On the stability of Alfvén discontinuity

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ABSTRACT

The stability of Alfvén discontinuities for the equations of ideal compressible magnetohydrodynamics (MHD) is studied. The Alfvén discontinuity is a characteristic discontinuity for the hyperbolic system of MHD equations but, as in the case of shock waves, there is a mass flux through its front. The Lopatinskii condition for a planar Alfvén discontinuity is tested numerically, and the domain in the space of parameters of the discontinuity where it is unstable is determined. In fact, in this domain the Alfvén discontinuity is not only unstable: the initial-boundary-value problem for corresponding linearized equations is ill-posed in the sense of Hadamard.

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I Introduction

Alfvén (or rotational) discontinuities represent a particular type of discontinuous solutions of the equations of the ideal compressible magnetohydrodynamics. They naturally arise when one tries to construct steady solutions of the magnetohydrodynamic (MHD) equations [1]. Alfvén discontinuities play an important role in solving the MHD Riemann problem and, as a consequence, in numerical simulations of compressible MHD flows [2]–[4]. They are used to model flows of astrophysical plasma and interpret observations of the behaviour of Earth's magnetosphere and interplanetary medium [5]–[8]. The simplest MHD flows with an Alfvén discontinuity are piecewise constant solutions of the MHD equations with a planar discontinuity surface. Such solutions exist but if solutions that are close to them do not, then flows with planar Alfvén discontinuities are not physically realizable. In this paper we study the linear stability of a planar Alfvén discontinuity. Earlier, the stability of *incompressible* Alfvén discontinuities had been studied in Reference 9 where it had been shown that they are always stable. Several authors (see Refs. 5 and 6 and the literature cited therein) studied interaction of an Alfvén discontinuity in a compressible medium with incident Alfvén and magnetosonic waves of small amplitude. However, their results cannot be interpreted in terms of the instability of the Alfvén discontinuity. Wu [10] had performed one-dimensional numerical simulations of the evolution of an Alfvén discontinuity in the framework of viscous and resistive magnetohydrodynamics and concluded that the dissipation results in instability of the discontinuity. However, his interpretation of the numerical results as an evidence for instability of Alfvén discontinuities in viscous and resistive magnetohydrodynamics is not indisputable (see, e.g., Ref. 11). In any case, the results of Ref. 10 say nothing about the stability of an Alfvén discontinuity in the framework of ideal magnetohydrodynamics. As far as we know, the only paper that directly deals with the stability of Alfvén discontinuities in compressible MHD flows is Reference 12, where an incorrect conclusion about instability of Alfvén discontinuities in the limit of magnetically dominated flows had been made.

In this paper, we re-examine the problem and show that for certain values of the parameters the Alfvén discontinuity is unstable and that the growth rate of the instability can be arbitrarily large, so that the initial-boundary-value problem for corresponding linearized equations is ill-posed in the sense of Hadamard. This implies that in unstable region of the parameter space Alfvén discontinuities as smooth surfaces cannot exist for any finite, however short, time interval.

We conclude this introduction with mathematical formulation of the problem. Throughout the paper we set the vacuum magnetic permeability $\mu_0 = 1$. Equations of the magnetohydrodynamics of an inviscid, compressible, perfectly conducting fluid can be written as

$$\frac{1}{\rho c^2}\frac{dp}{dt} + \nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$\rho\frac{d\mathbf{v}}{dt} - (\mathbf{h}\cdot\nabla)\mathbf{h} + \nabla\left(p + \frac{\mathbf{h}^2}{2}\right) = 0, \tag{2}$$

