PROPER COVERS OF AMPLE MONOIDS *

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Abstract

Proper ample monoids are described by means of a certain category acted upon on both sides by a cancellative monoid. Making use of this characterization, we show that every ample monoid S has a proper ample cover, which can be taken to be finite whenever S is finite.

Introduction

On an inverse monoid S, we define a unary operation $^+$ by $a^+ = a\,a^{-1}$ for $a \in S$ thus making S into a (2,1,0)-algebra. Similarly, there is a dual operation * given by $a^* = a^{-1}a$. Of course, S also has the unary operation $^{-1}$, but this will not play a significant role in the paper.

A *left ample monoid* can now be defined to be a monoid S with a unary operation $^+$ such that there is a $+$-embedding (that is, a (2,1,0)-algebra embedding) of S into an inverse monoid. As usual, $E(S)$ denotes the set of idempotents of S. It is immediate from the definition that, in a left ample monoid, the idempotents commute with each other, and so $E(S)$ is a subsemilattice of S.

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Similarly, we say that a monoid S with a unary operation * and a $*$-embedding into an inverse monoid is a *right ample monoid*. An *ample monoid* is one which is both left and right ample. We emphasise that the $+$ and $*$-embeddings need not be into the same inverse monoid.

It is clear that every inverse monoid is an ample monoid with the operations $^+$ and * defined as above. Cancellative monoids and full submonoids of inverse monoids are also ample.

On a left ample monoid S, there is a least congruence σ such that S/σ is a right cancellative monoid. We say that S is *left proper* if for all elements a and b of S such that $a^+ = b^+$ and $a \,\sigma\, b$, we have $a = b$. In the case of right ample monoids, there is the dual notion of *right proper*, and if S is an ample monoid, then S/σ is cancellative and S is said to be *proper* if it is both left and right proper.

It is well known that an inverse monoid is proper if and only if it is E-unitary (see, for example, [6, Proposition 5.9.1]). Example 3 of [2] shows that the corresponding statement does not hold for ample monoids.

In [4] we used techniques of Margolis and Pin [9] to describe proper left ample monoids. In the present paper our first objective is to extend these methods so that they can be applied to proper ample monoids. To do this we consider certain small categories with actions on both the right and the left by cancellative monoids. We introduce the formal definitions in Section 1 and use them in Section 2 to give a structure theorem for proper [E-unitary] ample monoids.

Let S be an ample monoid. We say that a proper ample monoid P is a *(proper) cover of* S if there is a map from P onto S which is simultaneously a $+$-homomorphism and a $*$-homomorphism of monoids that maps $E(P)$ isomorphically onto $E(S)$.

In [7] Lawson showed that every ample monoid S has a proper ample cover P. This is also a consequence of a more general result of Simmons [12]. However, the constructions of Lawson and Simmons always yield an infinite monoid P, even when S is finite. One purpose of the present paper is to show that finite ample monoids have finite proper covers. In Section 3, we present a proof of Lawson's result which does yield a finite cover P whenever S is finite. We thus provide a unified proof for the general and the finite cases. Our stronger result is crucial in [5], where we show that a bountiful finite semigroup belongs to the pseudovariety $\mathbf{A} \vee \mathbf{G}$ if and only if $\mathscr{H} \subseteq \mu$, where μ is the greatest congruence contained in the equivalence $\mathscr{H}^*$ (defined in Section 1).

1 Preliminaries

For basic notation and terminology regarding monoids and semigroups, we follow [6].

We start by giving an alternative characterization of ampleness.

On a monoid S a relation $\mathscr{R}^*$ is defined by the rule $a \mathscr{R}^* b$ if and only if a and b are $\mathscr{R}$-related in an extension of S. The dual of $\mathscr{R}^*$ is $\mathscr{L}^*$, and $\mathscr{H}^* = \mathscr{R}^* \cap \mathscr{L}^*$. From [10] and [11], we have

Lemma 1.1 *On a monoid S, for $a, b \in S$, the following are equivalent:*

a) $a \mathscr{R}^* b$;

b) *for all $x, y \in S$, $xa = ya$ if and only if $xb = yb$.*

This condition simplifies when one element is idempotent.

Lemma 1.2 *On a monoid S, for $a \in S$ and $e \in E(S)$, the following are equivalent:*

a) $a \mathscr{R}^* e$;

b) *$ea = a$ and for all $x, y \in S$, $xa = ya$ implies $xe = ye$.*

We recall that $\mathscr{R}^*$ is a left congruence and $\mathscr{R}$ is contained in $\mathscr{R}^*$. If S is regular, then $\mathscr{R}^* = \mathscr{R}$. The $\mathscr{R}^*$-class of an element $a \in S$ is denoted by R^*_a. If $E(S)$ is a semilattice, then R^*_a can contain at most one idempotent. When such an idempotent exists it is denoted by a^+. Similar results hold for $\mathscr{L}^*$, and when there is a unique idempotent in an $\mathscr{L}^*$-class L^*_a, it is denoted by a^*.

