ENLARGEMENTS, SEMIABUNDANCY AND UNIPOTENT MONOIDS[1]

John Fountain, Gracinda Gomes and Victoria Gould

Department of Mathematics
University of York
Heslington
York YO1 5DD UK
jbf1,varg1@york.ac.uk

Centro do Álgebra da Universidade de Lisboa
Avenida Prof Gama Pinto, 2
1699 Lisboa
Portugal
ggomes@alf1.cii.fc.ul.pt

ABSTRACT

The relation $\widetilde{\mathcal{R}}$ on a monoid S provides a natural generalisation of Green's relation $\mathcal{R}$. If every $\widetilde{\mathcal{R}}$-class of S contains an idempotent, S is *left semiabundant*; if $\widetilde{\mathcal{R}}$ is a left congruence then S *satisfies (CL)*. Regular monoids, indeed left abundant monoids, are left semiabundant and satisfy (CL). However, the class of left semiabundant monoids is much larger, as we illustrate with a number of examples.

This is the first of three related papers exploring the relationship between unipotent monoids and left semiabundancy. We consider the situations where the *power enlargement* or the *Szendrei expansion* of a monoid yields a left semiabundant monoid with (CL). Using the Szendrei expansion and the notion of the least unipotent monoid congruence σ on a

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monoid S, we construct functors $(\widetilde{\bullet})^{S\mathcal{R}} : \mathbf{U} \to \mathbf{F}$ and $F^{\sigma} : \mathbf{F} \to \mathbf{U}$ such that $(\widetilde{\bullet})^{S\mathcal{R}}$ is a left adjoint of F^{σ}. Here $\mathbf{U}$ is the category of unipotent monoids and $\mathbf{F}$ is a category of left semiabundant monoids with properties echoing those of F-inverse monoids.

1. Introduction

The relation $\widetilde{\mathcal{R}}$ is defined on a monoid S by the rule that $a \,\widetilde{\mathcal{R}}\, b$ if and only if a and b have the same set of idempotent left identities. Green's relation $\mathcal{R}$ is contained in $\widetilde{\mathcal{R}}$, indeed $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$ where $a \,\mathcal{R}^*\, b$ if and only if $a \,\mathcal{R}\, b$ in some oversemigroup of S. When restricted to the regular elements of S, $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}$ coincide with $\mathcal{R}$.

A monoid S is *left abundant* if every $\mathcal{R}^*$-class contains an idempotent and *left semiabundant* if every $\widetilde{\mathcal{R}}$-class contains an idempotent. Thus left abundant and left semiabundant monoids provide successive generalisations of regular monoids. If S is left (semi)abundant and the idempotents $E(S)$ of S commute, then S is *left (semi)adequate*. It is easy to see that in a left (semi)adequate monoid the idempotent in the $\mathcal{R}^*$-class ($\widetilde{\mathcal{R}}$-class) of $a \in S$ is unique; in either case we denote this element by a^+. There is little chance of ambiguity here as if S is left abundant then $\mathcal{R}^* = \widetilde{\mathcal{R}}$. The relation $\mathcal{R}^*$ is clearly a left congruence; $\widetilde{\mathcal{R}}$ need not be (for an example, see [6]). If $\widetilde{\mathcal{R}}$ is a left congruence, we say that S *satisfies condition (CL)*.

A left adequate monoid S in which

$$ae = (ae)^+a \qquad \text{(AL)}$$

for all $a \in S$ and $e \in E(S)$ is called *left ample* (formerly left type A [4]). If S is left semiadequate and satisfies (CL) and (AL) then S is *weakly left ample*.

Any regular monoid is left abundant and any inverse monoid is left ample. However, the classes of left abundant and left ample monoids are much larger than the classes of regular and inverse monoids. For example, any right cancellative monoid T and any Bruck-Reilly semigroup $BR(T,\theta)$ over T is left ample. More surprisingly, papers by a number of authors have shown that right cancellative monoids play a role in the general structure of left abundant, left adequate and left ample monoids analogous to that played by groups in the structure of regular and inverse monoids, see for example [3],[5].

As $\mathcal{R}^* = \widetilde{\mathcal{R}}$ on a left abundant monoid S [6], it is clear that every left abundant (left adequate, left ample) monoid is left semiabundant (left semiadequate, weakly left ample). For a simple example of a weakly left ample monoid that is not left ample, take any unipotent monoid T which is not right cancellative, or indeed any Bruck-Reilly extension of T. Further examples of left semiabundant and left semiadequate monoids are given in Section 3. These include endomorphism monoids of certain free algebras. The basic philosophy of this paper and the two which succeed it [9], [10] is that unipotent monoids play the role for left semiabundant and left semiadequate monoids with (CL), and for weakly left ample monoids, that right cancellative monoids fulfill for left abundant, left adequate and left ample monoids respectively.

As in [11] we define an *enlargement* to be a functor $\overline{(\bullet)}$ from the category of monoids $\mathbf{M}$ into some special category of monoids. If in addition there is a natural transformation η from $\overline{(\bullet)}$ to the identity functor, such that η_S is surjective for each monoid S, then in the terminology introduced by Birget and Rhodes [1], $\overline{(\bullet)}$ is an *expansion*. Thus an enlargement $\overline{(\bullet)}$ is an expansion if for each monoid S there is a surjective morphism $\eta_S : \overline{S} \to S$ such that if $\theta : S \to T$ is a morphism, then $\overline{\theta}\eta_T = \eta_S\theta$; moreover, $\overline{\theta}$ is onto if θ is.

Of course, in order for an enlargement $\overline{(\bullet)}$ to be of use we require at least that the structure of $\overline{S}$ be related in some reasonable way to that of S. The notion of expansion takes us in this direction; in this case certainly S is a morphic image of its expansion $\overline{S}$. However, there are enlargements which fail to be expansions, yet which have proved to be an interesting tool for classifying monoids. The example considered here is that of the *power enlargement*. This can, however, be modified to produce expansions which we call *Szendrei expansions*.

Power enlargements, related enlargements and Szendrei expansions have been studied by a number of authors. In [17] Sullivan characterises those monoids for which the power enlargement $\mathcal{P}(S)$ is regular, and Fountain and Gomes do the same for the Szendrei expansion $\widetilde{(\bullet)}^{S\mathcal{R}}$ in [5] (in fact they consider the left-right dual $\widetilde{(\bullet)}^{S\mathcal{L}}$). Monoids for which $\mathcal{P}(S)$ or $\widetilde{S}^{S\mathcal{R}}$ is left abundant are described in [11]. In particular, $\widetilde{T}^{S\mathcal{R}}$ is left abundant for any right cancellative monoid T. In this paper we characterise those monoids S for which $\mathcal{P}(S)$ or $\widetilde{S}^{S\mathcal{R}}$ is left semiabundant or left semiadequate with (CL). It emerges that the power enlargement yields nothing new, in the sense that $\mathcal{P}(S)$ is left semiabundant with (CL) if and only if $\mathcal{P}(S)$ is left abundant. The expansion $\widetilde{(\bullet)}^{S\mathcal{R}}$ is more interesting; in particular, $\widetilde{S}^{S\mathcal{R}}$ is weakly left ample if and only if S is a unipotent monoid or the two element chain.

In a subsequent paper [10] we show that if S is a weakly left ample monoid then the least unipotent monoid congruence σ on S is given by the rule that

$$a \ \sigma \ b \text{ if and only if } ea = eb \text{ for some } e \in E(S),$$

that is, σ has the same description as the least right cancellative congruence on a left ample monoid [3] or the least group congruence on an inverse monoid [14]. If S is weakly left ample then S is partially ordered by $\leq$ where $a \leq b$ if and only if $a = eb$ for some $e \in E(S)$; clearly $\leq$ restricted to $E(S)$ coincides with the usual partial ordering of $E(S)$ as a semilattice. In line with the terminology of [5] we say that a weakly left ample monoid in which the σ-class $[a]$ of any element $a \in S$ contains a maximum element $m(a)$, is *weakly left FA* if for all $a, b \in S$

$$m(a)^+ m(ab)^+ = (m(a)m(b))^+ \qquad \text{(FL)}.$$

We denote the category of weakly left FA monoids and appropriate morphisms by $\mathbf{F}$ and the category of unipotent monoids and monoid morphisms by $\mathbf{U}$. In the latter part of this paper we show that $\widetilde{S}^{S\mathcal{R}}$ is weakly left FA for a unipotent monoid S and construct functors $(\widetilde{\bullet})^{S\mathcal{R}} : \mathbf{U} \to \mathbf{F}$ and $F^\sigma : \mathbf{F} \to \mathbf{U}$ such that $(\widetilde{\bullet})^{S\mathcal{R}}$ is a left adjoint of F^σ. This works exactly as in the corresponding case for right cancellative monoids and left FA monoids [5]. The original result for groups and F-inverse monoids is due to Szendrei [18].

