FINITE ABUNDANT SEMIGROUPS IN WHICH THE IDEMPOTENTS FORM A SUBSEMIGROUP*

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Abstract. We consider certain abundant semigroups in which the idempotents form a subsemigroup, and which we call bountiful semigroups. We find a simple criterion for a finite bountiful semigroup to be a member of the join of the pseudovarieties of finite groups and finite aperiodic semigroups.

Introduction

In the 1970s Schützenberger posed the problem of finding a characterisation of the semigroups in the pseudovariety $\mathbf{A} \vee \mathbf{G}$ where $\mathbf{A}$ is the pseudovariety of all finite aperiodic semigroups, and $\mathbf{G}$ is the pseudovariety of all finite groups. McAlister answered this question for orthodox semigroups [13] by showing that a finite orthodox semigroup S is in $\mathbf{A} \vee \mathbf{G}$ if and only if $\mathscr{H}$ is a congruence on S. He subsequently generalised this result to obtain a characterisation of the regular semigroups in $\mathbf{A} \vee \mathbf{G}$ [14]. Recently, Steinberg [21] has shown that McAlister's results can be deduced from work of Rhodes and Tilson [18] and that this approach allows McAlister's theorems to be extended to a wider class of

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semigroups. Our purpose in this paper is to extend the result for orthodox semigroups in a different direction.

Recall that the relation $\mathscr{R}^*$ on a semigroup S is defined by $a\mathscr{R}^*b$ if and only if a and b are $\mathscr{R}$-related in an extension of S. The dual of $\mathscr{R}^*$ is $\mathscr{L}^*$, and $\mathscr{H}^* = \mathscr{R}^* \cap \mathscr{L}^*$. A semigroup is *abundant* if every $\mathscr{R}^*$-class and every $\mathscr{L}^*$-class contains an idempotent. We say that an abundant semigroup is *bountiful* if it is idempotent-connected (see Section 1 for the definition) and its idempotents form a subsemigroup. Let μ be the greatest congruence contained in $\mathscr{H}^*$. We now give the main result of the paper.

Theorem 1. *Let S be a finite bountiful semigroup. Then $S \in \mathbf{A} \vee \mathbf{G}$ if and only if $\mathscr{H} \subseteq \mu$.*

In the following section, we recall some basic results related to abundant semigroups. Following [1], we say that an idempotent-connected abundant semigroup in which the idempotents generate a regular subsemigroup is *concordant*. For a concordant semigroup S with set of idempotents $E(S)$, El Qallali and Fountain [2] gave a representation of S as a full subsemigroup of the Hall-Nambooripad semigroup $T_{\langle E(S)\rangle}$ with kernel μ. We use this to prove the main result of Section 2: if S is a concordant semigroup in $\mathbf{A} \vee \mathbf{G}$, then $\mathscr{H} \subseteq \mu$ on S. Our proof also requires the corresponding result for regular semigroups, due to McAlister [13]. In the final section, we complete the proof of the main theorem by showing that if S is a bountiful semigroup satisfying $\mathscr{H} \subseteq \mu$, then $S \in \mathbf{A} \vee \mathbf{G}$. Our proof is modelled on that for the orthodox case; we require a preliminary result to the effect that a finite bountiful semigroup has a finite bountiful E-unitary cover of a special type. The existence of bountiful E-unitary covers for bountiful semigroups has recently been established by Simmons [20], but his proof does not give finite covers for finite semigroups. Our proof mimics that of McAlister [13] for the orthodox case, and uses the fact that a finite ample semigroup has a finite proper cover, a result due to the authors [6]. That ample semigroups have proper covers was first shown by Lawson [11], and is also a consequence of Simmons' work on bountiful semigroups. However, their proofs do not give finite covers for finite semigroups.

1. Preliminaries

For basic semigroup notation and terminology, we follow [9]. In particular, $E(S)$ denotes the set of idempotents of a semigroup S. We recall the following alternative characterisation of $\mathscr{R}^*$ from [12] and [16] which we shall use without further mention.

