Primitive inverse congruences on categorical semigroups

John Fountain[*]
Department of Mathematics, University of York,
Heslington, York YO1 5DD, U.K.

Gracinda M. S. Gomes[†]
Centro de Álgebra da Universidade de Lisboa,
Av. Prof. Gama Pinto 2, 1699 Lisboa, Portugal

Abstract

We give an abstract description of the kernel of a proper primitive inverse congruence on a categorical semigroup. More specifically we show that it is a $*$-reflexive, $*$-unitary, $*$-dense subsemigroup and that on a given categorical semigroup there is a one-one correspondence between such subsemigroups and the proper primitive inverse congruences. Our results allow us to give a description of the minimum proper primitive inverse semigroup congruence on a strongly E^*-dense categorical semigroup.

INTRODUCTION

It is argued in [10] that when considering semigroups with zero, a case can be made for regarding the class of Brandt semigroups as the natural analogue of the class of groups. Another possible analogue is the related class of primitive inverse semigroups (with zero). The point is that in either class, any semigroup without zero-divisors is a group with an adjoined zero. These thoughts suggest that for semigroups with zero, Brandt semigroup congruences or primitive inverse semigroup congruences might play the role played by group congruences for general semigroups.

Three different, although necessarily equivalent, characterisations of the kernel of a group congruence have been given by Dubreil [3], Levi [15, 16] and Gomes [8]. An account of the approach of [3] and a brief description of that of [15, 16] is given in Chapter 10 of [2]. The properties used in the characterisations can be adapted to the case of semigroups with zero. However, it is not clear that the

adaptations of the three characterisations are equivalent and the appropiate one for semigroups with zero seems to be the approach of Gomes in [8].

After presenting the basic definitions in Section 1, we start Section 2 by showing how to associate a proper primitive inverse congruence with a strongly $*$-dense subsemigroup of a categorical semigroup. We then investigate the kernel of a proper primitive inverse congruence on a categorical semigroup and show that it is $*$-reflexive, $*$-unitary and $*$-dense. Conversely, any subsemigroup of a categorical semigroup which has these properties is the kernel of a proper primitive inverse congruence.

The question arises of when a semigroup with zero has a proper primitive inverse congruence. Preston [21], extending an earlier result of Munn [19], has shown that the semigroup must be categorical. A categorical inverse semigroup has a proper primitive inverse congruence, but categoricity is not sufficient in the general case. We conclude Section 2 with a result suggested by the referee giving necessary and sufficient conditions for the existence of a proper primitive inverse congruence on an arbitrary semigroup.

We consider E^*-dense semigroups in Section 3 and give a description of the minimum proper primitive inverse congruence on a strongly E^*-dense categorical semigroup which is an analogue of the description in [18] of the minimum group congruence on an E-dense semigroup.

In Section 4 we specialise to the case of categorical inverse semigroups and examine the connections between our approach and the kernel-trace approach of Petrich [20, Chapter III]. Finally, in Section 5 we give some analogues of results due to Edwards [4].

We would like to thank the referee for drawing our attention to [17] and suggesting Theorem 2.8.

1. Preliminaries

For the standard terminology and notation of semigroup theory we refer the reader to [12]. In particular, $E(S)$ denotes the set of idempotents of a semigroup S, and a subsemigroup T of S is *full* if $E(S) \subseteq T$. We adopt the usual convention that a semigroup with zero must contain at least two elements. By a *proper* congruence on a semigroup with zero we mean a congruence ρ such that $0\rho = \{0\}$. Proper congruences are also widely known as 0-*restricted* congruences (see for example [2]). We occasionally use the term 0-*subsemigroup* to emphasise that 0 is a member of the subsemigroup under consideration. The 0-subsemigroup generated by a subset A of a semigroup with zero will be denoted by $\langle A \rangle^0$.

Many concepts of semigroup theory can be modified to give closely related ideas which are more useful in the context of semigroups with zero. We now describe how the notions of dense, reflexive, and unitary subsets of a semigroup are modified. Let T be a subset of a semigroup S with zero. The set of non-zero

elements of T is denoted by T^*; in particular, $E^*(S)$ or just E^* is the set of non-zero idempotents of S.

We say that T is $*$-*dense* in S if for every non-zero element a of S there are elements $x, y \in S$ such that $ax, ya \in T^*$.

We define T to be $*$-*reflexive* if $ab \in T^*$ implies that $ba \in T^*$ for all $a, b \in S$.

Next, T is said to be $*$-*unitary* if for all elements a of S and t of T, we have $a \in T$ if at or ta is in T^*.

For an element a of S, define the sets $X_T(a)$ and $Y_T(a)$ by:

$$X_T(a) \quad = \quad \{b \in S \mid atb \in T^* \text{ for all } t \in T^1 \text{ such that } at \neq 0 \text{ or } tb \neq 0\},$$
$$Y_T(a) \quad = \quad \{b \in S \mid bta \in T^* \text{ for all } t \in T^1 \text{ such that } ta \neq 0 \text{ or } bt \neq 0\}.$$

Note that, in particular, if $b \in X_T(a)$, then $ab \in T^*$, and if $b \in Y_T(a)$, then $ba \in T^*$.

Now put $W_T^*(a) = X_T(a) \cap Y_T(a)$.

We say that T is *strongly* $*$-*dense* if T has no non-zero nilpotent elements and $W_T^*(a)$ is nonempty for all non-zero elements a of S. Note that by definition, a strongly $*$-dense subsemigroup is $*$-dense. Indeed, it satisfies the condition that for every non-zero element a of S there is an element b such that $ab, ba \in T^*$.

The following two lemmas are immediate from the definitions and will be used frequently without further mention.

Lemma 1.1. *Let T be a $*$-reflexive subset of a semigroup S with zero. Then the following conditions are equivalent:*

(1) *T is $*$-unitary,*
(2) *for all elements a of S and t of T, if $at \in T^*$, then $a \in T$,*
(3) *for all elements a of S and t of T, if $ta \in T^*$, then $a \in T$.*

Lemma 1.2. *Let T be a $*$-reflexive subset of a semigroup S with zero. Then the following conditions are equivalent:*

(1) *T is $*$-dense,*
(2) *for every non-zero element a of S, there is an element x of S such that $ax \in T^*$,*
(3) *for every non-zero element a of S, there is an element y of S such that $ya \in T^*$,*
(4) *for every non-zero element a of S, there is an element z of S such that $az \in T^*$ and $za \in T^*$.*

A semigroup S is *categorical* if it has a zero and for elements a, b and c of S such that $abc = 0$ we have $ab = 0$ or $bc = 0$. Categorical semigroups are also said to be categorical at zero (see [2]). Note that when we say that a semigroup is categorical it is implicit that the semigroup has a zero.

Lemma 1.3. *Let S be a categorical semigroup and T be a $*$-reflexive subsemigroup of S. Let a, b be elements of S with $ab \in T^*$. If $t \in T$ and $at \neq 0$ or $tb \neq 0$, then $atb \in T^*$.*

Proof. Let $a, b \in S$ and $t \in T$ be such that $ab \in T^*$ and $at \neq 0$. Then $ba \in T^*$ since T is $*$-reflexive, and $bat \neq 0$ since S is categorical. Hence $bat \in T^*$ and consequently, $atb \in T^*$ as required .

Similarly, if $tb \neq 0$, then we also have $atb \in T^*$. $\qquad\square$

Corollary 1.4. *If T is a $*$-reflexive, $*$-dense subsemigroup of a categorical semigroup S, then T is strongly $*$-dense.*

Proof. Since T is $*$-reflexive and $*$-dense, if a is a non-zero element of S, there is an element z such that $az, za \in T^*$. It is now immediate from the lemma that $z \in W_T^*(a)$.

