OPTIMAL CONSUMPTION UNDER INCOME UNCERTAINTY
An Example and a Conjecture

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This paper demonstrates that, under certain circumstances, the optimal consumption strategy of an individual with constant absolute risk-aversion, facing income uncertainty, is linear in wealth. Comparative static results are derived, and the paper concludes with a conjecture about other classes of utility functions.

A number of authors have examined the problem of the individual's choice of optimal consumption strategy when faced with income uncertainty, and several important general results have been obtained. [For a partial survey see Hey (1979)]. However, the existence of a significant special case appears to have gone unnoticed. As we demonstrate below, this special case has several interesting properties. Moreover, it immediately leads to a conjecture concerning the optimal consumption strategy in a broad class of other empirically relevant cases. Indeed, it suggests a way of determining an individual's attitude to risk through observations on his consumption behaviour.

To keep our analysis as simple as possible we confine attention to a stationary infinite-horizon problem of a form familiar in the literature [for two relatively recent examples, see Schechtman (1976) and Yaari (1976)]. Thus, we consider the case of an individual, who is a von Neumann-Morgenstern utility maximiser with an additively separable utility function, who accordingly wishes to maximise the expected value of total discounted lifetime utility as given by

\[ \sum_{t=1}^{\infty} \rho^{t-1} U(C_t), \]  

(1)

where \( \rho \) is the constant discount rate, and \( C_t \) denotes consumption in period \( t \). The individual is assumed to live an infinite number of periods. We assume that \( U' > 0 \) everywhere, and, when necessary, that \( U'' < 0 \) everywhere.

We assume that the individual can freely borrow and lend at a constant rate of interest \( (r - 1) \) where \( r \) is greater than one. The individual is, however, faced with
income uncertainty; denoting income in period \( t \) by \( Y_t \), we assume that \( Y_1, Y_2, \ldots \)
are identically and independently distributed random variables with a known prob-
bility density function. Moreover, we assume that the ‘rules of the game’ are such
that the individual must choose \( C_t \), consumption in period \( t \), before the uncertainty
about \( Y_t \) is resolved. (Alternative assumptions about relative timings lead to the
same qualitative conclusions.) Finally, we denote by \( W_t \) the wealth owned by the
individual at the beginning of period \( t \) (that is, before \( C_t \) has been consumed and
before \( Y_t \) has been received). Wealth in successive periods is linked by the equation
\[
W_{t+1} = r(Y_t - C_t + W_t) .
\]
This embodies the assumption that \( C_t \) is consumed, and \( Y_t \) received, at the beginning
of period \( t \).

The assumptions detailed above are sufficient to guarantee that the only factor
that can influence the optimal choice of consumption in any period is the amount
of wealth held at the beginning of that period. In particular, the optimal strategy is
time-independent: given an initial stock of wealth, the optimal consumption level
is the same whether the period is \( t \) or \( t + 1 \) or whatever. Accordingly, we denote the
optimal strategy by \( C^* \); thus
\[
C^* = C^*(W) ,
\]
where \( W \) is beginning-of-period wealth. To determine \( C^* \) we introduce the function
\( V(W) \) which measures the maximum expected discounted lifetime utility [as given
by (1)] as viewed from the beginning of a period in which the initial stock of
wealth is \( W \). The function \( V(\cdot) \) is clearly time-independent, and is determined by
the familiar recursive equation
\[
V(W) = \max_C (U(C) + \rho E[V[r(Y - C + W)]) ,
\]
where the expectation is taken with respect to the distribution of \( Y \). By definition,
\( C^* \) is the value of \( C \) at which the maximum on the right-hand side of (4) is achieved.
The first-order condition, found in the usual fashion, is
\[
U'(C^*) = \rho E[V'[r(Y - C^* + W)]].
\]
We assume that the second-order condition holds.

Replacing \( C \) by \( C^* \) in (4) we have definitionally that
\[
V(W) = U(C^*) + \rho E[V[r(Y - C^* + W)]].
\]
Now, if (6) is differentiated throughout with respect to \( W \) (noting that \( C^* \) is a func-
tion of \( W \)), and if the optimality condition (5) is used to simplify the resulting
expression, we get
\[
V'(W) = \rho E[V'[r(Y - C^* + W)] = U'(C^*).
\]
This splendidly simple result states the satisfying conclusion that the optimal con-
The actual solution to this depends upon the particular utility function and particular distribution function under consideration. We do not propose to investigate the general solution [properties of which have been explored by the writers referred to in Hey (1979)]; instead we examine an important special case. Suppose that the individual displays constant absolute risk-aversion; that is, his utility function \( U(\cdot) \) is such that
\[
U(x) = -\exp(-Rx),
\] (9)
where \( R(>0) \) denotes the Arrow–Pratt index of absolute risk aversion. In this case, it is straightforward to verify that, the solution (which can be shown to be unique) to the optimal consumption problem is of the form
\[
C^*(W) = a + bW. \tag{10}
\]
To demonstrate this assertion, substitute (10) into (8) using (9); this yields
\[
-\exp[-R(a + bW)] = rpE[\exp(-R[a + b(r(Y - a - bW + W)])])]. \tag{11}
\]
Since this must hold for all \( W \), the term in \( W \) must disappear; this condition requires that
\[
-Rb = -Rbr(1 - b),
\]
and hence that
\[
(1 - b)r = 1. \tag{12}
\]
Substituting (12) into (11) we get for a further condition that
\[
rpE[\exp[Rbr(a - Y)]] = 1.
\]
This can be further simplified using (12) to give
\[
rpE[\exp[R(r - 1)(a - Y)]] = 1. \tag{13}
\]
Hence (10) is the solution to (8) when the utility function is of the constant absolute risk-aversion form. Furthermore, the coefficients of (10), the optimal consumption strategy, are \( a \) and \( b \) as given by (13) and (12) respectively.