Lemma 1.1. *The following are equivalent for elements a, b of a semigroup S:*

(1) *$a\mathscr{R}^*b$,*
(2) *for all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$.*

This condition is simplified when one of the elements involved is an idempotent.

Corollary 1.2. *Let a be an element of a semigroup S, and $e \in E(S)$. Then the following are equivalent:*

(1) *$a\mathscr{R}^*e$,*
(2) *$ea = a$ and for all $x, y \in S^1$, $xa = ya$ implies $xe = ye$.*

We remark that $\mathscr{R}^*$ is a left congruence, and $\mathscr{L}^*$ is a right congruence. Also on any semigroup S we have $\mathscr{R} \subseteq \mathscr{R}^*$. It is well known and easy to see that if $a, b \in S$ are regular, then $a\mathscr{R}^*b$ if and only if $a\mathscr{R}b$. In particular, if S is regular, then $\mathscr{R}^* = \mathscr{R}$. If there is any danger of ambiguity, we use $\mathscr{R}^*(S)$, etc. to denote the relation $\mathscr{R}^*$ on S. The $\mathscr{R}^*$-class of $a \in S$ will be denoted by R_a^* or $R_a^*(S)$, and corresponding notation is used for $\mathscr{L}^*$- and $\mathscr{H}^*$-classes.

A semigroup in which each $\mathscr{R}^*$-class and each $\mathscr{L}^*$-class contains an idempotent is said to be *abundant*. From [2] we have the following lemma.

Lemma 1.3. *Let U be an abundant subsemigroup of an abundant semigroup S such that the idempotents of U form an order-ideal of those of S. Then*

$$\mathscr{R}^*(U) = \mathscr{R}^*(S) \cap (U \times U).$$

The next corollary is an easy consequence of the lemma and its dual.

Corollary 1.4. *Let S be an abundant semigroup. Then every full subsemigroup of S is abundant, and if $e \in E(S)$, then eSe is abundant.*

We say that a homomorphism $\varphi : S \to T$ of abundant semigroups is *good* if, for all elements a, b of S, we have $a\mathscr{R}^*(S)b$ implies $a\varphi\mathscr{R}^*(T)b\varphi$, and $a\mathscr{L}^*(S)b$ implies $a\varphi\mathscr{L}^*(T)b\varphi$. A congruence ρ on an abundant semigroup S is *good* if the natural homomorphism $S \to S/\rho$ is good. We remark that any homomorphism with regular domain is good.

Let S be an abundant semigroup and B be the subsemigroup generated by $E(S)$. We say that S is *idempotent-connected* (IC) when for each element a of S and some idempotents e, f in R_a^* and L_a^* respectively, there is a bijection $\alpha : \langle E(eBe)\rangle \to \langle E(fBf)\rangle$ satisfying $xa = a(x\alpha)$ for all $x \in \langle E(eBe)\rangle$. We remark that in [2] it is shown that the word "some" can be replaced by "all", and that the bijection α is an isomorphism. It is also worth mentioning that any regular semigroup S is IC since, for any $a \in S$ and idempotents e, f in R_a and L_a, there is an inverse a' with $aa' = e$, $a'a = f$ and we have an isomorphism $\alpha : \langle E(aa'Baa')\rangle \to \langle E(a'aBa'a)\rangle$ with the required property given by $x\alpha = a'xa$.

More details about IC abundant semigroups, and alternative formulations of the definition can be found in [1], [2], [10], and [20].

As mentioned in the introduction, the congruence μ on an abundant semigroup S is the largest congruence contained in $\mathscr{H}^*$. Hence if S is regular, μ is the largest congruence contained in $\mathscr{H}$, and so it is the maximum idempotent separating congruence on S. Thus our notation is consistent with the standard notation for regular semigroups. When more than one semigroup is involved, we write μ_S for the relation μ on S.

2. Concordant Semigroups

A *concordant* semigroup is an IC abundant semigroup in which the idempotents generate a regular semigroup. In this section we prove the following result.