If $t \in T^*$, then by $*$-reflexivity and $*$-denseness, $ty, yt \in T^*$ for some $y \in S$. By categoricity, $tyt \in T^*$ and so $yt^2 \in T^*$. Hence $t^2 \neq 0$. It follows that T is strongly $*$-dense. $\qquad\square$

A *Brandt* semigroup is an inverse completely 0-simple semigroup. An inverse semigroup S with zero is said to be *primitive* if every non-zero idempotent e in S is primitive, that is, for all $f \in E^*(S)$, if $e \leqslant f$, then $e = f$. For instance, Brandt semigroups are primitive and, in fact, every primitive inverse semigroup is a 0-direct union of Brandt semigroups (see for example, [20, Theorem III.4.3] or [22, Corollary 2]). From [2, Lemma 7.61] we have the following result.

Lemma 1.5. *A primitive inverse semigroup is categorical.*

The following lemma is a special case of Lemma 7.63(i) of [2].

Lemma 1.6. *Let S be a primitive inverse semigroup. Then, for $e \in E(S)$ and $s \in S$, if $es \neq 0$, then $es = s$. Similarly, $se \neq 0$ implies $se = s$.*

A proper congruence ρ on a semigroup S with zero is said to be a *primitive inverse (Brandt) congruence* if S/ρ is a primitive inverse (Brandt) semigroup. As we are concerned only with proper congruences, when we use the phrases "primitive inverse congruence" or "Brandt congruence" it is implicit that the congruence is proper.

By the *kernel* of a congruence ρ on a semigroup S we mean the subset $\text{Ker}\,\rho$ of S defined by

$$\text{Ker}\,\rho = \{a \in S \mid a\rho \in E(S/\rho)\} = \{a \in S \mid a\rho a^2\}.$$

Observe that if ρ is a proper congruence and a is a non-zero element of $\text{Ker}\,\rho$, then $a^2 \neq 0$ and consequently, $\text{Ker}\,\rho$ has no non-zero nilpotent elements.

2. The Characterisation

Throughout this section S will be a categorical semigroup. We re-emphasise that saying a semigroup is categorical implies that it has a zero. Our main aim is to characterise the kernels of proper primitive inverse congruences on S as the $*$-unitary, $*$-reflexive, $*$-dense subsemigroups of S. We then show that there is

an order-isomorphism between the set of all such subsemigroups of S and the set of all proper primitive inverse congruences on S.

First, given any strongly $*$-dense subsemigroup T of S we define a congruence ρ_T on S as follows. For all $a, b \in S$,

$(a, b) \in \rho_T$ if and only if $a = b = 0$ or $xa = bt \neq 0$ for some $x, t \in T$.

The following proposition is an analogue of [7, Corollary 3.2].

Proposition 2.1. *Let T be a strongly $*$-dense subsemigroup of S. Then the relation ρ_T is a primitive inverse congruence on S and $T \subseteq \operatorname{Ker} \rho_T$.*

Proof. Note that if ρ_T is a congruence, then by definition, it is a proper congruence.

For any non-zero element a of S there is an element x such that $ax, xa \in T^*$ since T is strongly $*$-dense. Now S is categorical so that $axa \neq 0$ and hence $(a, a) \in \rho_T$ and ρ_T is reflexive.

Now let a, b be non-zero elements of S such that $(a, b) \in \rho_T$. Then there are elements x, t of T such that $xa = bt \neq 0$. Let $a' \in W_T^*(a)$, $b' \in W_T^*(b)$. Since $x \in T$ and $xa \neq 0$ we have $a'bt = a'xa \in T^*$ and similarly, $xab' = btb' \in T^*$. Now by categoricity,
$$a(a'xa)(b'b) = (aa')(btb')b \neq 0$$
and $(a'xa)(b'b), (a'a)(btb') \in T$ so that $(b, a) \in \rho_T$ and ρ_T is symmetric.

To prove that ρ_T is transitive, let $a, b, c \in S^*$ be such that $(a, b), (b, c) \in \rho_T$. Let $x, t, u, v \in T$ be such that
$$xa = bt \neq 0 \qquad \text{and} \qquad ub = cv \neq 0.$$
Then $ubt \neq 0$ and so $uxa = ubt = cvt \neq 0$ where $ut, vt \in T$. Thus $(a, c) \in \rho_T$.

Next, we show that ρ_T is right compatible. Let $a, b, c \in S^*$ and suppose that $(a, b) \in \rho_T$. Then $xa = bt \neq 0$ for some $x, t \in T$. Clearly, if $ac = bc = 0$, then $(ac, bc) \in \rho_T$.

If $bc \neq 0$, let $b' \in W_T^*(b)$ and $c' \in W_T^*(c)$. Since $b'b \neq 0$ we have $b'xa = b'bt \neq 0$ and so $b'x \neq 0$. Also $bcc' \neq 0$ by categoricity so that $bcc'b' \in T^*$ and hence $bcc'b'x \in T^*$.

Now $b'b \in T^*$ and $b'bc \neq 0$ by categoricity, so $c'b'bc \in T^*$. It follows that $c'b'bt \neq 0$ and so $c'b'btc \in T^*$. In particular, $btc \neq 0$ so that $xac \neq 0$. Now by categoricity,
$$(bc)(c'b'btc) = (bcc'b'x)(ac) \neq 0$$
so that $(ac, bc) \in \rho_T$.

Since ρ_T is symmetric, it follows that if $ac \neq 0$, then we also have $(ac, bc) \in \rho_T$ and so ρ_T is right compatible. A similar argument shows that it is also left compatible.

Next we show that $T \subseteq \operatorname{Ker} \rho_T$. Certainly, $0 \in \operatorname{Ker} \rho_T$. Let t be a non-zero element of T. Then $t^2 \neq 0$ since T has no nilpotents and so $t^3 \neq 0$ by categoricity. Hence $(t, t^2) \in \rho_T$ and $t \in \operatorname{Ker} \rho_T$.

Now suppose that $(a^2, a) \in \rho_T$ for some non-zero element a of S. Then $aq \in T^*$ for some $q \in S$ and we have $a^2 q \, \rho_T \, aq$. Hence we have $t, x \in T$ such that

$$a(aqt) = x(aq) \neq 0.$$

Now $aqt, x \in T$ so that $a \, \rho_T \, aq$ and a is related to an element of T. Thus we have shown that

$$E(S/\rho_T) = \{t\rho_T \mid t \in T\}.$$

We now show that $E(S/\rho_T)$ is a semilattice. If $t, u \in T$ and $tu \neq 0$, then $(tu)^2 \neq 0$ so that $ut \neq 0$. Thus $u(tu) = (ut)u \neq 0$ so that $(tu, ut) \in \rho_T$. If $tu = 0$, then clearly, $ut = 0$ so that again $(tu, ut) \in \rho_T$. Hence $E(S/\rho_T)$ is a semilattice.

Next we show that S/ρ_T is regular and hence inverse. If a is a non-zero element of S, then $ax, xa \in T^*$ for some $x \in S$. Hence $(xa)^2 \in T$ and

$$ax(axa) = a(xa)^2 \neq 0$$

so that $(a, axa) \in \rho_T$ and S/ρ_T is regular.

