RELATIVELY FREE ALGEBRAS WITH WEAK EXCHANGE PROPERTIES

John Fountain and Victoria Gould

This is a revised version of the article that appears in the Journal of the Australian Mathematical Society **75** (2003), 1–30. *The authors are grateful to Professor George Bergman for correcting their original definition of 'constant subalgebra.*

Abstract. We consider algebras for which the operation PC of pure closure of subsets satisfies the exchange property. Subsets that are independent with respect to PC are *directly independent*. We investigate algebras in which PC satisfies the exchange property and which are relatively free on a directly independent generating subset. Examples of such algebras include independence algebras and finitely generated free modules over principal ideal domains.

2000 *Mathematics Subject Classification* primary 08B20, 08A05; secondary 08A60, 08A62.
Keywords and phrases: free generators, (direct) independence, exchange property.

Introduction

Following [3] and [12], we say that an algebra A (in the sense of universal algebra) is *relatively free* on a subset X if X generates A and if every function from X to A can be extended to an endomorphism of A. Such a set X is said to be a *free generating set* or *free basis* or simply a *basis* for A. More generally, a subset X of A is said to be A-*free* if every function from X to A can be extended to a morphism from $\langle X\rangle$ to A, where $\langle X\rangle$ is the subalgebra generated by X.

Our main purpose is to study certain relatively free algebras satisfying some axioms inspired by properties of free modules over Bezout domains. Recall that a *Bezout domain* is an integral domain (not necessarily commutative) in which all finitely generated left and right ideals are principal.

We call such algebras *basis algebras.* Other examples of basis algebras are provided by independence algebras, and by free S-acts where S is a cancellative principal left ideal monoid. Recall that for a monoid S, an S-*act* is a set A on which S acts unitarily on the left, that is, for all $s \in S$ and $a \in A$, there is a uniquely defined element sa of A such that $1a = a$ for all $a \in A$ and $(st)a = s(ta)$ for all $a \in A$ and $s, t \in S$. A *free* S-act is one on which S acts freely.

The notion of independence algebra was introduced by Narkiewicz [13] under the name of v^*-algebra. It was rediscovered in the nineties by Gould [7] who introduced the term 'independence algebra' and initiated a study of the endomorphism monoids of these algebras. In a recent study [2], Cameron and Szabo classify finite independence algebras. In the present paper we describe the fundamental properties of basis algebras. In two subsequent papers, we describe the endomorphism monoid of a basis algebra, and, for the case of an algebra of finite rank, we use the description to investigate the semigroup generated by the idempotent endomorphisms.

Underlying our investigations is the behaviour of two closure operators on an algebra A. One is the standard operator which, to each subset X of A, associates the subalgebra $\langle X \rangle$ generated by X, and which we call the *subalgebra operator*.

The other, which we denote by PC, maps X to the smallest 'pure' subalgebra of A containing X, when A is an appropriate type of algebra. If A is an independence algebra, then $\langle X \rangle = \mathrm{PC}(X)$ for all subsets X of A and the exchange property is satisfied. The latter is a powerful property which ensures, for instance, that a notion of rank may be defined. Strictly speaking, PC is not always a closure operator; however, in Section 1 we introduce 'weak exchange' algebras, defining them to be algebras for which PC is a closure operator such that the exchange property holds for PC.

Of particular importance in the algebras under consideration is the idea of an 'independent' set of elements. Many concepts of independence have been used in universal algebra (see, for example, [6]). One approach is to use independence relative to a closure operator on an algebra. We use the term 'independence' for independence relative to the subalgebra operator. For a weak exchange algebra, we have independence relative to PC, and we call this 'direct independence' (see [14]). A consequence of the exchange property for PC is that, in a weak exchange algebra A, all maximal directly independent sets have the same cardinality, known as the *rank* of A.

Another approach to independence is based on A-freeness, and we investigate the connections between this concept and our other two notions of independence in Section 2. In particular, we show that for certain algebras, A-free subsets are directly independent. These ideas are illustrated by a number of examples in Section 3.

Imposing the condition that directly independent subsets of a weak exchange algebra A are A-free leads, in Section 4, to the notion of a 'weak independence' algebra. We investigate the monoid of unary term operations of a weak independence algebra in Section 5 and use this to introduce torsion-free weak independence algebras. We can then characterise free generating sets, or bases, in such an algebra. In Section 6 we note that a subalgebra generated by a subset of a basis is pure; this is an important tool in the description of the endomorphism monoid.

We adopt the convention that a unary algebra is an algebra which has some basic operations all of which are unary. Using our description of the monoid of unary term operations, we classify those relatively free, torsion-free weak independence algebras which are term equivalent to unary algebras in Section 7. This allows us to show that such an algebra can be embedded in an independence algebra.

In the final section, we introduce basis algebras; these algebras have the property that a basis of a pure subalgebra may be extended to a basis of the whole algebra. We illustrate the ideas by characterising those basis algebras which are term equivalent to unary algebras, and in which all 2-generated subalgebras are relatively free.

1. Closure operators, the exchange property and purity

For the basic ideas of universal algebra we refer the reader to [3], [8] or [12]. However, there are substantial differences in terminology and notation in these books, and, in the interest of clarity, we begin by describing those adopted in this paper. By an *algebra* A we mean an algebra in the sense of universal algebra. Thus A comes equipped with a set (which may be empty) of *basic operations* all of which we assume to have finite arity. We permit 0 as an arity and refer to the images of nullary operations as *constants*. The operations on A derived from the basic operations and projections by composition are called *term operations*. It is convenient to allow $\emptyset$ to be a subalgebra in the case where A has no constants. Thus $\langle\emptyset\rangle$, the least subalgebra of A, is empty if and only if A has no constants. On the other hand, if A has constants, then $\langle\emptyset\rangle$ is the subalgebra of A they generate. We say that a subalgebra B of A is *nonconstant* if $B \neq \langle\emptyset\rangle$, and refer to $\langle\emptyset\rangle$ as the *constant subalgebra*. Throughout the paper we will use the notation κ_a for the constant function from A to itself with value $a \in A$. If A is an algebra with constants, then it is easy to see that κ_a is a unary term operation on A if and only if $a \in \langle\emptyset\rangle$.

The key to many properties of the algebras we consider in this paper is the interrelated behaviour of two closure operators. We will see that if A is an independence algebra, these closure operators coincide, but in general they are distinct.

Recall that a *closure operator* C on a set A is a function $C : \mathcal{P}(A) \to \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the set of all subsets of A, such that for all $X, Y \in \mathcal{P}(A)$,

(1) $X \subseteq C(X)$;
(2) if $X \subseteq Y$, then $C(X) \subseteq C(Y)$;
(3) $C(C(X)) = C(X)$.

A subset of A of the form $C(X)$ is said to be *closed*. The following is standard.

Lemma 1.1. *Let C be a closure operator on a set A, and X be a subset of A. Then A is closed, the intersection of any non-empty set of closed sets is closed, and $C(X)$ is the smallest closed subset of A containing X.*

A closure operator C is *algebraic* if for all $X \in \mathcal{P}(A)$,

$$C(X) = \bigcup\{C(Y) : Y \text{ is a finite subset of } X\}.$$

The canonical example of an algebraic closure operator is the subalgebra operator on an algebra A.

Let C be an algebraic closure operator on a set A. A subset X of A is said to be *C-independent*, or *independent with respect to C*, if $x \notin C(X \setminus \{x\})$ for all $x \in X$; it is *C-dependent* if it is not C-independent, that is, if there is an element x of X such that $x \in C(X \setminus \{x\})$.

The *exchange property* (EP) for a closure operator C on a set A is defined as follows:

(EP) for all $x, y \in A$ and $X \subseteq A$,

$$\text{if } x \in C(X \cup \{y\}) \text{ and } x \notin C(X), \text{ then } y \in C(X \cup \{x\}).$$

Algebraic closure operators which satisfy the exchange property are intimately connected with abstract dependence relations, and we now restate several fundamental results from [3, Section VII.2] in terms of algebraic closure operators. The first comes from the proof of Proposition VII.2.1 in [3] (see also [12, page 50 Exercise 6(a)]).

Lemma 1.2. *Let C be an algebraic closure operator on a set A. Then the following conditions are equivalent:*

1. *C satisfies the exchange property,*
2. *if X is a C-independent subset of A and $y \notin C(X)$, then $X \cup \{y\}$ is C-independent.*

Lemma 1.3. [3, Lemma VII.2.2] *Let C be an algebraic closure operator satisfying* (EP) *on a set A and let $Y \subseteq X \subseteq A$. Then the following conditions are equivalent:*

1. *Y is a maximal C-independent subset of X,*
2. *Y is C-independent and $C(Y) = C(X)$,*
3. *Y is minimal with respect to $C(Y) = C(X)$.*

If C is an algebraic closure operator on a set A, then it is easy to see that the union of a chain of C-independent sets is C-independent. Since $\emptyset$ is clearly C-independent, a Zorn's lemma argument gives that, for any subset X of A, there is a maximal C-independent subset of X.

Writing a slightly generalised version of Theorem VII.2.4 of [3] in terms of closure operators (see also [12, page 50 Exercise 6(b)]) we have the following result.

Theorem 1.4. *Let C be an algebraic closure operator satisfying* (EP) *on a set A, and let $X \subseteq Y \subseteq A$. If X is C-independent, then there is a C-independent subset Z with $X \subseteq Z \subseteq Y$ and $C(Z) = C(Y)$. Moreover, if Z and Z' are C-independent subsets of Y with $C(Z) = C(Z') = C(Y)$, then they have the same cardinality.*

In view of Lemma 1.3, a C-independent subset Z with $Z \subseteq Y$ and $C(Z) = C(Y)$ is a maximal C-independent subset of Y. Such a maximal C-independent subset is often called a C-basis of Y, but we reserve the term 'basis' for a free basis. In view of Lemma 1.3

and Theorem 1.4, we can define the C-*rank* of Y to be the cardinality of any maximal C-independent subset of Y. The following corollary is a straightforward consequence of Lemma 1.3 and Theorem 1.4.

Corollary 1.5. *Let C be an algebraic closure operator satisfying* (EP) *on a set A. If X and Y are subsets of A with $X \subseteq Y$, then*

(1) $C\text{-rank}(X) \leqslant C\text{-rank}(Y)$,
(2) $C\text{-rank}(X) = C\text{-rank}(C(X))$.

Given an algebra A, we now introduce a relation on A which leads to the operator PC, and we then determine for which algebras PC is an algebraic closure operator.