It follows from [2, Proposition 1.2] that

Theorem 1.3 *A monoid S is ample if and only if, for all $a \in S$ and $e \in E$,*

a) *every $\mathscr{R}^*$-class R^*_a contains a (unique) idempotent a^+;*

a′) *every $\mathscr{L}^*$-class L^*_a contains a (unique) idempotent a^*;*

b) $ae = (ae)^+ a$;

b′) $ea = a(ea)^*$;

c) *$E(S)$ is a semilattice.*

From [2, Lemma 1.3] and its dual, we conclude that, on an ample monoid S, there is a *minimum cancellative monoid congruence* σ defined by, for all $a, b \in S$,

$$\begin{aligned} a \,\sigma\, b \qquad & \text{if and only if } \; ae = be, \;\; \text{for some } e \in E(S) \\ & \text{if and only if } \; ea = eb, \;\; \text{for some } e \in E(S). \end{aligned}$$

An ample monoid is said to be *left* [*right*] *proper* if $\sigma \cap \mathscr{R}^* = \iota$ [$\sigma \cap \mathscr{L}^* = \iota$]. We say that it is *proper* if it is both left and right proper. It is clear that these definitions agree with those given in the introduction.

Observe that in an ample monoid any two idempotents are σ-related and $E(S)$ is contained in the identity of S/σ. Moreover, $E(S)$ is a σ-class if and only if S is E-unitary Also, we have that any proper ample monoid is E-unitary but the converse is not true [2, Example 3].

We now extend the notion of ample to categories as we did in [3] for the left ample (formerly left type-A) case.

Let $\mathscr{C}$ be a (small) category with set of *objects* $\operatorname{Obj}\mathscr{C}$ and set of *morphisms* $\operatorname{Mor}\mathscr{C}$. For all $u \in \operatorname{Obj}\mathscr{C}$, the set of morphisms with *domain* [*codomain*] u is denoted by $\operatorname{Mor}(u,-)$ [$\operatorname{Mor}(-,u)$]. We use additive notation for *composition* of morphisms and represent the *identity* at an object u by 0_u. For each object u, the set $\operatorname{Mor}(u,u)$ is a monoid with identity 0_u called the *local monoid* at u.

On $\operatorname{Mor}\mathscr{C}$, we define a relation $\mathscr{R}^*$ as follows, for all $p,q \in \operatorname{Mor}\mathscr{C}$,

$$(p,q) \in \mathscr{R}^* \text{ if and only if } [(\text{for all } s,t \in \operatorname{Mor}\mathscr{C})\ \ s+p=t+p \ \Leftrightarrow\ s+q=t+q]\,,$$

where we make the convention that writing $s+p=t+p$ implies that both $s+p$ and $t+p$ are defined. Dually we define a relation $\mathscr{L}^*$.

An element $p \in \operatorname{Mor}\mathscr{C}$ is said to be *idempotent* if $p+p$ is defined and $p=p+p$. A category is *locally idempotent* if every element of each local monoid is idempotent.

The following lemma is easily proved.

Lemma 1.4 *Let $\mathscr{C}$ be a category, $u \in \operatorname{Obj}\mathscr{C}$ and $p,q \in \operatorname{Mor}\mathscr{C}$. Then*

a) *if $p \in \operatorname{Mor}(u,-)$ and $(p,q) \in \mathscr{R}^*$, then $q \in \operatorname{Mor}(u,-)$;*

b) *if p is idempotent and $p \in \operatorname{Mor}(u,v)$, then $u=v$;*

c) *if p is idempotent, then*

$$(p,q) \in \mathscr{R}^* \textit{ if and only if } \begin{cases} q = p+q\ , & \textit{and} \\ \textit{for all } s,t \in \operatorname{Mor}\mathscr{C}, & s+q=t+q \ \Rightarrow\ s+p=t+p\ ; \end{cases}$$

d) *$\mathscr{R}^*$ is a left congruence on the partial semigroup $\operatorname{Mor}\mathscr{C}$.*

Clearly, the dual of Lemma 1.4 holds for $\mathscr{L}^*$.

A category $\mathscr{C}$ is said to be *ample* if, for all $u \in \operatorname{Obj}\mathscr{C}$, the set $E(\operatorname{Mor}(u,u))$ forms a semilattice; each $\mathscr{R}^*$-class R_p^* contains an idempotent p^+ which is necessarily unique; each $\mathscr{L}^*$-class L_p^* contains an idempotent p^* which is necessarily unique; and, for all $p,q \in \operatorname{Mor}\mathscr{C}$,

$$q+p^+ = (q+p^+)^+ + q$$

and

$$p^* + q = q + (p^*+q)^*.$$

Thus a category is ample if and only if it is both left and right ample (or type-A) [4]. Note that each local monoid of an ample category is ample, and also that in an ample category, $(p^+)^+ = p^+$ and $(p^*)^* = p^*$ for each morphism p.

An ample category is said to be *proper* if, for all $p,q \in \operatorname{Mor}(u,v)$,

$$p^+ = q^+ \ \Rightarrow\ p=q \qquad (\textit{left proper})$$

and

$$p^* = q^* \ \Rightarrow\ p=q \qquad (\textit{right proper}).$$

It follows immediately from the preceding paragraph that if an ample category is proper, then it is locally idempotent, for if p is in a local monoid, then p^+ is in the same monoid and so $p^+ = p$ follows from $(p^+)^+ = p^+$.

