MAXIMAL ORDERS IN COMPLETELY 0-SIMPLE SEMIGROUPS

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Abstract

Fountain, Gould and Smith introduced the concept of equivalence of orders in a semigroup and the notion of a maximal order. We examine these ideas in the context of orders in completely 0-simple semigroups with particular emphasis on abundant orders.

Introduction

In this paper we develop further the theory, introduced in [3], of equivalence and maximality of orders in a semigroup. In particular, we study maximal and abundant orders in completely 0-simple semigroups.

Two equivalence relations on the set of weak straight left orders in a semigroup Q were introduced in [3]. In general these relations are distinct but they coincide when Q is completely 0-simple. The two relations are denoted by $\equiv$ and $\sim$, and we define $\sim$ in Section 1. Two weak straight left orders are said to be equivalent if they are related by $\equiv$ and a maximal weak straight left order is one which is maximal in its $\equiv$-class. Of course, in a completely 0-simple semigroup we can use the relation $\sim$ to define these notions. Thus we examine weak straight left orders which are maximal in their $\sim$-classes. Such an investigation was initiated in [3] and we take it further in Section 2 where we introduce the inverse of a one-sided fractional one-sided S-ideal when S is a straight left order in a regular semigroup.

We use the results of Section 2 to study maximal orders in completely 0-simple semigroups. This is the subject of Section 3 where we note that if S is an order in a completely 0-simple semigroup Q, then the fractional S-ideals of Q form a semigroup $\mathcal{F}(S)$ under multiplication of subsets. Furthermore, for certain maximal orders, $\mathcal{F}(S)$ is a group.

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We review projective S-acts where S is a semigroup with zero in Section 4. Then in Section 5 we obtain more precise results relating the properties of $\mathcal{F}(S)$ and those of S when S is an abundant order in a completely 0-simple semigroup. These results are strengthened in Section 6 where we consider orders in Brandt semigroups. We conclude the paper with some examples in Section 7.

Some of the results of this paper were announced in [2].

1. Preliminaries

We refer the reader to [6] for standard concepts and facts concerning semigroups. In particular, details about Green's relations and completely 0-simple semigroups can be found there.

Let Q be a regular semigroup. A subset U of Q is *large* if it has non-empty intersection with each group $\mathcal{H}$-class of Q. If a is an element in a group $\mathcal{H}$-class of Q, then $a^\sharp$ denotes the inverse of a in H_a.

A *weak left order* in Q is a subsemigroup S of Q such that every element q of Q can be written as $q = a^\sharp b$ for some $a, b \in S$. *Weak right orders* are defined dually and S is a *weak order* in Q if it is both a weak left order and a weak right order.

A weak left order S in Q is *straight* if every element of Q can be written as $a^\sharp b$ where $a, b \in S$ and $a\mathcal{R}b$ in Q. *Weak straight right orders* and *weak straight orders* are defined in the obvious way.

An element a of a semigroup S is said to be *square-cancellable* if, for all elements x, y of S^1, we have $xa^2 = ya^2$ implies $xa = ya$, and $a^2x = a^2y$ implies $ax = ay$. A weak left order S in Q is a *left order* in Q if every square-cancellable element lies in a subgroup of Q. Similarly, one has right orders and orders. When Q is a completely 0-simple semigroup it is clear that every weak left or right order is a left or right order.

Two fundamental results from [5] on weak left orders which we will often use without further mention are the following.

Proposition 1.1. *If Q is a regular semigroup on which $\mathcal{H}$ is a congruence, then every weak left order in Q is straight.*

Proposition 1.2. *If S is a weak straight left order in a semigroup Q, then S has non-empty intersection with each $\mathcal{H}$-class of Q. Moreover, for each group $\mathcal{H}$-class of Q, the subsemigroup $S \cap H$ is a left order in H.*

Let Q be a regular semigroup and let Π index the group $\mathcal{H}$-classes of Q. Define the relation $\sim$ on the set of all large subsemigroups of Q by the rule that $S \sim T$ if and only if for all $\sigma \in \Pi$, there are elements a_σ, b_σ, c_σ, $d_\sigma \in H_\sigma$ such that for all π, $\theta \in \Pi$, $a_\pi S b_\theta \subseteq T$ and $c_\pi T d_\theta \subseteq S$. It is shown in [3] that $\sim$ is an equivalence relation. It is also shown that if S, T are large subsemigroups and $S \sim T$, then S is a weak straight left order if and only if T is a weak straight left order.

In [3] another equivalence relation $\equiv$ is defined on the set of large subsemigroups of Q and two large subsemigroups S and T are said to be *equivalent* if $S \equiv T$. A weak straight left order is *maximal* if it is a maximal member (under inclusion) of its $\equiv$-equivalence class. However, we shall be concerned with orders in completely 0-simple semigroups and, as observed in [3], in this case the relations $\sim$ and $\equiv$ coincide. Thus if S, T are (left) orders in a completely 0-simple semigroup Q, then S is equivalent to T if and only if $S \sim T$ and S is a maximal (left) order in Q if and only if S is maximal in its $\sim$-class.

2. Fractional Ideals

Let S be a large subsemigroup of a regular semigroup Q. Following [3] we define a subset I of Q to be a *left S-ideal* if

(i) $SI \subseteq I$, and

(ii) I is large in Q.

The notion of *right S-ideal* is obtained by replacing (i) by its dual; an *S-ideal* is a subset of Q which is both a left and a right S-ideal.

A *left fractional left S-ideal* of Q is a left S-ideal such that

(iii) for every group $\mathcal{H}$-class H of Q there is an element c of H such that $Ic \subseteq S$.

By replacing (iii) by its dual we obtain the notion of a *right fractional left S-ideal*; a *fractional left S-ideal* is a left S-ideal I for which both (iii) and its dual hold.