For an element a of an algebra A and a subset X of A we say that a *depends* on X and write $a \prec X$ if

$$a \in \langle \emptyset \rangle \text{ or } \langle a \rangle \cap \langle X \rangle \neq \langle \emptyset \rangle.$$

We remark that if $a \in X$, then $a \prec X$, and

$$a \prec X \text{ if and only if } a \prec \langle X \rangle.$$

Note also that if c is a constant, then $c \prec X$; moreover for any $a \in A$,

$$a \prec \emptyset \text{ if and only if } a \in \langle \emptyset \rangle.$$

For subsets X, Y of A we say that Y *depends* on X, and write $Y \prec X$, if $y \prec X$ for every $y \in Y$. We say that an algebra A satisfies condition (T) if for all $a \in A$ and $X, Y \subseteq A$,

$$\text{(T)} \qquad \text{if } a \prec X \text{ and } X \prec Y \text{ then } a \prec Y.$$

For any subset X of an algebra A, we put

$$\mathrm{PC}(X) = \{a \in A : a \prec X\}.$$

Notice that $\mathrm{PC}(\emptyset) = \langle \emptyset \rangle$.

Theorem 1.6. *The operator* PC *is an algebraic closure operator on an algebra A if and only if A satisfies condition* (T).

Further, if X is a subset of A and A satisfies (T), *then* $\mathrm{PC}(X)$ *is a subalgebra of A.*

Proof. Suppose that PC is an algebraic closure operator on A; let a be an element of A, and X, Y be subsets of A such that $a \prec X$ and $X \prec Y$. By definition, $a \in \mathrm{PC}(X)$ and $X \subseteq \mathrm{PC}(Y)$. Hence $\mathrm{PC}(X) \subseteq \mathrm{PC}(\mathrm{PC}(Y)) = \mathrm{PC}(Y)$, and so $a \in \mathrm{PC}(Y)$, that is, $a \prec Y$. Thus (T) holds as required.

Conversely, suppose that A satisfies condition (T). For a subset X of A and an element a of A, it is clear, from the definition of $\prec$ and the remarks above, that $X \subseteq \mathrm{PC}(X)$, and that $a \in \mathrm{PC}(X)$ if and only if $a \in \mathrm{PC}(F)$ for some finite subset F of X.

If $X \subseteq Y \subseteq A$ and $a \in \mathrm{PC}(X)$, then $a \prec X$ and $X \prec Y$, so that by (T), $a \prec Y$, that is, $a \in \mathrm{PC}(Y)$.

For any subset X of A, we have

$$\mathrm{PC}(X) \subseteq \mathrm{PC}(\mathrm{PC}(X)).$$

If $a \in \mathrm{PC}(\mathrm{PC}(X))$, then $a \prec \mathrm{PC}(X)$. By definition, $\mathrm{PC}(X) \prec X$ so that, again by (T), $a \prec X$, that is, $a \in \mathrm{PC}(X)$.

To see that $\mathrm{PC}(X)$ is a subalgebra, notice first that $\langle\emptyset\rangle \subseteq \mathrm{PC}(X)$. If t is an n-ary term operation for some $n \geqslant 1$ and $x_1, \ldots, x_n \in \mathrm{PC}(X)$, then $x_i \prec X$ for each i. Now $t(x_1, \ldots, x_n) \in \langle\{x_1, \ldots, x_n\}\rangle$ so that $t(x_1, \ldots, x_n) \prec \{x_1, \ldots, x_n\} \prec X$ and by (T), we have $t(x_1, \ldots, x_n) \prec X$, that is, $t(x_1, \ldots, x_n) \in \mathrm{PC}(X)$ as required. □

Let X be a subset of an algebra A which satisfies condition (T); an immediate consequence of the theorem is that $\langle X\rangle \subseteq \mathrm{PC}(X)$ and $\mathrm{PC}(X) = \mathrm{PC}(\langle X\rangle)$.

If A is an algebra for which the subalgebra operator satisfies (EP), we say that A *satisfies the exchange property*; if PC satisfies (EP), we say that A *satisfies the weak exchange property* (WEP). Note that the latter is equivalent to

(WEP) for all $x, y \in A$ and $X \in \mathcal{P}(A)$,

$$\text{if } \langle x\rangle \cap \langle X \cup \{y\}\rangle \neq \langle\emptyset\rangle \text{ and } \langle x\rangle \cap \langle X\rangle = \langle\emptyset\rangle, \text{ then } \langle y\rangle \cap \langle X \cup \{x\}\rangle \neq \langle\emptyset\rangle.$$

Definition 1. An algebra is a *weak exchange algebra* if it satisfies (T) and (WEP).

We remark that, for an algebra A with subalgebra B, the relation $\prec$ is unambiguous in the sense that, for any subset X of B and element $b \in B$, we have $b \prec X$ in B if and only if $b \prec X$ in A. Clearly then, if A satisfies (T), then so does B. Since the closed sets of the closure operator PC on B are the intersection of the closed sets of the closure operator PC on A with B, it is clear that if A has (WEP), then so does B. We have argued the following.

Lemma 1.7. *A subalgebra of a weak exchange algebra is a weak exchange algebra.*

We also note that any constant algebra is a weak exchange algebra.

The next lemma elucidates the relationship between the exchange property for an algebra and the weak exchange property.

Lemma 1.8. *Let A be an algebra which satisfies* (EP). *Then for any $a \in A$ and $X \subseteq A$,*

$$a \prec X \text{ if and only if } a \in \langle X\rangle.$$

Consequently, $\mathrm{PC}(X) = \langle X\rangle$ *for any $X \subseteq A$.*

Proof. If $a \prec X$, then either $a \in \langle\emptyset\rangle$ and so $a \in \langle X\rangle$, or $\langle a\rangle \cap \langle X\rangle \neq \langle\emptyset\rangle$. In the latter case, let $b \in \langle a\rangle \cap \langle X\rangle$ where $b \notin \langle\emptyset\rangle$. Then $b \in \langle\emptyset \cup \{a\}\rangle$ but $b \notin \langle\emptyset\rangle$ so that by (EP), $a \in \langle\emptyset \cup \{b\}\rangle = \langle b\rangle$. As $b \in \langle X\rangle$, we must have $a \in \langle X\rangle$ as required.

The converse is obvious. □

Corollary 1.9. *If an algebra A satisfies* (EP)*, then it is a weak exchange algebra.*

Proof. From Lemma 1.8, the subalgebra operator coincides with the operator PC. Thus A has (WEP), and Theorem 1.6 gives that (T) holds. □

In Section 3, we give examples of weak exchange algebras which do not satisfy (EP), and note, in particular, that the conclusion of Lemma 1.8 does not hold in general.

If X is a subset of any algebra A, we say that X is *directly independent* if for all $x \in X$,

$$x \not\prec X \setminus \{x\}.$$

Thus if A satisfies (T), so that PC is a closure operator, X is directly independent if and only if it is independent with respect to PC. Following [7] we say that X is *independent* if X is independent with respect to the subalgebra operator, that is, if for all $x \in X$,

$$x \notin \langle X \setminus \{x\} \rangle.$$

Clearly, a subset of an independent subset of an algebra is independent, and a similar statement holds for directly independent subsets. Indeed, a subset X of an algebra is (directly) independent if and only if every finite subset of X is (directly) independent.

The next result follows from the fact that, for a subset X and element a of an algebra A, if $a \in \langle X \rangle$, then $a \prec X$.

Lemma 1.10. *Every directly independent subset of an algebra is independent.*

The converse of Lemma 1.10 is not true as shown by the simple examples below. Here and throughout the paper we use (left) R-modules over a ring R and (left) S-acts over a monoid S to illustrate the concepts we introduce.

In an R-module M, the zero is the unique constant. A finite subset X of M is directly independent if and only if $X = \emptyset$ or $0 \notin X$ and for all families $\{r_x\}_{x \in X}$ of elements of the ring R,

$$\sum_{x \in X} r_x x = 0 \text{ implies that } r_x x = 0 \text{ for all } x \in X.$$

Thus if R is an integral domain and M is torsion-free (that is, if $r \in R$ and $m \in M \setminus \{0\}$ and $rm = 0$, then $r = 0$), then an arbitrary subset of M is directly independent if and only if it is R-linearly independent.

Note that, regarding $\mathbb{Z}$ as a $\mathbb{Z}$-module, $\{2, 3\}$ is independent but not directly independent.

In an S-act A a subset X is directly independent if and only if for all elements $x, y \in X$ and $s, t \in S$,

$$sx = ty \text{ implies that } x = y.$$

Regarding the set $\mathbb{N}$ of positive integers as an act over the multiplicative monoid $\mathbb{N}$, we again have that $\{2, 3\}$ is independent, but not directly independent.

Specialising Lemma 1.3 to the operator PC on a weak exchange algebra gives the following for directly independent subsets of such an algebra.

Corollary 1.11. *A directly independent subset X of a weak exchange algebra A is a maximal directly independent subset if and only if $a \prec X$ for all $a \in A$.*

If A is a weak exchange algebra, then by Theorem 1.4, we can define the rank of a subset X of A with respect to PC; it is the cardinality of any maximal directly independent subset of X. If A also satisfies (EP), we can also define the rank of X with respect to the subalgebra operator as the cardinality of any maximal independent subset of X. However, it follows from Lemma 1.8 that these two ranks are equal, and so there is no ambiguity when we refer to the rank of X with respect to PC as simply the *rank* of X.

Corollary 1.12. *Let X be a subset of a weak exchange algebra A. Then*

(1) $\operatorname{rank}(\langle X\rangle) = \operatorname{rank}(X) \leqslant |X|$,
(2) *if X is finite and* $\operatorname{rank}(\langle X\rangle) = |X|$, *then X is directly independent,*
(3) *if B is a subalgebra of A, then* $\operatorname{rank} B \leqslant \operatorname{rank} A$.

Proof. We have

$$X \subseteq \langle X\rangle \subseteq \mathrm{PC}(X)$$

so that, by Corollary 1.5,

$$\operatorname{rank}(X) \leqslant \operatorname{rank}(\langle X\rangle) \leqslant \operatorname{rank}(\mathrm{PC}(X)) = \operatorname{rank}(X).$$

Thus

$$\operatorname{rank}(\langle X\rangle) = \operatorname{rank}(X),$$

and (1) and (2) follow by Lemma 1.3.