After a section of preliminaries we concentrate in Section 3 on giving a selection of examples of left semiabundant and left semiadequate monoids. In Section 4 we consider the power enlargement $\mathcal{P}(S)$ of a monoid S. We show that if $\mathcal{P}(S)$ is left semiabundant with (CL) then S is a group G with at most two elements, or S is an ideal extension of a royal band B by such a group G, where for all $a \in G$ and $e \in B$, $ae = e$. It is known that this is equivalent to the left abundancy of $\mathcal{P}(S)$ [11].

In Section 5 we show that the Szendrei expansion $\widetilde{S}^{S\mathcal{R}}$ of a monoid S is left semiabundant with (CL) if and only if S is a unipotent monoid or S is an ideal extension of a royal band B by a unipotent monoid M such that for all $m \in M$ and $e \in B, em = e$. We deduce that $\widetilde{S}^{S\mathcal{R}}$ is weakly left ample if and only if S is a unipotent monoid or the two element chain.

In the final section we observe that if S is a unipotent monoid then $\widetilde{S}^{S\mathcal{R}}$ is weakly left FA. Consequently, $\widetilde{S}^{S\mathcal{R}}$ is *proper* in the sense that $\widetilde{\mathcal{R}} \cap \sigma = \iota$ (see [10]). We then define the functors $(\widetilde{\bullet})^{S\mathcal{R}}$ and F^σ and indicate that $(\widetilde{\bullet})^{S\mathcal{R}}$ is a left adjoint of F^σ. Much of the detail of the proof is omitted since it is essentially the same as that given in Section 4 of [5].

For further results on left semiabundant monoids and monoids satisfying the corresponding two-sided properties, see [2] (where these notions were introduced), [13] and [6].

2. Preliminaries

In this section we gather together some elementary facts and definitions which will be made use of later in the paper.

First we remark that the left-handed definitions given in the introduction may be dualised to obtain their right-handed versions. For example, the relation $\widetilde{\mathcal{L}}$ is defined on a monoid S by the rule that $a \mathrel{\widetilde{\mathcal{L}}} b$ if and only if a and b have the same set of idempotent right identities, that is,

$$ae = a \text{ if and only if } be = b$$

for all $e \in E(S)$. The monoid S is then *right semiabundant* if every $\widetilde{\mathcal{L}}$-class contains an idempotent. If S is both left and right (semi)abundant then we say that S is *(semi)abundant*. The definitions of a (semi)adequate monoid and a (weakly) ample monoid are formed in the same manner from the left- and right-handed versions.

The following easy lemma and its corollary will be used repeatedly.

Lemma 2.1. *Let S be a monoid, $a \in S$ and $e \in E(S)$. Then $a \mathrel{\widetilde{\mathcal{R}}} e$ if and only if $ea = a$ and for any $f \in E(S)$, if $fa = a$ then $fe = e$.*

The lemma is saying that an element a of a monoid S is $\widetilde{\mathcal{R}}$-related to an idempotent e if and only if e is minimum under the quasi-order induced by $\mathcal{R}$ on the set ${}_Ea$ of idempotent left identities of a. A trivial but useful consequence is that if ${}_Ea = \{1\}$, then $a \mathrel{\widetilde{\mathcal{R}}} 1$.

Corollary 2.2. *Let S be a monoid and $e, f \in E(S)$. Then $e \mathrel{\widetilde{\mathcal{R}}} f$ if and only if $e \mathrel{\mathcal{R}} f$.*

Many of the monoids which arise in this paper are ideal extensions of royal bands. We do not assume that bands are monoids. Recall that a monoid S is an *ideal extension* of I if I is an ideal of S; we make a slight departure from standard terminology by defining S to be an *ideal extension of I by M* if I is an ideal of S and $M = S \setminus I$. It is clear that if $M \neq \emptyset$ then M must contain the identity of S. A semigroup is *royal* if it cannot be generated by any of its proper subsets. Howie and Giraldes prove in [8] that a band B is royal precisely when its $\mathcal{J}$-order is a chain, if $J_e < J_f$ then $e < f$ for all $e, f \in B$, and each $\mathcal{J}$-class is a left zero semigroup or a right zero semigroup. It is then easy to see that a band B is royal if and only if for all $e, f \in B, ef \in \{e, f\}$.

The last lemma of the section is easy to check.

Lemma 2.3. *A monoid is left semiabundant with (CL) if and only if the monoid obtained by adjoining a zero is left semiabundant with (CL).*

3. Examples

We begin by noting that a unipotent monoid S is weakly ample; S is then left (right) ample if and only if S is right (left) cancellative.

It is easy to construct left semiabundant and left semiadequate monoids from a given unipotent monoid S. For example, let I, Λ be non-empty sets and let P be a $\Lambda \times I$ matrix over $S \cup \{0\}$ such that every column of P contains a unit of S. Then it is straightforward to show that the Rees matrix semigroup $\mathcal{M}^0(S; I, \Lambda; P)$, with an adjoined identity, is left semiabundant with (CL). Similarly, the Rees matrix semigroup $\mathcal{M}^0(S; I, I; Q)$ with an adjoined identity, where Q is the identity matrix, is semiadequate, indeed weakly ample. Further, for any endomorphism θ of S, the Bruck-Reilly semigroup $BR(S, \theta)$ over S is weakly ample. All these monoids fail to be left abundant unless S is right cancellative.

Other popular constructions yield other relevant examples. For instance, the Schützenberger product $M \diamond N$ of unipotent monoids M and N is semiadequate and satisfies (CR) ((CL)) if and only if M is left cancellative (N is right cancellative) [6]. For $M \diamond N$ to be right adequate we need in addition that N be left cancellative [6]. If M is left cancellative but not a group, then $M \diamond N$ fails to have (AR) [6]. On the other hand, the following table gives the multiplication in a four element monoid $S = \{1, e, a, b\}$ which is commutative, semiadequate, has (AL) and (AR), but fails to have (CL) and (CR).

.	1	e	a	b
1	1	e	a	b
e	e	e	b	b
a	a	b	e	e
b	b	b	e	e

The results of [7] show that the endomorphism monoid of any *basis algebra* (for the definition, see [7]) is abundant. Examples of basis algebras given in [7] are free R-modules on a finite set, where R is a principal ideal domain, and free S sets on a finite set, where S is a commutative cancellative principal ideal monoid.

We now consider the endomorphism ring of a free R-module of rank 2 over an integral domain R; in other words we look at the matrix ring $M_2(R)$.

Theorem 3.1. *Let R be an integral domain. Then $M_2(R)$ is semiabundant. If R is a unique factorisation domain which is not a principal ideal domain, then $M_2(R)$ is not abundant and (CL) and (CR) do not hold.*

Proof Let Q be the field of quotients of R. Clearly $M_2(R)$ is a subring of the matrix ring $M_2(Q)$ over the field Q; matrices in $M_2(R)$ thus have a well defined rank, namely, their rank in the over-ring $M_2(Q)$. Let $B \in M_2(R)$. If rank $B = 2$, then B is invertible in $M_2(Q)$, hence cancellable in $M_2(R)$, so that certainly $B \ \widetilde{\mathcal{R}} \ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If rank $B = 1$, then either I_2 is the only idempotent left identity of B, in which case $B \ \widetilde{\mathcal{R}} \ I_2$, or B has an idempotent left identity E, where rank $E = 1$. In the latter case $EB = B \neq 0$, where E and B may be regarded as elements of the completely 0-simple semigroup of matrices of $M_2(Q)$ consisting of those matrices of rank 1 together with the zero matrix (see [15]). It follows that $E \ \mathcal{R} \ B$ in $M_2(Q)$ and so $E \ \widetilde{\mathcal{R}} \ B$ in $M_2(R)$ (indeed $E \ \mathcal{R}^* \ B$ in $M_2(R)$). Thus $M_2(R)$ is left semiabundant.

