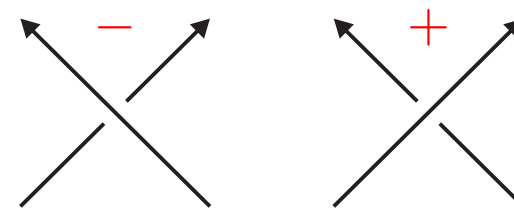


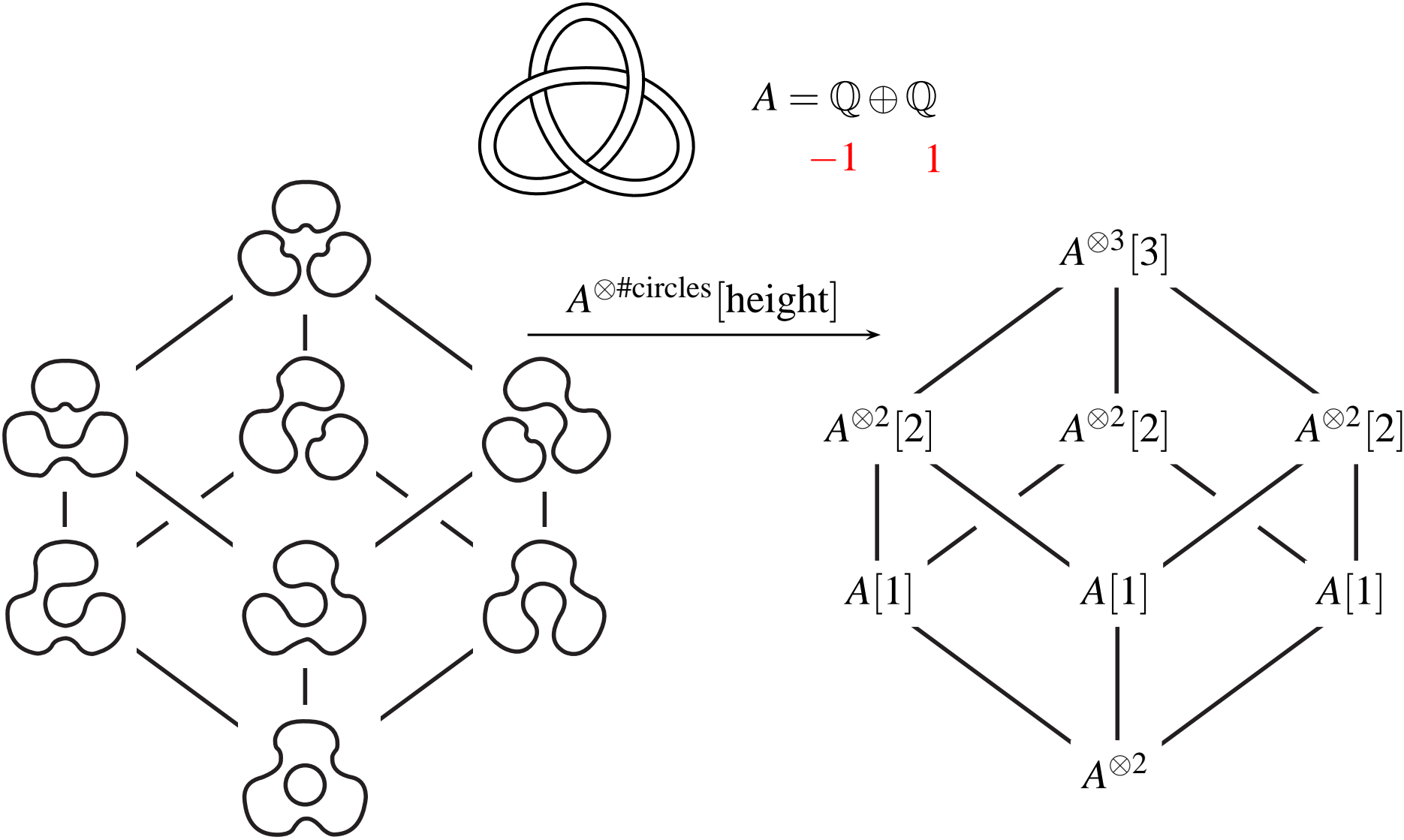
$$J\left(\text{trefoil}\right) = \frac{1}{(q + q^{-1})} \hat{J}\left(\text{trefoil}\right) \longleftarrow (-1)^{n_-} q^{n_+ - 2n_-} \left\langle \text{trefoil} \right\rangle$$