For any one-sided fractional one-sided S-ideal I of Q we define subsets $O_\ell(I)$ and $O_r(I)$ as follows:

$$O_\ell(I) = \{q \in Q \mid qI \subseteq I\},$$
$$O_r(I) = \{q \in Q \mid Iq \subseteq I\}.$$

Clearly, both $O_\ell(I)$ and $O_r(I)$ are subsemigroups of Q. If I is a one-sided fractional left S-ideal, then $S \subseteq O_\ell(I)$ and so $O_\ell(I)$ is large in Q. In fact, $O_r(I)$ is also large in Q: if H is a group $\mathcal{H}$-class of Q, then there is an element c of H such that $Ic \subseteq S$. Also, I is large in Q so that $I \cap H \neq \emptyset$. Let $d \in H \cap I$. Then $cd \in H$ and $Icd \subseteq Sd \subseteq SI \subseteq I$ so that $O_r(I)$ is large. We record these facts and others in the following result.

Proposition 2.1. *Let I be a left fractional left S-ideal of Q. Then*

1. *$O_r(I)$ is a large subsemigroup of Q, $O_r(I) \sim S$ and I is a right fractional right $O_r(I)$-ideal,*
2. *$O_\ell(I)$ is a large subsemigroup of Q, $O_\ell(I) \sim S$ and I is a left fractional left $O_\ell(I)$-ideal.*

Proof. Let Π index the group $\mathcal{H}$-classes of Q and for each $\pi \in \Pi$ let c_π, $d_\pi \in H_\pi$ be such that $Ic_\pi \subseteq I$ and $d_\pi \in H \cap I$. Then, for all θ, $\pi \in \Pi$,

$$Ic_\theta S d_\pi \subseteq Sd_\pi \subseteq SI \subseteq I$$

so that

$$c_\theta S d_\pi \subseteq O_r(I)$$

and

$$d_\theta O_r(I) c_\pi \subseteq IO_r(I)c_\pi \subseteq Ic_\pi \subseteq S.$$

Thus $S \sim O_r(I)$.

For each $\pi \in \Pi$ there is an element a_π in $S \cap H_\pi$ since S is large in Q. Now, for all θ, $\pi \in \Pi$,

$$a_\pi S a_\theta I \subseteq SI \subseteq I$$

so that

$$a_\pi S a_\theta \subseteq O_\ell(I)$$

and

$$a_\pi O_\ell(I) d_\theta c_\theta \subseteq a_\pi I c_\theta \subseteq a_\pi S \subseteq S.$$

Hence $S \sim O_\ell(I)$.

It is clear that I is a right $O_r(I)$-ideal and a left $O_\ell(I)$-ideal. For $\pi \in \Pi$ we also have $Ic_\pi I \subseteq SI \subseteq I$ so that $c_\pi I \subseteq O_r(I)$ and $Ic_\pi \subseteq S \subseteq O_\ell(I)$ as required. □

The following result is an easy consequence of the facts that $O_\ell(I) \sim S$ and that $S \subseteq O_\ell(I)$.

Corollary 2.2. *If S is a $\sim$-maximal weak straight left order and I is a left fractional left S-ideal, then $S = O_\ell(I)$.*

Given a large subsemigroup S of a regular semigroup Q and a one-sided fractional one-sided S-ideal I we define the *inverse* of I to be the set

$$\begin{aligned} I^{-1} &= \{q \in Q \mid IqI \subseteq I\} \\ &= \{q \in Q \mid Iq \subseteq O_\ell(I)\} \\ &= \{q \in Q \mid qI \subseteq O_r(I)\}. \end{aligned}$$

Lemma 2.3. *If I is a left fractional left S-ideal of Q, then I^{-1} is a right fractional right $O_\ell(I)$-ideal.*

Proof. First we have $I(I^{-1}O_\ell(I)) = (II^{-1})O_\ell(I) \subseteq O_\ell(I)^2 \subseteq O_\ell(I)$ so that $I^{-1}O_\ell(I) \subseteq I^{-1}$.

For any $\mathcal{H}$-class H of Q, there is an element $c \in H$ such that $Ic \subseteq S$. Since $S \subseteq O_\ell(I)$ it follows that $c \in I^{-1}$ and hence I^{-1} is large.

Now I is large so that $I \cap H \neq \emptyset$. Let $d \in I \cap H$. Then

$$dI^{-1} \subseteq II^{-1} \subseteq O_\ell(I)$$

as required. □

One might hope that $II^{-1} = S = I^{-1}I$, but this is not the case in general as we see from Example 7.2. We do, however, have that both II^{-1} and $I^{-1}I$ are $\sim$-related to S.

Proposition 2.4. *Let S be a large subsemigroup of a regular semigroup Q and let I be a left fractional left S-ideal of Q. Then*

1. *II^{-1} and $I^{-1}I$ are large subsemigroups of Q and $II^{-1} \sim S \sim I^{-1}I$,*
2. *I is a left fractional left II^{-1}-ideal and a right fractional right $I^{-1}I$-ideal,*
3. *I^{-1} is a right fractional right II^{-1}-ideal and a left fractional left $I^{-1}I$-ideal.*

Proof. (1) Since $II^{-1}I \subseteq I$ it is clear that II^{-1} and $I^{-1}I$ are subsemigroups of Q. They are both large because I and I^{-1} are large.

Let the group $\mathcal{H}$-classes of Q be indexed by Π and for each $\pi \in \Pi$ choose a_π, c_π, $d_\pi \in H_\pi$ with $a_\pi \in S$, $d_\pi \in I$ and $Ic_\pi \subseteq S$. Then for all π, $\theta \in \Pi$ we have $c_\theta \in I^{-1}$ and $d_\theta c_\theta \in H_\theta$ so that

$$a_\pi S d_\theta c_\theta \subseteq SII^{-1} \subseteq II^{-1}$$

and

$$a_\pi II^{-1} d_\theta c_\theta \subseteq a_\pi I c_\theta \subseteq I c_\theta \subseteq S.$$

Moreover,

$$d_\pi I^{-1} I c_\theta \subseteq I c_\theta \subseteq S \qquad \text{and} \qquad c_\pi S d_\theta \subseteq I^{-1} I.$$

Thus $II^{-1} \sim S \sim I^{-1}I$.

Parts (2) and (3) follow immediately from the definitions involved. □

3. Orders In Completely 0-simple Semigroups

We have already noted that in a completely 0-simple semigroup Q, every weak (left) order is actually a straight (left) order. Now, an easy consequence of Proposition 1.2 is that every non-zero ideal I of S meets each $\mathcal{H}$-class of Q. In particular, I is large in Q and consequently, I is a fractional S-ideal. Note that S itself is thus a fractional S-ideal. Recall from Section 1 that the maximal orders in Q are precisely the $\sim$-maximal orders. In view of these observations we have the following result from Proposition 4.1 of [3].