Suppose now that R is a unique factorisation domain which is not a principal ideal domain. We show that $M_2(R)$ is not left abundant.

Let $a, b \in R$ be such that $aR + bR$ is not a principal ideal. By factoring from a and b their highest common factor, we may assume that a and b are coprime. Let $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$. We show that I_2 is the only idempotent left identity of A.

Let $E = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ be an idempotent such that $EA = A$. Then

$$fb = (1-e)a, \ ga = (1-h)b. \tag{1}$$

As $aR + bR$ is *not* principal, none of $f, 1-e, g, 1-h$ can be units, in particular, $e \neq 0$ and $h \neq 0$.

From $E^2 = E$ we have that

$$\begin{aligned} e &= e^2 + fg, \quad f = ef + fh, \\ g &= ge + hg, \quad h = gf + h^2. \end{aligned} \tag{2}$$

Equation (1) gives that a is a factor of $1-h$, so that $1-h = ac$ for some $c \in R$. If $f \neq 0$, (2) gives that $1 = e + h$ and so $1 - e = h$. Using (1) again we have $fb = ha$ and so $h = bd$ for some $d \in R$. Now we have

$$1 = h + ac = bd + ac \in aR + bR,$$

a contradiction since $aR + bR$ is not equal to R.

Thus $f = 0$; then $e = 1$ by (1) so that by (2), $g = 0$ or $h = 0$. But $h \neq 0$, giving that $g = 0$ and again by (2), $h = 1$. Thus $E = I_2$. As commented in Section 2, we thus have $A \,\widetilde{\mathcal{R}}\, I_2$.

If $M_2(R)$ were left abundant, then we would have $\widetilde{\mathcal{R}} = \mathcal{R}^*$ [6]. But A is not $\mathcal{R}^*$-related to I_2 since

$$\begin{pmatrix} -b & a \\ -b & a \end{pmatrix} A = 0 = 0A.$$

This also shows that $\widetilde{\mathcal{R}}$ is not a left congruence, since $I_2 \,\widetilde{\mathcal{R}}\, A$ but $\begin{pmatrix} -b & a \\ -b & a \end{pmatrix} I_2 = \begin{pmatrix} -b & a \\ -b & a \end{pmatrix}$ is not $\widetilde{\mathcal{R}}$-related to $\begin{pmatrix} -b & a \\ -b & a \end{pmatrix} A = 0$. Thus (CL) does not hold.

Dual arguments give the right-handed results.

In the remainder of this section we consider the endomorphism monoid M of $\mathbf{F}$, where $\mathbf{F}$ is a free right S-set over a unipotent monoid S. We show that M is right semiabundant with (CR), but need not be left semiabundant or right abundant.

Let S be a fixed unipotent monoid and let I be a non-empty set. The free right S-set $\mathbf{F}$ on the set I may be described as follows. Let $\{x_i : i \in I\}$ be a set in one-one correspondence with I and let $\mathbf{F}$ be the disjoint union $\bigcup_{i \in I} x_i S$. Then S acts on the right of $\mathbf{F}$ in the obvious way. For each $i \in I$ we identify $x_i 1$ with x_i, so that x_i may be regarded as an element of $\mathbf{F}$. The endomorphism monoid of $\mathbf{F}$ consists of those maps $\alpha : \mathbf{F} \to \mathbf{F}$ such that $(ys)\alpha = y\alpha s$ for all $y \in \mathbf{F}$ and $s \in S$. Note that an endomorphism is determined by its effect on the subset $\{x_i : i \in I\}$ of $\mathbf{F}$.

Lemma 3.2. *An element α of $M = End\,\mathbf{F}$ is idempotent if and only if there is a non-empty subset J of I such that $Im\ \alpha = \bigcup_{i \in J} x_i S$, and $x_i \alpha = x_i$ for all $i \in J$.*

Proof Since $\alpha \in M$ is idempotent if and only if the restriction of α to Im α is the identity map, it is clear that if Im $\alpha = \bigcup_{i \in J} x_i S$ for some $\emptyset \neq J \subseteq I$ and $x_i \alpha = x_i$ for all $i \in J$, then $\alpha^2 = \alpha$.

Conversely, suppose that $\alpha^2 = \alpha$. Let $x_i t \in \text{Im } \alpha$. Then

$$x_i t = y\alpha = y\alpha^2 = (x_i t)\alpha = x_i \alpha t$$

for some $y \in \mathbf{F}$. Thus $x_i \alpha = x_i s$ for some $s \in S$. Now

$$x_i s = x_i \alpha = x_i \alpha^2 = (x_i s)\alpha = x_i \alpha s = x_i s^2$$

so that $s = s^2$ and as S is unipotent, $s = 1$. It follows that there is a non-empty subset J of I such that Im $\alpha = \bigcup_{i \in J} x_i S$ and as α is the identity map on Im α, $x_i \alpha = x_i$ for all $i \in J$.

Proposition 3.3. *The monoid M is right semiabundant with (CR).*

Proof For any $\alpha \in M$ the non-empty subset I_α of I is defined by

$$I_\alpha = \{i \in I : \text{ Im } \alpha \cap x_i S \neq \emptyset\}.$$

Let $\gamma \in M$ and fix $k \in I_\gamma$. Define $\epsilon \in M$ by

$$\begin{array}{ll} x_i\epsilon = x_i & \text{for all } i \in I_\gamma \\ x_i\epsilon = x_k & \text{for all } i \in I \setminus I_\gamma. \end{array}$$

Certainly $\epsilon^2 = \epsilon$ and as Im $\gamma \subseteq$ Im ϵ, $\gamma\epsilon = \gamma$. Suppose now that $\nu \in M, \nu^2 = \nu$ and $\gamma\nu = \gamma$. Then Im $\gamma \subseteq$ Im ν so that by Lemma 3.2, Im $\epsilon \subseteq$ Im ν and $\epsilon\nu = \epsilon$. The dual of Lemma 2.1 now gives that $\gamma \; \widetilde{\mathcal{L}} \; \epsilon$. Thus M is right semiabundant.

We claim that for $\alpha, \beta \in M$,

$$\alpha \; \widetilde{\mathcal{L}} \; \beta \text{ if and only if } I_\alpha = I_\beta.$$

If $I_\alpha = I_\beta$, then by the above argument we may find an idempotent η such that $\alpha \; \widetilde{\mathcal{L}} \; \eta$ and $\beta \; \widetilde{\mathcal{L}} \; \eta$, so that certainly $\alpha \; \widetilde{\mathcal{L}} \; \beta$.

Conversely, suppose that $\alpha \; \widetilde{\mathcal{L}} \; \beta$. Again by the above, there are idempotents $\tau, \kappa \in M$ such that $\alpha \; \widetilde{\mathcal{L}} \; \tau, \beta \; \widetilde{\mathcal{L}} \; \kappa$,

$$\text{Im } \tau = \bigcup_{i \in I_\alpha} x_i S \text{ and Im } \kappa = \bigcup_{i \in I_\beta} x_i S.$$

Since $\alpha \; \widetilde{\mathcal{L}} \; \beta$ we have that $\tau \; \widetilde{\mathcal{L}} \; \kappa$. It follows that $\tau\kappa = \tau$ and $\kappa\tau = \kappa$, which gives that Im $\tau =$ Im κ and so $I_\alpha = I_\beta$ and the claim is proven.

Suppose now that $\alpha, \beta, \gamma \in M$ and $\alpha \; \widetilde{\mathcal{L}} \; \beta$. Let $i \in I_{\alpha\gamma}$. Then $x_i s = y\alpha\gamma$ for some $y \in \mathbf{F}, s \in S$. Writing $y\alpha = x_j t$ we have $x_i s = x_j \gamma t$ and $x_j\gamma \in x_i S$. Also $j \in I_\alpha$ so that by the claim, $j \in I_\beta$ and $x_j u = z\beta$ for some $z \in \mathbf{F}, u \in S$. Thus $z\beta\gamma = x_j\gamma u$ and so $i \in I_{\beta\gamma}$. Together with the dual argument this gives that $I_{\alpha\gamma} = I_{\beta\gamma}$. By the claim, $\alpha\gamma \; \widetilde{\mathcal{L}} \; \beta\gamma$ so that condition (CR) holds.