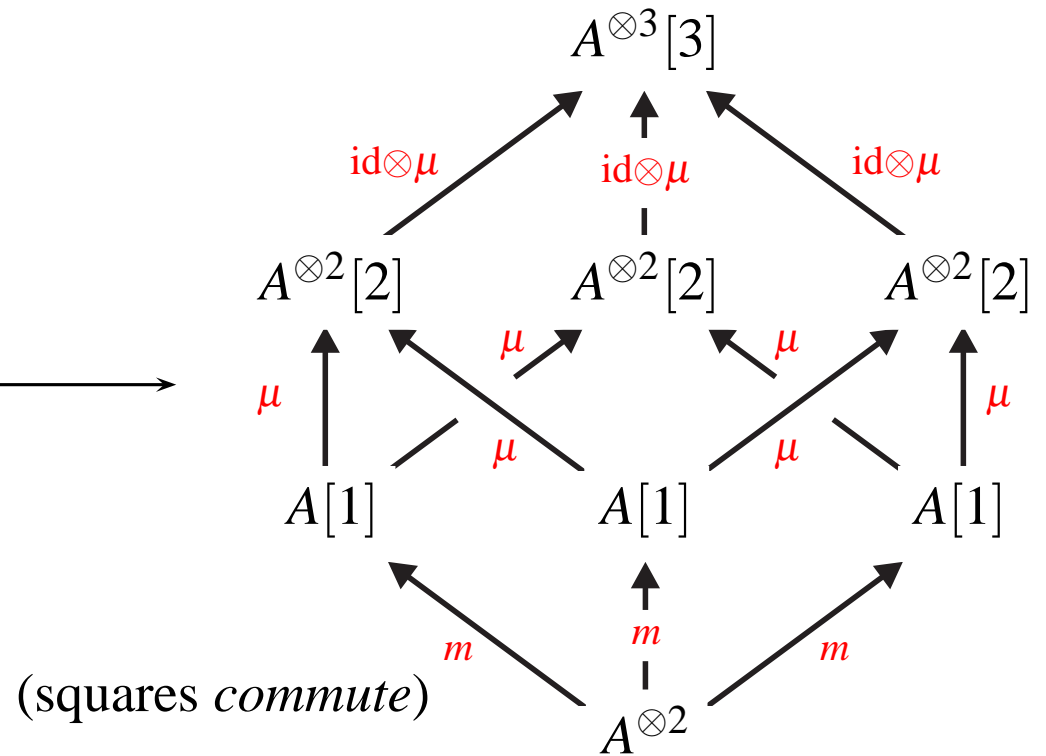
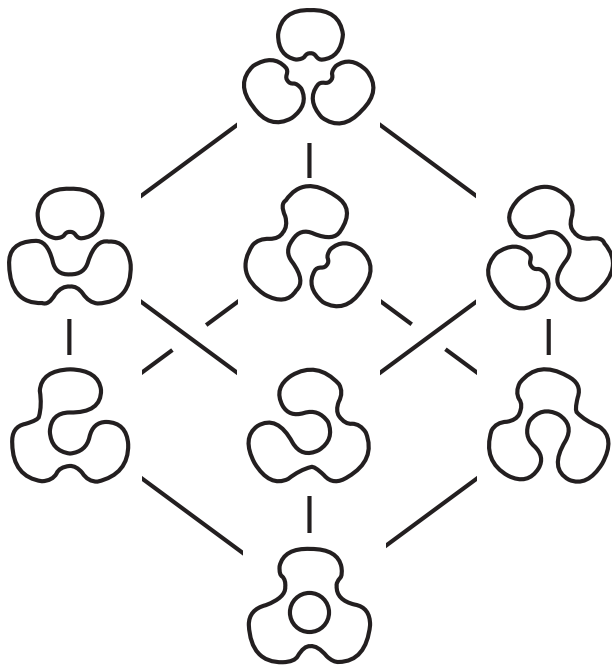
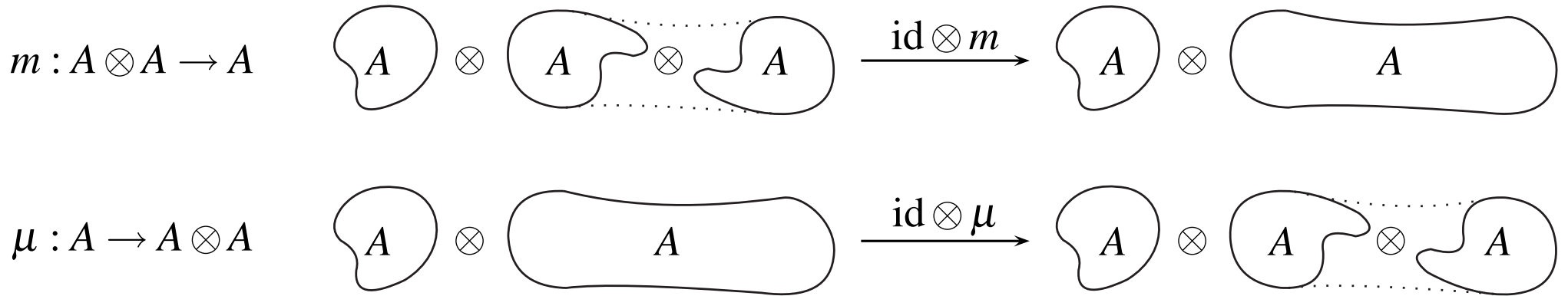
(Jones)
(unnormalized Jones)
(Kauffman bracket)