Proposition 2.1. *If S is a finite concordant semigroup in $\mathbf{A} \vee \mathbf{G}$, then $\mathscr{H} \subseteq \mu$.*

Our approach is to use a representation (due to El Qallali and Fountain [2]) of a concordant semigroup S in a 'fundamental' regular semigroup obtained from $\langle E(S)\rangle$ by a construction due to Hall [8]. First, we note the following alternative description of concordant semigroups.

Lemma 2.2. *An* IC *abundant semigroup is concordant if and only if the regular elements form a subsemigroup.*

Proof. This is immediate by Result 7 of [8]. $\square$

From [1, Lemma 2.4 and Theorem 2.5], we have the following two lemmas, the first being what we might call Lallement's lemma for concordant semigroups.

Lemma 2.3. *Let S be a concordant semigroup and $\varphi : S \to T$ be a good homomorphism. If $a \in S$ is such that $a\varphi \in E(T)$, then $a\varphi = h\varphi$ for some idempotent $h \in S$.*

Lemma 2.4. *Let S be a concordant semigroup. If $\varphi : S \to T$ is a surjective good homomorphism, then T is concordant.*

We make use of results from [8] and [2] for which we need the following semigroup constructed in [8] (see also [15]); given an idempotent generated regular semigroup $\langle E\rangle$ with set of idempotents E, Hall constructs a regular semigroup $T_{\langle E\rangle}$ such that $\langle E(T_{\langle E\rangle})\rangle$ is isomorphic to $\langle E\rangle/\mu_{\langle E\rangle}$. Moreover, $T_{\langle E\rangle}$ is *fundamental*, that is, the congruence $\mu_{T_{\langle E\rangle}}$ is trivial. The following theorem is due to Hall [8] in the regular case, and was extended to concordant semigroups in [2]. The construction of $T_{\langle E(S)\rangle}$ does not play a part in the arguments of this paper; all we need is that it exists, and some of its properties.

Theorem 2.5. *Let S be a concordant semigroup with set of idempotents E and let $\langle E\rangle$ be the subsemigroup generated by E. Then there is a good homomorphism $\alpha : S \to T_{\langle E\rangle}$ such that*

(1) *$\mu = \alpha\alpha^{-1}$, and*
(2) *$S\alpha$ is a full subsemigroup of $T_{\langle E\rangle}$.*

Moreover, every full subsemigroup of $T_{\langle E\rangle}$ is fundamental.

Note that as a consequence of the theorem, μ is a good congruence on S.

By Lemma 2.4, $S\alpha$ is concordant, and so, by Lemma 2.2, $\mathrm{Reg}(S\alpha)$ is a subsemigroup of $S\alpha$ which is obviously full. By the theorem, $S\alpha$ is a full subsemigroup of $T_{\langle E\rangle}$, and so $\mathrm{Reg}(S\alpha)$ is a full regular subsemigroup of $T_{\langle E\rangle}$. Hence $\mathrm{Reg}(S\alpha)$ is fundamental.

Now, if S is a member of $\mathbf{A} \vee \mathbf{G}$, then so is $\mathrm{Reg}(S\alpha)$, and so, by Proposition 1.6 of [13], $\mathscr{H}$ is a congruence on $\mathrm{Reg}(S\alpha)$. Thus, $\mathscr{H} = \mu$ on $\mathrm{Reg}(S\alpha)$ so that $\mathrm{Reg}(S\alpha)$ is $\mathscr{H}$-trivial.