Proposition 3.1. *Let S be an order in a completely 0-simple semigroup Q. Then S is a maximal order in Q if and only if for all non-zero ideals I of S and elements q of Q,*

$$qI \subseteq I \text{ implies } q \in S \quad \text{and} \quad Iq \subseteq I \text{ implies } q \in S.$$

We say that S is *closed* in Q if the ideal S satisfies the condition of the proposition.

We need one more definition. If S is an order in a completely 0-simple semigroup Q and if I is a fractional S-ideal, then we say that I is *invertible* if there is a fractional S-ideal $\bar{I}$ such that $I\bar{I} = S = \bar{I}I$. We now come to the main result of this section.

Theorem 3.2. *The fractional S-ideals of Q form a semigroup $\mathcal{F}(S)$ under multiplication of subsets. If S is an identity for $\mathcal{F}(S)$, then the following conditions are equivalent:*

1. *S is maximal and $S = II^{-1} = I^{-1}I$ for all non-zero ideals I of S,*
2. *every non-zero ideal of S is invertible and S is closed in Q,*
3. *$\mathcal{F}(S)$ is a group and S is closed in Q.*

Proof. If I, $J \in \mathcal{F}(S)$, then certainly IJ is large and an S-ideal. Furthermore, if H is a group $\mathcal{H}$-class of Q, then there are elements $c, d \in H$ with $cI \subseteq S$, $dJ \subseteq S$ and hence $dcIJ \subseteq S$. Similarly, there is an element x in H such that $IJx \subseteq S$. Thus $\mathcal{F}(S)$ is a semigroup.

Now suppose that S is an identity for $\mathcal{F}(S)$.

If (1) holds, then certainly every non-zero ideal of S is invertible and as S is maximal, it follows from Proposition 3.1 that S is closed in Q.

Suppose that (2) holds and let $I \in \mathcal{F}(S)$ and $q \in Q$ be such that $qI \subseteq I$. Now I is invertible and so there is a fractional S-ideal $\bar{I}$ such that $I\bar{I} = S = \bar{I}I$. Hence $qS = qI\bar{I} \subseteq I\bar{I} = S$ and so $q \in S$ since S is closed in Q. Similarly, $Iq \subseteq I$ implies $q \in S$ and so S is maximal by Proposition 3.1.

Since S is maximal, $S = O_\ell(I)$ by Corollary 2.2 and so I^{-1} is a right fractional right S-ideal by Lemma 2.3. By the left-right duals of Corollary 2.2 and Lemma 2.3, I^{-1} is also a left fractional left S-ideal and hence $I^{-1} \in \mathcal{F}(S)$. Since $S = O_\ell(I) = O_r(I)$, it follows from the definition of I^{-1} that $\bar{I} \subseteq I^{-1}$ and also that $II^{-1} \subseteq S$ and $I^{-1}I \subseteq S$. Thus $S = I^{-1}I = II^{-1}$. In particular, this is true when I is a non-zero ideal of S and so (1) holds.

It is clear that (2) follows from (3) and so to complete the proof we show that if (1) holds, then $\mathcal{F}(S)$ is a group. Let $J \in \mathcal{F}(S)$. Then $J^{-1} \in \mathcal{F}(S)$ because S is maximal and so $J^{-1} \cap S$ is an ideal of S. Put $K = J^{-1} \cap S$.

Let H be a non-zero group $\mathcal{H}$-class of Q. Then $H \cap J^{-1} \neq \emptyset$ and if q is an element of $H \cap J^{-1}$, then by Proposition 1.2, $q = a^\sharp b$ for some a, b in $S \cap H$. Now

$$b = aa^\sharp b = aq \in SJ^{-1} \subseteq J^{-1}$$

so that $b \in K$ and K is non-zero. Therefore, by assumption, $K^{-1}K = S = KK^{-1}$.

Further, $KJ \in \mathcal{F}(S)$ and $KJ \subseteq J^{-1}J \subseteq S$ so that KJ is a non-zero ideal of S and hence $(KJ)(KJ)^{-1} = S = (KJ)^{-1}(KJ)$. Since S is maximal, $(KJ)^{-1}$ is in $\mathcal{F}(S)$ and so $(KJ)^{-1}K \in \mathcal{F}(S)$. Also,

$$J((KJ)^{-1}K) = SJ(KJ)^{-1}K = K^{-1}KJ(KJ)^{-1}K = K^{-1}SK = K^{-1}K = S$$

and $((KJ)^{-1}K)J = S$ so that $(KJ)^{-1}K$ is an inverse of J in $\mathcal{F}(S)$. Thus $\mathcal{F}(S)$ is a group as required. □

Remark 3.3. *In the notation of the above proof we have $(KJ)^{-1}K = J^{-1}$ because from the proof of* (2) *implies* (1) *we see that if $I \in \mathcal{F}(S)$ has an inverse in $\mathcal{F}(S)$, then the inverse must be I^{-1}.*

Again, suppose that S is a maximal order in a completely 0-simple semigroup Q. By Corollary 2.2 and its left-right dual, $S = O_\ell(I) = O_r(I)$ for any fractional S-ideal I. Thus

$$I^{-1} = \{q \in Q \mid Iq \subseteq S\} = \{q \in Q \mid qI \subseteq S\}.$$

Since $II^{-1} \subseteq S$ we have $I \subseteq (I^{-1})^{-1}$. Further, if $I \subseteq S$, then $SI \subseteq I \subseteq S$ so that $S \subseteq I^{-1}$. Again, if $I \subseteq S$, then $(I^{-1})^{-1} \subseteq S$ since for $u \in (I^{-1})^{-1}$ we have $uS \subseteq uI^{-1} \subseteq S$ so that $u \in S$ by Proposition 3.1. We say that I is *reflexive* if $I = (I^{-1})^{-1}$. Notice that if $\mathcal{F}(S)$ is a group with identity S, then all fractional S-ideals are reflexive.

Recall that a proper ideal P of S is *prime* when for all ideals I, J of S, if $IJ \subseteq P$, then at least one of I, J is contained in P. It is not difficult to show that P is prime if and only if for all elements a, b of S, if $aSb \subseteq P$, then $a \in P$ or $b \in P$.