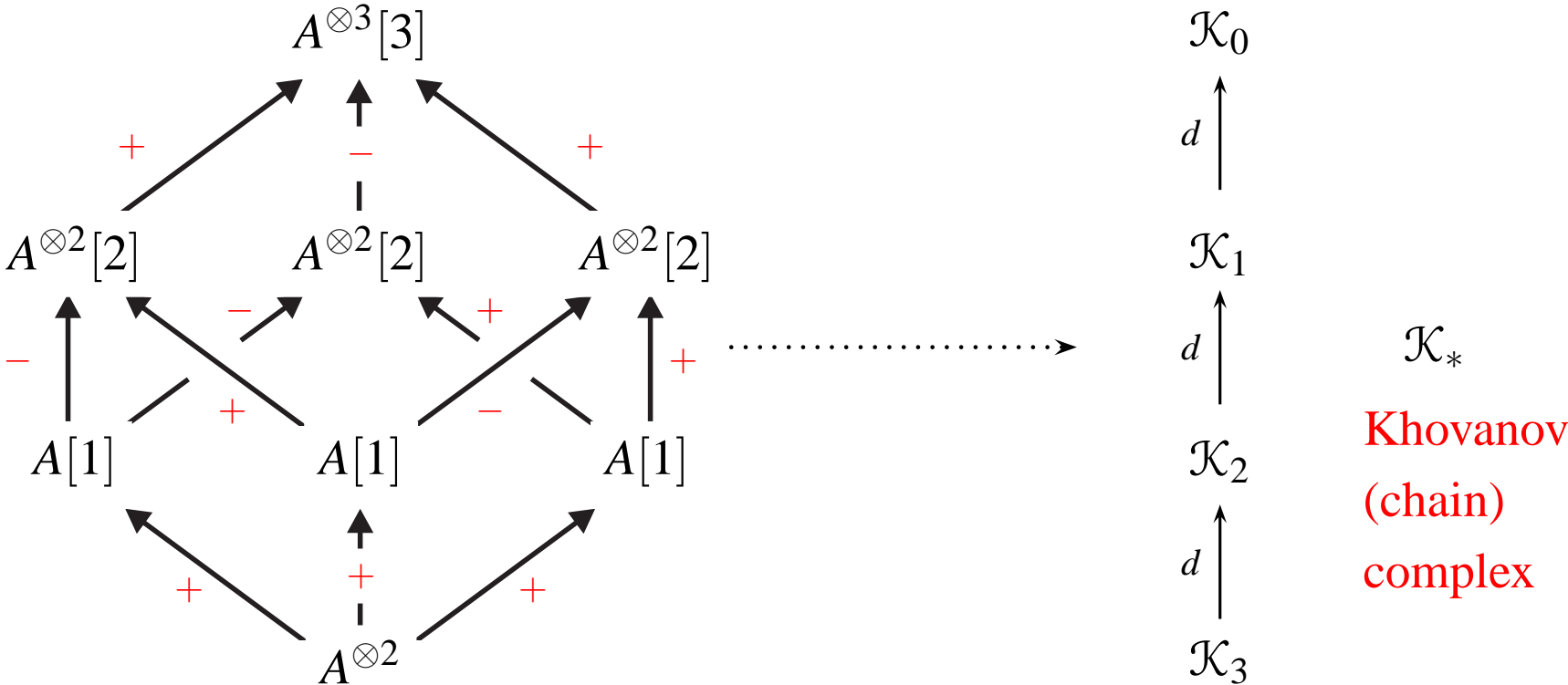


$$\sum \text{nodes} = \left\langle \text{trefoil} \right\rangle = -q^6 + q^2 + 1 + q^{-2}$$



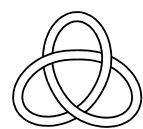






add \pm 's to edge maps so squares *anticommute*

Khovanov homology $KH_* \left(\text{trefoil}, \mathbb{Q} \right) = H_*(\mathcal{K}_*)$

	6	4	2	0	-2	$q\dim$
KH_0	\mathbb{Q}					q^6
KH_1			\mathbb{Q}			q^2
KH_2						0
KH_3				\mathbb{Q}	\mathbb{Q}	$1 + q^{-2}$

Euler characteristic $\chi(\mathcal{K}_*)$

$$= \sum (-1)^i q^{\dim KH_i} \left(\text{trefoil}, \mathbb{Q} \right)$$

$$= q^6 - q^2 - 1 - q^{-2}$$