Finally, (3) is immediate by Corollary 1.5. □

We conclude this section by introducing the notion of purity. We say that a subset X of an algebra A is *pure*, or *pure in* A, if $X = \mathrm{PC}(X)$, that is, if for each element a of A,

$$a \prec X \text{ implies that } a \in X.$$

Note that $\langle\emptyset\rangle$ is always pure, and that for an algebra A satisfying (T), the pure subsets are precisely the closed sets of the closure operator PC. We refer to $\mathrm{PC}(X)$ as the *pure closure* of X. In view of Theorem 1.6, if a subset is pure then it must be a subalgebra. The converse is not true in general, as examples in Section 3 illustrate. It follows from Lemma 1.8 that all subalgebras of any algebra satisfying (EP) are pure. In later sections we show that for certain weak exchange algebras, the pure subalgebras play a role analogous to that of arbitrary subalgebras of an independence algebra.

Remark 1. Let B be a subgroup of a torsion-free abelian group A. Recall that B is a pure subgroup of A if for every $b \in B$ and positive integer n, the equation $nx = b$ has a solution in B whenever it has a solution in A. Since A is torsion-free, such an equation can have at most one solution, and hence B is a pure subgroup if and only if it is pure in the sense of the above definition.

Corollary 1.13. *Let A be an algebra satisfying* (T)*. Then*

(1) *the intersection of pure subalgebras is pure,*
(2) *the pure closure of a subset X of A is the smallest pure subalgebra containing X,*
(3) *for any subalgebras B, C of A with $B \subseteq C$, if B is pure in C and C is pure in A, then B is pure in A.*

Proof. (1) and (2) are immediate consequences of Lemma 1.1.

Suppose that $B \subseteq C$, B is pure in C and C is pure in A. For any $a \in A$, if $a \prec B$, then $a \prec C$ by (T), so that $a \in C$ as C is pure in A. Hence $a \in B$, since $a \prec B$ and B is pure in C. □

It is worth mentioning that families of algebraic closure operators with the exchange property (called matroidal structures) have recently been used by Bergman [1] in a construction (from a given ring R) of a division ring D and homomorphism from R into D with specified kernel and such that D is generated by the image of R. The final section of [1] discusses the construction of matroidal structures on objects of varieties of algebras other than modules.

2. Independence, direct independence and freeness

In the previous section, we introduced the ideas of independence and direct independence, and noted that every directly independent subset of an algebra is independent, but that the converse is not true. As a consequence of Lemma 1.8, in an algebra satisfying the exchange property, the two notions coincide.

In this section we compare these concepts of independence with A-freeness. Recall from the introduction that a subset X of A is *A-free* if every function from X to A can be extended to a morphism from $\langle X \rangle$ to A.

First, we point out that $\emptyset$ is independent, directly independent and A-free in any algebra A. For an element a of A, the subset $\{a\}$ is independent if and only if it is directly independent, and this is the case if and only if $a \notin \langle \emptyset \rangle$.

From the corollary on page 197 of [8] and remarks on p.49 of the same book, we have the following.

Lemma 2.1. *If A is an algebra with $|A| > 1$, then every A-free subset is independent.*

We can strengthen the above in the case of an algebra with constants.

Lemma 2.2. *Let A be an algebra with constants such that $|A| > 1$. Then every A-free subset is directly independent.*

Proof. Let X be an A-free subset of A. There is nothing to prove if $X = \emptyset$, and so we may assume that $X \neq \emptyset$. If $x \in X \cap \langle \emptyset \rangle$, then $x\alpha = x$ for every morphism $\alpha : \langle X \rangle \to A$. But A has at least two elements so that we can define a function β on X with $x \neq x\beta$. Now β

cannot be extended to a morphism from $\langle X \rangle$ to A contradicting the fact that X is A-free. Hence $X \cap \langle \emptyset \rangle = \emptyset$.

Now suppose that $z \in \langle x \rangle \cap \langle X \diagdown \{x\} \rangle$. Then $z = t_1(x) = t_2(x_1, \ldots, x_n)$ for some terms t_1 and t_2 and elements $x_1, \ldots, x_n$ of $X \diagdown \{x\}$. Let $c \in \langle \emptyset \rangle$. Define a morphism β from $\langle X \rangle$ to A as follows. Put $x\beta = x$ and $v\beta = c$ for all $v \in X \diagdown \{x\}$. Then

$$z = t_1(x) = t_1(x\beta) = t_1(x)\beta = t_2(x_1, \ldots, x_n)\beta = t_2(x_1\beta, \ldots, x_n\beta) = t_2(c, \ldots, c).$$

Thus $\langle x \rangle \cap \langle X \diagdown \{x\} \rangle = \langle \emptyset \rangle$, so that X is directly independent. □

For algebras without constants we have the following.

Lemma 2.3. *Let A be an algebra without constants, and with no constant unary term operations if $|A| > 1$. Then every A-free set is directly independent.*

Proof. We begin by remarking that if $A = \{a\}$, then $\{a\}$ is directly independent. Suppose then that $|A| > 1$ and $\emptyset \neq X \subseteq A$ is A-free.

Let $z \in \langle x \rangle \cap \langle X \diagdown \{x\} \rangle$. Then $z = t_1(x) = t_2(x_1, \ldots, x_n)$ for some terms t_1 and t_2 and elements $x_1, \ldots, x_n$ of $X \diagdown \{x\}$. Let $a \in A$ and define morphisms α and β from $\langle X \rangle$ to A as follows. Put $y\alpha = a$ for all $y \in X$; put $x\beta = x$ and $v\beta = a$ for all $v \in X \diagdown \{x\}$. Then

$$\begin{aligned} t_1(a) &= t_1(x\alpha) = t_1(x)\alpha = t_2(x_1, \ldots, x_n)\alpha = t_2(x_1\alpha, \ldots, x_n\alpha) = t_2(a, \ldots, a) \\ &= t_2(x_1\beta, \ldots, x_n\beta) = t_2(x_1, \ldots, x_n)\beta = t_1(x)\beta = t_1(x\beta) = t_1(x) \\ &= z. \end{aligned}$$

Thus t_1 is a constant unary term operation. If A has none such, then $\langle x \rangle \cap \langle X \setminus \{x\} \rangle = \emptyset$ so that X is directly independent. □

Definition 2. An algebra A is an *independence algebra* if it satisfies (EP) and every maximal independent subset of A is A-free.

Clearly, every independent subset of an independence algebra A is A-free. Also, by Corollary 1.9, an independence algebra is a weak exchange algebra. We shall be concerned with those weak exchange algebras in which every directly independent subset is A-free.

3. Examples

We give some examples of algebras in which (T) holds (so that PC is a closure operator) and examples of weak exchange algebras. Most are inspired by the known examples of independence algebras listed in [2], although we shall see that not all the obvious generalisations of the examples of [2] give weak exchange algebras.

As noted in the Section 2, every independence algebra is a weak exchange algebra.

Recall that an integral domain or a monoid S is *left Ore* or *right reversible* if for any elements a, b of S (non-zero elements in the ring case), there are elements c, d of S such that $ca = db$ (and $ca \neq 0$ in the ring case).

We remark that in our examples involving modules over rings, we include the nullary operation with image $\{0\}$ amongst our basic operations, so that $\langle\emptyset\rangle = \{0\}$.

Example 1. Let M be a torsion-free (left) module M over a left Ore domain R. We show that (T) holds; since (WEP) clearly holds for any module over any ring, it will follow that M is a weak exchange algebra.

Let $x \in M$ and Y, Z be subsets of M such that $x \prec Y$ and $Y \prec Z$. Then either $x = 0$, so that certainly $x \prec Z$, or $x \neq 0$ and there are non-zero elements $r, a_1, \ldots, a_k$ in R and $y_1, \ldots, y_k$ in Y such that

$$rx = a_1y_1 + \cdots + a_ky_k.$$

Now $y_i \prec Z$ for each $i = 1, \ldots, k$, and so there are elements $b_1, \ldots, b_k$ of R such that b_iy_i is a non-zero element of RZ. By the left Ore condition, we have elements c_1, d_1 in R such that $c_1a_1 = d_1b_1 \neq 0$. Now $c_1rx \neq 0$ since R is an integral domain and M is torsion-free, and we have

$$c_1rx = c_1a_1y_1 + c_1a_2y_2 + \cdots + c_1a_ky_k = d_1b_1y_1 + c_1a_2y_2 + \cdots + c_1a_ky_k.$$

Similarly, there are elements c_2, d_2 in R such that $c_2c_1a_2 = d_2b_2 \neq 0$; $c_2c_1rx \neq 0$; and

$$c_2c_1rx = c_2d_1b_1y_1 + c_2c_1a_2y_2 + \cdots + c_2c_1a_ky_k = c_2d_1b_1y_1 + d_2b_2y_2 + c_2c_1a_3y_3 + \cdots + c_2c_1a_ky_k.$$

Continuing in this way, we obtain an expression for a non-zero multiple of x as a linear combination of $b_1y_1, \ldots, b_ky_k$ which is an element of RZ. Hence $x \prec Z$ and M satisfies condition (T).

Since M is a weak exchange algebra, every submodule (indeed, every subset) has a well-defined rank (the cardinality of a maximal directly independent subset of the submodule). Since direct independence is the same as R-linear independence in M, this rank coincides with the usual notion of rank for a module over a left Ore domain.

A submodule N of M is pure if $rx \in N, rx \neq 0$ implies that $x \in N$. Regarding $\mathbb{Z}$ as a $\mathbb{Z}$-module, the only pure submodules are $\{0\}$ and $\mathbb{Z}$. For, if $B \neq \{0\}$ is a pure submodule and $b \in B \setminus \{0\}$, then $b \cdot 1 \in B$ so that $1 \in B$ and consequently, $B = \mathbb{Z}$. In particular, $\mathrm{PC}(\{2\}) \neq \langle 2\rangle$.

Example 2. Let S be a left Ore monoid and let A be a (left) S-act. If $x \in A$ and Y, Z are subsets of A such that $x \prec Y$ and $Y \prec Z$, then there is an element $y \in Y$ and elements $s, t \in S$ such that $sx = ty$, and there is an element $z \in Z$ and $u, v \in S$ such that $uy = vz$. By the left Ore condition, $rt = wu$ for some $r, w \in S$ so that

$$rsx = rty = wuy = wvz,$$

and $x \prec Z$.

Thus A satisfies (T). It is routine to show that any act over any monoid satisfies (WEP), so that A is a weak exchange algebra.

A subact B of an S-act A is pure in A if $sa \in B$ implies $a \in B$ where $s \in S$ and $a \in A$. Not all subacts of such acts are pure, for example, regarding the multiplicative monoid $\mathbb{N}$ of positive integers as an act over itself, it is easy to see that the only non-empty pure subact is $\mathbb{N}$ itself.