If S is an order in a completely 0-simple semigroup, then by Theorem 4.1 of [4], 0 is a prime ideal of S. We now consider non-zero prime ideals of a maximal order S. A non-zero prime ideal P is said to be a *minimal* prime if the only prime ideal of S properly contained in P is 0.

Proposition 3.4. *Let S be a maximal order in a completely 0-simple semigroup and let P be a non-zero prime ideal of S. Then P is reflexive if and only if S is properly contained in P^{-1}. Furthermore, if P is reflexive, then P is a minimal prime of S.*

Proof. Suppose that P is reflexive and that $S = P^{-1}$. Then $SP^{-1} \subseteq S$ so that $S \subseteq (P^{-1})^{-1}$ and $P \neq (P^{-1})^{-1}$, a contradiction. Hence $S \subsetneqq P^{-1}$.

Conversely, if $S \subsetneqq P^{-1}$, then it follows from the observations above that $PP^{-1}(P^{-1})^{-1} \subseteq P$. But $PP^{-1} \subseteq S$ and $(P^{-1})^{-1} \subseteq S$ so that PP^{-1} and $(P^{-1})^{-1}$ are ideals of S. Since P is prime we have $PP^{-1} \subseteq P$ or $(P^{-1})^{-1} \subseteq P$. If $PP^{-1} \subseteq P$, then $P^{-1} \subseteq O_r(P) = S$, a contradiction. Hence $(P^{-1})^{-1} \subseteq P$ and so P is reflexive.

Now suppose that P is reflexive and that K is a non-zero prime ideal of S contained in P. Then $P^{-1}K \in \mathcal{F}(S)$ and $P^{-1}K \subseteq P^{-1}P \subseteq S$ so that $P^{-1}K$ is an ideal of S. Now $PP^{-1}K \subseteq SK \subseteq K$ and K is prime so that $P \subseteq K$ or $P^{-1}K \subseteq K$. But if the latter holds, then $P^{-1} \subseteq O_\ell(K) = S$, a contradiction since P is reflexive. Thus $P = K$ and P is a minimal prime. $\square$

4. Projective Acts

For a semigroup S, a *pointed left S-act* or *left S-act with zero* is a left S-act A with a distinguished element $\mathbf{0}$ satisfying $s\mathbf{0} = \mathbf{0}$ for all $s \in S$. Notice that if S itself has a zero and A is a pointed left S-act, then $0a = \mathbf{0}$ for all $a \in A$ so that there is little danger of confusion in using the symbol 0 for both the zero of S and the zero of A.

Of course, we may define *pointed right S-acts* and for each of the following definitions and results there is a left-right dual.

Notice that in a semigroup S with zero, the left ideals are pointed left S-acts with the semigroup zero being the zero of the act.

For the rest of this section, every semigroup has a zero, the term "S-act" means "pointed left S-act" and the action is assumed to be unitary if S is a monoid.

Note that if $\theta : A \to B$ is an S-morphism, that is, if $(sa)\theta = s(a\theta)$ for all $a \in A$, $s \in S$, then $0\theta = 0$.

Projective S-acts are defined in the usual way, that is, an S-act P is *projective* if for any S-acts A, B and S-morphisms $\alpha : P \to B$, $\theta : A \to B$ with θ surjective, there is an S-morphism $\beta : P \to A$ such that the triangle

$$\begin{tikzcd} & P \arrow[dl, "\beta"'] \arrow[d, "\alpha"] \\ A \arrow[r, "\theta"'] & B \end{tikzcd}$$

is commutative.

If S is a monoid, there are analogous results to those for projective acts without zero [7]. The proofs are essentially the same and we omit them. In our case, the coproduct is the 0-direct union rather than simply disjoint union. Thus a free S-act is a 0-direct union $\bigcup_{i \in I} Sx_i$ where each Sx_i is isomorphic to S and we have the following two propositions.

Proposition 4.1. *Let S be a monoid with zero and suppose that the S-act P is a 0-direct union of S-acts P_i $(i \in I)$. Then P is projective if and only if each P_i is projective.*

Proposition 4.2. *Let S be a monoid with zero. Then an S-act is projective if and only if it is a 0-direct union of cyclic S-acts each of which is isomorphic to an idempotent generated principal left ideal of S.*

If S is a semigroup without an identity, then clearly each S-act is also a unitary S^1-act and conversely, each unitary S^1-act becomes an S-act by restricting the action. Equally clearly, for S-acts A, B, an S-morphism from A to B is an S^1-morphism and vice versa. Thus an S-act is projective if and only if it is a projective S^1-act and we obtain a semigroup version of Proposition 4.1 by simply replacing the word "monoid" by "semigroup". By a cyclic S-act we mean an S-act of the form S^1c for some c in the act. Now we have the following semigroup version of Proposition 4.2.

Proposition 4.3. *Let S be a semigroup with zero. Then an S-act is projective if and only if it is a 0-direct union of cyclic S-acts each of which is isomorphic to an idempotent generated principal left ideal of S^1.*

In some cases, even if S is not a monoid we can replace S^1 in the statement of Proposition 4.3 by S.

Lemma 4.4. *Let A be an S-act where $S \neq S^1$ and suppose that for each element a of A there is an element s of S such that $sa = a$. Then A is not isomorphic to S^1.*

Proof. If $\theta : A \to S^1$ is an isomorphism, then $a\theta = 1$ for some $a \in A$. Now there is an element $s \in S$ such that $sa = a$ and so $s = s1 = s(a\theta) = (sa)\theta = a\theta = 1$, a contradiction. $\square$

The following proposition is now an immediate consequence of Lemma 4.4 and Proposition 4.3.