$$\mathbf{h}_t - \nabla \times (\mathbf{v}\times\mathbf{h}) = 0, \tag{3}$$

$$\nabla \cdot \mathbf{h} = 0, \tag{4}$$

$$\frac{ds}{dt} = 0. \tag{5}$$

Here $\rho = \rho(t, \mathbf{x})$ is the density, $\mathbf{v} = \mathbf{v}(t, \mathbf{x}) = (v_1, v_2, v_3)$ the velocity, $\mathbf{h} = \mathbf{h}(t, \mathbf{x}) = (h_1, h_2, h_3)$ the magnetic field, $p = p(t, \mathbf{x})$ the pressure, s the entropy per unit mass, t the time, $\mathbf{x} = (x_1, x_2, x_3)$ Cartesian coordinates, $d/dt = \partial_t + \mathbf{v}\cdot\nabla$ and $c^2 = \partial p/\partial\rho$ is the square of the sound speed. The equations (1)–(5) are supplemented by the equation of state of the fluid

$$p = p(\rho, s). \tag{6}$$

Note that Eq. (4) can be treated as an additional constraint on in initial data: if (4) is satisfied initially at $t = 0$, then, as a consequence of Eq. (3) it holds for all $t > 0$. Therefore, from now on, we drop Eq. (4).

Let $\Gamma(t)$ be a smooth surface of strong discontinuity for solutions of Eqs. (1)–(5). We assume that it can be described by the equation

$$x_1 = f(t, x_2, x_3). \tag{7}$$

Let

$$\mathbf{n} = \frac{(1, -\partial_2 f, -\partial_3 f)}{\sqrt{1 + (\partial_2 f)^2 + (\partial_3 f)^2}}$$

be the unit normal to Γ, $v_n \equiv \mathbf{v}\cdot\mathbf{n}$, $h_n \equiv \mathbf{v}\cdot\mathbf{n}$, $[g] = g^+ - g^-$ denote the jump of g across the discontinuity surface ($g^\pm \equiv g|_{(x_1 - f(t,x_2,x_3))\to\pm 0}$), and let

$$j = \rho\left(v_n - \frac{f_t}{\sqrt{1 + (\partial_2 f)^2 + (\partial_3 f)^2}}\right)$$

be the mass flux across the discontinuity. Note that the conservation of mass and the law that the magnetic flux through an arbitrary closed surface is zero (the absence of magnetic charges) imply that (see, e.g., Reference 2)

$$[j] = 0, \quad [h_n] = 0. \tag{8}$$

There are four types of MHD discontinuities (see, e.g., References 2 and 3): MHD shock waves ($[j] \neq 0$, $[\rho] \neq 0$), tangential discontinuities or current-vortex sheets ($j = 0$, $h_n = 0$), contact discontinuities ($j=0$, $h_n\neq 0$) and Alfvén (or rotational) discontinuities ($j\neq 0$, $[\rho]=0$). Discontinuities of the last three types are characteristic, i.e. the corresponding discontinuity surfaces are the characteristic surfaces of Eqs. (1)–(5).

In this paper we are interested in Alfvén discontinuities. The jump conditions for an Alfvén discontinuity are (see Reference 1)

$$
\begin{aligned}
&[p]=0,\quad [s]=0,\quad [h_n]=0,\quad [j]=0,\\
&[\mathbf{h}^2]=0,\quad \left[\mathbf{v}-\frac{\mathbf{h}}{\sqrt{\rho}}\right]=0,\quad j=h_n\sqrt{\rho}.
\end{aligned}
\tag{9}
$$

Note that $h_n\neq 0$ if $j\neq 0$. The condition $[\rho]=0$ automatically follows from the first two conditions in (9). Thus, in Alfvén discontinuity, all thermodynamic quantities are continuous at the discontinuity surface. The normal magnetic field and the normal velocity are also continuous. The direction of the magnetic field vector has a jump at $\Gamma(t)$, while its magnitude remains unchanged, i.e. the magnetic field vector rotates about the normal to the discontinuity surface.