For later use, we introduce 'acts with constants' as follows. Let A be an S-act which is the disjoint union of non-empty subacts B and C. For each $s \in S$, there is a unary operation λ_s on A given by the left action of s, and for each $c \in C$, we define a nullary operation ν_c with value c, so that $C = \langle \emptyset \rangle$. It is clear that A satisfies (WEP), and if S is left Ore, then condition (T) follows as above together with the fact that an element of C depends on any subset. Thus in this case, A is a weak exchange algebra.

Example 3. Let $\theta : S \to G$ be a surjective homomorphism from a monoid S onto a non-trivial group G. Then G can be regarded as an S-act where the action is given by $s \cdot g = (s\theta)g$. We claim that the S-act G is a weak exchange algebra. We have already mentioned that every act over every monoid satisfies (WEP). To see that G satisfies (T), suppose that $x \in G$ and Y, Z are subsets of G such that $x \prec Y$ and $Y \prec Z$. Then there is an element $y \in Y$ and elements $t, u \in S$ such that $t \cdot x = u \cdot y$, and there is an element $z \in Z$ and $v, s \in S$ such that $v \cdot y = s \cdot z$. Let $\bar{u} \in S$ be such that $\bar{u}\theta = (u\theta)^{-1}$. Then

$$\bar{u}t \cdot x = \bar{u} \cdot (t \cdot x) = \bar{u} \cdot (u \cdot y) = \bar{u}u \cdot y = (\bar{u}u)\theta y = y,$$

so $(v\bar{u}t) \cdot x = v \cdot (\bar{u}t \cdot x) = v \cdot y = s \cdot z$, and hence $x \prec Z$. Thus G is a weak exchange algebra.

A specific example of this case is given as follows. Take G to be the free group $FG(X)$ on a non-empty set X, and S to be the free monoid on the set $X \cup X^{-1}$. We can obtain $FG(X)$ from S by factoring out the (monoid) congruence $\sim$ on S generated by

$$\{(xx^{-1}, 1) : x \in X\} \cup \{(x^{-1}x, 1) : x \in X\}.$$

We now take θ to be the natural homomorphism from S onto $S/\sim$.

We can mimic the construction of affine algebras as given in [2, Example 3.2], but, as we see, the algebras constructed are not, in general, weak exchange algebras.

Example 4. Let R be a left Ore domain and M be a torsion-free (left) R-module. We define an algebra which we call $\mathrm{Aff}(M)$ as follows.

For each element c of $R \setminus \{0, 1\}$ define a binary operation μ_c on M by the rule:

$$\mu_c(x, y) = x + c(y - x).$$

Define a ternary operation τ on M by the rule:

$$\tau(x, y, z) = x + y - z.$$

Then $\mathrm{Aff}(M)$ is the algebra with universe M and operations τ and μ_c for all $c \in R \setminus \{0, 1\}$.

We remark that if R is the field with two elements, then τ is the only basic operation. On the other hand, if R is a division ring with more than two elements, then, choosing $e \in R \setminus \{0,1\}$, $c = e^{-1}$ and $d = (1-c)^{-1}$, we have

$$\tau(x, y, z) = \mu_c(\mu_d(z, y), \mu_e(z, x)),$$

so that, as in Example 3.2 of [2], the basic operations of $\mathrm{Aff}(M)$ can be taken to be the μ_c for $c \in R \setminus \{0,1\}$.

Note that a subset of $\mathrm{Aff}(M)$ is a subalgebra if and only if it is empty or is a coset of a submodule of M. In particular, every singleton subset of M is a subalgebra, and, in general, if $X \subseteq M$, then, for any $x \in X$, the subalgebra generated by X is the coset $x+RY$ where $Y = \{x'-x : x' \in X \setminus \{x\}\}$. The endomorphisms of $\mathrm{Aff}(M)$ are the mappings $\alpha : M \to M$ for which there is an element $m_0 \in M$ and an R-linear map $\theta : M \to M$ such that $m\alpha = m_0 + m\theta$.

Since $\langle x \rangle = \{x\}$, we have that, in this case, $x \prec Y$ means $x \in \langle Y \rangle$ so that (T) holds. Moreover, the closure operator PC coincides with the subalgebra operator. However, (EP) does not hold, in general. For example, in $\mathrm{Aff}(\mathbb{Z})$ we have $6 \in \langle \{0,2\} \rangle = 0 + 2\mathbb{Z}$ and $6 \notin \langle 0 \rangle = \{0\}$, but $2 \notin 0 + 6\mathbb{Z} = \langle \{0,6\} \rangle$. In fact, it is easy to see that $\mathrm{Aff}(M)$ is a weak exchange algebra if and only if R is a division ring.

Proposition 3.1. *Let R be a left Ore domain and M be a torsion-free R-module. Then the following conditions are equivalent:*

(1) $\mathrm{Aff}(M)$ *satisfies* (EP),
(2) $\mathrm{Aff}(M)$ *is a weak exchange algebra,*
(3) $\mathrm{Aff}(M)$ *is an independence algebra,*
(4) *R is a division ring.*

Proof. If R is a division ring, then by Example 3.2 of [2], condition (3) holds. Every independence algebra is a weak exchange algebra; moreover, since the closure operator PC coincides with the subalgebra operator, (1) is a consequence of (2).

Suppose that $\mathrm{Aff}(M)$ satisfies (EP), and let a and m be non-zero elements of R and M respectively. Then $\langle \{0, am\} \rangle = Ram$ and $a^2m \in Ram$, but $a^2m \notin \langle 0 \rangle$, so that, by the exchange property, $am \in \langle \{0, a^2m\} \rangle$, that is, $am \in Ra^2m$. Hence $am = ba^2m$ for some $b \in R$, and since M is torsion-free, $a = ba^2$. Thus $1 = ba$ since R is an integral domain, and it follows that R is a division ring. □

4. Weak Independence Algebras

We introduce weak independence algebras which are defined as follows.

Definition 3. A *weak independence* algebra is a weak exchange algebra A in which every directly independent subset of A is A-free.

Lemma 4.1. *If B is a subalgebra of a weak independence algebra A, then B is a weak independence algebra.*

Proof. From Lemma 1.7, B is a weak exchange algebra. Since direct independence does not depend upon the subalgebra in question, if $X \subseteq B$ is directly independent, then X is A-free, and consequently B-free. □

Clearly, independence algebras are weak independence algebras, as are constant algebras. Further examples of weak independence algebras are given in the next section. Here we obtain some elementary results connecting direct independence and morphisms.

Proposition 4.2. *Let B be a subalgebra of a weak independence algebra A and $\theta\colon B \to A$ be a morphism. Then*

(1) *if θ is one-one and $X \subseteq B$ is directly independent, then $X\theta$ is directly independent,*
(2) *if Y is a directly independent subset of $B\theta$ and $Z \subseteq B$ is such that $Z\theta = Y$ and θ is one-one on Z, then Z is directly independent.*

Proof. (1) Let $x \in X$. If $\langle x\theta\rangle \cap \langle X\theta \setminus \{x\theta\}\rangle \neq \langle\emptyset\rangle$, then

$$t(x\theta) = t'(x_1\theta, \ldots, x_n\theta) \notin \langle\emptyset\rangle$$

for some term operations t, t' and elements $x_1, \ldots, x_n$ of $X \setminus \{x\}$. Thus

$$t(x)\theta = t'(x_1, \ldots, x_n)\theta \notin \langle\emptyset\rangle$$

and since θ is one-one, $t(x) = t'(x_1, \ldots, x_n) \notin \langle\emptyset\rangle$, contradicting the direct independence of X.

(2) Certainly $Z \cap \langle\emptyset\rangle = \emptyset$, since $Z\theta \cap \langle\emptyset\rangle\theta = Y \cap \langle\emptyset\rangle = \emptyset$. If $z \in Z$ and $\langle z\rangle \cap \langle Z \setminus \{z\}\rangle \neq \langle\emptyset\rangle$, then

$$t(z) = t'(z_1, \ldots, z_n) \notin \langle\emptyset\rangle$$

for some term operations t, t' and elements $z_1, \ldots, z_n$ of $Z \setminus \{z\}$. Hence

$$t(z\theta) = t(z)\theta = t'(z_1, \ldots, z_n)\theta = t'(z_1\theta, \ldots, z_n\theta).$$

Now $z\theta \in Y$ so that $\{z\theta\}$ is directly independent and hence A-free. Thus there is a morphism $\alpha\colon \langle z\theta\rangle \to A$ with $(z\theta)\alpha = z$, and

$$t(z\theta)\alpha = t((z\theta)\alpha) = t(z) \notin \langle\emptyset\rangle$$

so that $t(z\theta) \notin \langle\emptyset\rangle$. Hence

$$\langle z\theta\rangle \cap \langle Y \setminus \{z\theta\}\rangle \neq \langle\emptyset\rangle,$$

contradicting the direct independence of Y. □

Lemma 4.3. *Let X be a directly independent subset of a weak independence algebra A, and let $\alpha\colon X \to A$ be one-one. If $X\alpha$ is directly independent, then the morphism $\overline{\alpha}\colon \langle X\rangle \to A$ which extends α is one-one.*

Proof. Clearly, $\operatorname{Im}\overline{\alpha} = \langle X\alpha\rangle$ and so we may regard $\overline{\alpha}$ as a morphism from $\langle X\rangle$ onto $\langle X\alpha\rangle$. Let $\beta : X\alpha \to X$ be the inverse of α. Since $X\alpha$ is directly independent and A is a weak independence algebra, we can extend β to a morphism $\overline{\beta} : \langle X\alpha\rangle \to \langle X\rangle$. It is clear that $\overline{\alpha}$ and $\overline{\beta}$ are mutually inverse and so $\overline{\alpha}$ is one-one. □

As a simple consequence, we have the following result.

Corollary 4.4. *Let A be a weak independence algebra. If X and Y are directly independent subsets of A of the same cardinality, then the subalgebras $\langle X\rangle$ and $\langle Y\rangle$ are isomorphic.*

In particular, any two cyclic subalgebras different from the constant subalgebra are isomorphic.