minor miracle: KH_* an invariant (after a bit of nudging)

Q						
		Q				
		Q				
				Q		
					Q	Q

Q							
		Q					
		Q					
			Q	Q			
			Q		Q		
					Q+Q		
						Q	
						Q	Q

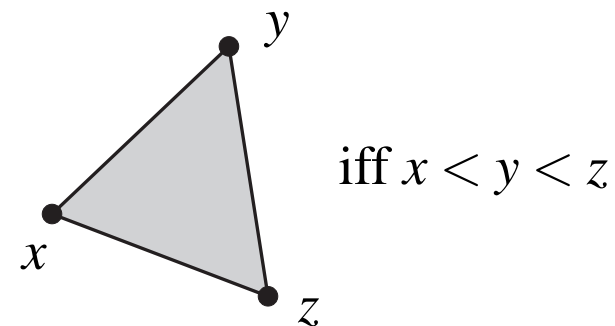
$$KH_* \left(\text{Knot} \right)$$

- $\text{Jones} \left(\text{Knot} \right) = \text{Jones} \left(\text{Knot} \right)$

- **FUNCTORIAL!!**

$$KH_* \left(\text{Knot} \right)$$

- poset $P \longrightarrow |P|$ order (simplicial) complex.



- **poset homology** = simplicial homology of $|P|$
ie: $H_*(P, R) := H_*(|P|, R) =$ homology of chain complex