It is easy to see that if S is not right cancellative then M is not right abundant. For if S is not right cancellative there exist $u, a, b \in S$ with $au = bu$ and $a \neq b$. Define $\alpha, \beta, \gamma \in M$ by $x_i\alpha = x_i u, x_i\beta = x_i a$ and $x_i\gamma = x_i b$ for all $i \in I$. Then $\alpha\beta = \alpha\gamma$ and from the proof of Proposition 3.3, $\alpha \; \widetilde{\mathcal{L}} \; \iota$, where ι is the identity of M. But $\beta \neq \gamma$ so that M cannot be right abundant, since in a right abundant monoid $\mathcal{L}^* = \widetilde{\mathcal{L}}$.

Proposition 3.4. *If S is not left cancellative and $|I| \geq 2$, then M is not left semiabundant.*

Proof We may assume that $I = \{1,2\} \cup J$ where $J \cap \{1,2\} = \emptyset$. Suppose there exist $b, u, v \in S$ with $bu = bv$ and $u \neq v$. Define $\alpha \in M$ by

$$x_1\alpha = x_2bu,\ x_2\alpha = x_2b,\ x_j\alpha = x_j$$

for all $j \in J$. Let $\delta, \epsilon \in M$ be given by

$$x_1\delta = x_2u,\ x_i\delta = x_i = x_i\epsilon,\ x_1\epsilon = x_2v$$

for all $i \neq 1$. Then

$$x_1\delta\alpha = (x_2u)\alpha = x_2\alpha u = x_2bu = x_1\alpha$$

and similarly, $x_1\epsilon\alpha = x_1\alpha$. Thus $\delta\alpha = \alpha = \epsilon\alpha$ and certainly $\delta^2 = \delta, \epsilon^2 = \epsilon$. Suppose that $\mu^2 = \mu$ and $\alpha\ \widetilde{\mathcal{R}}\ \mu$. Then $\delta\mu = \mu = \epsilon\mu$ so that $\mu \neq \iota$. Since $\mu\alpha = \alpha$ we have that $x_j\mu = x_j$ for all $j \in J$ and $x_1\mu, x_2\mu \in x_1S \cup x_2S$. Thus there are two possibilities for μ. One is that $x_2\mu = x_1p$ and $x_1\mu = x_1$ for some $p \in S$. The other is that $x_1\mu = x_2q$ and $x_2\mu = x_2$ for some $q \in S$.

In the first case we would have

$$x_1 = x_1\mu = x_1\delta\mu = x_2\mu u = x_1pu$$

and also $x_1 = x_1pv$, so that $pu = pv = 1$. Since S is unipotent, $up = vp = 1$ and it follows that $u = v$, a contradiction.

In the second,

$$x_2u = x_2\mu u = (x_2u)\mu = x_1\delta\mu = x_1\mu$$

and similarly, $x_2v = x_1\mu$. Again this yields the contradiction that $u = v$. We conclude that α is not $\widetilde{\mathcal{R}}$-related to any idempotent and M is not left semiabundant.

Even if S is commutative and cancellative, M need not be left abundant.

Proposition 3.5. *Suppose that S is commutative and cancellative, and there are elements $a, b \in S$ with $a \notin bS$ and $b \notin aS$. If $|I| \geq 2$, then M is not left abundant.*

Proof We assume that $I = \{1,2\}$; a simple adjustment to the argument, along the lines of the previous proposition, gives the more general result.

Define $\alpha \in M$ by

$$x_1\alpha = x_2a,\ x_2\alpha = x_2b.$$

Suppose that $\epsilon^2 = \epsilon$ and $\epsilon\alpha = \alpha$. If $x_1\epsilon = x_2u$, then

$$x_2a = x_1\alpha = x_1\epsilon\alpha = (x_2u)\alpha = x_2\alpha u = x_2bu,$$

so that $a \in bS$, a contradiction. If $x_2\epsilon = x_1 v$, then

$$x_2 b = x_2\alpha = x_2\epsilon\alpha = (x_1 v)\alpha = x_1\alpha v = x_2 a v,$$

giving that $b \in aS$, again a contradiction. We conclude from Lemma 3.2 that $\alpha \,\widetilde{\mathcal{R}}\, \iota$.

Define $\mu, \nu \in M$ by

$$x_1\mu = x_1 b,\ x_2\mu = x_2,\ x_1\nu = x_2 a,\ x_2\nu = x_2.$$

Then $\mu \neq \nu$ but $\mu\alpha = \nu\alpha$. Thus α is not $\mathcal{R}^*$-related to ι and so M is not left abundant.

4. The power enlargement

Consider the functor $\mathcal{P} : \mathbf{M} \to \mathbf{M}$ such that for each $S \in Ob(\mathbf{M})$,

$$\mathcal{P}(S) = \{X : X \subseteq S\}$$

with multiplication given by $X \cdot Y = XY$, and for each $S, T \in Ob(\mathbf{M})$ and monoid morphism $\theta : S \to T$, $\mathcal{P}(\theta) : \mathcal{P}(S) \to \mathcal{P}(T)$ is defined by

$$X\mathcal{P}(\theta) = \{x\theta : x \in X\}$$

for each $X \in \mathcal{P}(S)$. The monoid $\mathcal{P}(S)$ is called the *power monoid* of the monoid S. Related enlargements are the functors $\mathcal{P}', \mathcal{PF}, \mathcal{PF}'$ and $\mathcal{PF}'_1$ which have the same action on morphisms but for $S \in Ob(\mathbf{M})$, $\mathcal{P}'(S), (\mathcal{PF}(S), \mathcal{PF}'(S), \mathcal{PF}'_1(S))$ is the submonoid of $\mathcal{P}(S)$ consisting of all non-empty subsets of S, (all finite subsets of S, all finite non-empty subsets of S, all finite subsets of S containing 1). These enlargements are of course not expansions, since in general S is not a morphic image of the corresponding enlarged monoid. Nevertheless, $\mathcal{P}$ and $\mathcal{P}'$ in particular have been widely studied, as have the analogous enlargements of semigroups (see for example [16], [17]).

This section describes those monoids S for which $\mathcal{P}(S)$ is (left) semiabundant, or (left) semiadequate, with (CL). We remark that if $\mathcal{P}(S)$ is left semiadequate with (CL) then $\mathcal{P}(S)$ is forced to be inverse, hence certainly is weakly left ample.

Proposition 4.1. *Let S be a monoid such that $\mathcal{P}'(S)$ is left semiabundant with (CL). Then S is a group G with at most two elements, or S is an ideal extension of a royal band B by such a group G, where for each $a \in G$ and $e \in B$, $ae = e$.*

Proof Write $\mathcal{E}$ for $E(\mathcal{P}'(S))$ and note that for $Y \in \mathcal{P}'(S)$ and $F \in \mathcal{E}$, if $Y \,\widetilde{\mathcal{R}}\, F$ and $1 \in Y$, then $F \subseteq Y$.

We first show that if $x \in S$ then $x^2 = 1$ or $x^2 = x$. If $x = 1$ then there is nothing to show. Suppose that $x \neq 1$ and let $X \in \mathcal{E}$ be such that $\{1,x\}\ \widetilde{\mathcal{R}}\ X$; by the remark above, $X \subseteq \{1,x\}$. If $x \in X$ then $X\{1,x\} = \{1,x\}$ gives $x^2 \in \{1,x\}$ as required. We proceed to rule out the possibility that $X = \{1\}$.