Finally, we show that S/ρ_T is primitive. Let $t\rho_T, u\rho_T$ be non-zero idempotents of S/ρ_T and suppose that $t\rho_T \leqslant u\rho_T$ so that $(t, ut) \in \rho_T$. Then $xt = uty \neq 0$ for some elements x, y of T. We can assume that $t, u \in T$ and so we have $(t, u) \in \rho_T$ since $ty \in T$.

Therefore, S/ρ_T is a primitive inverse semigroup. $\qquad\square$

Our next result, which corresponds to [8, Lemma 5], shows that every primitive inverse congruence on S is of the form ρ_K for some $*$-unitary, $*$-reflexive, $*$-dense subsemigroup K of S.

Proposition 2.2. *Let ρ be a primitive inverse congruence on S. Then $\operatorname{Ker}\rho$ is a $*$-unitary, $*$-reflexive, $*$-dense subsemigroup of S and $\rho = \rho_{\operatorname{Ker}\rho}$.*

Proof. Let $K = \operatorname{Ker}\rho$. Suppose that $a, b \in S$ and $ab \in K^*$. Then $(a\rho)(b\rho) = (ab)\rho$ is a non-zero idempotent of S/ρ. Since S/ρ is primitive inverse, $(ba)\rho = (b\rho)(a\rho)$ is also a non-zero idempotent. Thus $ba \in K^*$ and K is $*$-reflexive.

If a is a non-zero element of S, then $a\rho$ is a non-zero element of S/ρ and has an inverse $(a\rho)^{-1}$ in S/ρ. Let $(a\rho)^{-1} = x\rho$. Then $(ax)\rho = (a\rho)(x\rho) = (a\rho)(a\rho)^{-1}$ is a non-zero idempotent of S/ρ so that $ax \in K^*$. Thus K is $*$-dense.

If $a, b \in S$ and $ab, b \in K^*$, then $(a\rho)(b\rho)$ and $b\rho$ are non-zero idempotents of S/ρ. Since all idempotents in S/ρ are primitive it follows that $a\rho \mathcal{L} b\rho$ and so $(a\rho)(b\rho) = a\rho$. Thus $a\rho$ is a non-zero idempotent. Hence K is $*$-unitary as required.

If $a, b \in S^*$ and $(a, b) \in \rho$, then since S/ρ is inverse we have $(a\rho)^{-1} = (b\rho)^{-1} = x\rho$ for some non-zero element x of S. Now $xa, bx \in K$ and $(bx)a = b(xa) \neq 0$ so that $(a, b) \in \rho_K$.

Conversely, if $(a, b) \in \rho_K$ and a, b are non-zero, then $ua = bv \neq 0$ for some $u, v \in K$. In S/ρ, the element $(u\rho)(a\rho)$ is non-zero and $u\rho \in E(S/\rho)$ so that by

Lemma 1.6, $(u\rho)(a\rho) = a\rho$. Similarly, $(b\rho)(v\rho) = b\rho$ so that $(a, b) \in \rho$. Thus $\rho = \rho_K$ as required. $\qquad\square$

Note that a non-zero intersection of $*$-reflexive, $*$-unitary subsemigroups of S is again $*$-reflexive and $*$-unitary. Let T be a non-zero subsemigroup of S and suppose that T is contained in a $*$-reflexive, $*$-unitary subsemigroup. Let T_∞ be the least $*$-reflexive, $*$-unitary subsemigroup of S containing T. Note that if T is $*$-dense, then T_∞ is also $*$-dense since $T \subseteq T_\infty$.

We can construct T_∞ from T in the following way. First, let A be a non-zero 0-subsemigroup of S which is contained in a $*$-reflexive, $*$-unitary 0-subsemigroup B. Let
$$u^*(A) = \{x \in S \mid xy \text{ or } yx \text{ is in } A^* \text{ for some } y \in A^1\}.$$
Since B is $*$-unitary we have $A^* \subseteq u^*(A) \subseteq B$ and so the 0-subsemigroup $\langle u^*(A)\rangle^0$ generated by $u^*(A)$ is contained in B. Let
$$r^*(A) = \{x \in S \mid x = yz \text{ for some } y, z \in S^1 \text{ such that } zy \in A^*\}.$$
Note that if $x \in r^*(A)$, then $x \neq 0$. This is immediate if $y = 1$ or $z = 1$. If $y, z \in S$, then $zy \in B^*$ so that $yz \in B^*$ and $yz \neq 0$. Observe also that since B is $*$-reflexive, $A \subseteq \langle r^*(A)\rangle^0 \subseteq B$.

Now let $T_0 = T$ and for $k \geqslant 0$, let $T_{2k+1} = \langle u^*(T_{2k})\rangle^0$ and $T_{2k+2} = \langle r^*(T_{2k+1})\rangle^0$. Then we have an ascending chain
$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$$
and $T' = \bigcup_{k \geqslant 0} T_k$ is a subsemigroup of S. Since $T_0 \subseteq T_\infty$, it follows by induction and the above remarks that $T_k \subseteq T_\infty$ for all k. Thus $T' \subseteq T_\infty$.

If $ab \in T'$ and $ab \neq 0$, then $ab \in T^*_{2k+1}$ for some k. Thus $ba \in r^*(T_{2k+1})$ so that ba is a non-zero element of T' and hence T' is $*$-reflexive.

If ab, b are non-zero elements of T', then we have $ab, b \in T^*_{2k}$ for some k and so $a \in u^*(T_{2k})$. It follows that T' is $*$-unitary and consequently, $T_\infty = T'$.

Now we show that if T is a strongly $*$-dense subsemigroup of S, then T_∞ is the kernel of the primitive inverse congruence ρ_T.

Proposition 2.3. *If T is a strongly $*$-dense subsemigroup of S, then $T_\infty = \operatorname{Ker}\rho_T$ and $\rho_T = \rho_{T_\infty}$.*

Proof. By Proposition 2.1, T is contained in $\operatorname{Ker}\rho_T$ and by Proposition 2.2, $\operatorname{Ker}\rho_T$ is $*$-unitary and $*$-reflexive. Hence, by definition, $T_\infty \subseteq \operatorname{Ker}\rho_T$.

On the other hand, if $a \in \operatorname{Ker}\rho_T$, then as in the proof of Proposition 2.1, $(a, t) \in \rho_T$ for some $t \in T$. If a is non-zero, then there are elements x, v of T such that $ax = vt \neq 0$. But $vt \in T$ and $T \subseteq T_\infty$ so that $ax \in T^*_\infty$ and hence $a \in T^*_\infty$ since T_∞ is $*$-unitary.

Thus $T_\infty = \operatorname{Ker}\rho_T$ and so by Proposition 2.2, $\rho_T = \rho_{T_\infty}$. $\qquad\square$

We can now put our results together to obtain the main theorem of the section.

Theorem 2.4. *The mappings $T \mapsto \rho_T$ and $\rho \mapsto \operatorname{Ker} \rho$ are mutually inverse order isomorphisms between the set of $*$-unitary, $*$-reflexive, $*$-dense subsemigroups of S and the set of all primitive inverse congruences on S.*

Proof. If T is a $*$-unitary, $*$-reflexive, $*$-dense subsemigroup of S, then ρ_T is a primitive inverse congruence by Proposition 2.1. By Proposition 2.3, $\operatorname{Ker} \rho_T = T_\infty$ and $T = T_\infty$ since T is $*$-unitary, $*$-reflexive and $*$-dense.

On the other hand, if ρ is a primitive inverse congruence on S, then $\operatorname{Ker} \rho$ is $*$-unitary, $*$-reflexive and $*$-dense, and $\rho = \rho_{\operatorname{Ker} \rho}$ by Proposition 2.2.