As in [4] we say that *a monoid B acts on the right on a category $\mathscr{C}$* if there exist maps $\operatorname{Obj}\mathscr{C} \times B \to \operatorname{Obj}\mathscr{C}$ and $\operatorname{Mor}\mathscr{C} \times B \to \operatorname{Mor}\mathscr{C}$ where we write xb for the result of the action of the monoid element b on an object or morphism x, and the following conditions are satisfied:

a) if $p \in \operatorname{Mor}(u, v)$, then $pb \in \operatorname{Mor}(ub, vb)$;

b) for all $b, b_1, b_2 \in B$, $p, q \in \operatorname{Mor}\mathscr{C}$, $u \in \operatorname{Obj}\mathscr{C}$,

- **i)** $(p + q)b = pb + qb$;
- **ii)** $(pb_1)b_2 = p(b_1b_2)$;
- **iii)** $p\,1 = p$;
- **iv)** $0_u b = 0_{ub}$.

When B is right cancellative, we always assume that the action is *right cancellative*, that is, it satisfies the following conditions:

c) for all $p, q \in \operatorname{Mor}\mathscr{C}$ and $b \in B$,

$$pb = qb \quad \Rightarrow \quad p = q;$$

d) for all $u \in \operatorname{Obj}\mathscr{C}$ and $b \in B$, if $q \in \operatorname{Mor}(w, ub)$, then $w = vb$ for some object v and $q = pb$ for some $p \in \operatorname{Mor}(v, u)$.

Dually, we can define the notion of a left action of a left cancellative monoid A on a category $\mathscr{C}$, and we always assume that such an action is left cancellative.

If monoids A and B act on $\mathscr{C}$ simultaneously, we require the actions to be *compatible*, that is, for all $p \in \operatorname{Mor}\mathscr{C}$, $a \in A$ and $b \in B$,

$$(ap)b = a(pb).$$

In this case we denote $(ap)b$ by apb.

If A and B are cancellative and $\mathscr{C}$ is ample, it follows from the proof of [12, Proposition 25] or is easily proved directly that for all $p \in \operatorname{Mor}\mathscr{C}$, $a \in A$ and $b \in B$,

$$(ap)^+ = ap^+; \quad (ap)^* = ap^*; \quad (pb)^* = p^*b \quad \text{and} \quad (pb)^+ = p^+b.$$

An ample category $\mathscr{C}$ acted upon on the left by a cancellative monoid A and on the right by a cancellative monoid B is said to be an *ample (A, B)-category* if $\operatorname{Obj}\mathscr{C} = A \times B$ and, for all $(u, v) \in \operatorname{Obj}\mathscr{C}$, $a \in A$ and $b \in B$,

$$a(u, v) = (au, v) \quad \text{and} \quad (u, v)b = (u, vb).$$

We note that an ample (A, B)-category is locally idempotent if and only if $\mathscr{M} = \operatorname{Mor}((1, 1), (1, 1))$ is a semilattice. For, if $a \in A$, $b \in B$ and $q \in \operatorname{Mor}((a, b), (a, b))$, then $(a, b) = a(1, 1)b$ so that $q = apb$ for some $p \in \mathscr{M}$. Hence $q + q = apb + apb = a(p + p)b = apb = q$.

2 Proper ample monoids

The approach in this section is a slight variation on that of Simmons [12] where he obtains much more general results. For completeness, we give proofs of our structure theorems for ample, E-unitary ample and proper ample monoids. Rather than using the general notion of double semidirect product as in [12], we use the special case of the ample (A, B)-categories introduced at the end of the preceding section.

We start by showing how to associate a monoid $C_{(1,1)}$ to an (A, B)-category. Given an ample (A, B)-category $\mathscr{C}$, let

$$C_{(1,1)} = \{(b, p, a)\colon\ b \in B,\ a \in A,\ p \in \mathrm{Mor}((1, b), (a, 1))\}$$

and, for $(b, p, a), (d, q, c) \in C_{(1,1)}$, define

$$(b, p, a)\,(d, q, c) = (bd,\ pd + aq,\ ac).$$

Notice that this operation is well defined since if p and q are in $\mathrm{Mor}((1, b), (a, 1))$ and $\mathrm{Mor}((1, d), (c, 1))$ respectively, then $pd \in \mathrm{Mor}((1, bd), (a, d))$ and $aq \in \mathrm{Mor}((a, d), (ac, 1))$ so that $pd + aq$ is defined and belongs to $\mathrm{Mor}((1, bd), (ac, 1))$.

Theorem 2.1 *Let A and B be cancellative monoids. If $\mathscr{C}$ is an ample (A, B)-category, then $C_{(1,1)}$ is an ample monoid. Moreover, if $\mathscr{C}$ is locally idempotent, then $C_{(1,1)}$ is E-unitary, and if $\mathscr{C}$ is proper, then so is $C_{(1,1)}$.*

Proof: Throughout the proof, we denote $\mathrm{Mor}((1, 1), (1, 1))$ by $\mathscr{M}$.