Corollary 4.5. *Let t be a unary term operation on a weak independence algebra A. If a, b are nonconstant elements of A, then the elements $t(a), t(b)$ are either both constants or both nonconstants.*

Proof. The elements $t(a), t(b)$ correspond under the isomorphism between $\langle a\rangle$ and $\langle b\rangle$. □

5. Unary Term Operations and Torsion-freeness

Let A be an algebra and let T_1 be the set of all unary term operations on A. Clearly, T_1 is a monoid under composition of functions. We let

$$T_C = \{\kappa_c : c \in \langle\emptyset\rangle\}$$

so that if A has no constants, then $T_C = \emptyset$. We show that if A is a nonconstant weak independence algebra then T_C is a prime ideal of T_1 and $T_1^* = T_1 \setminus T_C$ is a right cancellative left Ore submonoid of T_1. We use T_1^* to introduce a notion of torsion-freeness which generalises that for acts over cancellative monoids [11], and discuss several properties of torsion-free weak independence algebras.

Using the fact that, in a weak independence algebra A, the directly independent sets are A-free, the proof of the following lemma is straightforward.

Lemma 5.1. *Let s, t be n-ary term operations on a weak independence algebra A. If there is a directly independent subset $\{x_1, \dots, x_n\}$ of A such that $s(x_1, \dots, x_n) = t(x_1, \dots, x_n)$, then $s(a_1, \dots, a_n) = t(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in A$.*

When we apply this to $s, t \in T_1$, we have: if $s(x) = t(x)$ for some $x \in A\setminus\langle\emptyset\rangle$, then $s = t$.

Proposition 5.2. *If A is a weak independence algebra with constants, such that $A \neq \langle\emptyset\rangle$, then, for $t \in T_1$, the following are equivalent*:

(1) $t = \kappa_c$ *for some* $c \in A$,
(2) $t(a) \in \langle\emptyset\rangle$ *for all* $a \in A$,
(3) $t(x) \in \langle\emptyset\rangle$ *for some* $x \in A\setminus\langle\emptyset\rangle$.

Proof. If (3) holds, let $t(x) = c \in \langle\emptyset\rangle$. Now $\kappa_c \in T_1$ and, since $t(x) = \kappa_c(x)$, we have $t = \kappa_c$ by Lemma 5.1. The remaining implications are clear. □

Proposition 5.3. *Let A be a nonconstant weak independence algebra. Then T_C is a prime ideal of the monoid T_1, and $T_1^* = T_1 \setminus T_C$ is a right cancellative, left Ore submonoid.*

Proof. If A has no constants, then $T_C = \emptyset$ and is certainly an ideal of T_1. Otherwise, let $\kappa_c \in T_C$ and $s \in T_1$. Then $\kappa_c s = \kappa_c \in T_C$ and $s\kappa_c = \kappa_{s(c)} \in T_C$, so that T_C is an ideal.

To say that the ideal T_C is prime is equivalent to saying that T_1^* is a submonoid of T_1. Clearly, the identity function is in T_1^*. Let $t, u \in T_1^*$. Then, calling upon Proposition 5.2 for the case where A has constants, $t(a), u(a) \in A \setminus \langle\emptyset\rangle$ for all $a \in A \setminus \langle\emptyset\rangle$. Hence $tu(a) \in A \setminus \langle\emptyset\rangle$ for all $a \in A \setminus \langle\emptyset\rangle$, that is, $tu \in T_1^*$.

Let $r, s, t \in T_1^*$ and suppose that $rt = st$. Then $rt(a) = st(a)$ for all $a \in A$. As $t \in T_1^*$, we have that $t(a) \notin \langle\emptyset\rangle$ for some $a \in A$, again using Proposition 5.2 in the case where $\langle\emptyset\rangle \neq \emptyset$. Hence $r = s$ by Lemma 5.1, and so T_1^* is right cancellative.

If $r, s \in T_1^*$, and $x \in A \setminus \langle\emptyset\rangle$, then, by Proposition 5.2, $r(x)$ and $s(x)$ are not in $\langle\emptyset\rangle$. By Corollary 1.12, $\langle x\rangle$ has rank 1, and so $\{r(x), s(x)\}$ cannot be directly independent. Hence

$$\langle r(x)\rangle \cap \langle s(x)\rangle \neq \langle\emptyset\rangle.$$

Let z be an element in the intersection with $z \notin \langle\emptyset\rangle$. Then $z = t(r(x)) = u(s(x))$ for some $t, u \in T_1^*$, and so, by Lemma 5.1, $tr = us$ and T_1^* is left Ore. □

Let A be an act over a cancellative monoid S. Specialising the definition of torsion-freeness for general acts (see [11, p.218]), A is *torsion-free* if for any $x, y \in A$ and any $c \in S$, the equality $cx = cy$ implies $x = y$.

We extend this to weak independence algebras in the following definition. In view of Proposition 5.2, our definition insists that any unary term operation either has image contained in the constant subalgebra, or is injective.

Definition 4. A nonconstant weak independence algebra A is *torsion-free* if for any $t \in T_1^*$ and any elements a, b of A,

(TF) if $t(a) = t(b)$, then $a = b$.

We declare a constant algebra to be torsion-free.

The following lemma is immediate.

Lemma 5.4. *Let A be a torsion-free weak independence algebra with no constants. If $|A| > 1$ then A has no constant unary term operations.*

Consequent upon the definition, if A is a nonconstant torsion-free weak independence algebra, the elements of T_1^* are left cancellable in T_1. Hence, in view of Proposition 5.3, T_1^* is cancellative, and so being torsion-free means that A is torsion-free as a T_1^*-act. We record the fact that T_1^* is cancellative in the following corollary.

Corollary 5.5. *Let A be a nonconstant torsion-free weak independence algebra. Then the monoid T_1^* is cancellative and left Ore.*

We proceed to investigate A-free subsets of torsion-free weak independence algebras, starting with the following simple observation.

Lemma 5.6. *If A is a nonconstant torsion-free weak independence algebra, then the A-free sets in A are precisely the directly independent subsets.*

Proof. We need only argue that A-free subsets are directly independent.

If A has constants, then as A is nonconstant, $|A| > 1$ so the result follows from Lemma 2.2.

Suppose on the other hand that A has no constants. If $|A| > 1$, then by Lemma 5.4, A has no constant unary term operations. Lemma 2.3 gives the result. □

We wish to show that a B-free subset of a subalgebra B of a torsion-free weak independence algebra is A-free. First, we need to argue that torsion-freeness is inherited by subalgebras.

Lemma 5.7. *Let B be a nonconstant subalgebra of a torsion-free weak independence algebra A. Then B is a torsion-free weak independence algebra.*

Proof. We have already observed in Lemma 4.1 that B is a weak independence algebra. If A has no constants, then clearly B is torsion-free. On the other hand if A has constants and t is a unary term operation such that $t|_B \neq \kappa_c|_B$ for any $c \in \langle \emptyset \rangle$, then $t \in T_1^*$ and it follows that B is torsion-free. □

Corollary 5.8. *Let B be a nonconstant subalgebra of a torsion-free weak independence algebra A. Then the following are equivalent for any $X \subseteq B$:*

(1) *X is A-free,*
(2) *X is directly independent as a subset of A,*
(3) *X is directly independent as a subset of B,*
(4) *X is B-free.*

Lemma 5.9. *Let A be a nonconstant torsion-free weak independence algebra X and Y be disjoint non-empty subsets such that $X \cup Y$ is A-free. Then*

$$\langle X \rangle \cap \langle Y \rangle = \langle \emptyset \rangle.$$

Proof. If $p \in \langle X \rangle \cap \langle Y \rangle$, then there are term operations t, s and elements $x_1, \dots, x_n \in X$ and $y_1, \dots, y_m \in Y$ such that

$$p = t(x_1, \dots, x_n) = s(y_1, \dots, y_m).$$

For any $a \in A$, there is a morphism $\alpha : \langle X \cup Y \rangle \to A$ with $x_i\alpha = a$ and $y_j\alpha = y_j$ for $i = 1, \dots, n$ and $j = 1, \dots, m$. Now

$$t(a, \dots, a) = t(x_1\alpha, \dots, x_n\alpha) = t(x_1, \dots, x_n)\alpha = p\alpha$$

$$= s(y_1, \dots, y_m)\alpha = s(y_1\alpha, \dots, y_m\alpha) = s(y_1, \dots, y_m) = p.$$

If A has constants, then we must have that $p \in \langle \emptyset \rangle$. On the other hand if A has no constants then $|A| > 1$ so that by Lemma 5.4, A has no constant unary term operations. Thus no such p exists and $\langle X \rangle \cap \langle Y \rangle = \emptyset$ as required. □

We can prove the following result for more general algebras.

Proposition 5.10. *Let A be an algebra and let X, Y and Z be pairwise disjoint non-empty subsets such that $X \cup Y \cup Z$ is A-free. Then*

$$\langle X \cup Y \rangle \cap \langle X \cup Z \rangle = \langle X \rangle.$$

Proof. If u is an element of $\langle X \cup Y \rangle \cap \langle X \cup Z \rangle$, then there are term operations t, s and elements $x_1, \dots, x_n, x'_1, \dots, x'_m$ in X, $y_1, \dots, y_h$ in Y and $z_1, \dots, z_k$ in Z such that

$$u = t(x_1, \dots, x_n, y_1, \dots, y_h) = s(x'_1, \dots, x'_m, z_1, \dots, z_k).$$

Let $v \in \langle X \rangle$. Then there is a morphism $\alpha : \langle X \cup Y \cup Z \rangle \to A$ with $x\alpha = x$ for $x \in X$, $y\alpha = y$ for $y \in Y$ and $z\alpha = v$ for $z \in Z$. Now

$$\begin{aligned} u &= t(x_1, \dots, x_n, y_1, \dots, y_h) = t(x_1\alpha, \dots, x_n\alpha, y_1\alpha, \dots, y_h\alpha) \\ &= t(x_1, \dots, x_n, y_1, \dots, y_h)\alpha = u\alpha = s(x'_1, \dots, x'_m, z_1, \dots, z_k)\alpha \\ &= s(x'_1\alpha, \dots, x'_m\alpha, z_1\alpha, \dots, z_k\alpha) = s(x'_1, \dots, x'_m, v, \dots, v). \end{aligned}$$

Thus $u \in \langle X \rangle$ and so

$$\langle X \cup Y \rangle \cap \langle X \cup Z \rangle = \langle X \rangle.$$

□

We illustrate the notion by characterising those S-acts and faithful R-modules which are torsion-free weak independence algebras.

First, let S be a monoid and A be an S-act. Each $s \in S$ induces a unary term operation λ_s given by $\lambda_s(a) = sa$ for all $a \in A$. Then $T_1 = T_1^* = \{\lambda_s : s \in S\}$ and $\lambda : S \to T_1^*$ defined by $s\lambda = \lambda_s$ is a surjective homomorphism with $S/\ker\lambda \cong T_1^*$.