Thus the two mappings are mutually inverse. It is straightforward to verify that they are order-preserving. $\qquad\square$

An immediate consequence of the theorem is the following corollary.

Corollary 2.5. *Let ρ, θ be primitive inverse congruences on S. Then $\rho = \theta$ if and only if $\operatorname{Ker} \rho = \operatorname{Ker} \theta$.*

We now turn our attention to Brandt congruences. We note first that if a semigroup R has a Brandt congruence, then by [19, Theorem 1.1], R is categorical and also, any two non-zero ideals of R have non-zero intersection. Following [9] we say that such a semigroup is *strongly categorical*.

It is not difficult to show that if R is a strongly categorical semigroup and if τ is any proper congruence on R, then R/τ is strongly categorical. Hence, if ρ is a primitive inverse congruence on R, then R/ρ is a Brandt semigroup. The following corollary is thus an immediate consequence of Theorem 2.4.

Corollary 2.6. *Let R be a strongly categorical semigroup. Then the mappings $T \mapsto \rho_T$ and $\rho \mapsto \operatorname{Ker} \rho$ are mutually inverse order isomorphisms between the set of $*$-unitary, $*$-reflexive, $*$-dense subsemigroups of R and the set of all Brandt congruences on R.*

Having characterised proper primitive inverse congruences by their kernels we now consider when such congruences exist. We start by quoting the following result [17, Proposition 1].

Proposition 2.7. *Let S be a semigroup with zero . Then*
$$\rho = \{(a,b) \in S \times S \mid \text{for all } s,t \in S^1, \; sat = 0 \text{ if and only if } sbt = 0\}$$
is a proper congruence on S. If τ is any proper congruence on S, then $\tau \subseteq \rho$.

Borrowing terminology from ring theory, we say that a semigroup S with zero is *semiprime* if $aSa \neq 0$ for every non-zero element a of S. A homomorphism $\alpha \colon S \to T$ of semigroups with zero is *proper* if $0\alpha^{-1} = \{0\}$, that is, α is proper if and only if the congruence it induces is proper.

Theorem 2.8. *Let S be a semigroup with zero and let*
$$U = \{a \in S \mid a^2 \neq 0\} \cup \{0\}.$$

Then the following statements are equivalent:

  (1) *S has a proper primitive inverse homomorphic image,*
  (2) *S is categorical, semiprime, and for all a, b in S,*

$$a^2 \neq 0, b^2 \neq 0, ab \neq 0 \ \text{together imply } ba \neq 0,$$

  (3) *S is categorical and U is a $*$-dense subsemigroup,*
  (4) *S is categorical and U is a $*$-unitary, $*$-reflexive, $*$-dense subsemigroup.*

Proof. Suppose first that S has a proper primitive inverse homomorphic image, say θ is a proper homomorphism onto a primitive inverse semigroup. Then by [2, Lemma 7.62], S is categorical. If a is a non-zero element of S, then $a\theta \neq 0$ so that if $b\theta$ is the inverse of $a\theta$, then $aba \neq 0$ since θ is proper, and hence S is semiprime.

A homomorphic image of an inverse semigroup is inverse [6, Theorem 5.1.4] and it follows easily from Lallement's Lemma [6, Lemma 2.4.4] that any non-zero idempotent in a homomorphic image of a primitive inverse semigroup is primitive. Thus the class of primitive inverse semigroups is closed under homomorphic images and so by Proposition 2 of [17], S/ρ is primitive inverse. Now let $a, b \in S$ be such that a^2, b^2, and ab are non-zero. By categoricity, it follows that $sa^2t \neq 0$ if and only if $sat \neq 0$, and hence $a\rho$ is idempotent. Similarly, $b\rho$ is also idempotent and so $(a\rho)(b\rho) = (b\rho)(a\rho)$ since S/ρ is inverse. Thus $ba \neq 0$ and (2) holds.

Now suppose that (2) holds and let $a, b \in U^*$ so that a^2 and b^2 are non-zero. If $ab \neq 0$, then $ba \neq 0$ by assumption and so by categoricity, $abab \neq 0$, whence $ab \in U$ and U is a subsemigroup.

Let s be a non-zero element of S. Then $sSs \neq 0$ since S is semiprime. Let $b \in S$ be such that $sbs \neq 0$. By categoricity, $sbsb \neq 0$ and $bsbs \neq 0$ so that $sb, bs \in U$ and U is $*$-dense in S.

Next, suppose that (3) holds. If $ab \in U^*$, then $(ab)^2 \neq 0$ so that $ba \neq 0$, and hence by categoricity, $(ba)^2 \neq 0$, that is, $ba \in U^*$. Thus U is $*$-reflexive.

If $s \in S$, $u \in U$ and $su \in U^*$, then $u^2 \neq 0$ and $(su)^2 \neq 0$ so that by categoricity, $(us)^2 \neq 0$, that is, $us \in U$. Hence $su^2s \in U$ and by categoricity again, $su^2s \neq 0$. Thus $(su^2s)^2 \neq 0$ so that $s^2 \neq 0$, that is, $s \in U$ and therefore U is $*$-unitary.

Finally, if (4) holds, then by Corollary 1.4 and Proposition 2.1, ρ_U is a proper primitive inverse congruence on S. □

3. E^*-DENSE SEMIGROUPS

A semigroup S with zero is E^*-*dense* if for every non-zero element a of S there are elements b, c of S such that $ab, ca \in E^*(S)$. An E^*-dense semigroup is obviously E-dense and the notion is an analogue for the class of semigroups with zero of the concept of E-denseness in the class of all semigroups. An E-dense semigroup is said to be E-inversive in [2]; E^*-dense semigroups are called 0-inversive in [14] and weakly regular in [17].

The analogy between E-dense and E^*-dense semigroups is far from perfect. For example, it is known that every E-dense semigroup has a minimum group congruence. The existence of such a congruence was noted in [11] and an explicit description given in [18, Proposition 9]. In contrast, as we have already noted, the work of Munn and Preston shows that the existence of a (Brandt) primitive inverse congruence on a semigroup forces the semigroup to be (strongly) categorical. For an inverse semigroup, Preston (Munn) shows that (strong) categoricity is sufficient and Fountain and Hayes [6] prove a similar result for an E^*-dense semigroup in which the idempotents commute with each other. However, we have seen that categoricity by itself is not sufficient for a general semigroup with zero to have a primitive inverse congruence and the same is true for E^*-dense semigroups.

Let T be a semigroup with zero such that $E(T)$ is a commutative subsemigroup. Then, by Theorem 3 of [14], T is primitive inverse if and only if it is E^*-dense and satisfies the following weak cancellation law:

if $a, b, x, y \in T$, then $ax = bx \neq 0$ and $ya = yb \neq 0$ together imply $a = b$.

Using this result it is easy to see that if I is the set of proper primitive inverse congruences on an E^*-semigroup S, and if $\sigma = \bigcap\{\tau : \tau \in I\}$, then, provided that I is not empty, σ is the minimum proper primitive inverse congruence on S.

We find necessary and sufficient conditions different from those of Theorem 2.8 for an E^*-dense semigroup S to have a (Brandt) primitive inverse congruence. In fact, we show that S has such a congruence if and only if it is (strongly) categorical and $D(S)$ is strongly $*$-dense in S where $D(S)$ is the least full, weakly self-conjugate subsemigroup of S. We say that an E^*-dense semigroup S is *strongly E^*-dense* if $D(S)$ is a strongly $*$-dense subsemigroup of S. As noted above, a strongly E^*-dense semigroup has a minimum proper primitive inverse congruence and, as an analogue of Proposition 9 of [18], we give an explicit description this congruence.