First, it is routine to show that the binary operation defined on $C_{(1,1)}$ is associative with identity element $(1, 0_{(1,1)}, 1)$.

Since both A and B are cancellative monoids, we get

$$E(C_{(1,1)}) = \{(1, p, 1)\colon\ p \in E(\mathscr{M})\}$$

and so it is clear that $E(C_{(1,1)})$ is isomorphic to $E(\mathscr{M})$, which is a semilattice since $\mathscr{C}$ is ample.

Given $p \in \mathrm{Mor}((1, b), (a, 1))$ there exists a unique idempotent p^+ in its $\mathscr{R}^*$-class and $p^+ \in \mathrm{Mor}((1, b), (1, b))$. As the action of B on $\mathscr{C}$ is right cancellative and $(1, b) = (1, 1)b$, there exists a unique $p_0^+ \in \mathscr{M}$ such that $p^+ = p_0^+ b$. Again by the right cancellativity of the action, p_0^+ is also idempotent. Now

$$(1, p_0^+, 1)\,(b, p, a) = (b, p_0^+ b + p, a) = (b, p, a).$$

Suppose that, for some (d_1, p_1, c_1) and $(d_2, p_2, c_2) \in \mathrm{Mor}\,\mathscr{C}$,

$$(d_1, p_1, c_1)\,(b, p, a) = (d_2, p_2, c_2)\,(b, p, a).$$

Then

$$(d_1 b,\ p_1 b + c_1 p,\ c_1 a) = (d_2 b,\ p_2 b + c_2 p,\ c_2 a)$$

and so

$$d_1 b = d_2 b\,, \qquad c_1 a = c_2 a \qquad \text{and} \qquad p_1 b + c_1 p = p_2 b + c_2 p.$$

Since A and B are right cancellative, $d_1 = d_2$ and $c_1 = c_2$. Thus $p_1 b + c_1 p = p_2 b + c_1 p$. Now

$$(c_1 p)^+ = c_1 p^+ \qquad \text{and so} \qquad p_1 b + c_1 p^+ = p_2 b + c_1 p^+.$$

But $p^+ = p_0^+ b$ and the actions are compatible, so by the right cancellativity of the action of B, we get

$$p_1 + c_1 p_0^+ \;=\; p_2 + c_1 p_0^+.$$

Therefore

$$(d_1, p_1, c_1)\,(1, p_0^+, 1) = (d_2, p_2, c_2)\,(1, p_0^+, 1)$$

and so the idempotent $(1, p_0^+, 1)$ is $\mathscr{R}^*$-related to (b, p, a).

Next, for $(1, r, 1) \in E(C_{(1,1)})$ and $(b, p, a) \in C_{(1,1)}$, we have

$$\begin{aligned}
((b,p,a)\,(1,r,1))^+\;(b,p,a) &= (b,\; p + ar,\; a)^+\;(b,p,a)\\
&= \big(1,\; (p+ar)_0^+,\; 1\big)\;(b,p,a)\\
&= \big(b,\; (p+ar)_0^+ b + p,\; a\big)\\
&= \big(b,\; (p+ar)^+ + p,\; a\big)\\
&= (b,\; p+ar,\; a)\\
&= (b,p,a)\,(1,r,1).
\end{aligned}$$

Thus $C_{(1,1)}$ is left ample. The proof of the dual properties for $\mathscr{L}^*$ is similar. We conclude that $C_{(1,1)}$ is an ample monoid.

Now suppose that $\mathscr{C}$ is locally idempotent and let $(b,p,a), (1,q,1) \in C_{(1,1)}$ be such that $(b,p,a)(1,q,1)$ is idempotent, that is, $(b, p+aq, a)$ is idempotent. Then $b = 1$ and $a = 1$ so that p is in $\mathscr{M}$. By assumption, p is idempotent and hence so is (b,p,a). Thus $C_{(1,1)}$ is E-unitary.

It remains to show that $C_{(1,1)}$ is proper whenever $\mathscr{C}$ is proper. Suppose that $\mathscr{C}$ is proper and that $(b,p,a), (d,q,c) \in C_{(1,1)}$ are σ-related and $\mathscr{R}^*$-related. Then $(1, p_0^+, 1) = (1, q_0^+, 1)$, and so $p_0^+ = q_0^+$.

On the other hand, there is an idempotent $(1, r, 1)$ such that

$$(b,q,a)\,(1,r,1) = (d,q,c)\,(1,r,1).$$

Thus $b = d$ and $a = c$. Hence $p, q \in \mathrm{Mor}((1,b),(a,1))$ and $p^+ = p_0^+ b = q_0^+ b = q_0^+ d = q^+$. As $\mathscr{C}$ is left proper, $p = q$. Therefore $\mathscr{R}^* \cap \sigma = \iota$ in $C_{(1,1)}$.

Similarly, we can show that $\mathscr{L}^* \cap \sigma = \iota$ in $C_{(1,1)}$, and so $C_{(1,1)}$ is proper. ∎

Our next task is to prove that any [proper, E-unitary] ample monoid is isomorphic to a suitable monoid $C_{(1,1)}$.