Now let $e \in E(S)$. Then $e\alpha \in E(S\alpha)$. By [5, Lemma 1.12], the $\mathscr{H}^*$-class $H^*_{e\alpha}(S\alpha)$ of $e\alpha$ in $S\alpha$ is a cancellative subsemigroup of $S\alpha$; but $S\alpha$ is finite, so $H^*_{e\alpha}(S\alpha)$ is a group, and hence coincides with the $\mathscr{H}$-class of $e\alpha$ in $S\alpha$. Clearly, $H^*_{e\alpha}(S\alpha) \subseteq \mathrm{Reg}(S\alpha)$, and it follows easily that it is also the $\mathscr{H}^*$-class of $e\alpha$ in $\mathrm{Reg}(S\alpha)$. But $\mathrm{Reg}(S\alpha)$ is regular, so $\mathscr{H}^*$ coincides with $\mathscr{H}$ on $\mathrm{Reg}(S\alpha)$, and since $\mathrm{Reg}(S\alpha)$ is $\mathscr{H}$-trivial, we see that $H^*_{e\alpha}(S\alpha) = \{e\alpha\}$. Thus all subgroups of $S\alpha$ are trivial, that is, $S\alpha \in \mathbf{A}$. By [17, Proposition 3.4.2], $S\alpha$ is $\mathscr{H}$-trivial, and since homomorphisms map $\mathscr{H}$-related elements to $\mathscr{H}$-related elements, it follows that, on S, we have $\mathscr{H} \subseteq \alpha\alpha^{-1} = \mu$. This completes the proof of Proposition 2.1.

3. Bountiful Semigroups

A *bountiful* semigroup is an IC abundant semigroup in which the idempotents form a subsemigroup. Thus a bountiful semigroup is concordant. In this section, we prove that if S is a bountiful semigroup with $\mathscr{H} \subseteq \mu$, then $S \in \mathbf{A} \vee \mathbf{G}$. Our proof relies on a result from [6] which we now explain.

An *ample* semigroup is a bountiful semigroup in which the subsemigroup of idempotents is commutative. In an ample semigroup S, each $\mathscr{R}^*$-class and each $\mathscr{L}^*$-class contains a unique idempotent. For $a \in S$, we denote the idempotent in R_a^* by a^+, and that in L_a^* by a^*. Thus we may regard an ample semigroup as a (2,1,1)-algebra with unary operations $+$ and $*$. We note that a semigroup homomorphism $\theta : S \to T$ of ample semigroups is good if and only if it is a (2,1,1)-algebra homomorphism.

As is well known, every semigroup S has a minimum cancellative congruence, and we denote this by σ. For regular or finite semigroups, σ is, of course, the minimum group congruence. An explicit description of σ on a bountiful semigroup is given in the next proposition which is due to Simmons [20, Proposition 6].

Proposition 3.1. *If S is a bountiful semigroup, then the minimum cancellative congruence σ on S is given by:*

$$a\sigma b \text{ if and only if } ea = bf \text{ for some } e, f \in E(S).$$

An ample semigroup is *proper* if $\mathscr{R}^* \cap \sigma = \iota = \mathscr{L}^* \cap \sigma$. It follows from [4], and is not difficult to show, that a proper ample semigroup is E-unitary, and, of course, an inverse semigroup is E-unitary if and only if it is proper. However, the semigroup of Example 3 in [4] is easily seen to be ample; it is noted in [4] that it is not proper but is E-unitary.

A proper ample semigroup P is a *proper cover* of an ample semigroup S if there is a surjective good homomorphism $\alpha : P \to S$ such that α maps $E(P)$ isomorphically onto $E(S)$. The homomorphism α is called a *covering homomorphism*. From [6], we have

Theorem 3.2. *Every ample semigroup S has a proper cover which can be taken to be finite if S is finite.*

As mentioned in the introduction, this result without the finiteness condition was obtained by Lawson [11] and Simmons [20].

Let S be a bountiful semigroup. For $e \in E(S)$, we denote the $\mathscr{D}$-class of e in $E(S)$ by $E(e)$. The relation δ on S is defined by the rule:

$$a\delta b \text{ if and only if } b = eaf \text{ for some } e \in E(h), f \in E(k)$$
$$\text{where } h \in E(S) \cap R_a^* \text{ and } k \in E(S) \cap L_a^*.$$

It is shown in [3] that if δ is a congruence, then it is the minimum ample good congruence on S, and that $\mathscr{H}^* \cap \delta = \iota$. Ronghua [19] and Guo [7] independently proved that δ is a congruence on any bountiful semigroup. Putting these results together, we have the following proposition.