A *weak inverse* of an element a of a semigroup is an element b such that $bab = b$. The set of all weak inverses of a is denoted by $W(a)$ and if a is non-zero, then $W^*(a)$ is the set of all non-zero weak inverses of a.

Lemma 3.1. *Let S be a semigroup with zero, P be a primitive inverse semigroup and $\alpha : S \to P$ be a proper morphism. If a is a non-zero element of S and $a' \in W^*(a)$, then $a'\alpha$ is the inverse of $a\alpha$ in P.*

Proof. Both $a\alpha$ and $a'\alpha$ are non-zero and since $a'aa' = a'$, it follows that $(a\alpha)(a'\alpha) = (aa')\alpha$ is a non-zero idempotent of P. Similarly, $(a'\alpha)(a\alpha) \in E^*(P)$. Since P is primitive inverse, it follows that $a\alpha \mathcal{L}(a'\alpha)(a\alpha)$ and hence that $a'\alpha = (a\alpha)^{-1}$. □

We quote the following result from [6]. Its easy proof can be extracted from [10] and it summarises some elementary properties of E^*-dense semigroups.

Proposition 3.2. *Let S be a semigroup with zero and $E = E(S)$. Then the following conditions are equivalent:*

(1) *S is E^*-dense,*

(2) *for every non-zero element a of S, there is an element b of S such that $ab \in E^*$ and $ba \in E^*$,*

(3) *for every non-zero element a of S, there is an element c of S such that $ac \in E^*$,*

(4) *for every non-zero element a of S, there is an element d of S such that $da \in E^*$,*

(5) *every non-zero element of S has a non-zero weak inverse.*

A subsemigroup T of S is said to be *closed under weak conjugation* or *weakly self-conjugate* if $ata', a'ta \in T$ for all $t \in T$, $a \in S$ and $a' \in W(a)$. Clearly, in a semigroup with zero, the intersection of a family of weakly self-conjugate subsemigroups contains zero and is weakly self-conjugate. Hence we can define $D(S)$ to be the least (under inclusion) full, weakly self-conjugate subsemigroup of S.

There is a well known construction of $D(S)$ [1] which we now describe. First, for any subsemigroup T of S, put

$$q(T) = \{ata', a'ta \mid t \in T, a, a' \in S^1, a'aa' = a'\}$$

and $Q(T) = \langle q(T) \rangle$. Now put $D_0(S) = \langle E(S) \rangle$ and for each non-negative integer i, put $D_{i+1} = Q(D_i(S))$. Clearly, we have an ascending chain

$$D_0(S) \subseteq D_1(S) \subseteq ... \subseteq D_i(S) \subseteq ...$$

and $D' = \bigcup_{i \geqslant 0} D_i(S)$ is a subsemigroup of S. In fact, it is not difficult to see that $D' = D(S)$.

Lemma 3.3. *Let S be a categorical semigroup, P be a primitive inverse semigroup and let $\alpha: S \to P$ be a proper surjective morphism. Then $D(S)\alpha \subseteq E(P)$.*

Proof. Clearly, $D_0(S)\alpha = \langle E(S) \rangle \alpha \subseteq E(P)$. Suppose that $D_i(S)\alpha \subseteq E(P)$ and that $a \in S^1$, $t \in D_i(S)$ and $a' \in W(a)$. Then either $ata' = 0$ in which case $(ata')\alpha \in E(P)$ or $a' \in W^*(a)$. If $a = 1$, then a' is idempotent so that $ta' \in D_i(S)$ and $(ata')\alpha \in E(P)$. Otherwise, by Lemma 3.1, $a'\alpha = (a\alpha)^{-1}$. Hence $(ata')\alpha = (a\alpha)(t\alpha)(a\alpha)^{-1}$ which belongs to $E(P)$ since $t\alpha \in E(P)$ by assumption and P is inverse. Similarly, $(a'ta)\alpha \in E(P)$ and so $q(D_i(S))\alpha \subseteq E(P)$. Hence $D_{i+1}(S)\alpha \subseteq E(P)$ and the lemma follows by induction. $\square$

We now define a subsemigroup T of a categorical E^*-dense semigroup S to be *weakly $*$-self-conjugate* if it contains 0 and for all non-zero elements a of S and all $a' \in W^*(a)$ and $t \in T^1$, if one of at or ta' is non-zero, then $ata' \in T^*$, and if one of ta or $a't$ is non-zero, then $a'ta \in T^*$.

It is clear that if T is weakly $*$-self-conjugate, then it is weakly self-conjugate. It is also clear that if, in addition, T contains no non-zero nilpotents, then T is

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strongly $*$-dense. The converse of the last assertion, when T is full and weakly self-conjugate, is a consequence of the following proposition.

Proposition 3.4. *Let S be a categorical E^*-dense semigroup, P be a primitive inverse semigroup and $\alpha : S \to P$ be a proper surjective morphism. If T is a non-zero weakly self-conjugate full subsemigroup of S such that $T\alpha \subseteq E(P)$, then T has no non-zero nilpotents and is weakly $*$-self-conjugate.*

Proof. Since α is proper, we have $T^*\alpha \subseteq E^*(P)$ and hence if $t \in T^*$, then $t^2 \neq 0$ and so T has no non-zero nilpotents.

Let a be a non-zero element of S and $a' \in W^*(a)$. Then certainly $aa' \neq 0$. Suppose that $t \in T$ and $at \neq 0$. Then $(at)\alpha$ is non-zero, $t\alpha \in E^*(P)$, and by Lemma 3.1, $a'\alpha = (a\alpha)^{-1}$ so that $(ta')\alpha = (t\alpha)(a\alpha)^{-1} = ((a\alpha)(t\alpha))^{-1}$ is non-zero. Hence $ta' \neq 0$ and by categoricity, $ata' \neq 0$. But $ata' \in T$ since T is weakly self-conjugate and so $ata' \in T^*$.

Similar arguments give the other three conditions for T to be weakly $*$-self-conjugate. $\qquad\square$

Corollary 3.5. *Let T be a weakly self-conjugate full subsemigroup of a categorical E^*-dense semigroup S. Then T is strongly $*$-dense if and only if it has no non-zero nilpotents and is weakly $*$-self-conjugate.*

Proof. If T is strongly $*$-dense, then by definition, T has no non-zero nilpotents and by Proposition 2.1, ρ_T is a proper primitive inverse congruence on S with $T \subseteq \operatorname{Ker} \rho_T$. Hence by Proposition 3.4, T is weakly $*$-self-conjugate.

The converse is clear. $\qquad\square$

Corollary 3.6. *Let S be a categorical E^*-dense semigroup and suppose that S has a proper primitive inverse congruence ρ. Then $D(S)$ is strongly $*$-dense.*

Proof. By Lemma 3.3, $D(S) \subseteq \operatorname{Ker} \rho$ and since ρ is proper, $t\rho \in E^*(S/\rho)$ for any non-zero element t of $D(S)$. The result now follows from Proposition 3.4 and Corollary 3.5. $\qquad\square$

In general, if S is a categorical E^*-dense semigroup, then $D(S)$ need not be strongly $*$-dense. For instance, it is easy to produce examples of idempotent generated completely 0-simple semigroups (which are categorical) which contain non-zero nilpotent elements. For such a semigroup S we have $D(S) = S$ so that $D(S)$ is not strongly $*$-dense.

On the other hand, if S is E^*-dense and the idempotents of S form a subsemigroup, then $D(S)$ is strongly $*$-dense. To prove this we use the following result.