The basic state whose stability is investigated is the simplest piecewise constant solution of Eqs. (1)–(5) given by

$$
\rho=\rho_0,\quad s=s_0,\quad p=p_0,\quad \mathbf{v}=\mathbf{V}=\begin{cases}\mathbf{V}^+, & x_1>0\\ \mathbf{V}^-, & x_1<0\end{cases},\quad \mathbf{h}=\mathbf{H}=\begin{cases}\mathbf{H}^+, & x_1>0\\ \mathbf{H}^-, & x_1<0\end{cases}. \tag{10}
$$

The discontinuity surface is the plane $x_1=0$, and the solution represents a planar Alfvén discontinuity in the reference frame moving with the front. The jump conditions (9) imply that

$$
[H_1]=0,\quad V_1^{\pm}=\frac{H_1^{\pm}}{\sqrt{\rho_0}},\quad [H_2^2+H_3^2]=0,\quad \left[V_{2,3}-\frac{H_{2,3}}{\sqrt{\rho_0}}\right]=0. \tag{11}
$$

The problem considered in the rest of the paper is the stability of the discontinuous MHD flow given by Eq. (10).

II Stability analysis

A. Linearized problem. Let $x_1=\tilde f(t,x_2,x_3)$ be the equation of the perturbed discontinuity surface, and let $\tilde{\mathbf{v}}$, $\tilde{\mathbf{h}}$, $\tilde p$ and $\tilde s$ be the perturbations of the corresponding quantities. Assuming that the perturbations are small, we linearize Eqs. (1)–(5). As a result, we have

$$
\begin{aligned}
&\tilde p_t+\mathbf{V}\cdot\nabla\tilde p+\rho_0c_0^2\,\nabla\cdot\tilde{\mathbf{v}}=0,\\
&\rho_0\left(\tilde{\mathbf{v}}_t+(\mathbf{V}\cdot\nabla)\tilde{\mathbf{v}}\right)-(\mathbf{H}\cdot\nabla)\tilde{\mathbf{h}}+\nabla\left(\tilde p+\mathbf{H}\cdot\tilde{\mathbf{h}}\right)=0,\\
&\tilde{\mathbf{h}}_t+(\mathbf{V}\cdot\nabla)\tilde{\mathbf{h}}-(\mathbf{H}\cdot\nabla)\tilde{\mathbf{v}}+\mathbf{H}\,\nabla\cdot\tilde{\mathbf{v}}=0,\\
&\tilde s_t+\mathbf{V}\cdot\nabla\tilde s=0\quad\text{for}\quad \mathbf{x}\in\mathbb{R}^3_{\pm}.
\end{aligned}
\tag{12}
$$

Here $\mathbb{R}^3_+$ ($\mathbb{R}^3_-$) is the half space bounded by the plane $x_1 = 0$ and corresponding to positive (negative) x_1. Since the equation for $\tilde{\mathbf{h}}$ implies that

$$(\nabla \cdot \tilde{\mathbf{h}})_t + \nabla \cdot \left(\mathbf{V}(\nabla \cdot \tilde{\mathbf{h}})\right) = 0,$$

the condition $\nabla \cdot \tilde{\mathbf{h}} = 0$ is treated as a restriction on the initial data for $\tilde{\mathbf{h}}$. Linearization of the jump conditions (9) yields

$$\begin{aligned}
&[\tilde{s}] = [\tilde{p}] = 0, \quad [\tilde{h}_1] = [H_2]\partial_2 \tilde{f} + [H_3]\partial_3 \tilde{f}, \quad [\mathbf{H}\cdot\tilde{\mathbf{h}}] = 0,\\
&\tilde{v}_1 - \frac{1}{\sqrt{\rho_0}}\left(\tilde{h}_1 - \frac{\tilde{\rho}}{2\rho_0}H_1\right) = \tilde{f}_t + \left(V_2 - \frac{H_2}{\sqrt{\rho_0}}\right)\partial_2\tilde{f} + \left(V_3 - \frac{H_3}{\sqrt{\rho_0}}\right)\partial_3\tilde{f},\\
&\left[\tilde{v}_2 - \frac{1}{\sqrt{\rho_0}}\left(\tilde{h}_2 - \frac{\tilde{\rho}}{2\rho_0}H_2\right)\right] = 0, \quad \left[\tilde{v}_3 - \frac{1}{\sqrt{\rho_0}}\left(\tilde{h}_3 - \frac{\tilde{\rho}}{2\rho_0}H_3\right)\right] = 0 \quad \text{at} \quad x_1 = 0.
\end{aligned} \tag{13}$$