Proposition 5.11. *Let A be an S-act. Then A is a torsion-free weak independence algebra if and only if $S/\ker\lambda$ is cancellative and left Ore, and A satisfies the following two conditions for all $s, t \in S$:*

1. λ_s *is injective,*
2. *if $sa \neq ta$ for some $a \in A$, then $sx \neq tx$ for all $x \in A$.*

Proof. If A is a torsion-free weak independence algebra, then by Corollary 5.5, $S/\ker\lambda$ is cancellative and left Ore. Condition (1) is immediate from the definition of torsion-free. Let $s, t \in S$ and suppose that $sa \neq ta$ for some $a \in A$. If there is an element x such that $sx = tx$, then $\lambda_s(x) = \lambda_t(x)$. Now A has no constants, and so $\{x\}$ is directly independent. Hence, by Lemma 5.1, $\lambda_s = \lambda_t$ so that $sa = \lambda_s(a) = \lambda_t(a) = ta$, a contradiction. Thus condition (2) holds.

Conversely, if $S/\ker\lambda$ is cancellative and left Ore, then it follows from Example 2 that A is a weak exchange algebra.

Let $X \subseteq A$ be directly independent, so that $sx \neq ty$ for any $s, t \in S$ and $x, y \in X$ with $x \neq y$. Then $\langle X\rangle$ is a disjoint union of cyclic subacts, and condition (2) guarantees that X is A-free. Hence A is a weak independence algebra.

That A is torsion-free (as a weak independence algebra) is immediate from condition (1). □

Recall that an S-act A is *faithful* if the homomorphism λ is injective; thus an S-act is faithful if and only if S is isomorphic to $T_1 = T_1^*$. As in [11], we say that A is *strongly faithful* if, for $s, t \in S$ we have $s \neq t$ implies $sx \neq tx$ for all $x \in A$. Hence a faithful act A is strongly faithful exactly when condition (2) of Proposition 5.11 holds. Thus the following is an immediate consequence of the proposition.

Corollary 5.12. *Let A be a faithful S-act. Then A is a torsion-free weak independence algebra if and only if S is cancellative and left Ore, and A is a torsion-free strongly faithful act.*

We now give the analogous result for faithful modules. Recall that an R-module is *faithful* if $rM \neq 0$ for all non-zero elements r of R.

Proposition 5.13. *A non-trivial faithful R-module M is a torsion-free weak independence algebra if and only if R is a left Ore domain and M is a torsion-free R-module.*

Proof. Suppose that M is a torsion-free weak independence algebra. Since R acts faithfully, the multiplicative monoid of R is isomorphic to the monoid T_1 of unary term operations, and it follows that the non-zero elements of R form a monoid isomorphic to T_1^*. Since M is a torsion-free weak independence algebra, R is a left Ore domain by Corollary 5.5. If $r \in R$, $m \in M \setminus \{0\}$ and $rm = 0$, then by Proposition 5.2, $rn = 0$ for all $n \in M$. Thus $r = 0$ as R acts faithfully. Hence M is a torsion-free module.

Conversely, suppose that R is a left Ore domain and M is a torsion-free module. Example 1 gives that M is a weak exchange algbra. We have pointed out earlier that if X is a directly independent subset of M, then $\langle X\rangle$ is a free R-module and so X is certainly M-free. Thus M is a weak independence algebra, and clearly torsion-freeness in the sense of Definition 4 follows from torsion-freeness as an R-module. □

We now give several further examples of torsion-free weak independence algebras, and we also consider two examples of weak independence algebras which are not torsion-free.

Example 5. We saw in Example 3 that, given a homomorphism θ from a monoid S onto a non-trivial group G, we can regard G as an S-set where the action is given by $s \cdot g = (s\theta)g$, that is, $\lambda_s(g) = (s\theta)g$. Thus λ_s is clearly injective for each $s \in S$, and $\lambda_s = \lambda_t$ if and only if $s\theta = t\theta$, so that $S/\ker\lambda \cong G$. It is easy to verify that condition (2) of Proposition 5.11 holds, and so, by this proposition, the S-act G is a torsion-free weak independence algebra.

Example 6. Let S be a cancellative left Ore monoid, and let A be an S-act 'with constants' as in Example 2 so that A is the disjoint union of nonempty subacts B and C, and C is the set of constants. We know from Example 2 that A is a weak exchange algebra. If A is torsion-free (as an S-act) and B is strongly faithful, then any directly independent subset X of A must be contained in B and as in the proof of Proposition 5.11, $\langle X\rangle$ is the disjoint union of cyclic subacts. Since B is strongly faithful, it follows that $\langle X\rangle$ is a free S-act, and hence certainly A-free. Clearly, $T_1^* = \{\lambda_s : s \in S\}$ and as A is a torsion-free act, A is torsion-free as a weak independence algebra.

Example 7. We have already remarked that independence algebras are weak independence algebras. Now let t be a unary term operation on an independence algebra A and suppose that $t(x) \notin \langle\emptyset\rangle$ for some $x \in A$. For any element a of A, there is an endomorphism α of A with $x\alpha = a$ since $\{x\}$ is independent. As A is an independence algebra, (EP) gives that $\langle x\rangle = \langle t(x)\rangle$ so that $x = s(t(x))$ for some unary term operation s. Now

$$a = x\alpha = (s(t(x)))\alpha = s(t(x\alpha)) = s(t(a))$$

and it follows easily that A is torsion-free.

Example 8. Let M be a cancellative left Ore monoid with $|M| > 1$ and trivial group of units, and let Mx be the free M-act on $\{x\}$. Let $A = Mx \cup \{b, c\}$ be the algebra with nullary operations ν_b, ν_c (with values b, c respectively), and a unary operation t_m for each element m of M, where $t_m(m'x) = (mm')x$ for all $m' \in M$, and

$$t_1(b) = b, \quad t_1(c) = c, \quad \text{and} \quad t_m(b) = t_m(c) = c \text{ for } m \in M \setminus \{1\}.$$

It is easy to see that M is a weak exchange algebra. The only directly independent subsets of A are the singleton subsets of Mx, and these are precisely the A-free subsets, so that A is a weak independence algebra. Clearly, A is not torsion-free, but it is worth noting that we do have $t(a) = t(b)$ implies $a = b$ for nonconstant elements a, b and nonconstant unary operations t.

Example 9. Let M be a left Ore, right cancellative monoid which is not left cancellative. Then M, regarded as an M-act has no constants, and it is a weak independence algebra which is not torsion-free. The left Ore condition ensures that the directly independent sets are the singletons. If $m, n \in M$, then, by right cancellation, there is a well defined M-morphism θ from Mm to M given by $(sm)\theta = sn$. Thus, every singleton subset is M-free, and so M is a weak independence algebra. It is not torsion-free because it is not left cancellative.

Monoids with the appropriate properties exist, for example, the opposite monoid M of the additive monoid P of all ordinals less than ϵ_0 (the least ϵ-number). For, P is left cancellative, and if α, β are any members of P, then there is an ordinal γ in P greater than both α and β. Now, $\omega^\gamma \in P$ and $\alpha + \omega^\gamma = \omega^\gamma = \beta + \omega^\gamma$ so that P is right Ore but not right cancellative.

6. Purity in torsion-free weak independence algebras

We now turn our attention to determining when a subset of a torsion-free weak independence algebra A is a free generating set for A.

We start by demonstrating the purity of a subalgebra associated with a pair of endomorphisms. This will also play a role in subsequent papers describing the endomorphism monoids of certain torsion-free weak independence algebras.

Lemma 6.1. *If α, β are endomorphisms of a torsion-free weak independence algebra A, then*

$$S_{\alpha,\beta} = \{a \in A : a\alpha = a\beta\}$$

is a pure subalgebra of A.

Proof. It is easy to see that $S_{\alpha,\beta}$ is a subalgebra. Suppose that $a \prec S_{\alpha,\beta}$. Then, either $a \in \langle\emptyset\rangle$ so that $a \in S_{\alpha,\beta}$, or $t(a) \in S_{\alpha,\beta}$ with $t(a) \notin \langle\emptyset\rangle$ for some term operation t. In this case, we have

$$t(a\alpha) = t(a)\alpha = t(a)\beta = t(a\beta),$$

and so, since A is torsion-free, $a\alpha = a\beta$ and $a \in S_{\alpha,\beta}$.

Thus $S_{\alpha,\beta}$ is pure. □

Corollary 6.2. *If α is an idempotent endomorphism of a torsion-free weak independence algebra A, then* $\operatorname{Im}\alpha$ *is pure in A.*

Proof. It is enough to note that since α is idempotent, $\operatorname{Im}\alpha = S_{I,\alpha}$ where I is the identity automorphism of A. □

Proposition 6.3. *Let X be a subset of a torsion-free weak independence algebra A. If X is directly independent, then for any subset Y of X, the subalgebra $\langle Y\rangle$ is pure in $\langle X\rangle$.*

Proof. If $Y = \emptyset$, there is nothing to prove. Otherwise, choose $y_0 \in Y$; since A is a weak independence algebra, there is a morphism $\alpha : \langle X\rangle \to A$ with $y\alpha = y$ for all $y \in Y$ and $x\alpha = y_0$ for all $x \in X \setminus Y$. Clearly, we may regard α as an endomorphism of $\langle X\rangle$, and

$\operatorname{Im}\alpha = \langle Y \rangle$. Now $\langle X \rangle$ is torsion-free and a weak independence algebra, so by Corollary 6.2, $\langle Y \rangle$ is pure in $\langle X \rangle$. □

We now consider relatively free torsion-free weak independence algebras. We remark that, as we see from Examples 8 and 9, torsion-freeness is not a consequence of being relatively free, and, of course, not all torsion-free weak independence algebras are relatively free, an example being the group of rationals regarded as a $\mathbb{Z}$-module.

Let A be an algebra. A *basis* of A is an A-free set $X \subseteq A \setminus \langle \emptyset \rangle$ such that X generates A. ¿From the definition of torsion-free weak independence algebra and Lemma 5.6, a basis in such an algebra A is the same thing as a generating set which is directly independent. Moreover, from Corollary 1.11, a basis is certainly a PC-basis. We remark that $\langle \emptyset \rangle$ has unique basis $\emptyset$.

The following is an immediate consequence of Proposition 6.3.

Corollary 6.4. *Let A be a relatively free torsion-free weak independence algebra. If Y is a subset of a basis of A, then $\langle Y \rangle$ is pure in A.*

The next result gives several characterisations of the notion of basis in a relatively free torsion-free weak independence algebra.