Proposition 3.7. *If S is an E-dense semigroup in which $E(S)$ is a subsemigroup, then $E(S) = D(S)$.*

Proof. The result follows from [5, Proposition 2.1] but for completeness we include a short proof from [7, Lemma 8.14]. Let $a \in S$, $a' \in W(a)$ and $e \in E(S)$. Then $a'aa' = a'$ and $a'a \in E(S)$ so that

$$(aea')^2 = aea'aea' = aea'aea'aa' = a(ea'a)^2a' = a(ea'a)a' = aea'.$$

Thus $aea' \in E(S)$. Similarly, $a'ea \in E(S)$ and so $E(S)$ is weakly self-conjugate. Hence $E(S) = D(S)$. □

Proposition 3.8. *Let S be a categorical E^*-dense semigroup such that $E(S)$ is a subsemigroup. Then $E(S) = D(S)$ and $E(S)$ is strongly $*$-dense.*

Proof. Certainly, S is E-dense and so by Proposition 3.7, $E(S) = D(S)$.

It is clear that $E(S)$ contains no non-zero nilpotent elements. Further, by Proposition 3.2, $W^*(a)$ is nonempty for every non-zero element a of S. It follows from this and [10, Lemma 1.6] that $E(S)$ is strongly $*$-dense. □

We now give an analogue of [18, Proposition 9].

Theorem 3.9. *Let S be a categorical E^*-dense semigroup. Then S has a proper primitive inverse congruence if and only if $D(S)$ is strongly $*$-dense. Moreover, if $D(S)$ is strongly $*$-dense, then the congruence $\rho_{D(S)}$ is the minimum proper primitive inverse congruence on S.*

Proof. If S has a proper primitive inverse congruence, then by Corollary 3.6, $D(S)$ is strongly $*$-dense.

Conversely, if $D(S)$ is strongly $*$-dense, then by Proposition 2.1, the congruence $\rho_{D(S)}$ is proper and primitive inverse.

Now suppose that $D(S)$ is strongly $*$-dense and let τ be a proper primitive inverse congruence on S. Let $K = \operatorname{Ker}\tau$. Clearly, $E(S) \subseteq K$. Suppose that $k \in K^1$ and $a \in S$. Let $a' \in W(a)$. If $k = 1$, then aka' and $a'ka$ are idempotents and so belong to K. Otherwise, when $aka' \neq 0$, we have $(aka')\tau = (a\tau)(k\tau)(a\tau)^{-1}$ by Lemma 3.1. Now $(a\tau)(k\tau)(a\tau)^{-1}$ is idempotent and hence $aka' \in K$. Similarly, $a'ka \in K$ and K is weakly self-conjugate. Thus $D(S) \subseteq K$ and so $\rho_{D(S)} \subseteq \rho_K$. But $\rho_K = \tau$ by Proposition 2.2 and consequently, $\rho_{D(S)}$ is the minimum primitive inverse congruence on S. □

Of course, if we suppose that S is strongly categorical, then we can replace "primitive inverse" by "Brandt" in the above theorem.

The following corollary strengthens [10, Theorem 2.1].

Corollary 3.10. *Let S be a (strongly) categorical E^*-dense semigroup whose idempotents form a subsemigroup. Then S has a proper (Brandt) primitive inverse congruence and the relation*

$$\beta = \{(a,b) \mid a = b = 0 \text{ or } ea = bf \neq 0 \text{ for some } e, f \in E(S)\}$$

is the minimum proper primitive inverse (Brandt) congruence on S.

Proof. By Proposition 3.8, $E(S) = D(S)$ and hence the result follows from the theorem. $\square$

When $E(S)$ is actually a semilattice we can recover the following result of Fountain and Hayes [6, Theorem 2.2] which generalises [19, Theorem 2.7] and [22, Theorem 5].

Corollary 3.11. *Let S be a (strongly) categorical E^*-dense semigroup with $E(S)$ a semilattice. Then the relation*

$$\beta = \{(a,b) \mid a = b = 0 \text{ or } ea = eb \neq 0 \text{ for some } e \in E(S)\}$$

is the minimum primitive inverse (Brandt) congruence on S.

Proof. For the moment, denote the congruence of Corollary 3.10 by β'. By Corollary 3.10, it suffices to prove that $\beta' = \beta$. Let $a, b \in S$ be such that $ea = bf \neq 0$ for some $e, f \in E$. Then $eaf = ebf \neq 0$. By [6, Lemma 2.1], it follows that there is an idempotent k such that $k(ea) = k(eb) \neq 0$, that is, $(ke)a = (ke)b \neq 0$ and $(a,b) \in \beta$. Thus $\beta' \subseteq \beta$.

Now suppose that $ea = eb \neq 0$ and let $a' \in W^*(a)$ and $b' \in W^*(b)$. Then by [10, Lemma 1.6], $f = a'ea \in E^*(S)$ and since $bf \neq 0$ follows from $af \neq 0$, we also have $bfb' \in E^*(S)$. Put $h = fb'b$. Then

$$ah = afb'b = aa'eab'b = aa'e^2ab'b = eaa'eab'b = eafb'b = ebfb'b \neq 0$$

and $ebfb'$ is idempotent so that $(a,b) \in \beta'$ and the corollary follows. $\square$

We conclude this section with the following result for categorical E^*-dense semigroups.

Proposition 3.12. *Let S be a categorical E^*-dense semigroup and let T be a weakly $*$-self-conjugate full subsemigroup of S. If T is $*$-unitary, then it is also $*$-reflexive.*

Proof. Suppose that $a, b \in S$ and $ab \in T^*$. By Proposition 3.2, a has a non-zero weak inverse a'. Also, $a'(ab) \neq 0$ and since T is weakly $*$-self-conjugate, we have $a'aba \in T^*$. But $a'a$ is idempotent and hence belongs to T and so $ba \in T^*$ since T is $*$-unitary. Thus T is $*$-reflexive as claimed. $\square$

4. Proper primitive congruences on inverse semigroups

We can, of course, specialise the results of Section 2 to obtain a characterisation of primitive inverse congruences on categorical inverse semigroups. However, there is also a well known theory of congruences on arbitrary inverse semigroups (see [20, Chapter III]) in which congruences are characterised by congruence pairs.

Recall that a subsemigroup K of an inverse semigroup S is *self-conjugate* if $a^{-1}Ka \subseteq K$ for all $a \in S$ and that K is a *normal* subsemigroup of S if it is a

full, self-conjugate, inverse subsemigroup. A congruence τ on $E(S)$ is *normal* if for any $e, f \in E(S)$ and $a \in S$ we have $(a^{-1}ea, a^{-1}fa) \in \tau$ whenever $(e, f) \in \tau$.

The pair (K, τ) is a *congruence pair* for S if K is a normal subsemigroup of S, τ is a normal congruence on $E(S)$ and the following two conditions are satisfied for all $a \in S$, $e \in E(S)$, $k \in K$:

(i) If $ae \in K$ and $(e, a^{-1}a) \in \tau$, then $a \in K$,

(ii) $(kk^{-1}, k^{-1}k) \in \tau$.

Given a congruence pair (K, τ) on S, the relation $\rho_{(K,\tau)}$ defined by

$$a \, \rho_{(K,\tau)} \, b \text{ if and only if } a^{-1}a \, \tau \, b^{-1}b \text{ and } ab^{-1} \in K$$

is a congruence on S with kernel K and which restricts to τ on $E(S)$.

Our aim in this section is to show the connections between the kernel-trace approach and that of Section 2 to primitive inverse congruences on inverse semigroups.