A number of important implications flow from these results. First, the consumption function is linear in wealth. Second, the marginal propensity to consume (out of wealth) depends upon \( r \) alone; in particular, changes in the individual’s discount rate \( \rho \), in the individual’s index of absolute risk aversion \( R \), and in the distribution of income have absolutely no effect on the marginal propensity to consume. These results are, of course, significant for the aggregation problem — since \( r \) might be pre-
sumed the same for all individuals, aggregation of first-differenced consumption functions can proceed immediately. Third, and in contrast, the intercept of the consumption does depend upon \( \rho, R \) and the distribution of income (in addition to \( r \)).

Looking in detail at these results, we note first of all that (12) implies that \( b \) will lie between 0 and 1 (for \( r > 1 \)), and that \( b \) is an increasing function of \( r \). The polar cases are of interest: if \( r \) is unity then \( b \) will be zero; and as \( r \) approaches infinity \( b \) approaches unity.

We now explore the determination of the intercept \( a \), and the effect on it of various parameter changes. We note first from (13), that if \( Y \) takes only non-negative values then \( a \) must be positive if \( \rho \) is less than unity. If, on the contrary, \( \rho \) is greater than unity, then a negative value for \( a \) is possible (though unlikely). The condition as to whether \( \rho \) is greater or less than unity is a condition that recurs frequently in the results that follow: it has an obvious economic interpretation. [If \( \rho \) is greater (less) than unity then the rate of return is sufficiently large (small) for the discounted future worth of a pound saved to be larger (smaller) than the current worth of a pound.]

From (13) it is clear that an increase in \( \rho \) leads to a decrease in \( a \). Also a rightward shift of the income distribution by some amount leads to an increase in \( a \) by the same amount. Moreover, a Rothschild-Stiglitz increase in risk in the income distribution necessitates a decrease in \( a \). (Since \( \exp[R(r - 1)(a - Y)] \) is a convex function of \( Y \), its expected value increases when \( Y \) undergoes such an increase in risk [see Rothschild and Stiglitz (1970)]; thus \( a \) must fall to keep (13) satisfied.) These results accord with intuition. However, the effects of changes in \( R \) and \( r \) on \( a \) are ambiguous, as can be illustrated with the following specific distribution.

Suppose that \( Y \) is normally distributed with mean \( \mu \) and variance \( \sigma^2 \). Using standard results, (13) becomes

\[
\rho \exp[R(r - 1)(a - \mu) + \frac{1}{2} R^2 (r - 1)^2 \sigma^2] = 1,
\]

which solves to yield

\[
a = \mu - \frac{1}{2} R(r - 1) \sigma^2 - \log(\rho)/[R(r - 1)].
\] (14)

From (14) it is clear that if \( \rho \) is less than unity, then increases in \( R \) and \( r \) both lead to decreases in \( a \). However, if \( \rho \) is greater than unity, it is possible (though unlikely) for the opposite result to hold. We suspect that for most empirically relevant cases, \( a \) is a decreasing function of \( r \) (and \( R \)). Combined with our earlier result on \( b \) this implies that an increase in \( r \) will lead to a rotation of the consumption function (in an anticlockwise direction) around a point in the positive quadrant.

This concludes our simple example. Its crucial feature is that the assumed constancy of the absolute risk-aversion index implies a linear consumption function. This immediately suggests our conjecture, which is as follows: if the utility function displays decreasing (increasing) absolute risk-aversion, then the consumption function is convex (concave), that is, its slope is increasing (decreasing) everywhere. As yet, we have been unable to find a proof, though the structure of the mathematics
looks promising. Is there anyone out there who can help?

Clearly, if our conjecture is correct, then evidence on the shape of (cross-section) consumption functions (as functions of wealth) contains information as to attitudes to risk. If, as generally appears to be the case, such functions are concave, then this could shed doubt on the commonly accepted notion that most individuals display decreasing absolute risk-aversion.

References