$$C_n(P, R) = \bigoplus_{x_0 < \dots < x_n} R$$

with differential $d : C_n(P, R) \rightarrow C_{n-1}(P, R)$

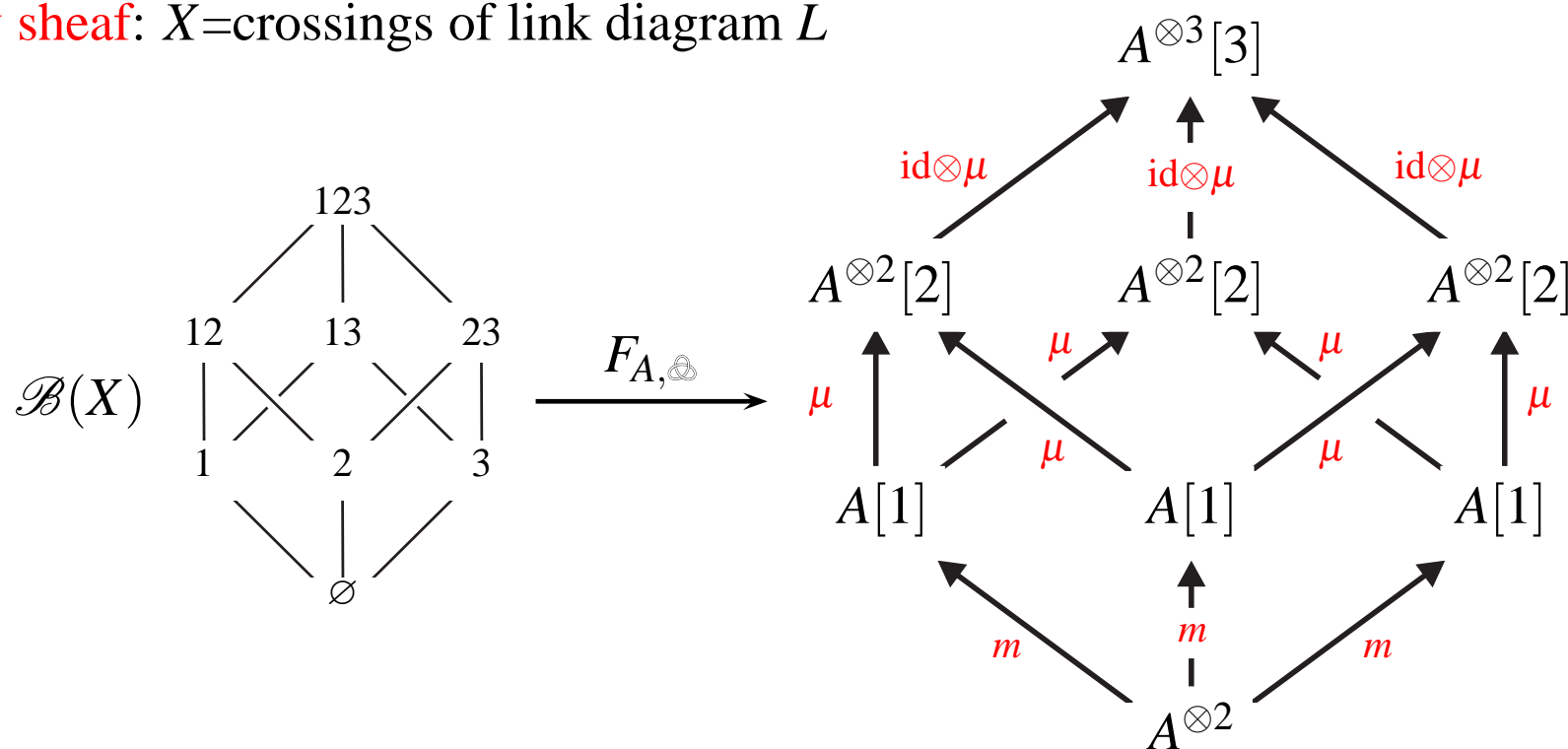
$$\lambda \cdot (x_0 < \dots < x_n) \xrightarrow{d} \sum_{j=0}^n (-1)^j \lambda \cdot (x_0 < \dots < \hat{x}_j < \dots < x_n)$$

- Eg: [Folkman-Björner] P finite geometric lattice

$$\tilde{H}_n(P \setminus \{0, 1\}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{|\mu(0,1)|} & n = \text{rk}P - 2, \\ 0 & \text{otherwise.} \end{cases}$$

- $P \xrightarrow{F} R\text{-mod}$ (covariant) functor (= **pre-cosheaf of modules over P**)