We start by pointing out that a $*$-dense, $*$-unitary subsemigroup T of any semigroup S with zero is full. For, if $e \in E^*(S)$, then $eb \in T^*$ for some $b \in S$. But $e(eb) = eb$ and T is $*$-unitary so that $e \in T$. On the other hand, if S is inverse, then it is clear that any full subsemigroup is $*$-dense. Hence the kernels of primitive inverse congruences on an inverse semigroup are the full, $*$-unitary, $*$-reflexive subsemigroups. We also point out that an easy consequence of a subsemigroup of an inverse semigroup being full and $*$-unitary is that it is an inverse subsemigroup.

Now let T be a subsemigroup of an inverse semigroup with zero. We say that T is $*$-*self-conjugate* if for all $a \in S$ and $t \in T^*$, if $ta^{-1} \neq 0$, then $ata^{-1} \in T^*$. The next proposition relates this concept to others we have introduced earlier.

Proposition 4.1. *Let T be a full subsemigroup of a categorical inverse semigroup S. Then the following conditions are equivalent:*

(1) *T is $*$-self-conjugate,*

(2) *T is weakly $*$-self-conjugate,*

(3) *T is weakly self-conjugate and contains no non-zero nilpotents,*

(4) *T is self-conjugate and contains no non-zero nilpotents,*

(5) *T is strongly $*$-dense.*

Proof. If (1) holds, let $t \in T^*$. Then $tt^{-1} \neq 0$ so that $t^2t^{-1} \neq 0$ since T is $*$-self-conjugate. Hence $tt^{-1}t^{-1}t = t^{-1}ttt^{-1} \neq 0$ by categoricity and hence if $a \in S$ and $at \neq 0$, then $att^{-1}t^{-1}t \neq 0$. By [10, Lemma 1.6], $att^{-1}t^{-1}ta' \neq 0$ for any $a' \in W^*(a)$. Hence $ta' \neq 0$ and so $ata' \neq 0$ by categoricity. Now

$$ata' = ata'aa' = ata'aa^{-1}aa' = (a(ta'a)a^{-1})aa'$$

which is in T since T is full and $*$-self-conjugate.

Similar arguments give the other three conditions needed for weak $*$-self-conjugacy.

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If (2) holds, then clearly, T is weakly self-conjugate. Also, if $t \in T^*$, then $tt^{-1} \neq 0$ and $t^{-1} \in W^*(t)$ so that $t^2 t^{-1} \neq 0$ since T is weakly $*$-self-conjugate. In particular, $t^2 \neq 0$ and so T contains no non-zero nilpotents.

That (3) implies (4) is immediate.

Suppose that (4) holds and let $a \in S$, $t \in T$ be such that $ta^{-1} \neq 0$. Then $at^{-1} = (ta^{-1})^{-1} \neq 0$ and $t^2 \neq 0$ so that

$$att^{-1}t^{-1}t = at^{-1}ttt^{-1} \neq 0$$

and in particular, $at \neq 0$. Hence $ata^{-1} \neq 0$ and T is $*$-self-conjugate, that is (1) holds.

Thus (1)–(4) are equivalent and it is now immediate from Corollary 3.5 that (5) is equivalent to these conditions. $\qquad\square$

If T is a full, inverse subsemigroup of a categorical inverse semigroup and T is $*$-self-conjugate, then T is said to be $*$-*normal*. As an immediate consequence of Proposition 4.1 we have the following result.

Corollary 4.2. *A subsemigroup of a categorical inverse semigroup is $*$-normal if and only if it is normal and contains no non-zero nilpotent elements.*

Next we note a connection between $*$-self-conjugacy and $*$-reflexivity.

Lemma 4.3. *If T is a full, $*$-unitary subsemigroup of a categorical, inverse semigroup S, then T is $*$-reflexive if and only if it is $*$-self-conjugate.*

Proof. If T is $*$-self-conjugate, then by Proposition 4.1 it is weakly $*$-self-conjugate. Hence, by Proposition 3.12, it is $*$-reflexive.

Conversely, if T is $*$-reflexive and $t \in T^*, a \in S$ are such that $ta^{-1} \neq 0$, then $ta^{-1}a \neq 0$. Now $ta^{-1}a \in T$ since T is full and hence $ata^{-1} \in T^*$ as required. $\quad\square$

The following alternative characterisation of kernels of primitive inverse congruences on categorical inverse semigroups now follows from Theorem 2.4 and the results of this section.

Proposition 4.4. *A subsemigroup of a categorical inverse semigroup is the kernel of a primitive inverse congruence if and only if it is a $*$-unitary, $*$-normal subsemigroup.*

Let S be a categorical inverse semigroup. We define a relation τ on $E(S)$ by the rule that

$$(e, f) \in \tau \text{ if and only if } e = f = 0 \text{ or } ef \neq 0.$$

Proposition 4.5. *If T is a $*$-unitary, $*$-normal subsemigroup of S, then the pair (T, τ) is a congruence pair for S and $\rho_T = \rho_{(T,\tau)}$.*

Proof. It is easy to see that τ is a congruence on $E(S)$ in view of the fact that S is categorical. If $e, f \in E(S)$ and $ef \neq 0$, then for any $a \in S$ such that $ae \neq 0$, we have

$$aea^{-1}afa^{-1} = aa^{-1}aefa^{-1} = aefa^{-1} \neq 0$$

by categoricity and hence τ is a normal congruence on $E(S)$.

Let $a \in S$ and $e \in E(S)$ with $ae \in T$ and $(e, a^{-1}a) \in \tau$. If $e = 0$, then $a = 0$ and $a \in T$. If $e \neq 0$, then $a^{-1}ae \neq 0$ and since T is $*$-unitary, it follows that $a \in T$.

If $t \in T^*$, then $t^2 \neq 0$ and it follows easily that $t^{-1}t\tau tt^{-1}$. Thus (T, τ) is a congruence pair.

If $a\rho_T b$ and $a \neq 0$, then $xa = bt \neq 0$ for some $x, t \in T$. Hence $xab^{-1} = btb^{-1}$ and since T is $*$-normal, $xab^{-1} \in T$. As $x \in T$ and T is also $*$-unitary we have $ab^{-1} \in T$. Now $a^{-1}ab^{-1}b \neq 0$ and so $(a, b) \in \rho_{(T,\tau)}$.

Conversely, if $(a, b) \in \rho_{(T,\tau)}$ and $a \neq b$, then $a^{-1}a\tau b^{-1}b$ and $ab^{-1} \in T$. Hence $a^{-1}ab^{-1}b \neq 0$ so that, in particular, $ab^{-1}b \neq 0$. It follows that $a\rho_T b$ since $b^{-1}b$ and ab^{-1} are both in T.

Thus $\rho_T = \rho_{(T,\tau)}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 4.6. *If T is a $*$-unitary, $*$-normal subsemigroup of a categorical inverse semigroup S, then for all $a, b \in S$,*

$$(a, b) \in \rho_T \text{ if and only if } a = b = 0 \text{ or } ab^{-1} \in T^*.$$

Proof. If $(a, b) \in \rho_T$, then $(a, b) \in \rho_{(T,\tau)}$ so that, if $a \neq 0$, then $ab^{-1} \in T^*$.

Conversely, if $ab^{-1} \in T^*$, then $ab^{-1}b \neq 0$ and hence $(a, b) \in \rho_T$. $\qquad$ $\square$

5. Joins with primitive inverse congruences

Recently, Edwards [4] proved that given a group congruence γ and an arbitrary congruence ρ on a semigroup, the join of γ with ρ is exactly $\gamma \circ \rho \circ \gamma$. As a consequence, any group congruence γ on a semigroup S is a dually right modular element of the lattice of all congruences on S in the sense of Jones [13].