Proposition 4.5. *Let S be a semigroup without identity. Let P be an S-act and suppose that for each $p \in P$ there is an element s of S such that $sp = p$. Then P is projective if and only if it is a 0-direct union of cyclic S-acts each of which is isomorphic to an idempotent generated left ideal of S.*

5. Abundant Orders In Completely 0-simple Semigroups

Recall that the relation $\mathcal{R}^*$ is defined on a semigroup S by the rule that $a\mathcal{R}^*b$ if and only if $a\mathcal{R}b$ in some oversemigroup of S. The relation $\mathcal{L}^*$ is defined dually. We remark that if S is regular, then $\mathcal{R} = \mathcal{R}^*$ and $\mathcal{L} = \mathcal{L}^*$. A semigroup S is *abundant* if each $\mathcal{R}^*$-class and each $\mathcal{L}^*$-class contains an idempotent. Note that an idempotent is a left (right) identity for its $\mathcal{R}^*$-class ($\mathcal{L}^*$-class). Further information about $\mathcal{R}^*$, $\mathcal{L}^*$ and abundant semigroups can be found in [1]. In particular, if a semigroup is abundant, then all its principal one-sided ideals are projective. The converse is true when S is a monoid but not generally. For example, it is easy to see that every principal ideal of the infinite monogenic semigroup is projective but this semigroup is not abundant. If a semigroup is such that all its one-sided ideals are projective, then it is said to be *hereditary.* We are more concerned with what we call *weakly hereditary* semigroups, that is, abundant semigroups in which every two-sided ideal is projective as a left act and a right act.

We now investigate abundant orders in completely 0-simple semigroups. First, we have the following result which is an immediate consequence of Lemma 2.10 and Proposition 5.2 of [4].

Lemma 5.1. *If S is an abundant order in a completely 0-simple semigroup Q, then the relations $\mathcal{R}^*$ and $\mathcal{L}^*$ on S are the restrictions of the relations of $\mathcal{R}$ and $\mathcal{L}$ on Q respectively.*

Lemma 5.2. *If I is a fractional S-ideal where S is an abundant order in a completely 0-simple semigroup Q, then $O_\ell(I)$ is an abundant order in Q.*

Proof. Since $S \subseteq O_\ell(I)$, it is clear that $O_\ell(I)$ is an order and since S is abundant, it follows easily from Lemma 5.1 that $O_\ell(I)$ is abundant. □

For a fractional S-ideal I as in the lemma, we recall from Section 2 that, by the definition of I^{-1},

$$I^{-1}I \subseteq O_r(I) = \{q \in I \mid Iq \subseteq I\}.$$

Further, by Proposition 2.1, I is a fractional left $O_\ell(I)$-ideal so that, in particular, I is an $O_\ell(I)$-act. With this notation we can now prove the following result.

Proposition 5.3. *If $I^{-1}I = O_r(I)$, then I is a projective $O_\ell(I)$-act.*

Proof. Let M, N be $O_\ell(I)$-acts and suppose that $\psi : M \to N$ and $\varphi : I \to N$ are $O_\ell(I)$-morphisms with ψ surjective.

For each $\mathcal{L}$-class L of Q choose an idempotent e_L in $L \cap S$. It follows from Lemma 5.1 that such a choice is possible since S is large in Q and is abundant. Now $S \subseteq O_r(I) = I^{-1}I$ so that $e_L \in I^{-1}I$. For each L, choose elements $h_L \in I^{-1}$, $k_L \in I$ such that $h_L k_L = e_L$. Finally, for each L, choose an element $m_L \in M$ such that $m_L\psi = k_L\varphi$.

We now define a function $\theta : I \to M$ as follows. First, $0\theta = 0$. Next, for a non-zero element x of I we have $x \in L$ for some $\mathcal{L}$-class L of Q and we put

$$x\theta = xh_L m_L.$$

Note that $xh_L \in II^{-1} \subseteq O_\ell(I)$ so that we have $x\theta \in M$ as required.

Let $t \in O_\ell(I)$ and let $x \in I \cap L$ as above. If $tx = 0$, then

$$(tx)\theta = 0 = 0m_L = txh_L m_L = t(x\theta).$$

If $tx \neq 0$, then

$$(tx)\theta = txh_L m_L = t(x\theta)$$

and hence θ is an $O_\ell(I)$-morphism.

Furthermore, $0\theta\psi = 0 = 0\varphi$ and

$$x\theta\psi = (xh_L m_L)\psi = (xh_L)(m_L\psi) = xh_L(k_L\varphi) = (xh_L k_L)\varphi = (xe_L)\varphi = x\varphi$$

so that $\theta\psi = \varphi$. Thus I is projective as claimed. □

The following simple lemma gives some properties of abundant orders in completely 0-simple semigroups not enjoyed by all orders.

Lemma 5.4. *Let S be an abundant order in a completely 0-simple semigroup Q. Then $\mathcal{F}(S)$ is a monoid with identity S and S is closed in Q.*

Proof. Let $q \in Q$. Since S is large and abundant, it follows from Lemma 5.1 that there are idempotents e, f in S such that $qe = q$ and $fq = q$. If $I \in \mathcal{F}(S)$, then $SI \subseteq I$ and $IS \subseteq I$ and it follows that $SI = I = IS$. Thus $\mathcal{F}(S)$ is a monoid.

If $q \in Q$, then we have just seen that $q \in qS$ and $q \in Sq$. It follows that S is closed in Q. □

We can now prove the following result which corrects Proposition 6.3 of [2].

Theorem 5.5. *If S is an abundant order in a completely 0-simple semigroup Q, then $\mathcal{F}(S)$ is a group if and only if S is maximal, weakly hereditary and $S = II^{-1} = I^{-1}I$ for all non-zero ideals I of S.*

Proof. If $\mathcal{F}(S)$ is a group, then it has identity S by Lemma 5.4. Also S is closed in Q and so by Theorem 3.2, S is maximal and $S = II^{-1} = I^{-1}I$ for all non-zero ideals I of S. If I is a non-zero ideal of S, then it follows from the maximality of S that $S = O_\ell(I) = O_r(I)$. Hence by Proposition 5.3 and its dual, I is projective as a left S-act and as a right S-act. Thus S is weakly hereditary.

For the converse note that since S is the identity of $\mathcal{F}(S)$, it follows from Theorem 3.2 that $\mathcal{F}(S)$ is a group. □

The next proposition tells us about the nature of prime ideals in a maximal, weakly hereditary order.