Theorem 2.2 *A monoid M with unary operations * and $^+$ is ample if and only if it is $(2,1,1,0)$-isomorphic to a monoid $C_{(1,1)}$ associated to an ample (A,B)-category, where A and B are cancellative monoids. Moreover, it is proper [E-unitary] if and only if the (A,B)-category is proper [locally idempotent].*

Proof: In view of the previous theorem it remains to show the direct part.

Suppose that M is an ample monoid. Let σ be the least cancellative congruence on M and put $A=B=M/\sigma$. We define the *derived* (A,B)*-category* $\mathscr{D}$ of the canonical epimorphism $\theta\colon M \to M/\sigma$ given by $a\theta = a\sigma$ as follows:

Obj $\mathscr{D} = A\times B$ and, for $(a_1,b_1),(a_2,b_2) \in A\times B$,

$$\mathrm{Mor}\,((a_1,b_1),(a_2,b_2)) = \{((a_1,b_1),m,(a_2,b_2))\colon\ a_1(m\theta) = a_2,\ b_1 = (m\theta)b_2\}$$

with composition defined by

$$((a_1,b_1),m,(a_2,b_2)) + ((a_2,b_2),n,(a_3,b_3)) = ((a_1,b_1),mn,(a_3,b_3))\,.$$

The action of A on the left is defined as follows:

for $a \in A$ and $((a_1,b_1),m,(a_2,b_2)) \in \mathrm{Mor}\,\mathscr{D}$,

$$a\,((a_1,b_1),m,(a_2,b_2)) = ((aa_1,b_1),m,(aa_2,b_2))\,.$$

Similarly, we define the action of B on the right.

It is routine to prove that the composition is well defined and associative. Given (a,b) in Obj $\mathscr{D}$, we have $0_{(a,b)} = ((a,b),1_M,(a,b))$, where 1_M is the identity of M.

Let $(a,b) \in \mathrm{Obj}\,\mathscr{D}$. Then, since $E(M)$ is a semilattice, so is

$$E\,(\mathrm{Mor}((a,b),(a,b))) = \{((a,b),m,(a,b))\colon\ m \in E(M)\}\,.$$

To prove that $\mathscr{D}$ is left ample is also routine after noticing that for any $((a_1,b_1),m,(a_2,b_2)) \in \mathrm{Mor}\,\mathscr{D}$

$$((a_1,b_1),m,(a_2,b_2))^+ = \big((a_1,b_1),m^+,(a_1,b_1)\big)\,.$$

A dual argument shows that $\mathscr{D}$ is right ample, and thus $\mathscr{D}$ is ample.

To conclude that $\mathscr{D}$ is an (A,B)-category, it remains to analyse the properties of the actions, but these are straightforward to verify.

Next we show that M and $C_{(1,1)}$ are isomorphic as $(2,1,1,0)$-algebras.

Observe that

$$\begin{aligned} C_{(1,1)} &= \{(b,((1,b),m,(a,1))\,,a)\colon\ a\in A,\ \ b\in B,\ \ b=m\theta=a\} \\ &= \{(m\theta,((1,m\theta),m,(m\theta,1))\,,m\theta)\colon\ m\in M\} \end{aligned}$$

and define $\psi\colon\ M \to C_{(1,1)}$ by putting

$$m\psi = (m\theta,((1,m\theta),m,(m\theta,1))\,,m\theta)\,.$$

Clearly ψ is both injective and onto. To prove that ψ is a monoid homomorphism is straightforward.

To show that ψ respects the unary operation $^+$, we consider $m \in M$ and $p = ((1,m\theta),m,(m\theta,1))$. Then

$$\begin{aligned} (m\psi)^+ &= (m\theta,((1,m\theta),m,(m\theta,1))\,,m\theta)^+ \\ &= (1,p_0^+,1), \end{aligned}$$

where $p^+ = ((1, m\theta), m^+, (1, m\theta))$ and $p_0^+ = ((1,1), m^+, (1,1))$. Hence $(m\psi)^+ = m^+\psi$.

Similarly, one can show that ψ respects the operation *. Therefore ψ is a $(2,1,1,0)$-isomorphism, as required.

Suppose that M is E-unitary. Then $E(M)$ is the identity of M/σ and, for any object (a,b) of $\mathscr{D}$,

$$\begin{aligned}
\mathrm{Mor}\,((a,b),(a,b)) &= \{((a,b), m, (a,b)) \colon\ m\theta = 1\} \\
&= \{((a,b), m, (a,b)) \colon\ m \in E(M)\} \\
&\simeq E(M).
\end{aligned}$$

Thus $\mathscr{D}$ is locally idempotent.

As in [4, Theorem 2.2], we can show that $\mathscr{D}$ is left proper [and dually that it is right proper], whenever M is left [right] proper. ∎

3 Proper covers of ample monoids

This section is devoted to proving that any [finite] ample monoid has a [finite] proper ample cover. We use relational morphisms to associate an ample (A,B)-category with an ample monoid M in such a way that the proper ample monoid $C_{(1,1)}$ is a cover of M.