In this section we show that very similar arguments yield a similar result for proper primitive inverse congruences on categorical semigroups. As in the case of group congruences, this result can be used to give an explicit description of the join of a primitive inverse congruence and an arbitrary proper congruence and a description of the kernel of such a join.

Proposition 5.1. *Let γ and ρ be proper congruences on a categorical semigroup S. If γ is primitive inverse, then $\gamma \vee \rho = \gamma \circ \rho \circ \gamma$.*

Proof. We modify the proof of [4, Theorem 1]. As there, we need only show that the relation $\gamma \circ \rho \circ \gamma$ is transitive.

Let a, b, c be non-zero elements of S with $(a, b), (b, c) \in \gamma \circ \rho \circ \gamma$. Then there are elements x, y, u, v of S such that $a\gamma x\rho y\gamma b$ and $b\gamma u\rho v\gamma c$. Now $y\gamma = b\gamma = u\gamma$ in the primitive inverse semigroup S/γ so that if $s \in S$ is such that $s\gamma = (y\gamma)^{-1}$,

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then $(us)\gamma$ and $(sy)\gamma$ are non-zero idempotents of S/γ. By categoricity, $usy \neq 0$ and as $(usy, usx) \in \rho$ we have $usx \neq 0$. It now follows from Lemma 1.6 that $usx\,\gamma\,x$.

Similarly, $vsy\,\gamma\,v$ and hence $(a, usx) \in \gamma$, $(usx, vsy) \in \rho$ and $(vsy, c) \in \gamma$ so that $(a, c) \in \gamma \circ \rho \circ \gamma$. $\qquad\square$

An example is given in [4] to show that in general, $\rho \vee \theta \neq \rho \circ \theta \circ \rho$ for arbitrary congruences ρ, θ on a semigroup S. By adjoining a zero to S, we see that this is also the case for proper congruences on a categorical semigroup.

The following lemma is another illustration of the similarity between groups and primitive inverse semigroups.

Lemma 5.2. *If ρ, σ are proper congruences on a primitive inverse semigroup S, then $\rho \circ \sigma = \sigma \circ \rho$.*

Proof. Since $ef = 0$ for distinct idempotents e, f of S, it follows that any proper congruence is idempotent separating. The lemma now follows from Theorem 7.56 of [2]. $\qquad\square$

In fact, on a Brandt semigroup the only congruence which is not proper is the universal congruence [21]. It follows easily from this and the lemma that any two congruences on a primitive inverse semigroup commute.

Corollary 5.3. *Let ρ and σ be proper primitive inverse congruences on a categorical semigroup S. Then $\rho \vee \sigma = \rho \circ \sigma \circ \rho = \sigma \circ \rho \circ \sigma$.*

Moreover, if $\rho \cap \sigma$ is a primitive inverse congruence on S, then $\rho \vee \sigma = \rho \circ \sigma = \sigma \circ \rho$.

Proof. The first part is immediate by Proposition 5.1. Let $\beta = \rho \cap \sigma$ and suppose that β is primitive inverse. Then ρ/β and σ/β are (proper) congruences on the primitive inverse semigroup S/β so that by Lemma 5.2, they commute. Hence ρ and σ commute. $\qquad\square$

Proposition 5.4. *Let γ be a primitive inverse congruence and ρ be an arbitrary proper congruence on a categorical semigroup S. Then, for all $a, b \in S$, $(a, b) \in \gamma \vee \rho$ if and only if $a = b = 0$ or $xa\,\rho\,bt$ for some $x, t \in \mathrm{Ker}\,\gamma$ with $xa \neq 0$.*

Proof. The proof is a simple modification of that of Theorem 6 of [4]. Let $K = \mathrm{Ker}\,\gamma$ and let a, b be non-zero elements of S such that $(a, b) \in \gamma \vee \rho$. Then by Theorem 5.1, there are elements c, d of S such that $a\gamma c\rho d\gamma b$. By Proposition 2.2, there are elements x, t, h, k of K such that

$$xa = ct \neq 0 \quad \text{and} \quad hd = bk \neq 0.$$

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As γ and ρ are proper and $xa \neq 0$, we have $xb \neq 0$ so that $xhd = xbk \neq 0$ and in particular, $xh \neq 0$. Hence $xh \in K^*$ and so $hx \in K^*$ since K is $*$-reflexive. Now $(hct, hdt) \in \rho$ and $hxa = hct$, $hdt = bkt$ so that $(hxa, bkt) \in \rho$. Also, $hx, kt \in K$ and $hxa \neq 0$ as required.

Conversely, suppose that a, b are non-zero elements of S and that $xa\rho bt$ for some $x, t \in K$ with $xa \neq 0$. Since $x, t \in K$ and S/γ is primitive, it follows from Lemma 1.6 that $xa\gamma a$ and $bt\gamma b$. Thus $(a, b) \in \gamma \vee \rho$ as required. $\qquad\square$

Finally, we give an analogue of Theorem 7 of [4]. First, we define the $*$-*closure* of a subset A of a semigroup S with zero to be the subset $A\omega^*$ where

$$A\omega^* = \{x \in S \mid ax \in A^* \text{ for some } a \in A\}.$$

Proposition 5.5. *If γ is a primitive inverse congruence and ρ is any proper congruence on a categorical semigroup S, then $\mathrm{Ker}(\gamma \vee \rho) = ((\mathrm{Ker}\,\gamma)\rho)\omega^* \cup \{0\}$.*

Proof. Let $K = \mathrm{Ker}(\gamma \vee \rho)$. Now $\gamma \vee \rho$ is proper so that if $a \in K^*$, then $a^2 \in K^*$. By Proposition 5.4, $xa^2\rho at$ for some $x, t \in \mathrm{Ker}\,\gamma$ with $xa \neq 0$. Now $\mathrm{Ker}\,\gamma$ is strongly $*$-dense and so there is an element z of S such that az, za are non-zero and in $\mathrm{Ker}\,\gamma$. By categoricity, $xaz \neq 0$. Also $xaz \in \mathrm{Ker}\,\gamma$ so that zxa is a non-zero element of $\mathrm{Ker}\,\gamma$ by $*$-reflexivity. Furthermore, $zxa^2\rho zat$ and $zat \in \mathrm{Ker}\,\gamma$ is non-zero and hence $a \in ((\mathrm{Ker}\,\gamma)\rho)\omega^*$.

Conversely, if $a \in ((\mathrm{Ker}\,\gamma)\rho)\omega^*$, then there is an element h of $(\mathrm{Ker}\,\gamma)\rho$ such that $ha \in (\mathrm{Ker}\,\gamma)\rho$ and $ha \neq 0$. Let $p, q \in \mathrm{Ker}\,\gamma$ be such that $h\rho p$ and $ha\rho q$. Now $(ha)^2\rho q^2\gamma q$ and $q \neq 0$ since $ha \neq 0$. Hence $(ha)^2 \neq 0$ and since $(pa)^2\rho(ha)^2$ we have $(pa)^2 \neq 0$. But $(pa)^2\gamma a^2$ since $p \in \mathrm{Ker}\,\gamma$ and consequently, $(a^2, ha) \in \gamma \vee \rho$. Since $ha\rho pa$ and $pa\gamma a$ we obtain $(a^2, a) \in \gamma \vee \rho$. Thus $a \in K$. $\qquad\square$

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