Note that the entropy perturbation $\tilde{s}(t,\mathbf{x})$ is a solution of a separate problem

$$\tilde{s}^{\pm}_t + \mathbf{V}^{\pm}\cdot\nabla\tilde{s}^{\pm} = 0 \quad \text{for} \quad \mathbf{x}\in\mathbb{R}^3_{\pm}, \quad [\tilde{s}] = 0 \quad \text{at} \quad x_1 = 0,$$

and cannot result in instability. Therefore, in what follows, we let

$$\tilde{s}(t,\mathbf{x}) \equiv 0.$$

On applying the Galilean transformation

$$t' = t, \quad x_1' = x_1, \quad x_2' = x_2 - \left(V_2 - \frac{H_2}{\sqrt{\rho_0}}\right)t, \quad x_3' = x_3 - \left(V_3 - \frac{H_3}{\sqrt{\rho_0}}\right)t,$$

to Eqs. (12) and (13) and introducing the dimensionless quantities

$$x = \frac{x_1'}{L}, \quad t = \frac{c_0}{L}t', \quad p = \frac{\tilde{p}}{\rho_0 c_0^2}, \quad \mathbf{v} = \frac{\tilde{\mathbf{v}}}{c_0}, \quad \mathbf{h} = \frac{\tilde{\mathbf{h}}}{c_0\sqrt{\rho_0}}, \quad f = \frac{\tilde{f}}{L},$$

we rewrite the linearized equations and boundary conditions in the form

$$\begin{aligned}
&p_t + \hat{\mathbf{H}}\cdot\nabla p + \nabla\cdot\mathbf{v} = 0,\\
&\mathbf{v}_t + (\hat{\mathbf{H}}\cdot\nabla)(\mathbf{v}-\mathbf{h}) + \nabla\left(p + \hat{\mathbf{H}}\cdot\mathbf{h}\right) = 0,\\
&\mathbf{h}_t - (\hat{\mathbf{H}}\cdot\nabla)(\mathbf{v}-\mathbf{h}) + \hat{\mathbf{H}}\,\nabla\cdot\mathbf{v} = 0 \quad \text{for} \quad \mathbf{x}\in\mathbb{R}^3_{\pm},
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&[p] = 0, \quad [h_1] = [\hat{H}_2]\partial_2 f + [\hat{H}_3]\partial_3 f, \quad [\hat{\mathbf{H}}\cdot\mathbf{h}] = 0,\\
&f_t = v_1 - h_1 + \hat{H}_1\frac{p}{2}, \quad \left[v_{2,3} - h_{2,3} + \hat{H}_{2,3}\frac{p}{2}\right] = 0 \quad \text{at} \quad x_1 = 0.
\end{aligned} \tag{15}$$

Here $\hat{\mathbf{H}} \equiv \mathbf{H}/c_0\sqrt{\rho_0}$.

B. Reduced problem. Equations (14) and boundary conditions (15) represent the hyperbolic system of linear first-order equations for seven unknowns p, $\mathbf{v}$, $\mathbf{h}$ with characteristic boundary. Following Reference 12, we reduce this system to a hyperbolic problem with non-characteristic boundary which does not involve the front perturbation f. Let

$$r = \hat{\mathbf{H}} \cdot \mathbf{h}, \quad \mathbf{w} = \mathbf{v} - \mathbf{h} + p\,\hat{\mathbf{H}}. \tag{16}$$

Then, it follows from Eqs. (14) that p, r and $\mathbf{w}$ satisfy the equations

$$\begin{aligned} &p_t + \nabla \cdot \mathbf{w} = 0, \\ &r_t - (\hat{\mathbf{H}} \cdot \nabla)(\hat{\mathbf{H}} \cdot \mathbf{w}) + \hat{\mathbf{H}}^2\, \nabla \cdot \mathbf{w} = 0, \\ &\mathbf{w}_t + 2(\hat{\mathbf{H}} \cdot \nabla)\mathbf{w} - \hat{\mathbf{H}}(\hat{\mathbf{H}} \cdot \nabla p) + \nabla(p + r) = 0. \end{aligned} \tag{17}$$