Let M be an ample monoid. Recall that a [proper] ample monoid P is said to be a [*proper*] *cover* of M if there exists a $(2,1,1,0)$-morphism θ from P onto M that separates idempotents. We remark that the latter condition is equivalent to the restriction of θ to $E(P)$ being an isomorphism onto $E(M)$.

For any subset X of a left ample monoid T, we put $X^+ = \{x^+ : x \in X\}$. A *relational morphism* $\tau : M \multimap T$ of left ample monoids M and T is a relational morphism of $(2,1,0)$-algebras, that is, τ maps M to $\mathscr{P}(T)$ and, for all $m, n \in M$, we have

(1) $m\tau \neq \emptyset$,

(2) $(m\tau)^+ \subseteq m^+\tau$,

(3) $(m\tau)(n\tau) \subseteq (mn)\tau$,

(4) $1_T \in 1_M\tau$.

These conditions ensure that the graph of τ, which we denote by $\mathrm{gr}(\tau)$, is a $(2,1)$-subalgebra of the direct product $M \times T$, and that the projection of $\mathrm{gr}(\tau)$ to M is surjective. In particular, $\mathrm{gr}(\tau)$ is a left ample submonoid of $M \times T$. We also note that if T is right cancellative, then $1_T \in e\tau$ for every idempotent e of M. For, $e\tau$ is not empty, and if $t \in e\tau$, then $1_T = t^+ \in (e\tau)^+ \subseteq e\tau$.

Given a relational morphism $\tau : M \multimap T$ from a left ample monoid M to a right cancellative monoid T, we say that τ is *left proper* if, for all $m, n \in M$,

$$m\tau \cap n\tau \neq \emptyset \ \text{ implies } \ n^+m = m^+n.$$

We remark that, as noted in [1], a left proper relational morphism $\tau : M \multimap T$ is *idempotent pure* in the sense that if $1_T \in m\tau$, then m is idempotent.

For a right ample monoid M and a left cancellative monoid T, we have the dual notion of a *right proper* relational morphism τ and such a τ is also idempotent pure.

In the following lemma, we give an alternative characterisation of a left proper relational morphism which looks more like the definition of a left proper ample category.

Lemma 3.1 *A relational morphism $\tau : M \multimap T$ from a left ample monoid M to a right cancellative monoid T is left proper if and only if*

$$m\tau \cap n\tau \neq \emptyset \quad \textit{and } m^+ = n^+ \textit{ imply } \quad m = n.$$

Proof: If τ is left proper, and if $m\tau \cap n\tau \neq \emptyset$ and $m^+ = n^+$, then $m = m^+m = n^+m = m^+n = n^+n = n$.

Conversely, suppose that the condition of the lemma holds and that $m\tau \cap n\tau \neq \emptyset$. Since $1 \in e\tau$ for any idempotent in M, we have

$$m\tau \cap n\tau \subseteq (n^+\tau)(m\tau) \cap (m^+\tau)(n\tau) \subseteq (n^+m\tau) \cap (m^+n\tau)$$

so that the latter is nonempty. Now $(n^+m)^+ = m^+n^+ = n^+m^+ = (m^+n)^+$, and so the condition gives $n^+m = m^+n$. ∎

Now let M be an ample monoid, A, B be cancellative monoids, $\tau_1 : M \multimap A$ be a relational morphism of right ample monoids, and $\tau_2 : M \multimap B$ be a relational morphism of left ample monoids. We construct an ample (A,B)-category $\mathscr{X} = \mathscr{X}_{(\tau_1,\tau_2)}$. Let Obj $\mathscr{X} = A \times B$. For $(a_1,b_1),(a_2,b_2) \in A\times B$,

$$\mathrm{Mor}\,((a_1,b_1),(a_2,b_2)) = \{((a_1,b_1),m,(a_2,b_2))\colon\ a_2 \in a_1(m\tau_1),\ b_1 \in (m\tau_2)b_2\}\,.$$

Composition is given as in the derived (A,B)-category of Section 2 as follows: for $((a_1,b_1),m,(a_2,b_2))$, $((a_2,b_2),n,(a_3,b_3)) \in \mathrm{Mor}\,\mathscr{X}$,

$$((a_1,b_1),m,(a_2,b_2)) + ((a_2,b_2),n,(a_3,b_3)) = ((a_1,b_1),mn,(a_3,b_3))\,.$$

It is routine to check that the composition is well defined and associative. For any idempotent e of M and elements a and b of A and B respectively, we have $a = a1_A \in a(e\tau_1)$ and $b = 1_Bb \in (e\tau_2)b$, and it follows that

$$E\,(\mathrm{Mor}\,((a,b),(a,b))) \;=\; \{((a,b),e,(a,b))\colon\ e \in E(M)\}\,.$$

It is obvious then that $E(\mathrm{Mor}((a,b),(a,b)))$ is isomorphic to the semilattice $E(M)$.