Proposition 5.6. *Let S be an abundant order in a completely 0-simple semigroup Q such that $\mathcal{F}(S)$ is a group. Then a proper ideal I of S is prime if and only if it is a maximal ideal.*

Proof. If I is a maximal ideal of S and $JK \subseteq I$ for some ideals J, K with $J \nsubseteq I$, then $I \cup J = S$ by the maximality of I. Let $a \in K$. Then $a = ea$ for some idempotent e of S since S is abundant. Thus $a \in (I \cup J)K \subseteq I$ and so $K \subseteq I$. Hence I is prime.

Conversely, suppose that I is prime and let W be an ideal of S with $I \subsetneqq W$. Since S is a maximal order in Q we have $S = O_\ell(J) = O_r(J)$ for any ideal J of S and so $W^{-1} \subseteq I^{-1}$. Consequently, $W^{-1}I \subseteq I^{-1}I = S$, and therefore $W(W^{-1}I) \subseteq SI = I$. Now I is prime and W is not contained in I, so $W^{-1}I \subseteq I$. Thus $W^{-1} \subseteq O_\ell(I) = S$ and so $W^{-1} = S$. Hence $W = SW = W^{-1}W = S$ and so I is a maximal ideal of S. □

Our next objective is to illustrate further the way in which the properties of the group $\mathcal{F}(S)$ and those of S are related. First we need the following three lemmas.

Lemma 5.7. *Let S be an abundant order in a completely 0-simple semigroup Q. Then S is 0-simple if and only if $S = Q$.*

Proof. Since S is abundant it contains an idempotent which is necessarily primitive. Hence, if S is 0-simple, then it is completely 0-simple. Thus S is regular and so by Lemma 5.1, elements of S which are $\mathcal{R}$-related or $\mathcal{L}$-related in Q are similarly related in S. If $q \in Q$, then $q = a^{\sharp}b$ for some elements a, b of S. Now $a\mathcal{H}a^2$ in Q so that $a\mathcal{H}a^2$ in S. Consequently, if H is the $\mathcal{H}$-class of a in S, then H is a group. Hence $a^{\sharp} \in S$ and $q \in S$ so that $S = Q$. □

Lemma 5.8. *Let S be an abundant order in a completely 0-simple semigroup Q such that $\mathcal{F}(S)$ is a group. If I, J are non-zero proper ideals of S and $I \subseteq J$, then there is an ideal A of S such $I = JA$.*

Proof. Put $A = J^{-1}I$. Then $J^{-1}I \in \mathcal{F}(S)$ and $A \subseteq J^{-1}J = S$ so that A is an ideal of S. Also
$$I = SI = (JJ^{-1})I = JA.$$
□

Lemma 5.9. *Let S be an abundant order in a completely 0-simple semigroup Q such that $\mathcal{F}(S)$ is a group. If M, N are maximal ideals of S, then*
$$MN = M \cap N = NM.$$

Proof. If 0 is a maximal ideal, then there is nothing to prove. Assume that M, N are distinct and non-zero. Then they are both large so that if H is a non-zero group $\mathcal{H}$-class of Q, then there are elements m and n in $M \cap H$ and $N \cap H$ respectively. Thus $mn \neq 0$ and $mn \in M \cap N$ since $MN \subseteq M \cap N$. Hence $M \cap N$ is non-zero and contained in M and consequently, by Lemma 5.8, $M \cap N = MA$ where $A = M^{-1}(M \cap N)$ is an ideal of S. Now N is prime, $M \nsubseteq N$ and $MA \subseteq N$ so that $A \subseteq N$. Thus $M \cap N = MA \subseteq MN$ and $M \cap N = MN$ as required. □

Theorem 5.10. *Let S be an abundant order in a completely 0-simple semigroup Q with $S \neq Q$ and $\mathcal{F}(S)$ a group. Then the following conditions are equivalent:*

1. *S satisfies the ascending chain condition for ideals,*
2. *$\mathcal{F}(S)$ is abelian and every non-zero proper ideal of S can be written as a product of maximal ideals of S.*

Proof. Suppose that (1) holds. Since $S \neq Q$, we may assume, by Lemma 5.7, that S does have a proper non-zero ideal I. Since S satisfies the ascending chain condition for ideals, $I \subseteq M_1$ for some maximal ideal M_1. As in the proof of Lemma 5.8, $M_1^{-1}I$ is an ideal of S; also $I \subseteq M_1^{-1}I$ since $S \subseteq M_1^{-1}$. If $I = M_1^{-1}I$, then $S = II^{-1} = M_1^{-1}II^{-1} = M_1^{-1}S = M_1^{-1}$ so that $M_1 = SM_1 = M_1^{-1}M_1 = S$, a contradiction. Hence $I \subsetneqq M_1^{-1}I$.

If $M_1^{-1}I = S$, then $I = M_1$. Otherwise, $M_1^{-1}I \subseteq M_2$ for some maximal ideal M_2 and we obtain
$$I \subsetneqq M_1^{-1}I \subsetneqq M_2^{-1}M_1^{-1}I \subseteq S.$$

Continuing in this way we see that since S satisfies the ascending chain condition for ideals, $S = M_n^{-1} \dots M_1^{-1}I$ for some maximal ideals $M_1, \dots, M_n$. Hence $I = M_1 \dots M_n$.

Now let $J \in \mathcal{F}(S)$ with $J \neq S$. Putting $K = J^{-1} \cap S$ we have, as in the proof of Theorem 3.2, that KJ and K are non-zero ideals of S. By Remark 3.3, $(KJ)^{-1}K = J^{-1}$ and since

$\mathcal{F}(S)$ is a group, $J = K^{-1}(KJ)$. Since all non-zero proper ideals are products of maximal ideals, it follows that the group $\mathcal{F}(S)$ is generated by the maximal ideals of S. Hence, by Lemma 5.9, $\mathcal{F}(S)$ is abelian.

Now suppose that (2) holds and let I, J be non-zero proper ideals of S with $I \subseteq J$. By assumption, $I = M_1 \dots M_s$ and $J = N_1 \dots N_t$ for some maximal ideals $M_1, \dots, M_s, N_1, \dots, N_t$. Thus $M_1 \dots M_s \subseteq N_i$ for $i = 1, \dots, t$. By the primeness of N_i one of the M_j's is contained in N_i. We may assume that M_1 is contained in N_1 and since M_1 is a maximal ideal, $M_1 = N_1$. By Lemma 5.8, there is an ideal A of S such that

$$M_1 \dots M_s = I = JA = N_1 \dots N_t A.$$

Cancelling, we have $M_2 \dots M_s = N_2 \dots N_t A$. Continuing in this way we see that $t \leqslant s$ and that $M_i = N_i$ for $i = 1, \dots, t$. The ascending chain condition now follows easily. □

6. Orders In Brandt Semigroups

A *Brandt* semigroup is an inverse completely 0-simple semigroup. When we specialise to the case of orders in Brandt semigroups, two of the results of the previous section can be strengthened and simplified. This is partly because we have a converse of Proposition 5.3 and also because all maximal orders are abundant.