- Eg: **Khovanov sheaf**: X =crossings of link diagram L



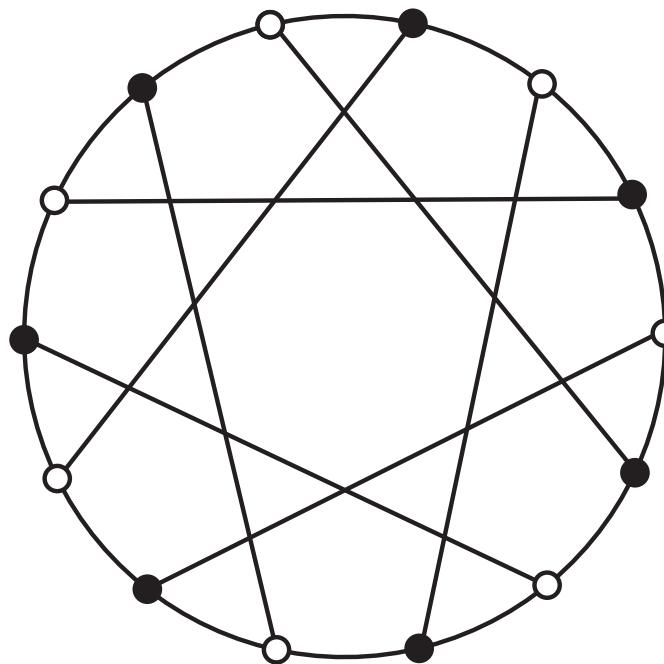
- Eg: **sheaf on a building**: poset of flags in V (a finite dimensional k -space)

$$(V_1 \subset \cdots \subset V_n) \leq (U_1 \subset \cdots \subset U_m) \Leftrightarrow \text{each } V_i = \text{some } U_j$$

sheaf:

$$F(V_1 \subset \cdots \subset V_n) = V_n \text{ and } F[(V_1 \subset \cdots \subset V_n) \leq (U_1 \subset \cdots \subset U_m)] = V_n \hookrightarrow U_m$$

V 3-dimensional over $k = \mathbb{F}_2$

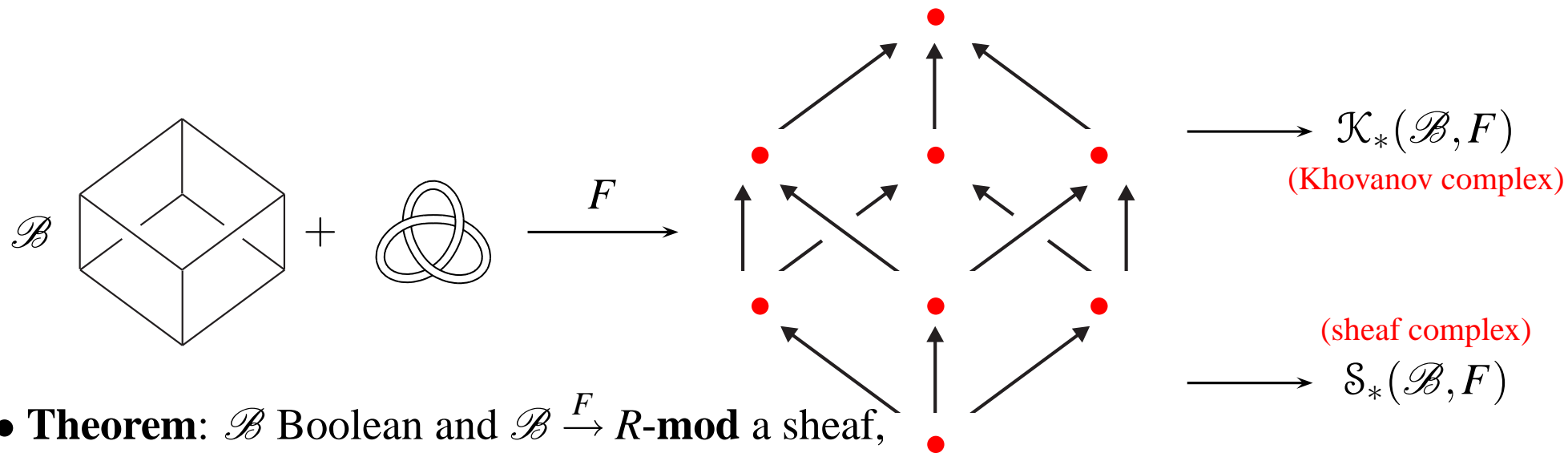


- $P \xrightarrow{F} R\text{-mod}$ sheaf
- **sheaf homology** $\mathcal{H}_*(P, F) =$ homology of chain complex

$$\mathcal{S}_n(P, F) = \bigoplus_{x_0 < \dots < x_n} F(x_0)$$

with differential $d : \mathcal{S}_n(P, F) \rightarrow \mathcal{S}_{n-1}(P, F)$

$$\begin{aligned} \lambda \cdot (x_0 < \dots < x_n) \xrightarrow{d} & F(x_0 < x_1)(\lambda) \cdot (\widehat{x}_0 < x_1 < \dots < x_n) \\ & + \sum_{j=1}^n (-1)^j \lambda \cdot (x_0 < \dots < \widehat{x}_j < \dots < x_n) \end{aligned}$$