Now let $p = ((a_1,b_1),m,(a_2,b_2)) \in \mathrm{Mor}\,\mathscr{X}$. Then $m^*, m^+ \in E(M)$ and consequently, $((a_1,b_1),m^+,(a_1,b_1))$ and $((a_2,b_2),m^*,(a_2,b_2))$ are idempotents of Mor $\mathscr{X}$. The proof that the first is $\mathscr{R}^*$-related to p, and the second is $\mathscr{L}^*$-related to p, follows as in the case of the derived (A,B)-category $\mathscr{D}$. Also, as with $\mathscr{D}$, we prove that $\mathscr{X}$ is ample and define an action on the right [left] of $\mathscr{D}$ by B $[A]$ as follows: for all $(a_1, b_1) \in \operatorname{Obj} \mathscr{X}$, $a \in A$, $b \in B$, $((a_1, b_1), m, (a_2, b_2)) \in \operatorname{Mor} \mathscr{C}$,

$$\begin{aligned} (a_1, b_1)b \;&=\; (a_1, b_1 b) \;\text{ and } a(a_1, b_1) \;=\; (aa_1, b_1); \\ ((a_1, b_1), m, (a_2, b_2))\, b \;&=\; ((a_1, b_1 b), m, (a_2, b_2 b))\,; \\ a\,((a_1, b_1), m, (a_2, b_2)) \;&=\; ((aa_1, b_1), m, (aa_2, b_2))\,. \end{aligned}$$

Notice that the actions are well defined. For example, the action of B is well defined because if $b_1 \in (m\tau_2)b_2$, then $b_1 b \in (m\tau_2)b_2 b$.

It is routine to verify the properties of the actions and we conclude that $\mathscr{X}$ is an ample (A, B)-category.

If τ_1 is left proper and τ_2 is right proper, then the (A, B)-category $\mathscr{X}$ is proper. To show that $\mathscr{X}$ is left proper, suppose that $p, q \in \operatorname{Mor}((a_1, b_1), (a_2, b_2))$ and $p^+ = q^+$. Then $p = ((a_1, b_2), m, (a_2, b_2))$ and $q = ((a_1, b_1), n, (a_2, b_2))$, for some $m, n \in M$, and $m^+ = n^+$. Now, $a_2 = a_1 x$, for some $x \in m\tau_1$ and $a_2 = a_1 y$, for some $y \in n\tau_1$. Since A is cancellative, $x = y$ and so $m\tau_1 \cap n\tau_1 \neq \emptyset$. As $m^+ = n^+$ and τ_1 is left proper, it follows by Lemma 3.1 that $m = n$. Thus $p = q$, as required. Similarly, $\mathscr{X}$ is right proper, and so it is proper.

Theorem 3.2 *Any [finite] ample monoid has a [finite] proper ample cover.*

Proof: Let M be an ample monoid. Since M is left ample, it follows from [2, Proposition 1.2] that there is a $^+$-embedding (of monoids) $\rho : M \to \mathscr{I}(M)$ into the symmetric inverse monoid on M which maps an element m of M to ρ_m where $\rho_m : Mm^+ \to Mm$ is given by $x\rho_m = xm$.

Let $X = M$ if M is finite, and if M is infinite, let $X = M \cup N$ where N is a set of the same cardinality as M and disjoint from M. Then $\mathscr{I}(M)$ is an inverse subsemigroup of $\mathscr{I}(X)$ and so we may regard ρ as a $+$-embedding (of semigroups) of M into $\mathscr{I}(X)$. It follows from [8, Proposition 2.1.1] that, for every element φ of $\mathscr{I}(M)$ there is a unit (that is, a bijection) α of $\mathscr{I}(X)$ such that $\varphi \leqslant \alpha$ where $\leqslant$ is the natural partial order on $\mathscr{I}(X)$.

Dually, as M is right ample, we have a $*$-embedding of semigroups $\lambda : M \to \mathscr{I}(X)^*$ mapping an element m of M to the partial bijection $\lambda_m : m^*M \to mM$ given by $\lambda_m(x) = mx$.

Denote by $\mathscr{G}(X)$ the group of units of $\mathscr{I}(X)$, that is, the symmetric group on X with the maps written on the right of their arguments. Its left-right dual $\mathscr{G}(X)^*$ is the group of units of $\mathscr{I}(X)^*$.

Now let

$$H_\rho = \{\alpha \in \mathscr{G}(X) \colon\ \rho_m \leqslant \alpha \text{ for some } m \in M\}$$

and

$$H_\lambda = \{\alpha \in \mathscr{G}(X)^* \colon\ \lambda_m \leqslant \alpha \text{ for some } m \in M\}\,.$$

If $\alpha, \beta \in H_\rho$, then $\rho_m \leqslant \alpha$ and $\rho_n \leqslant \beta$, for some $m, n \in M$. Hence $\rho_{mn} = \rho_m\, \rho_n \leqslant \alpha\beta$ and so $\alpha\beta \in H_\rho$. Also $I_X \in H_\rho$ since $\rho_{1_M} \leqslant I_X$. Thus H_ρ is a cancellative monoid, and if M is finite, then H_ρ is a group.

Similarly, H_λ is a cancellative monoid which is a group if M is finite.