Proposition 3.3. *A bountiful semigroup S is a subdirect product of S/μ and S/δ.*

An *E-unitary cover* of a bountiful semigroup S is an E-unitary bountiful semigroup T together with a surjective good homomorphism $\psi\colon T \to S$ which maps $E(T)$ isomorphically onto $E(S)$; ψ is called a *covering homomorphism.* We show that any bountiful semigroup has an E-unitary cover on which $\sigma \cap \mu = \iota$. The latter property holds for all orthodox semigroups by [13, Lemma 2.2], and so our result is simply a generalisation of the existence of E-unitary orthodox covers for orthodox semigroups due independently to McAlister [13], Szendrei [22] and Takizawa [23]. As mentioned in the introduction, Simmons has established the existence of E-unitary covers for bountiful semigroups, but the covers he constructs are always infinite.

Theorem 3.4. *Let S be a bountiful semigroup. Then S has an E-unitary cover T such that $\sigma \cap \mu = \iota$ on T, and T can be taken to be finite if S is finite.*

Our approach is inspired by that for orthodox semigroups in [13]. We start with the construction of T, and then in a series of lemmas show that T has the desired properties.

Let S be a bountiful semigroup. Then S/δ is ample, and so by Theorem 3.2, it has a proper cover V with covering homomorphism α say. Let

$$T = \{(s\mu, v) \in S/\mu \times V : v\alpha = s\delta\}.$$

It is easy to see that T is a subsemigroup of the direct product $S/\mu \times V$. We show, in a sequence of lemmas, that T is a cover of the required type.

Lemma 3.5. *The congruence δ is idempotent pure.*

Proof. If $s \in S$ is such that $s\delta$ is idempotent, then since δ is good, we have $s\delta = h\delta$ for some $h \in E(S)$ by Lemma 2.3. From the definition of δ, we have $s = ihj$ where i, j are idempotents. As $E(S)$ is a subsemigroup, we see that s is idempotent. □

Lemma 3.6. *The idempotents of T form a subsemigroup.*

Proof. If $e \in E(S)$ and $v \in E(V)$ with $v\alpha = e\delta$, then clearly, $(e\mu, v) \in E(T)$.

On the other hand, if $(s\mu, v) \in T$ is idempotent, then $(s\mu)^2 = s\mu$ and $v^2 = v$. From the latter we have $(s\delta)^2 = s\delta$ so that by Lemma 3.5, s is idempotent. It follows that

$$E(T) = \{(e\mu, v) \in S/\mu \times E(V) : e \in E(S) \text{ and } v\alpha = e\delta\}.$$

Since $E(S)$ and $E(V)$ are subsemigroups of S and V respectively, it follows that $E(T)$ is a subsemigroup of T. □

Lemma 3.7. *Let $(s\mu, u) \in T$. If $e \in E(S) \cap R^*_s$, then $(e\mu, u^+) \in T$.*

Proof. Since $(s\mu, u) \in T$, we have $s\delta = u\alpha$. Now α preserves $\mathscr{R}^*$, and so $u^+\alpha \mathscr{R}^* u\alpha$, that is, $u^+\alpha \mathscr{R}^* s\delta$. As δ is also good and S/δ is ample, this gives $u^+\alpha = e\delta$ and the lemma follows. □

Next, we describe $\mathscr{R}^*$ in T.

Lemma 3.8. *Let $(A, u), (B, v) \in T$. Then $(A, u)\mathscr{R}^*(B, v)$ if and only if $A\mathscr{R}^*B$ in S/μ and $u\mathscr{R}^*v$ in V.*

Proof. Let $(A,u),(B,v) \in T$, and suppose that $A\mathscr{R}^*B$ and $u\mathscr{R}^*v$. If $(X,x),(Y,y) \in T$ and $(X,x)(A,u) = (Y,y)(A,u)$, then $XA = YA$ and $xu = yu$ so that $XB = YB$ and $xv = yv$, that is, $(X,x)(B,v) = (Y,y)(B,v)$. Similarly, $(X,x)(A,u) = (A,u)$ implies $(X,x)(B,v) = (B,v)$. Together with the opposite implications, this gives $(A,u)\mathscr{R}^*(B,v)$.