The boundary conditions for p, r and $\mathbf{w}$, obtained from Eqs. (15), are

$$[p] = 0, \quad [r] = 0, \quad \left[\mathbf{w} - \frac{p}{2}\,\hat{\mathbf{H}}\right] = 0 \quad \text{at} \quad x_1 = 0. \tag{18}$$

It is clear that if the reduced problem has an exponentially growing (with time) solution, then so does the original problem, and this implies that the basic state is unstable. On the other hand, it can be shown that if the reduced problem has no exponentially growing solutions, then the basic state is stable [13]. Keeping this in mind, below we focus on the normal mode analysis of the reduced problem (17)–(18).

It is convenient to introduce the dimensionless parameter

$$\beta = \frac{1}{|\hat{\mathbf{H}}|^2} = \frac{\rho_0 c_0^2}{|\mathbf{H}|^2}$$

and re-scale the dependent and independent variables as follows

$$t \to \beta^{1/2} t, \quad \mathbf{w} \to \beta^{-1/2}\mathbf{w}, \quad r \to \beta^{-1} r.$$

Also, let θ be the angle that the magnetic field makes with the normal to the front and ϕ the angle of rotation of the magnetic field across the front of the Alfvén discontinuity, and let $\mathbf{a}^{\pm}$, $\mathbf{b}^{\pm}$, $\mathbf{c}^{\pm}$ be the orthogonal basis vectors in $\mathbb{R}^3_{\pm}$, given by

$$\begin{aligned} &\mathbf{a}^+ = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi), && \mathbf{a}^- = (\cos\theta, \sin\theta, 0), \\ &\mathbf{b}^+ = (-\sin\theta, \cos\theta\cos\phi, \cos\theta\sin\phi), && \mathbf{b}^- = (-\sin\theta, \cos\theta, 0), \\ &\mathbf{c}^+ = (0, -\sin\phi, \cos\phi), && \mathbf{c}^- = (0, 0, 1). \end{aligned}$$

Note that $\mathbf{a}^+$ and $\mathbf{a}^-$ are parallel to $\mathbf{H}^+$ and $\mathbf{H}^-$ respectively (see Fig. 1). Then $\mathbf{w}^\pm$ can be written as

$$\mathbf{w}^\pm = w_a^\pm \mathbf{a}^\pm + w_b^\pm \mathbf{b}^\pm + w_c^\pm \mathbf{c}^\pm,$$

and the reduced system becomes

$$\begin{aligned}
&p_t + \mathbf{a}\cdot\nabla w_a + \mathbf{b}\cdot\nabla w_b + \mathbf{c}\cdot\nabla w_c = 0,\\
&r_t + \mathbf{b}\cdot\nabla w_b + \mathbf{c}\cdot\nabla w_c = 0,\\
&w_{at} + 2\mathbf{a}\cdot\nabla w_a + (\beta-1)\mathbf{a}\cdot\nabla p + \mathbf{a}\cdot\nabla r = 0,\\
&w_{bt} + 2\mathbf{a}\cdot\nabla w_b + \beta\mathbf{b}\cdot\nabla p + \mathbf{b}\cdot\nabla r = 0,\\
&w_{ct} + 2\mathbf{a}\cdot\nabla w_c + \beta\mathbf{c}\cdot\nabla p + \mathbf{c}\cdot\nabla r = 0 \quad \text{in} \quad \mathbb{R}^3_\pm;\\
&[p] = 0, \quad [r] = 0, \quad \left[\left(w_a - \frac{p}{2}\right)\mathbf{a} + w_b\mathbf{b} + w_c\mathbf{c}\right] = 0 \quad \text{at} \quad x_1 = 0,
\end{aligned} \tag{19}$$

or, in matrix form,

$$\mathbf{U}_t + \sum_{k=1}^{3} A_k \mathbf{U}_{x_k} = 0 \quad \text{in} \quad \mathbb{R}^3_\pm, \tag{20}$$

where $\mathbf{U} = (p, r, w_a, w_b, w_c)$ and

$$A_k = \begin{pmatrix} 0 & 0 & a_k & b_k & c_k \\ 0 & 0 & 0 & b_k & c_k \\ (\beta-1)a_k & a_k & 2a_k & 0 & 0 \\ \beta b_k & b_k & 0 & 2a_k & 0 \\ \beta c_k & c_k & 0 & 0 & 2a_k \end{pmatrix}. \tag{21}$$