Theorem 6.5. *Let A be a torsion-free weak independence algebra with $A \neq \langle \emptyset \rangle$. For a subset X of A, the following conditions are equivalent:*

(1) *X is a basis of A,*
(2) *X is directly independent and $\langle X \rangle = A$,*
(3) *X is a maximal directly independent subset and $\langle X \rangle$ is pure,*
(4) *X is a maximal directly independent subset and $\langle X' \rangle$ is pure for every finite subset X' of X,*
(5) *X is a minimal generating set and $\langle X' \rangle$ is pure for every finite subset X' of X.*

Proof. We have already noted that (1) and (2) are equivalent.

Suppose that $A = \langle X \rangle$ and X is directly independent. Certainly, $\langle X \rangle$ is pure, and, by Proposition 6.3, for every finite subset X' of X, the subalgebra $\langle X' \rangle$ is pure.

Further, every element of A depends on X so that by Corollary 1.11, X is a maximal directly independent set. Thus both (3) and (4) follow from (2).

To see that (5) is also a consequence, note that if Y is a proper subset of X and $\langle Y \rangle = A$, then for any $x \in X \diagdown Y$ we have $x \in \langle Y \rangle$ contradicting the direct independence of X.

If (3) holds, then $\langle X \rangle = A$ by Corollary 1.11, and so we have condition (2). If (4) holds, then, for any element a of A, we have $a \prec X$. Hence $a \prec X'$ for some finite subset X' of X. Since $\langle X' \rangle$ is A-pure, $a \in \langle X' \rangle$ and so $\langle X \rangle = A$ and (2) holds.

Finally, if (5) holds and X is not directly independent, then $x \prec X \diagdown \{x\}$ for some $x \in X$. Hence $x \prec X'$ for some finite subset X' of $X \diagdown \{x\}$ so that $x \in \langle X' \rangle$ by the purity of $\langle X' \rangle$.

Hence $A = \langle X \setminus \{x\} \rangle$ contradicting the minimality of X. Thus X is directly independent, and hence (2) holds. □

From the theorem we see that if a directly independent set generates a torsion-free weak independence algebra, it must be a maximal directly independent subset. Since the rank of a weak independence algebra is the cardinality of any maximal directly independent subset, it follows that all bases of a relatively free nonconstant torsion-free weak independence algebra A have the same cardinality, and that this is the rank of A. Thus we can rephrase Corollary 1.12 as follows.

Corollary 6.6. *Let A be a nonconstant relatively free torsion-free weak independence algebra. Then*

(1) *if X is a basis of A, then $|X| = \operatorname{rank} A$,*
(2) *if X is a finite subset of A, and if $\operatorname{rank}(\langle X \rangle) = |X|$, then X is a basis for $\langle X \rangle$.*

7. Unary Algebras

We follow the convention that an algebra is *unary* if its set of basic operations is not empty and consists entirely of unary operations.

Two algebras on the same underlying set are said to be *term equivalent* if their sets of n-ary term operations are the same for each positive integer n. An algebra which has only nullary and unary basic operations is term equivalent to a unary algebra, and, in view of Proposition 5.2, two such algebras which are weak independence algebras (each having nonconstant elements) are term equivalent if and only if they have the same non-empty subalgebra generated by constants and the same nonconstant unary term operations, or have no constants and the same unary term operations.

Our object in this section is to classify certain weak independence algebras with only nullary or unary basic operations (namely, those which are relatively free and torsion-free) up to term equivalence,

Let T be a cancellative left Ore monoid with $T \neq \{1\}$. Let X be a non-empty set, F_X be the free T-act on X and C be a torsion-free T-act. Put $A = F_X \cup C$; then, for each $t \in T$, there is a unary operation λ_t on A given by the left action of T, and for each $c \in C$, we define a nullary operation ν_c with value c. Then A is an 'act with constants' as in Example 2, and since free acts are strongly faithful, A is a torsion-free weak independence algebra by Example 6.

It is clear that A is relatively free with basis X.

We call an algebra constructed in this way, a *standard weak independence algebra over T*.

Theorem 7.1. *Let A be a nonconstant torsion-free weak independence algebra which is term equivalent to a unary algebra. If A is relatively free, then A is term equivalent to a standard weak independence algebra.*

Proof. Let $C = \langle \emptyset \rangle$, and, using establised notation, let T_1 be the monoid of unary term operations on A. Put $T = T_1^*$ and $K = T_C$. To show that A is term equivalent to a standard weak independence algebra, we have to construct such an algebra over T with underlying set A, and set of constants C.

Certainly, the action of T on A, defined by $t \cdot a = t(a)$, makes A into a T-act with subact C. Moreover, because A is a torsion-free algebra, C is a torsion-free T-act.

By Corollary 5.5, T is cancellative and left Ore. Let X be a basis for A, and put $F = A \setminus C$. We complete the proof by showing that F is the free T-act on X. Certainly, $X \subseteq F$, and it follows from Proposition 5.2 that F is a T-subact of A. Now X generates A, and since A is term equivalent to a unary algebra, every element of F can be written as $t(x)$ for some $x \in X$ and $t \in T$. Thus X generates F as a T-act. Now suppose that $sx = ty$ for some $s, t \in T$ and $x, y \in X$. Then $\langle x \rangle \cap \langle y \rangle \neq C$, but X is directly independent, and so $x = y$. It now follows from Lemma 5.1 that $s = t$. Thus, by [11, Definition I.5.11], F is the free T-act on X. □

We can use the theorem to show that an algebra of the type under consideration can be embedded in a special way in an independence algebra. First, we need a lemma which must be well known but does not appear to be written down anywhere.

Lemma 7.2. *Let T be a cancellative, left Ore monoid, and let G be its group of quotients. If C is torsion-free T-act, then C can be embedded as a T-subact in a G-act.*

Proof. It follows from [9, Corollary 8.1.9] that the tensor product $G \otimes_T C$ is a (left) G-act. Of course, $D = G \otimes_T C$ is also a T-act and there is a T-morphism θ from C to D given by $c\theta = 1 \otimes c$. We complete the proof of the lemma by showing that θ is one-one.

If $1 \otimes c = 1 \otimes c'$, then, by [9, Proposition 8.1.8], $c = c'$ or there are elements $g_1, \ldots, g_{n-1}$ in G, $c_1, \ldots, c_{n-1}$ in C and $s_1, \ldots, s_n, t_1, \ldots, t_{n-1}$ in T such that

$$\begin{aligned} 1 &= g_1 s_1 & s_1 c &= t_1 c_1 \\ g_1 t_1 &= g_2 s_2 & s_2 c_1 &= t_2 c_2 \\ &\vdots\ \vdots & &\vdots\ \vdots \\ g_i t_i &= g_{i+1} s_{i+1} & s_{i+1} c_i &= t_{i+1} c_{i+1} \\ &\vdots\ \vdots & &\vdots\ \vdots \\ g_{n-1} t_{n-1} &= 1 s_n & s_n c_{n-1} &= c'. \end{aligned}$$

In the latter case, we choose n to be as small as possible. Since T is left Ore, there are elements u, v in T such that $ut_1 = vs_2$ so that if $n > 2$, we have a shorter sequence of equations

$$
\begin{aligned}
1 &= (g_1u^{-1})(us_1) & (us_1)c &= (vt_2)c_2 \\
(g_1u^{-1})(vt_2) &= g_3s_3 & s_3c_2 &= t_3c_3 \\
g_3t_3 &= g_4s_4 & s_4c_3 &= t_4c_4 \\
\vdots\ &\ \vdots & \vdots\ &\ \vdots \\
g_{n-1}t_{n-1} &= 1s_n & s_nc_{n-1} &= c'.
\end{aligned}
$$

Thus we may assume that $n = 2$ and that our sequence of equations is

$$
\begin{aligned}
1 &= g_1s_1 & s_1c &= t_1c_1 \\
g_1t_1 &= s_2 & s_2c_1 &= c'.
\end{aligned}
$$

Hence $t_1 = s_1s_2$ so that $s_1c = t_1c_1 = s_1s_2c_1 = s_1c'$, and since C is torsion-free, we have $c = c'$ as required. □

We now recall the following terminology. For an algebra A and positive integer n, let $T_n(A)$ be the set of n-ary term operations on A and put $T(A) = \bigcup_{n=1}^{\infty} T_n(A)$. We say that an algebra A is a *reduct* of an algebra B if $A \subseteq B$ and for every n, each element of $T_n(A)$ is the restriction to A^n of some member of $T_n(B)$.

Corollary 7.3. *Let A be a nonconstant torsion-free weak independence algebra. If A is relatively free and term equivalent to a unary algebra, then A is a reduct of an independence algebra.*

Proof. It is enough to prove the result for a standard unary-nullary torsion-free weak independence algebra $A = F_X \cup C$ over T. Let G be the group of left quotients of T. Then, by Lemma 7.2, C can be regarded as a T-subact of a G-act D. Let $\overline{F}_X$ be the free G-set on X and consider the standard weak independence algebra $B = \overline{F}_X \cup D$ over G. In fact, B is an independence algebra, and, clearly, A is a reduct of B. □

Corollary 7.4. *Let A be a nonconstant relatively free torsion-free weak independence algebra. If A is finite and term equivalent to a unary algebra, then A is an independence algebra.*

Proof. As in Corollary 7.3, we may take T to be the monoid T_1^*, which is finite and cancellative. □

8. Basis algebras

After defining various classes of basis algebras, and giving examples, we consider their elementary properties, and then characterise the algebras in the different classes which are term equivalent to a unary algebra. We start with the following definition.

Definition 5. A *basis algebra* A is a torsion-free weak independence algebra which satisfies the following condition:

(PEP) if P, Q are pure subalgebras in A with $P \subseteq Q$, and X is a basis for P, then there is a basis Y for Q with $X \subseteq Y$.

Since $\langle \emptyset \rangle$ is a pure subalgebra of A with basis $\emptyset$, it follows that if P is a pure subalgebra of A, then it has a basis (and so is relatively free). In particular, a basis algebra is a relatively free algebra. We remark that $\langle \emptyset \rangle$ is always a basis algebra.

We may regard (PEP) as a converse to Proposition 6.3 which says that if a subalgebra has a basis which can be extended to a basis of A, then it is pure.

Our first examples of basis algebras are provided by independence algebras; they are basis algebras because the exchange property guarantees that every independent subset of a subalgebra can be extended to a basis for that subalgebra. The next lemma gives more examples.