Proposition 6.1. *Let S be a maximal order in a Brandt semigroup Q. Then S is abundant.*

Proof. Put $T = E(Q) \cup S$ where $E(Q)$ is the semilattice of idempotents of Q. If $e \in E(Q)$ and $s \in S$, then $es = 0$ or $e\mathcal{R}s$ in Q so that $es = s$. Similarly, $se = 0$ or $se = s$. Thus T is a subsemigroup of Q and hence it is an order in Q. Since S is large we can choose an element in $S \cap H$ for each group $\mathcal{H}$-class H of Q and it is then easy to verify that $S \sim T$. Since $S \subseteq T$ and S is maximal, we have $S = T$ so that $E(Q) \subseteq S$ and S is abundant. □

We now give the converse of Proposition 5.3.

Proposition 6.2. *Let S be a maximal order in a Brandt semigroup Q and let $I \in \mathcal{F}(S)$. If I is a projective $O_\ell(I)$-act, then $I^{-1}I = O_r(I)$.*

Proof. Since S is abundant we must have $E(Q) \subseteq S$ and hence $O_\ell(I)$ is also full since $S \subseteq O_\ell(I)$. Thus if $i \in I$, then there is an element $e \in O_\ell(I)$ such that $ei = i$. By Proposition 4.5, I is a 0-direct union of cyclic $O_\ell(I)$-acts each of which is isomorphic to an idempotent generated principal left ideal of $O_\ell(I)$, say

$$I = \bigcup_{\lambda \in \Lambda} O_\ell(I)c_\lambda$$

for some index set Λ and non-zero elements c_λ of I. If $c_\lambda \mathcal{L} c_\mu$ in Q, then $c_\lambda = qc_\mu$ for some $q \in Q$. Now $q = a^\sharp b$ for some a, $b \in S$ with $a\mathcal{R}b$ in Q. Thus $ac_\lambda = aa^\sharp bc_\mu = bc_\mu \neq 0$ and as $S \subseteq O_\ell(I)$ we have

$$O_\ell(I)c_\lambda \cap O_\ell(I)c_\mu \neq 0.$$

Hence $\lambda = \mu$. Since I is large in Q, there is a c_λ in each $\mathcal{L}$-class of Q and so Λ indexes the $\mathcal{L}$-classes (and hence also the $\mathcal{R}$-classes) of Q.

Let c_μ^{-1} be the inverse of c_μ and note that for $x \in Q$ we have $xc_\mu^{-1}c_\mu \neq 0$ if and only if $x \in L_\mu$. Hence $c_\lambda c_\mu^{-1} = 0$ if $\lambda \neq \mu$. Consequently,

$$Ic_\mu^{-1} = \bigcup_{\lambda \in \Lambda} O_\ell(I)c_\lambda c_\mu^{-1} = O_\ell(I)c_\mu c_\mu^{-1} \subseteq O_\ell(I)$$

so that $c_\mu^{-1} \in I^{-1}$.

Clearly, $I^{-1}I \subseteq O_r(I)$. Let p be a non-zero element of $O_r(I)$ and suppose that $p \in R_\mu \cap L_\lambda$. Then $c_\mu p \in I$ since $c_\mu \in I$ and I is a right $O_r(I)$-act and so $p = c_\mu^{-1}c_\mu p \in I^{-1}I$. Thus $O_r(I) = I^{-1}I$ as required. □

Armed with these results we can now give the promised strengthening of some results in Section 5. First, corresponding to Theorem 5.5 we have the following theorem.

Theorem 6.3. *Let S be an abundant order in a Brandt semigroup Q. Then $\mathcal{F}(S)$ is a group if and only if S is maximal and weakly hereditary.*

Proof. If $\mathcal{F}(S)$ is a group, then, by Theorem 5.5, S is maximal and weakly hereditary.

Conversely, if S is maximal and weakly hereditary, then $O_\ell(I) = O_r(I) = S$ by maximality and hence by Proposition 6.2 and its left-right dual, $S = II^{-1} = I^{-1}I$ for all non-zero ideals I of S. It now follows from Theorem 3.2 that $\mathcal{F}(S)$ is a group. □

Next, corresponding to Theorem 5.10 we have the following result.

Theorem 6.4. *Let S be a maximal order in a Brandt semigroup Q with $S \neq Q$. Then the following conditions are equivalent:*

(1) *Every ideal of S is projective as a left S-act and right S-act and S satisfies the ascending chain condition for ideals,*
(2) *$\mathcal{F}(S)$ is an abelian group and every non-zero proper ideal of S can be written as a product of maximal ideals of S.*

Proof. By Proposition 6.1, S is abundant and so if (1) holds, then S is weakly hereditary so that by Theorem 6.3, $\mathcal{F}(S)$ is a group. That (2) follows from (1) is now immediate by Theorem 5.10.

Conversely, if (2) holds, then S is weakly hereditary by Theorem 6.3 and satisfies the ascending chain condition for ideals by Theorem 5.10. □

7. Examples

We conclude the paper with some examples to illustrate the theorems of Sections 3, 5 and 6. Note that if S is an order in a completely simple semigroup, then S^0 is an order in Q^0 and S is maximal or abundant or weakly hereditary if and only if S^0 has the same property. Thus the examples we give of orders in abelian groups are relevant to the theory we have developed.

We note that for a commutative semigroup, being weakly hereditary is the same as being hereditary and we recall the criterion from [3] for an order in an abelian group to be maximal.

Proposition 7.1. *A commutative cancellative semigroup C is a maximal order in its group of quotients G if and only if it satisfies the following condition:*
if $a \in C$, $g \in G$ are such that $ag^n \in C$ for all $n \geqslant 1$, then $g \in C$.