Assume that $X = \{1\}$. As $\{1,x\}\ \widetilde{\mathcal{R}}\ \{1\}$ and $\mathcal{P}'(S)$ has (CL), we have $\{1,x\}^n\ \widetilde{\mathcal{R}}\ \{1\}$ for all $n \in \mathbb{N}$. If x has finite order $m \in \mathbb{N}$ then

$$\{1,x\}^m = \{1,x,x^2,...,x^{m-1}\} \in \mathcal{E}$$

so that from $\{1,x\}^m\ \widetilde{\mathcal{R}}\ \{1\}$ we obtain $\{1,x\}^m = \{1\}$ and $x = 1$, a contradiction. Thus x has infinite order and for any distinct $n,m \in \mathbb{N}$, $x^n \neq x^m$. Putting $N = \{1,x,x^2,...\}$ and $M = N \setminus \{x\}$ we have $N \in \mathcal{E}$ and $N \neq M$. From $\{1,x\}\ \widetilde{\mathcal{R}}\ \{1\}$ and (CL) we have $M\{1,x\}\ \widetilde{\mathcal{R}}\ M\{1\}$ so that $N\ \widetilde{\mathcal{R}}\ M$ and $NM = M$. As $1 \in M$ it follows that $N = M$, a contradiction. Thus $X = \{1\}$ is not possible.

For the remainder of the proof, let $G = \{x \in S : x^2 = 1\}$ and $B = S \setminus G$ so that $B = E(S) \setminus \{1\}$. It is easy to see that G is the $\mathcal{R}$-class of 1 so that as $\mathcal{R}$ is a left congruence, G is a submonoid. Indeed G is a subgroup of S and as every element of G has order 1 or 2, G is commutative.

In fact, G has at most two elements. To see this, let $s,t \in G \setminus \{1\}$. We know that $\{1,s,t\}\ \widetilde{\mathcal{R}}\ F = F^2$ for some $F \subseteq \{1,s,t\}$. If $s \in F$ then from $F\{1,s,t\} = \{1,s,t\}$ we have $st \in \{1,s,t\}$ so that as G is a group, $s = t$. Similarly for $t \in F$. The remaining option is that $F = \{1\}$. From $\{1,s,t\}\ \widetilde{\mathcal{R}}\ \{1\}$ we then have $\{1,s,t\}^2\ \widetilde{\mathcal{R}}\ \{1\}$; but $\{1,s,t\}^2$ is a submonoid so that $\{1,s,t\}^2 \in \mathcal{E}$ and so $\{1,s,t\}^2 = \{1\}$, a contradiction. Thus $s = t$ and G is either trivial or is the two element group.

If $B = \emptyset$ then we conclude that $S = G$ is a group with at most two elements. Suppose that $B \neq \emptyset$. We show that B is an ideal of S. Let $e \in B$ and $a \in G$. If $ea \in G$ then $ea\ \mathcal{R}\ 1$ so that $e1 = 1$, a contradiction. Thus $ea \in B$ and B is a right ideal. As G is in fact the $\mathcal{H}$-class of 1 the same argument gives that B is a left ideal. Thus B is a band and an ideal of S.

If $G = \{1\}$, clearly $ae = a$ for any $a \in G$ and $e \in B$. Consider the possibility that $G = \{1,a\}$ where $a \neq 1$. We show that for any $e \in B, ae = e$. Let K be an idempotent $\widetilde{\mathcal{R}}$-related to $\{1,a,e\}$ so that $K \subseteq \{1,a,e\}$. Suppose that $a \notin K$. From $K\{1,a,e\} = \{1,a,e\}$ and the fact that B is an ideal, we must have that $1 \in K$ so that $K = \{1\}$ or $K = \{1,e\}$. Since B is a band ideal, it is a routine matter to check that $\{1,a,e\}^3$ is a submonoid and we deduce that $K = \{1\}$ is not possible. Thus $K = \{1,e\}$. Now $\{1,a,e\}\ \widetilde{\mathcal{R}}\ \{1,e\}$ gives $\{1,e\}\{1,a,e\} = \{1,a,e\}$ and so $ea = e$. It follows that $\{1,a,e,ae\}$ is a submonoid. Now using

(CL) we have that

$$\{1, ae, e\}\{1, a, e\} \widetilde{\mathcal{R}} \{1, ae, e\}\{1, e\}$$

and so $\{1, a, e, ae\} \widetilde{\mathcal{R}} \{1, ae, e\}$. But $\{1, a, e, ae\} \in \mathcal{E}$ so that

$$\{1, a, e, ae\}\{1, ae, e\} = \{1, ae, e\}.$$

Hence $a \in \{1, ae, e\}$, a contradiction. Thus $a \notin K$ is not possible and from $a \in K$ and $K\{1, a, e\} = \{1, a, e\}$ we have $ae \in \{1, a, e\}$ so that $ae = e$ as required.

It remains to prove that B is a royal band. According to the remark before Lemma 2.3, it suffices to show that $ef \in \{e, f\}$ for all $e, f \in B$. If $e = f$ this is clear. Let e, f be distinct elements of B and let $L \in \mathcal{E}$ be such that $L \ \widetilde{\mathcal{R}} \ \{1, e, f\}$. The possibility that $L = \{1\}$ is ruled out by that fact that $\{1, e, f\}^3$ is a submonoid. So $e \in L$ or $f \in L$, giving $ef \in \{e, f\}$ or $fe \in \{e, f\}$, respectively. We must prove that $ef \in \{e, f\}$ whenever $e \notin L$ and $f \in L$. In this case we must have $L = \{1, f\}$ and $fe \in \{e, f\}$. Then $\{1, e, f, ef\} \in \mathcal{E}$ and from

$$\{1, f\} \widetilde{\mathcal{R}} \{1, e, f\}$$

and (CL) we have

$$\{1, ef\}\{1, f\} \widetilde{\mathcal{R}} \{1, ef\}\{1, e, f\}$$

and so $\{1, f, ef\} \widetilde{\mathcal{R}} \{1, e, f, ef\}$. Then $\{1, e, f, ef\}\{1, f, ef\} = \{1, f, ef\}$ so that $e \in \{1, f, ef\}$ and $e = ef$.

Proposition 4.1 and Lemma 2.3 together with Theorem 3.2 of [11] give the following result.

Theorem 4.2. *The following are equivalent for a monoid S:*
(i) $\mathcal{P}(S)$ is left abundant;
(ii) $\mathcal{P}(S)$ is left semiabundant with (CL);
(iii) S is a group G with at most two elements, or an ideal extension of a royal band B by such a group G, where for each $a \in G$ and $e \in B$, $ae = e$.

Corollary 4.3. *The following are equivalent for a monoid S:*
(i) $\mathcal{P}(S)$ is abundant;
(ii) $\mathcal{P}(S)$ is semiabundant with (CL) and (CR);
(iii) S is a group G with at most two elements, or an ideal extension of a royal band B by such a group G, where for each $a \in G$ and $e \in B$, $ae = e = ea$;
(iv) $\mathcal{P}(S)$ is regular.

Proof The equivalence of (i) and (ii) is immediate from Theorem 4.2. The equivalence of (iii) and (iv) is proved by Sullivan in [17]. Theorem 4.2 also gives that (iii) implies (i) and an easy argument as in Corollary 3.4 of [11] gives the opposite implication.

A simple example given in [11] shows that $\mathcal{P}(S)$ can be left but not right abundant.

Note that in the final result of this section, conditions (iii) and (iv) are symmetric. Thus conditions (i) and (ii) may be replaced by their left-right duals or by their two-sided versions.

Corollary 4.4. *The following are equivalent for a monoid S:*
(i) $\mathcal{P}(S)$ is left adequate;
(ii) $\mathcal{P}(S)$ is left semiadequate with (CL);
(iii) S is a group G with at most two elements, or S is an ideal extension of a chain C by such a group G;
(iv) $\mathcal{P}(S)$ is inverse.

Proof Theorem 4.2 gives that (i) and (ii) are equivalent. If (iii) holds then the fact that G is a group acting on C gives that $c = ac = ca$ for all $c \in C$ and $a \in G$. From Corollary 3.5 of [11], conditions (i), (iii) and (iv) are now equivalent.