• **Theorem:** \mathcal{B} Boolean and $\mathcal{B} \xrightarrow{F} R\text{-mod}$ a sheaf,

$$KH_*(\mathcal{B}, F) \cong \tilde{\mathcal{H}}_{*-1}(\mathcal{B} \setminus 1, F)$$

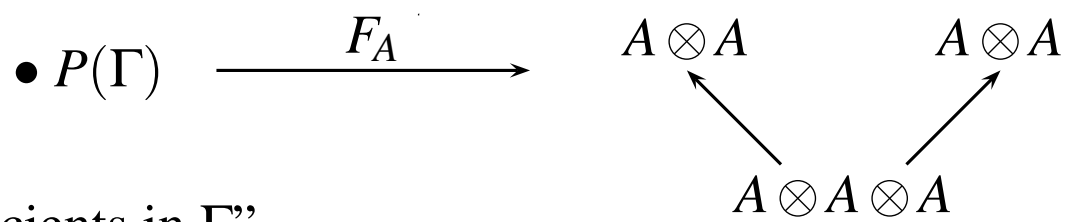
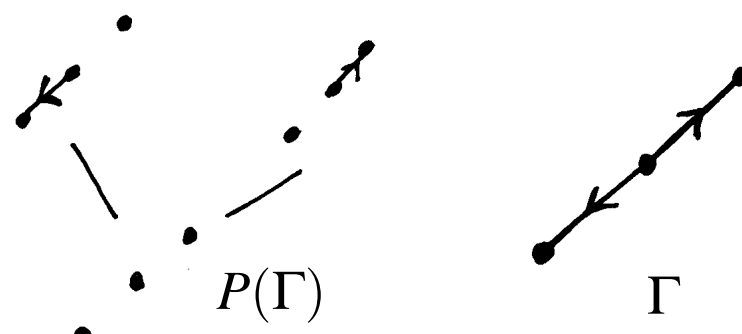
• [generally: one can define a “cellular” homology $H_*^{\text{cell}}(P, F)$:

Theorem: P “cellular” poset and $P \xrightarrow{F} R\text{-mod}$ a sheaf, then

$$H_*^{\text{cell}}(P, F) \cong \mathcal{H}_*(P, F)$$

Eg: $P =$ geometric lattices, cell posets regular CW-complexes, Cohen-Macaulay posets, ...]

- $A =$ associative R -algebra.
- $P(\Gamma) =$ quiver poset of directed graph Γ .



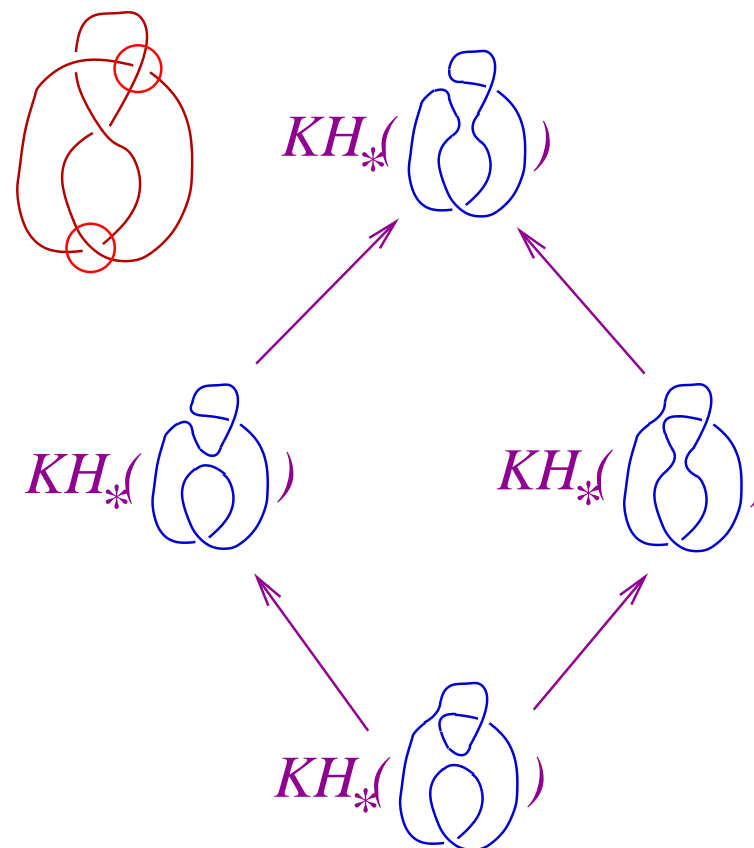
- “homology of A with coefficients in Γ ”
 $:= \mathcal{H}_*(P(\Gamma), F_A)$