Lemma 8.1. *A relatively free torsion-free weak independence algebra which is term equivalent to a unary algebra is a basis algebra.*

Proof. By Theorem 7.1, it is enough to prove the result for a standard weak independence algebra A over a cancellative left Ore monoid T. Let $A = F_X \cup C$ be as in Section 7. It is easy to see that the pure subalgebras of A all have the form

$$B = \bigcup_{y \in Y} Ty \cup C$$

where $Y \subseteq X$. Moreover, a basis of B is of the form $\{u_y y : y \in Y\}$ where each u_y is a unit of T. That (PEP) holds is now immediate. □

Next, we show that relatively free subalgebras of basis algebras are also basis algebras.

Proposition 8.2. *Let B be a relatively free subalgebra of a basis algebra A. Then*

(1) $B \cong \mathrm{PC}(B)$,
(2) *B is a basis algebra.*

Proof. If $B = \langle \emptyset \rangle$, then $B = \mathrm{PC}(B)$, and we have already remarked that $\langle \emptyset \rangle$ is a basis algebra.

Suppose therefore that $B \neq \langle \emptyset \rangle$. For (1), we note that by Corollary 1.5, B and $\mathrm{PC}(B)$ have the same rank. They are both relatively free, so that if X and Y are bases for B and $\mathrm{PC}(B)$ respectively, then there is a bijection from X to Y which extends to an isomorphism from $\langle X \rangle = B$ to $\langle Y \rangle = \mathrm{PC}(B)$.

In view of (1), to prove (2), it is enough to show that $\mathrm{PC}(B)$ is a basis algebra. We have already remarked that a subalgebra of a nonconstant torsion-free weak independence algebra is a torsion-free weak independence algebra. If P and Q are pure subalgebras of

PC(B), then by Corollary 1.13, P and Q are pure subalgebras of A, so that (PEP) holds for PC(B) since it holds for A. □

Further examples of basis algebras are provided by free modules of finite rank over Bezout domains.

Example 10. As mentioned in the introduction, a *Bezout domain* is an integral domain in which finitely generated left and right ideals are principal. We remark that a Bezout domain is the same thing as an integral domain which is a *Hermite ring* in the sense of [10].

Now let R be a Bezout domain and F be a finitely generated free left R-module. By [4, Proposition 1.1.4], every finitely generated submodule of F is free.

Suppose that B is a pure submodule of F. This means that if r is a non-zero element of R and $ra \in B$ where $a \in F$, then $a \in B$. Then F/B is finitely generated and torsion-free, and hence, by [5, Proposition 1.1.9], free. Thus the exact sequence $0 \to B \to F \to F/B \to 0$ splits, so that B is a direct summand of F and hence finitely generated. Consequently, B is free. Thus if C is also a pure submodule and $B \subseteq C$, then B is a direct summand of C, and any basis of B can be extended to one for C. That is, (PEP) holds and F is a basis algebra.

As noted above, as well as being a basis algebra, such a free module F has the additional property that every finitely generated submodule is free; if R is a principal ideal domain, then by the corollary of [4, Proposition 1.1.4], *every* submodule is finitely generated, and hence free. Not all basis algebras share these properties, for example, it follows from Lemma 8.1 that the $\mathbb{N}$-act $\mathbb{N}$, considered in Example 2, is a basis algebra, but it is clear that the subact generated by $\{2, 3\}$ is not free. These observations lead to the following definitions.

Let κ be a cardinal. A basis algebra A is *κ-free* if every subalgebra of A having a generating set of cardinality at most κ is relatively free, that is, has a basis. (This terminology is inspired by the term right α-fir in [4].) We say that A is *semihereditary* if it is n-free for all positive integers n; *hereditary* if it is κ-free for $\kappa = |A|$; and *stable* if it is κ-free for $\kappa = \operatorname{rank} A$.

The terminology, *semihereditary* and *hereditary*, is justified by the fact that, by Proposition 8.2, relatively free subalgebras of a basis algebra are themselves basis algebras. As we have just observed, finitely generated free modules over a Bezout domain (principal ideal domain) are semihereditary (hereditary). Also, since the class of independence algebras is closed under taking subalgebras, independence algebras are hereditary basis algebras.

Returning to the general case, we remark that any basis algebra is 1-free, but not necessarily 2-free, as seen above. The next result tells us when a basis algebra is 2-free.

Proposition 8.3. *A basis algebra is* 2*-free if and only if it satisfies the following condition:*
(FGC) *any two elements of a cyclic subalgebra generate a cyclic subalgebra.*

Proof. Let C be a cyclic subalgebra of a basis algebra which is 2-free. Then C is free of rank 1. If B is a subalgebra of C generated by two elements, then B is free by assumption. Also $\operatorname{rank} B \leqslant \operatorname{rank} C$ by Corollary 1.12, so that B is cyclic.

Now suppose that A is a basis algebra which satisfies condition (FGC), and let a, b be elements of A. Now, by Corollaries 1.5 and 1.12,

$$\operatorname{rank}\{a, b\} = \operatorname{rank}\langle\{a, b\}\rangle = \operatorname{rank} \mathrm{PC}\langle\{a, b\}\rangle.$$

Since A is a basis algebra, $\mathrm{PC}\langle\{a, b\}\rangle$ is a relatively free subalgebra of A. If $\mathrm{PC}\langle\{a, b\}\rangle$ is cyclic, then by assumption $\langle\{a, b\}\rangle$ is cyclic, and hence relatively free. Otherwise, $\mathrm{PC}\langle\{a, b\}\rangle$ has rank 2, and hence $\operatorname{rank}\{a, b\} = 2$. Thus $\{a, b\}$ is directly independent, and hence A-free. Thus the subalgebra $\langle\{a, b\}\rangle$ is relatively free, and it follows that A is 2-free. □

As an easy consequence, we have the following.

Corollary 8.4. *If A is a 2-free basis algebra, then every finitely generated subalgebra of a cyclic subalgebra is cyclic.*

We conclude by characterising those semihereditary or hereditary basis algebras which are term equivalent to unary algebras.

Proposition 8.5. *Let A be a relatively free torsion-free weak independence algebra which is term equivalent to a unary algebra, and let T_1^* be the monoid of nonconstant unary operations on A. Then the following conditions are equivalent:*

(1) *A is a semihereditary basis algebra,*
(2) *A is a 2-free basis algebra,*
(3) *A satisfies condition* (FGC)*,*
(4) *every finitely generated left ideal of T_1^* is principal.*

Proof. By Theorem 7.1, it is enough to prove the proposition for a standard weak independence algebra A over a cancellative left Ore monoid T. Let $A = F_X \cup C$ be as in Section 7, and note that $T_1^* \cong T$.

By Lemma 8.1, A is a basis algebra.

If (1) holds, then every finitely generated subalgebra is free, and so A is certainly 2-free.

Conditions (2) and (3) are equivalent by Proposition 8.3.

Suppose that (3) holds and let $s, t \in T$. Choose $x \in X$ and consider $sx, tx \in Tx$. Then sx, tx are in the cyclic subalgebra $\langle x\rangle = Tx \cup C$, and so, by condition (FGC), the subalgebra $\langle sx, tx\rangle$ is cyclic. Let $\langle sx, tx\rangle = \langle y\rangle = Ty \cup C$. Now y is in one of Tsx, Ttx and it follows that $Ts \cup Tt = Ts$ or $Ts \cup Tt = Tt$. Hence every finitely generated left ideal of T is principal.

Now suppose that (4) holds, and let

$$B = \bigcup_{i \in I} T s_i x_i \cup C$$

where I is finite and $\{x_i : i \in I\} \subseteq X$. For any $i \in I$, let

$$I_i = \{j \in I : x_j = x_i\}$$

and let $J \subseteq I$ be a set of representatives from the sets I_i. Each I_i is finite so that the left ideal

$$T_i = \bigcup_{j \in I_i} T s_j$$

is principal by assumption, with generator t_i say. Now

$$B = \bigcup_{i \in J} T t_i x_i \cup C$$

which has basis $\{t_i x_i : i \in J\}$, and hence is relatively free. Thus A is semihereditary. $\square$

Similar arguments to those for Proposition 8.5 give the following proposition.

Proposition 8.6. *Let A be a relatively free torsion-free weak independence algebra which is term equivalent to a unary algebra, and let T_1^* be the monoid of nonconstant unary operations on A. Then the following conditions are equivalent:*

(1) *A is a hereditary basis algebra,*
(2) *A satisfies the following condition:*
  (C) *every subalgebra of a cyclic subalgebra is cyclic,*
(3) *every left ideal of T_1^* is principal.*

Acknowledgements

The authors would like to thank Professor George Bergman for correcting their original definition of 'constant subalgebra'. We would are also grateful to the communicating editor of our original article and two anonymous referees for suggestions which led to improvements in the exposition of the paper.

References

[1] G.M. Bergman, 'Constructing division rings as module-theoretic direct limits', *Trans. Amer. Math. Soc.*, **354** (2002), 2079–2114.
[2] P.J. Cameron and C. Szabó, 'Independence algebras', *J. London Math. Soc.*, **61** (2000), 321–334.
[3] P. M. Cohn, *Universal algebra* (Harper & Row, New York, 1965).
[4] P. M. Cohn, *Free rings and their relations* (Academic Press, London, 1971).
[5] P. M. Cohn, *Free rings and their relations*, 2nd edition (Academic Press, London, 1985).
[6] K. Głazek, 'Some old and new problems in independence theory', *Colloq. Math.* **42** (1979), 127–189.
[7] V. Gould, 'Independence algebras', *Algebra Universalis* **33** (1995), 294–318.

[8] G. Grätzer, *Universal algebra* (Van Nostrand, Princeton, N.J., 1968).
[9] J. M. Howie, *Fundamentals of semigroup theory* (Oxford University Press, Oxford, 1995).
[10] I. Kaplansky, 'Elementary divisors and modules', *Trans. Amer. Math. Soc.* **66** (1949), 464–491.
[11] M. Kilp, U. Knauer and A.V. Mikhalev, *Monoids, acts and categories* (Walter de Gruyter, Berlin, 2000).
[12] R. N. McKenzie, G. F. McNulty, and W. T. Taylor, *Algebra, lattices, varieties* (Wadsworth, Monterey, CA, 1983).
[13] W. Narkiewicz, 'Independence in a certain class of abstract algebras', *Fund. Math.* **50** (1961/62), 333–340.
[14] J. Schmidt, 'Mehrstufige Austauschstrukturen', *Z. Math. Logik Grundlagen Math.* **2** (1956), 233–249.

Department of Mathematics,
University of York,
Heslington,
York YO10 5DD, U.K.
e-mail : `jbf1@york.ac.uk`
e-mail : `varg1@york.ac.uk`