5. The Szendrei expansion

Let S be a monoid. Then $\widetilde{S}^{S\mathcal{R}}$ is the monoid with underlying set

$$\widetilde{S}^{S\mathcal{R}} = \{(X, x) : X \in \mathcal{PF}'_1(S), x \in X\}$$

and multiplication given by

$$(X, x)(Y, y) = (X \cup xY, xy);$$

the identity of $\widetilde{S}^{S\mathcal{R}}$ is $(\{1\}, 1)$. If $\theta : S \to T$ is a monoid morphism, then defining $\widetilde{\theta}^{S\mathcal{R}} : \widetilde{S}^{S\mathcal{R}} \to \widetilde{T}^{S\mathcal{R}}$ by $(X, x)\widetilde{\theta}^{S\mathcal{R}} = (X\theta, x\theta)$ for all $(X, x) \in \widetilde{S}^{S\mathcal{R}}$, it is easy to see that $(\widetilde{\bullet})^{S\mathcal{R}}$ is an expansion, where for any monoid S, $\eta_S : \widetilde{S}^{S\mathcal{R}} \to S$ is given by $(X, x)\eta_S = x$. The monoid $\widetilde{S}^{S\mathcal{L}}$ is defined dually. The expansions $\widetilde{S}^{S\mathcal{R}}$ and $\widetilde{S}^{S\mathcal{L}}$ were introduced by Szendrei in [18], hence as in [5] we call them *Szendrei expansions*. Szendrei showed that if G is a group then the expansions $\widetilde{G}^{S\mathcal{R}}$ and $\widetilde{G}^{S\mathcal{L}}$ coincide with the Birget-Rhodes expansions $\widetilde{G}^{\mathcal{R}}$ and $\widetilde{G}^{\mathcal{L}}$ respectively; it was already known that $\widetilde{G}^{\mathcal{R}}$ and $\widetilde{G}^{\mathcal{L}}$ are isomorphic [1]. As remarked in [5], the expansions $\widetilde{S}^{S\mathcal{R}}$ and $\widetilde{S}^{\mathcal{R}}$ are in general *not* isomorphic. The expansion $\widetilde{S}^{S\mathcal{R}}$ is of particular use when S is a right cancellative monoid; in this case $\widetilde{S}^{S\mathcal{R}}$ is a left ample monoid of a special kind called *left FA* [5].

In this section we characterise those monoids S for which $\widetilde{S}^{S\mathcal{R}}$ is left semiabundant, or left semiadequate, with (CL). As in the case for $\mathcal{P}(S)$, if $\widetilde{S}^{S\mathcal{R}}$ is left semiadequate with (CL) then $\widetilde{S}^{S\mathcal{R}}$ is weakly left ample; however $\widetilde{S}^{S\mathcal{R}}$ is not forced to be inverse or even left adequate.

Observe that for any monoid S,

$$\{(X,1) : X \in \mathcal{PF}_1'(S)\}$$

is a semilattice isomorphic to $\mathcal{PF}_1'(S)$ under union. Denoting the set of idempotents of $\widetilde{S}^{S\mathcal{R}}$ by $\mathcal{E}$, it is immediate that for any element $(X,e) \in \widetilde{S}^{S\mathcal{R}}$

$$(X,e) \in \mathcal{E} \text{ if and only if } eX \subseteq X \text{ and } e \in E(S).$$

Theorem 5.1. *Let S be a monoid. Then $\widetilde{S}^{S\mathcal{R}}$ is left semiabundant with (CL) if and only if S is a unipotent monoid or an ideal extension of a royal band B by a unipotent monoid M such that for all $e \in B$ and $m \in M, em = e$.*

Proof We begin by assuming that $\widetilde{S}^{S\mathcal{R}}$ is left semiabundant with (CL). We make use of a series of subsidiary lemmas.

Lemma 5.2. *Let $(X,x),(Y,y) \in \widetilde{S}^{S\mathcal{R}}$. Then $(X,x)\ \widetilde{\mathcal{R}}\ (Y,y)$ implies that $x\ \widetilde{\mathcal{R}}\ y$.*

Proof Suppose that $(X,x)\ \widetilde{\mathcal{R}}\ (Y,y)$. We want to prove that $ex = x$ if and only if $ey = y$ for all $e \in E(S)$. Let $e \in E(S)$ with $ex = x$. Put $Z = eX \cup X$ so that $Z \in \mathcal{PF}_1'(S)$ and $e \in Z$ so that $(Z,1),(Z,e) \in \widetilde{S}^{S\mathcal{R}}$. Also $eZ \subseteq Z$, giving $(Z,e) \in \mathcal{E}$. Since $\widetilde{S}^{S\mathcal{R}}$ satisfies (CL) we have

$$(Z,1)(X,x)\ \widetilde{\mathcal{R}}\ (Z,1)(Y,y)$$

and so

$$(Z,x)\ \widetilde{\mathcal{R}}\ (Z \cup Y,y).$$

Notice now that $(Z,e)(Z,x) = (Z,x)$ so that also $(Z,e)(Z \cup Y,y) = (Z \cup Y,y)$ and in particular, $ey = y$. Together with the dual argument this gives that $x\ \widetilde{\mathcal{R}}\ y$.

Lemma 5.3. *If $(X,x) \in \widetilde{S}^{S\mathcal{R}}$ and $(X,x)\ \widetilde{\mathcal{R}}\ (U,u)$ where $(U,u) \in \mathcal{E}$, then $X = U$.*

Proof Lemma 2.1 gives that $(U,u)(X,x) = (X,x)$ so that certainly $U \subseteq X$. On the other hand, $(X,1) \in \mathcal{E}$ and $(X,1)(X,x) = (X,x)$ so that $(X,1)(U,u) = (U,u)$. This gives that $X \subseteq U$ and so $X = U$ as required.

Let $x \in S \backslash E(S)$. As $\widetilde{S}^{S\mathcal{R}}$ is left semiabundant and $(\{1, x\}, x) \in \widetilde{S}^{S\mathcal{R}}$, there is some $(U, u) \in \mathcal{E}$ such that $(\{1, x\}, x) \;\widetilde{\mathcal{R}}\; (U, u)$. By Lemma 5.3, $U = \{1, x\}$ so that as $u \in U$ we have $u = 1$ or $u = x$. As $x \in S \setminus E(S)$ only $u = 1$ is possible and Lemma 5.2 then gives that $x \;\widetilde{\mathcal{R}}\; 1$. It follows that S is the disjoint union of M and B where M is the $\widetilde{\mathcal{R}}$-class of 1 and $B = E(S) \setminus \{1\}$. Clearly then S is left semiabundant.

Lemma 5.4. *The monoid S satisfies condition (CL).*

Proof Let $x, y, z \in S$ with $x \;\widetilde{\mathcal{R}}\; y$.

Suppose first that $x \;\widetilde{\mathcal{R}}\; y \;\widetilde{\mathcal{R}}\; 1$. We have

$$(\{1, x, y\}, x) \;\widetilde{\mathcal{R}}\; (U, u)$$

for some $(U, u) \in \mathcal{E}$. By Lemma 5.3, $U = \{1, x, y\}$ and as $u^2 = u \in U$ the only possibility is that $u = 1$. Thus $(\{1, x, y\}, x) \;\widetilde{\mathcal{R}}\; (\{1, x, y\}, 1)$ and the same argument gives that $(\{1, x, y\}, y) \;\widetilde{\mathcal{R}}\; (\{1, x, y\}, 1)$. As $(\{1, z\}, z) \in \widetilde{S}^{S\mathcal{R}}$ and $\widetilde{S}^{S\mathcal{R}}$ has (CL),

$$(\{1, z\}, z)(\{1, x, y\}, x) \;\widetilde{\mathcal{R}}\; (\{1, z\}, z)(\{1, x, y\}, y)$$

which in view of Lemma 5.2 yields $zx \;\widetilde{\mathcal{R}}\; zy$.

The other possibility is that $x, y \in B$, hence $x, y \in E(S)$. By Corollary 2.2 $x \;\widetilde{\mathcal{R}}\; y$ gives that $x \;\mathcal{R}\; y$; certainly then $zx \;\mathcal{R}\; zy$ and as $\mathcal{R} \subseteq \widetilde{\mathcal{R}}$ we have $zx \;\widetilde{\mathcal{R}}\; zy$ as required.