Our first example gives a maximal, weakly hereditary order which satisfies the ascending chain condition for ideals.

Example 7.1. It is noted in Example 2.2 of [3] that $C = \{a^k \mid k \in \mathbb{Z},\, k \geqslant 0\}$ is a maximal order in the infinite cyclic group G with generator a. It is easy to verify that the fractional C-ideals are precisely the sets I_m where $m \in \mathbb{Z}$ and $I_m = \{a^k \mid k \geqslant m\}$. Thus C satisfies the ascending chain condition for ideals and so by Theorem 6.4, $\mathcal{F}(C)$ is an abelian group generated by the unique maximal ideal of C. In fact, $I_m I_n = I_{m+n}$ so that the group $\mathcal{F}(C)$ is isomorphic to $\mathbb{Z}$.

The next example shows that an order can be maximal and abundant but not weakly hereditary.

Example 7.2. Let $S = \{x \in \mathbb{R} \mid x \geqslant 0\}$. Then S is a maximal order in the additive group $\mathbb{R}$. It can be verified that the fractional S-ideals are the sets

$$I_a = \{x \in \mathbb{R} \mid x \geqslant a\} \quad \text{and} \quad K_a = \{x \in \mathbb{R} \mid x > a\}$$

where $a \in \mathbb{R}$. The semigroup $\mathcal{F}(S)$ (with operation addition) is a chain of two groups $A \cup B$ where $A = \{I_a \mid a \in \mathbb{R}\}$ and $B = \{K_a \mid a \in \mathbb{R}\}$.

Although S is maximal and $S = I_0$ is the identity of $\mathcal{F}(S)$, condition (1) of Theorem 3.2 does not hold because $K_a^{-1} = K_{-a}$ and $K_a + K_{-a} = K_0 \neq S$.

We now give a simple example of a non-maximal order S for which $\mathcal{F}(S)$ is a group.

Example 7.3. In the notation of Example 7.1, let $S = I_1$. Then S is an order in G but is not maximal since it is equivalent to and strictly contained in C. The fractional S-ideals are the same as those of C so that $\mathcal{F}(S)$ is a group but the identity is C rather than S. Since S is not abundant, this example does not contradict Theorem 6.3.

The next example is a maximal, weakly hereditary order in a non-commutative Brandt semigroup.

Example 7.4 Let G, C be as in Example 7.1 and let P be the 2×2 identity matrix. Put $S = \mathcal{M}^0(C; \mathbf{2}, \mathbf{2}; P)$ and $Q = \mathcal{M}^0(G; \mathbf{2}, \mathbf{2}; P)$ where $\mathbf{2} = \{1, 2\}$. Then it can be verified that the fractional S-ideals are the sets $\mathcal{M}^0(I_m; \mathbf{2}, \mathbf{2}; P)$ where $m \in \mathbb{Z}$ and that $\mathcal{F}(S)$ is a group isomorphic to $\mathbb{Z}$. Since S is a full semigroup of Q, it is certainly abundant and so by Theorem 6.3, S is maximal and weakly hereditary. We could also deduce that S is maximal from Proposition 2.4 of [3].

We now give a collection of inequivalent maximal orders in the multiplicative group of positive rationals. Each is abundant but not hereditary.

Example 7.5. First, let $T = \{x \in \mathbb{Q} \mid 1 \leqslant x\}$. Then T is an order in $\mathbb{Q}^+$ and using Proposition 7.1, it is easy to verify that T is maximal. It is straightforward to show that a fractional T-ideal is one of $I_a = \{x \in \mathbb{Q} \mid x > a\}$ for some $a \in \mathbb{R}^+$ or $J_a = \{x \in \mathbb{Q} \mid x \geqslant a\}$

for some $a \in \mathbb{Q}^+$. The principal ideals of T are those J_a with $a \geqslant 1$ and since J_a is isomorphic to T (as a T-act) for any positive rational a, it is clear that T is abundant. However, for any real a with $a \geqslant 1$, the ideal I_a cannot be written as a disjoint union of principal ideals and so by Corollary 3.8 of [7], T is not weakly hereditary. The monoid $\mathcal{F}(T)$ is a chain of two groups with the group of units consisting of $\{J_a \mid a \in \mathbb{Q}^+\}$. Notice that T is semihereditary, that is, every finitely generated ideal is projective.

Next note that by Proposition 7.1, the order $\mathbb{P}$ consisting of the positive integers is maximal. It is not even semihereditary since, for example, the ideal $2\mathbb{P} \cup 3\mathbb{P}$ is not projective. It is easy to see that T and $\mathbb{P}$ are not equivalent.

Finally, for each prime p, let $V_p = \{\frac{n}{p^k} \mid n \in \mathbb{P},\ k \in \mathbb{P} \cup \{0\}\}$. Clearly, each V_p is an order in $\mathbb{Q}^+$ and, again using Proposition 7.1, we see that each V_p is maximal. No V_p is semihereditary since for distinct primes p, q, r the ideal generated by q and r is not projective. It is easy to see that V_p and V_q are inequivalent if $p \neq q$ and also that no V_p is equivalent to either T or $\mathbb{P}$.

References

[1] J. Fountain, *Abundant semigroups,* Proc. London Math. Soc. **44** (1982), 103–129.

[2] J. Fountain, *Maximal orders in semigroups,* in Semigroups, Automata and Languages (J. Almeida, G. M. S. Gomes, and P. V. Silva, eds.), World Scientific, 1995, pp. 111–124.

[3] J. Fountain, V. Gould and P. Smith, *Maximal orders in semigroups,* Comm. Alg., to appear.

[4] J. Fountain and M. Petrich, *Completely* 0*-simple semigroups of quotients*, J. Algebra **101** (1986), 365–402.

[5] V. Gould, *Orders in semigroups*, in Contributions to General Algebra **5**, Proceedings of the Salzburg Conference, May 29–June 1, 1986, Verlag Hölder-Pichler-Tempsky, Vienna, 1987, pp. 163–169.

[6] J. M. Howie, *Fundamentals of semigroup theory*, Oxford University Press, 1995.

[7] U. Knauer, *Projectivity of acts and Morita equivalence of monoids*, Semigroup Forum **3** (1972), 359–370.