Conversely, suppose that $(A,u)\mathscr{R}^*(B,v)$ and let $A = r\mu$, $B = s\mu$. Now let e, f, u^+, v^+ be idempotents in the $\mathscr{R}^*$-classes of r, s, u and v respectively. Put $E = e\mu$ and $F = f\mu$. By Lemma 3.7, $(E,u^+),(F,v^+) \in T$, and, since μ is good, $A\mathscr{R}^*E$ and $B\mathscr{R}^*F$. Hence,by the first part, $(A,u)\mathscr{R}^*(E,u^+)$ and $(B,v)\mathscr{R}^*(F,v^+)$ so that $(E,u^+)\mathscr{R}^*(F,v^+)$. But these elements are idempotents, and so

$$(E,u^+)(F,v^+) = (F,v^+) \text{ and } (F,v^+)(E,u^+) = (E,u^+).$$

Comparing coordinates gives $E\mathscr{R}F$ whence $A\mathscr{R}^*B$, and, since V is ample, $u^+ = v^+$ so that $u\mathscr{R}^*v$. $\square$

Notice that the second part of the proof shows that every element of T is $\mathscr{R}^*$-related to an idempotent. Similarly, each element is $\mathscr{L}^*$-related to an idempotent so that we have the following corollary.

Corollary 3.9. *T is abundant.*

To show that T is bountiful, we use the following characterisation of idempotent connected abundant semigroups [1, Lemma 2.3].

Lemma 3.10. *Let S be an abundant semigroup. Then S is* IC *if and only if it satisfies the following two conditions for all $a \in S$:*

(1) *for some $h \in E(S) \cap R^*_a$ and all $f \leqslant h$, there exists $b \in S$ such that $fa = ab$:*
(2) *for some $k \in E(S) \cap L^*_a$ and all $e \leqslant k$, there exists $c \in S$ such that $ae = ca$.*

We note that the elements b and c can be taken to be idempotent, for if S is IC and $B = \langle E(S)\rangle$, then there is a connecting isomorphism $\beta\colon \langle E(hBh)\rangle \to \langle E(kBk)\rangle$ and b and c may be chosen to be $f\beta$ and $e\beta^{-1}$ respectively. Furthermore, the discussion in [1] shows that the word "some" in conditions (1) and (2) of the lemma may be replaced by the word "all". The proof of the following lemma uses both versions.

Lemma 3.11. *T is bountiful.*

Proof. By Corollary 3.9 and Lemma 3.6, T is abundant and its idempotents form a subsemigroup. All that remains is to show that T is IC.

Let $(s\mu, v) \in T$ so that $s\delta = v\alpha$, and let $h \in E(S)$ be $\mathscr{R}^*$-related to s. By Lemma 3.7, $(h\mu, v^+) \in T$, and by Lemma 3.8, $(h\mu, v^+)\mathscr{R}^*(s\mu, v)$. We show that condition (1) of Lemma 3.10 holds using $(h\mu, v^+)$ as the idempotent in the $\mathscr{R}^*$-class of $(s\mu, v)$. First, we remark that since $(h\mu, v^+) \in T$, we have

$$h\delta = v^+\alpha. \tag{1}$$

Let $(e\mu, f) \in E(T)$ where $e \in E(S)$, $f \in E(V)$ and $f\alpha = e\delta$. If $(e\mu, f) \leqslant (h\mu, v^+)$, then clearly $e\mu \leqslant h\mu$ and $f \leqslant v^+$. Hence $e\mu = (eh)\mu = (he)\mu$, and so, using (1),

$$(eh)\delta = (e\delta)(h\delta) = (f\alpha)(v^+\alpha) = (fv^+)\alpha = f\alpha.$$

Similarly, $(he)\delta = f\alpha$ so that $e\delta = (eh)\delta = (he)\delta$. As $\mu \cap \delta = \iota$, we obtain $e = eh = he$, that is, $e \leqslant h$.

Now S is bountiful, so by Lemma 3.10 and the remarks following it, there is an idempotent $k \in S$ with $es = sk$.