Lemma 5.5. *The set M is a unipotent submonoid and if $B \neq \emptyset$ then B is an ideal.*

Proof Since M is the $\widetilde{\mathcal{R}}$-class of 1 and S has (CL), M is a unipotent submonoid.

Suppose $B \neq \emptyset$. Let $e \in B$ and $x \in S$. If $ex \in M$ then $ex \;\widetilde{\mathcal{R}}\; 1$ so that as $e(ex) = ex$ we have that $e1 = 1$, a contradiction. Thus $ex \in B$ and B is a right ideal.

Considering the element xe, the fact that $ex \in E(S)$ gives that $(xe)^2 \in E(S)$. If xe is in the monoid M then unipotency gives that $(xe)^2 = 1$ from which we obtain the contradiction $1e = 1$. Thus $xe \in B$ so that B is an ideal.

At this stage we know that either $S = M$ is a unipotent monoid or S is an ideal extension of a band B by a unipotent monoid M.

Lemma 5.6. *The band B is royal.*

Proof In view of the remarks concerning royal bands in Section 2, it is enough to show that for any $e, f \in B$ the product $ef \in \{e, f\}$.

Let $e, f \in B$; we know from Lemmas 5.2 and 5.3 that there exists $(\{1, e, f\}, x) \in \mathcal{E}$ such that

$$(\{1, e, f\}, e) \, \widetilde{\mathcal{R}} \, (\{1, e, f\}, x)$$

where $x \, \widetilde{\mathcal{R}} \, e$. Since $x \in \{1, e, f\}$ we have $x = e$ or $x = f$. If $x = f$ then $ef = f \in \{e, f\}$. On the other hand, if $x = e$ then idempotency of $(\{1, e, f\}, e)$ gives $e\{1, e, f\} \subseteq \{1, e, f\}$ so that $ef \in \{e, f\}$ as required.

Lemma 5.7. *For all $e \in B$ and $m \in M$, $em = e$.*

Proof With $e \in B$ and $m \in M$, Lemmas 5.2 and 5.3 give that $(\{1, e, m\}, e)$ must be the (only) idempotent in its $\widetilde{\mathcal{R}}$-class. From $(\{1, e, m\}, e) \in \mathcal{E}$ we have that $e\{1, e, m\} \subseteq \{1, e, m\}$ so that $em = e$.

Lemma 5.7 completes the first half of the proof of Theorem 5.1.

For the converse, we suppose that S is a unipotent monoid M, or an ideal extension of a royal band B by such a monoid, where for each $e \in B$ and $m \in M, em = e$. This structure, together with the unipotency of M, gives that M is the $\widetilde{\mathcal{R}}$-class of 1.

Let $(X, x) \in \widetilde{S}^{S\mathcal{R}}$. It is easy to check from the given conditions on S that if $x \in B$ then $(X, x) \in \mathcal{E}$. Suppose that $x \in M$. Certainly $(X, 1) \in \mathcal{E}$; we claim that $(X, x) \, \widetilde{\mathcal{R}} \, (X, 1)$. In view of Lemma 2.1 it is enough to show that if $(Y, y) \in \mathcal{E}$ and $(Y, y)(X, x) = (X, x)$, then $(Y, y)(X, 1) = (X, 1)$.

Let $(Y, y) \in \mathcal{E}$ with $(Y, y)(X, x) = (X, x)$. Clearly $Y \subseteq X$ and as $yx = x, x \in M$ and $y \in E(S)$ it follows that $y = 1$. Thus $(Y, y)(X, 1) = (Y, 1)(X, 1) = (Y \cup X, 1) = (X, 1)$ as required.

We have shown that $\widetilde{S}^{S\mathcal{R}}$ is left semiabundant; it remains to show that (CL) holds.

Let $(X, x), (Y, y), (Z, z) \in \widetilde{S}^{S\mathcal{R}}$ with $(X, x) \, \widetilde{\mathcal{R}} \, (Y, y)$. If $x \in B$ then as indicated above, $(X, x) \in \mathcal{E}$. Then $(X, x)(Y, y) = (Y, y)$ gives that $xy = y$ so that $y \in B$ also and $(Y, y) \in \mathcal{E}$. As both (X, x) and (Y, y) are idempotent we actually have $(X, x) \; \mathcal{R} \; (Y, y)$ and as $\mathcal{R}$ is a left congruence contained in $\widetilde{\mathcal{R}}$, we obtain $(Z, z)(X, x) \, \widetilde{\mathcal{R}} \, (Z, z)(Y, y)$.

The other case is where $x, y \in M$. We know that

$$(X, 1) \, \widetilde{\mathcal{R}} \, (X, x) \, \widetilde{\mathcal{R}} \, (Y, y) \, \widetilde{\mathcal{R}} \, (Y, 1)$$

from which we obtain $X = Y$. If $z \in M$ then

$$(Z, z)(X, x) = (Z \cup zX, zx) \, \widetilde{\mathcal{R}} \, (Z \cup zX, 1) =$$

$$(Z \cup zY, 1) \, \widetilde{\mathcal{R}} \, (Z \cup zY, zy) = (Z, z)(Y, y).$$

Finally, if $z \in B$ then

$$(Z, z)(X, x) = (Z \cup zX, zx) = (Z \cup zX, z)$$

$$= (Z \cup zY, z) = (Z \cup zY, zy) = (Z, z)(Y, y).$$

Hence $\widetilde{\mathcal{R}}$ is a left congruence, that is, (CL) holds.

In a subsequent paper [10] we show that the *graph expansion* of (a monoid presentation of) a monoid S is weakly left ample if and only if S is unipotent. The same result holds for the Szendrei expansion $(\widetilde{\bullet})^{S\mathcal{R}}$, with one exception.

Corollary 5.8. *The following are equivalent for a monoid S:*
(i) $\widetilde{S}^{S\mathcal{R}}$ is left semiadequate with (CL);
(ii) S is a unipotent monoid, or the two element chain;
(iii) $\widetilde{S}^{S\mathcal{R}}$ is weakly left ample.

Proof First we prove that (i) implies (ii). By Theorem 5.1, if $\widetilde{S}^{S\mathcal{R}}$ is left semiadequate with (CL), then S is a unipotent monoid, or an ideal extension of a royal band B by a unipotent monoid M such that for all $e \in B$ and $m \in M, em = e$. Suppose the latter condition holds. We use the fact that $\mathcal{E} = E(\widetilde{S}^{S\mathcal{R}})$ is a semilattice to show that S is the two element chain.

Let $e \in B$ and $m \in M$, so that $em = e$. Since $(\{1, m\}, 1), (\{1, e\}, e) \in \mathcal{E}$ we have that

$$(\{1, m\}, 1)(\{1, e\}, e) = (\{1, e\}, e)(\{1, m\}, 1)$$

and so $\{1, m, e\} = \{1, e\}$, forcing m to be 1. Thus M is trivial. Further, if $f \in B$ then $(\{1, f\}, f) \in \mathcal{E}$ and

$$(\{1, e\}, e)(\{1, f\}, f) = (\{1, f\}, f)(\{1, e\}, e)$$

so that $\{1, e, ef\} = \{1, f, fe\}$ and $ef = fe$. Since B is a royal band, $ef = fe \in \{e, f\}$ and this gives that $e = f$. Thus $S = \{1, e\}$ is the two element chain.

To prove that (ii) implies (iii), suppose first that S is a unipotent monoid. From Theorem 5.1, $\widetilde{S}^{S\mathcal{R}}$ is left semiabundant with (CL). Also

$$\mathcal{E} = \{(X, 1) : X \in \mathcal{PF}'_1(S)\};$$

so that (as remarked at the beginning of this section) $\mathcal{E}$ is a semilattice. In the proof of Theorem 5.1 we showed that if $(X, x) \in \widetilde{S}^{S\mathcal{R}}$ where $x \in M$, then $(X, x) \; \widetilde{\mathcal{R}} \; (X, 1)$. Thus in the case to hand where $M = S$ we have that $(X, x)^+ = (X, 1)$ for any $(X, x) \in \widetilde{S}^{S\mathcal{R}}$. It is now easy to check that

$$((X, x)(Y, 1))^+(X, x) = (X, x)(Y, 1)$$

for all $(X, x) \in \widetilde{S}^{S\mathcal{R}}$ and $(Y, 1) \in \mathcal{E}$. Hence $\widetilde{S}^{S\mathcal{R}}$ is weakly left ample.