• **Corollary** [Turner-Wagner]:

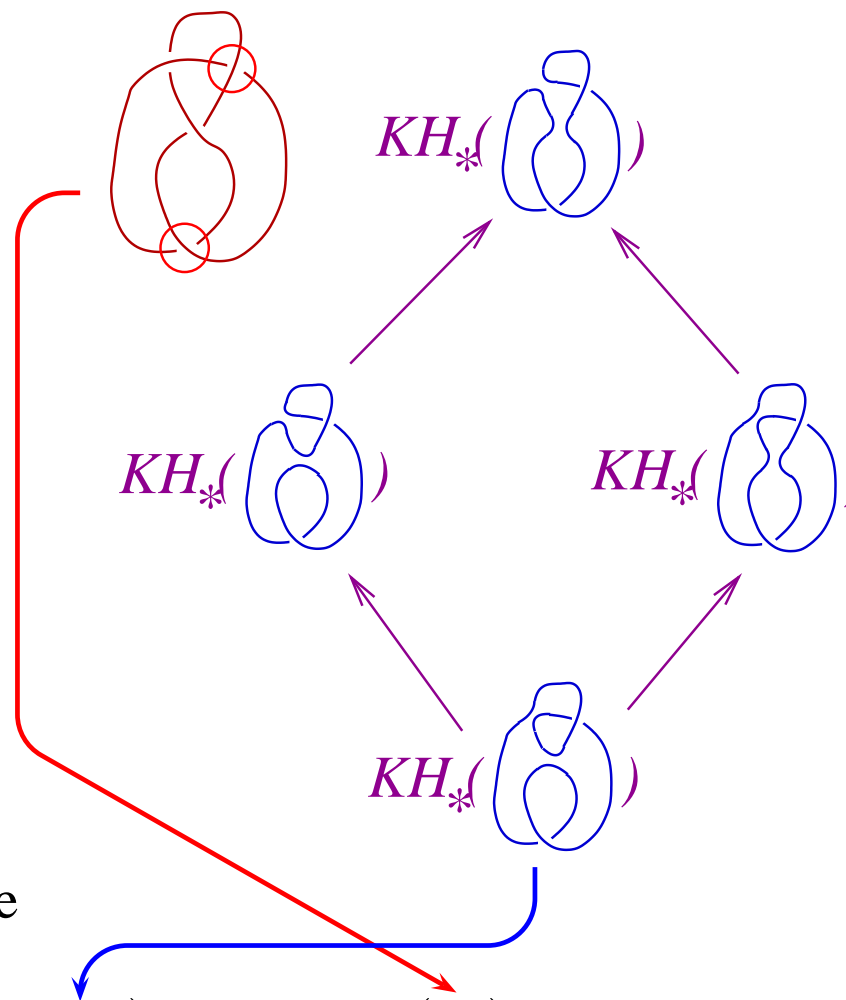
$$\mathcal{H}_i(P(n\text{-gon}), F_A) \cong HH_i(A), \quad (0 \leq i \leq n - 1)$$

($HH_*(A) =$ Hochschild homology)

- Take an N -crossing link diagram D and fix k crossings.
- Resolve each of the remaining crossings as usual.
- Put the resulting 2^{N-k} diagrams on a Boolean lattice \mathcal{B} .
- Define a sheaf on \mathcal{B} by taking $KH_*(-)$.



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Theorem: There is a spectral sequence

$$E_{p,q}^2 = KH_p(\mathcal{B}, KH_q) \implies KH_{p+q}(\quad)$$