Since V is a proper cover of S/δ, there is an idempotent u in V with $u\alpha = k\delta$ so that $(k\mu, u) \in E(T)$.

Now

$$(fv)\alpha = (f\alpha)(v\alpha) = (e\delta)(s\delta) = (es)\delta = (sk)\delta = (s\delta)(k\delta) = (v\alpha)(u\alpha) = (vu)\alpha.$$

As α preserves $*$ and is one-one on $E(V)$, we get $(fv)^* = (vu)^* = v^*u$. Hence, since V is ample,

$$vu = vv^*u = v(fv)^* = fv.$$

Now we have $(k\mu, u) \in T$ and

$$(e\mu, f)(s\mu, v) = ((es)\mu, fv) = ((sk)\mu, vu) = (s\mu, v)(k\mu, u),$$

and so condition (1) of Lemma 3.10 holds for T. Similarly, condition (2) also holds, and so by the lemma, T is IC, and hence bountiful. □

Lemma 3.12. *T is E-unitary.*

Proof. Let $(s\mu, v) \in T$ so that $s\delta = v\alpha$, and let $(e\mu, f) \in E(T)$ where $e \in E(S)$ and $e\delta = f\alpha$. Suppose that $(s\mu, v)(e\mu, f) \in E(T)$. Then $vf \in E(V)$ and so $v \in E(V)$ since V is proper and hence E-unitary.

Also $s\delta = v\alpha$ is an idempotent of S/δ and so, by Lemma 3.5, s is idempotent. Thus $(s\mu, v) \in E(T)$ and it follows that T is E-unitary. □

We will find the following lemma useful; it is essentially part of Proposition 2.1 of [2].

Lemma 3.13. *Let a, b be elements of an abundant semigroup S. Then the following are equivalent:*

(1) $a \mu_S b$,
(2) $ae \mathscr{R}^* be$ *and* $ea \mathscr{L}^* eb$ *for all* $e \in E(S)$.

Lemma 3.14. *If $(r\mu, u), (s\mu, v) \in T$ and $((r\mu, u), (s\mu, v)) \in \sigma \cap \mu$, then $(r\mu, u) = (s\mu, v)$.*

Proof. Let $e \in E(S)$ and f be any idempotent in V such that $(e\mu, f) \in E(T)$. Then, by Lemma 3.13 we have

$$(r\mu, u)(e\mu, f)\mathscr{R}^*(s\mu, v)(e\mu, f) \text{ and } (e\mu, f)(r\mu, u)\mathscr{L}^*(e\mu, f)(s\mu, v).$$

By Lemma 3.8 and its dual, $(r\mu)(e\mu)\mathscr{R}^*(s\mu)(e\mu)$ and $(e\mu)(r\mu)\mathscr{L}^*(e\mu)(s\mu)$. It follows from Lemma 2.3 that every idempotent of S/μ is of the form $e\mu$ for an idempotent e of S, and so, by Lemma 3.13 again, we have $(r\mu, s\mu) \in \mu_{S/\mu}$. Hence by Theorem 2.5, $\mu_{S/\mu} = \iota$ so that $r\mu = s\mu$.

Now $\mu \subseteq \mathscr{R}^*$, so that $(r\mu, u)\mathscr{R}^*(s\mu, v)$. By Lemma 3.8, we have $u\mathscr{R}^*v$. As $(r\mu, u)\sigma(s\mu, v)$, we have $(e\mu, f)(r\mu, u) = (s\mu, v)(h\mu, k)$ for some idempotents $(e\mu, f)$ and $(h\mu, k)$ of T, so that $fu = vk$, and, since V is ample, it follows that $u\sigma v$. Now $(u, v) \in \mathscr{R}^* \cap \sigma$ and V is proper, so $u = v$. Hence $(r\mu, u) = (s\mu, v)$. □

The proof of Theorem 3.4 is completed by the next lemma.