On the other hand, if S is the two element chain, then as shown in Corollary 4.6 of [11], $\widetilde{S}^{S\mathcal{R}}$ is the three element chain, which being inverse is certainly weakly left ample.

To complete the proof notice that (iii) implies (i) is clear.

6. The category **F**

At the beginning of the previous section we indicated that the functor $\widetilde{(\bullet)}^{S\mathcal{R}} : \mathbf{M} \to \mathbf{M}$ is an expansion. Moreover, for a *unipotent* monoid S, the expansion $\widetilde{S}^{S\mathcal{R}}$ is weakly left ample.

We may regard weakly left ample monoids as algebras of type $(2,1,0)$ with unary operation given by $a \mapsto a^+$, where a^+ is the unique idempotent in the $\widetilde{\mathcal{R}}$-class of a. The corresponding approach for left ample monoids is well established [5], [12]. Following the lead given in [2] and [3] for left ample monoids we consider a partial order $\leq$ and a congruence σ on a weakly left ample monoid as defined in the Introduction.

Lemma 6.1. [10] *Let S be a weakly left ample monoid. Then the relation σ is the least unipotent monoid congruence on S.*

If each σ-class of a weakly left ample monoid S contains a maximum element under $\leq$, then we define a second unary operation m on S by the rule that for any $a \in S, m(a)$ is the greatest element in the σ-class $[a]$ of a. As in the Introduction we say that a weakly left ample monoid S is *proper* if $\widetilde{\mathcal{R}} \cap \sigma = \iota$ and *weakly left FA* if every σ-class contains a maximum element such that for all $a, b \in S$,

$$m(a)^+ m(ab)^+ = (m(a)m(b))^+ \qquad \text{(FL)}.$$

We remark that an F-inverse monoid or a left FA monoid is weakly left FA [5]. The reader looking at the reference [5] should be careful not to confuse the notion of weakly left FA with the left-right dual of the notion weak right FA as defined in that paper.

Lemma 6.2. *Let S be a weakly left ample monoid such that every σ-class contains a maximum element. Then S is proper. Further, S is weakly left FA if and only if*

$$m(a)^+ m(ab) = m(a)m(b)$$

for all $a, b \in S$.

Proof The proof that S is proper is essentially the dual of that given in [5] in the right ample case.

If S is weakly left FA then for any $a, b \in S$

$$m(a)^+ m(ab) \;\widetilde{\mathcal{R}}\; m(a)^+ m(ab)^+ = (m(a)m(b))^+ \;\widetilde{\mathcal{R}}\; m(a)m(b)$$

and as $m(ab)\ \sigma\ ab\ \sigma\ m(a)m(b)$ and $m(a)^+\ \sigma\ 1$ we have also that

$$m(a)^+m(ab)\ \sigma\ m(a)m(b).$$

We know that S is proper and so $m(a)^+m(ab) = m(a)m(b)$ as required.

Conversely, if $m(a)^+m(ab) = m(a)m(b)$ for all $a, b \in S$, then the fact that $\widetilde{\mathcal{R}}$ is a left congruence yields (FL).

In a subsequent paper [10] we study proper weakly left ample monoids by means of graph expansions. The graph expansion of a unipotent monoid is proper weakly left ample but need not be weakly left FA. However the Szendrei expansion of a unipotent monoid *is* weakly left FA. The proof of this result and indeed of the remaining statements of this paper are omitted, since they are exactly analogous to those of Section 4 of [5], where Fountain and Gomes consider the left Szendrei expansion of a left cancellative monoid.

Proposition 6.3. *Let S be a unipotent monoid. Then for any (X, x) and $(Y, y) \in \widetilde{S}^{S\mathcal{R}}$,*
(i) $(X, x)\ \widetilde{\mathcal{R}}\ (Y, y)$ if and only if $X = Y$;
(ii) $(X, x)\ \sigma\ (Y, y)$ if and only if $x = y$.

Further, S is weakly left FA and for any $(X, x) \in \widetilde{S}^{S\mathcal{R}}$,

$$(X, x)^+ = (X, 1) \text{ and } m(X, x) = (\{1, x\}, x).$$

Let $\mathbf{U}$ denote the category of unipotent monoids and monoid morphisms and let $\mathbf{F}$ denote the category of weakly left FA monoids and morphisms, *regarded as algebras of type* $(2, 1, 1, 0)$. The two unary operations are $^+$ and $m(\ )$. Using Proposition 6.3 it is easy to check that if S and T are unipotent monoids and $\theta : S \to T$ is a monoid morphism, then the monoid morphism $\widetilde{\theta}^{S\mathcal{R}} : \widetilde{S}^{S\mathcal{R}} \to \widetilde{T}^{S\mathcal{R}}$, defined in the previous section, is in fact a $(2, 1, 1, 0)$-morphism. Consequently, $(\widetilde{\bullet})^{S\mathcal{R}} : \mathbf{U} \to \mathbf{F}$ is a functor.

Let $F^\sigma : \mathbf{F} \to \mathbf{U}$ be defined on objects by $SF^\sigma = S/\sigma$ and on morphisms by $\theta F^\sigma = \theta^\sigma$ where if $\theta \in \mathrm{Mor}_{\mathbf{F}}(S, T)$ then $\theta^\sigma \in \mathrm{Mor}_{\mathbf{U}}(S/\sigma, T/\sigma)$ is given by $[s]\theta^\sigma = [s\theta]$. The characterisation of σ ensures that θ^σ is well defined. It follows that $F^\sigma : \mathbf{F} \to \mathbf{U}$ is a functor. We claim that $(\widetilde{\bullet})^{S\mathcal{R}}$ is a left adjoint of F^σ.

To see this, first consider a monoid morphism $\zeta : T \to S/\sigma$ where $T \in \mathrm{Ob}(\mathbf{U})$ and $S \in \mathrm{Ob}(\mathbf{F})$. For $t \in T$ let $m(t\zeta) = m(a)$ where $t\zeta = [a]$. We then define $\overline{\zeta} : \widetilde{T}^{S\mathcal{R}} \to S$ by

$$(X, x)\overline{\zeta} = m(x_1\zeta)^+ \dots m(x_k\zeta)^+ m(x\zeta)$$

where $X = \{x_1, \dots, x_k\}$. An argument as in [5], making use of Lemma 6.2 above, gives that $\overline{\zeta}$ is a morphism in $\mathbf{F}$.

Again with $T \in \mathrm{Ob}(\mathbf{U})$ and $S \in \mathrm{Ob}(\mathbf{F})$, suppose that $\psi : \widetilde{T}^{S\mathcal{R}} \to S$ is a morphism in $\mathbf{F}$. Then $\psi' : T \to S/\sigma$ given by $t\psi' = [(X,t)\psi]$ for any $(X,t) \in \widetilde{T}^{S\mathcal{R}}$ is well defined and is a monoid morphism.

Lemma 6.4. *For any $T \in Ob(\mathbf{U})$, $S \in Ob(\mathbf{F})$, and morphisms $\zeta \in Mor_{\mathbf{U}}(T, S/\sigma)$ and $\psi \in Mor_{\mathbf{F}}(\widetilde{T}^{S\mathcal{R}}, S)$, then with the notation established above,*

$$(\overline{\zeta})' = \zeta \text{ and } \overline{\psi'} = \psi.$$

Lemma 6.4 provides us with a bijection

$$\mathrm{Mor}_{\mathbf{F}}(\widetilde{T}^{S\mathcal{R}}, S) \to \mathrm{Mor}_{\mathbf{U}}(T, S/\sigma)$$

for any $T \in \mathrm{Ob}(\mathbf{U})$ and $S \in \mathrm{Ob}(\mathbf{F})$. This is the hard part of showing that $(\widetilde{\bullet})^{S\mathcal{R}}$ is a left adjoint of F^{σ}; the rest of the proof involves checking that the relevant diagrams commute. We leave this to the reader.

Theorem 6.5. *The functor $(\widetilde{\bullet})^{S\mathcal{R}}$ is a left adjoint of the functor F^{σ}.*

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