Next, we define relational morphisms $\tau_\rho\colon M \multimap H_\rho$ and $\tau_\lambda\colon M \multimap H_\lambda$ by

$$m\tau_\rho = \{\alpha \in H_\rho\colon \rho_m \leqslant \alpha\} \text{ and } m\tau_\lambda = \{\alpha \in H_\lambda\colon \lambda_m \leqslant \alpha\}.$$

We show that τ_ρ is a left proper relational morphism of left ample monoids. We have already observed that for each $m \in M$ there is a unit α of $\mathscr{I}(X)$ with $\rho_m \leqslant \alpha$ so that $\alpha \in m\tau_\rho$ and $m\tau_\rho$ is not empty. If $e \in E(M)$, then ρ_e is idempotent in $\mathscr{I}(X)$, so $I_X \in e\tau_\rho$ since I_X is the greatest idempotent in $\mathscr{I}(X)$. In particular, $I_X \in 1_M\tau_\rho$, and $(m\tau_\rho)^+ = \{I_X\} \subseteq m^+\tau_\rho$ for all $m \in M$.

Let $m, n \in M$, $\alpha \in m\tau_\rho$ and $\beta \in n\tau_\rho$. Then $\rho_m \leqslant \alpha$ and $\rho_n \leqslant \beta$, and, as above, $\rho_{mn} \leqslant \alpha\beta$ so that $\alpha\beta \in (mn)\tau_\rho$. Thus $(m\tau_\rho)\,(n\tau_\rho) \subseteq (mn)\tau_\rho$.

Finally, suppose that $m, n \in M$ are such that $m\tau_\rho \cap n\tau_\rho \neq \emptyset$. Let $\alpha \in m\tau_\rho \cap n\tau_\rho$ so that $\rho_m \leqslant \alpha$ and $\rho_n \leqslant \alpha$. Then we have $\rho_m = \rho_m\rho_m^{-1}\alpha = \rho_m^+\alpha$ and $\rho_n = \rho_n\rho_n^{-1}\alpha = \rho_n^+\alpha$. Using the fact that ρ is a $^+$-embedding, we have

$$\rho_{m^+n} = \rho_{m^+}\rho_n = \rho_m^+\rho_n^+\alpha = \rho_n^+\rho_m^+\alpha = \rho_n^+\rho_m = \rho_{n^+m}$$

so that $m^+n = n^+m$ since ρ is injective.

In an exactly similar way, τ_λ is a right proper relational morphism of right ample monoids.

We now consider the (H_ρ, H_λ)-category $\mathscr{X} = \mathscr{X}_{(\tau_\rho,\tau_\lambda)}$ associated with the relational morphisms τ_ρ and τ_λ. We know that this is a proper ample category, and that the associated monoid $C_{(1,1)}$ is a proper ample monoid. In fact, $C_{(1,1)}$ is a proper ample cover of M.

Define the map $\theta : C_{(1,1)} \to M$ by

$$(b, ((1, b), m, (a, 1)), a)\,\theta = m.$$

It is clear that θ is onto since, for any $m \in M$, there exist $a \in m\tau_\rho$ and $b \in m\tau_\lambda$ and so

$$p = ((1, b), m, (a, 1)) \in \operatorname{Mor}\mathscr{X}, \ (b, p, a) \in C_{(1,1)} \text{ and } (b, p, a)\theta = m.$$

Also θ maps the identity of $C_{(1,1)}$ to the identity of M.

Let $\gamma_1 = (b_1, ((1, b_1), m_1, (a_1, 1)), a_1)$ and $\gamma_2 = (b_2, ((1, b_2), m_2, (a_2, 1)), a_2)$ be arbitrary elements of $C_{(1,1)}$. Then

$$\gamma_1\gamma_2 = (b_1b_2, ((1, b_1b_2), m_1m_2, (a_1a_2, 1)), a_1a_2)$$

and so

$$(\gamma_1\gamma_2)\theta = m_1m_2 = (\gamma_1\theta)\,(\gamma_2\theta).$$

Now

$$\gamma_1^+\theta = \big(1, \big((1, 1), m_1^+, (1, 1)\big), 1\big)\,\theta = m_1^+ = (\gamma_1\theta)^+$$

and similarly

$$\gamma_1^*\theta = (\gamma_1\theta)^*.$$

It remains to show that θ separates idempotents. But this is clear, since

$$E(C_{(1,1)}) = \{(1, ((1,1), e, (1,1)), 1) \colon e \in E(M)\}.$$

Therefore is a proper ample cover of M.

To conclude, we observe that if M is finite, then certainly H_ρ and H_λ are finite, and hence $C_{(1,1)}$ is finite. ∎

We briefly consider semigroups as opposed to monoids. It is clear how to extend the definitions of ample, proper and proper cover from monoids to semigroups. Thus if S is an ample semigroup, we say that a proper ample semigroup $\widehat{S}$ is a *proper cover* of S if there is a $(2,1,1)$-morphism θ from $\widehat{S}$ onto S that separates idempotents.

Corollary 3.3 *Every ample semigroup has a proper ample cover.*

Proof: Let S be an ample semigroup. The monoid S^1 is also ample and by the last theorem it has a proper cover T by means of a $(2,1,1,0)$-morphism $\theta \colon T \to S^1$. We can show that $S\theta^{-1}$ is a $(2,1,1)$-subalgebra of T and so it is also a proper ample semigroup and it is a cover of S. ∎

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