Lemma 3.15. *The mapping $\theta : T \to S$ given by $(s\mu, v)\theta = s$ is a well defined good homomorphism onto S which maps $E(T)$ isomorphically onto $E(S)$.*

Proof. First, we note that if $(r\mu, u) = (s\mu, v)$, then $r\delta = u\alpha = v\alpha = s\delta$ so that $(r, s) \in \mu\cap\delta$, and so, by the remarks preceding Proposition 3.3, $r = s$. Thus θ is well defined. It is clear that θ is a homomorphism. If $s \in S$, then, since α is surjective, there is an element $v \in V$ such that $v\alpha = s\delta$, so that $(s\mu, v) \in T$ and θ is surjective; this also shows that θ maps $E(T)$ onto $E(S)$.

To see that θ is one-one on $E(T)$, let $(e\mu, v), (f\mu, w) \in E(T)$ and suppose that $(e\mu, v)\theta = (f\mu, w)\theta$. Then $e = f$ so that certainly $e\mu = f\mu$. Also, $v, w \in E(V)$ and $v\alpha = e\delta = f\delta = w\alpha$ so that $v = w$ since α is idempotent separating.

Finally, to see that θ is good, suppose that $(r\mu, u), (s\mu, v) \in T$ with $(r\mu, u)\mathscr{R}^*(s\mu, v)$. By Lemma 3.8, $r\mu\mathscr{R}^*s\mu$ and $u\mathscr{R}^*v$. From the latter, we get $r\delta\mathscr{R}^*s\delta$ since $r\delta = u\alpha$, $s\delta = v\alpha$ and α is good. Let e, f $E(S)$ be such that $e\mathscr{R}^*r$ and $f\mathscr{R}^*s$. Then $e\mu\mathscr{R}^*f\mu$ and $e\delta\mathscr{R}^*f\delta$. It follows from this that $(ef, f), (fe, e) \in \mu \cap \delta$. Hence $ef = f$ and $fe = e$ so that $e\mathscr{R}^*f$ and thus $r\mathscr{R}^*s$ as desired.

Similarly, θ preserves $\mathscr{L}^*$ and so θ is good. □

Having proved Theorem 3.4, it is now easy to prove the following result which completes the proof of Theorem 1.

Proposition 3.16. *If S is a finite bountiful semigroup with $\mathscr{H} \subseteq \mu$, then $S \in \mathbf{A} \vee \mathbf{G}$.*

Proof. Let T be a finite E-unitary cover of S with $\mu \cap \sigma = \iota$ and covering map $\theta : T \to S$. Now $\mu \cap \sigma = \iota$, and so T can be embedded (as a subdirect product) in $T/\mu \times T/\sigma$. Since T/σ is cancellative and finite, we have $T/\sigma \in \mathbf{G}$.

Now $\mathscr{H} \subseteq \mu$ on S, so S/μ has only trivial subgroups, and hence $S/\mu \in \mathbf{A}$.

We claim that $T/\mu \cong S/\mu$. To see this, it is enough to show that for $a, b \in T$ we have $a\mu_T b$ if and only if $(a\theta)\mu_S(b\theta)$. Using Lemma 3.13 and the fact that θ is good, it is easy to see that $a\mu_T b$ implies $(a\theta)\mu_S(b\theta)$.

Conversely, if $(a\theta)\mu_S(b\theta)$, then, again by Lemma 3.13, we have that $(ae)\theta\mathscr{R}^*(be)\theta$ for all $e \in E(T)$. Let $f, h \in E(T)$ be in the $\mathscr{R}^*$-classes of ae and be respectively. Since θ is good, we obtain $f\theta\mathscr{R}^*h\theta$. But these elements are idempotent, so $(fh)\theta = (f\theta)(h\theta) = h\theta$ and $(hf)\theta = f\theta$. Now T is bountiful, so fh and hf are idempotents, and so $fh = h$ and $hf = f$ since θ is idempotent separating. Hence $f\mathscr{R}h$ and so $ae\mathscr{R}^*be$. Similarly, $ea\mathscr{L}^*eb$ for all $e \in E(T)$ so that $a\mu_T b$.

Hence the claim is proved, so that $T/\mu \in \mathbf{A}$, and thus $S \in \mathbf{A} \vee \mathbf{G}$, since S divides $T/\mu \times T/\sigma$. □

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