Boundary conditions (18) can be written as

$$\mathbf{U}^+ = B\mathbf{U}^- \quad \text{at} \quad x_1 = 0, \tag{22}$$

where

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1-(\mathbf{a}^+\cdot\mathbf{a}^-)}{2} & 0 & (\mathbf{a}^+\cdot\mathbf{a}^-) & (\mathbf{a}^+\cdot\mathbf{b}^-) & (\mathbf{a}^+\cdot\mathbf{c}^-) \\ -\frac{(\mathbf{b}^+\cdot\mathbf{a}^-)}{2} & 0 & (\mathbf{b}^+\cdot\mathbf{a}^-) & (\mathbf{b}^+\cdot\mathbf{b}^-) & (\mathbf{b}^+\cdot\mathbf{c}^-) \\ -\frac{(\mathbf{c}^+\cdot\mathbf{a}^-)}{2} & 0 & (\mathbf{c}^+\cdot\mathbf{a}^-) & (\mathbf{c}^+\cdot\mathbf{b}^-) & (\mathbf{c}^+\cdot\mathbf{c}^-) \end{pmatrix}.$$

C. Normal mode analysis. We seek solutions of the reduced problem in the form of normal modes

$$\mathbf{U} = \begin{cases} \hat{\mathbf{U}}^+ e^{k(\sigma t + \lambda^+ x_1 + i(\mathbf{q}\cdot\mathbf{x}))}, & x_1 > 0 \\ \hat{\mathbf{U}}^- e^{k(\sigma t + \lambda^- x_1 + i(\mathbf{q}\cdot\mathbf{x}))}, & x_1 < 0 \end{cases}.$$

Here k is a positive real constant (the magnitude of the wave vector in the plane parallel to the front of the discontinuity), $\mathbf{q} = (0, \cos\psi, \sin\psi)$ (ψ is the angle between the wave vector and the x_2 axis), $\lambda^\pm$ are constants satisfying the conditions

$$\mathrm{Re}(\lambda^+) < 0 \quad \text{and} \quad \mathrm{Re}(\lambda^-) > 0, \quad \text{if} \quad \mathrm{Re}(\sigma) > 0. \tag{23}$$

Substituting these in Eqs. (20), we obtain

$$\hat{A}^+ \mathbf{U}_0^+ = 0, \quad \hat{A}^- \mathbf{U}_0^- = 0,$$

where

$$\hat{A}^\pm = \sigma I + \lambda^\pm A_1^\pm + i \left(\cos\psi A_2^\pm + \sin\psi A_3^\pm \right). \tag{24}$$

These linear systems have nontrivial solutions if

$$\det \hat{A}^+ = 0, \quad \det \hat{A}^- = 0.$$

Hence, we obtain the dispersion relations for λ^+ and λ^-:

$$F^+(\lambda^+, \sigma) = 0, \quad F^-(\lambda^-, \sigma) = 0.$$

where

$$F^\pm(\lambda^\pm, \sigma) = K_1^\pm \left\{ \beta\sigma K_2^\pm \left[1 - (\lambda^\pm)^2 \right] + (K_1^\pm)^2 \left[1 - (\lambda^\pm)^2 + (K_1^\pm)^2 \right] \right\},$$

are polynomials of degree 5 in $\lambda^\pm$. Here

$$K_1^\pm = \sigma + N^\pm, \quad K_2^\pm = \sigma + 2N^\pm,$$
$$N^+ = \lambda^+ \cos\theta + i \sin\theta \cos(\phi - \psi), \quad N^- = \lambda^- \cos\theta + i \sin\theta \cos\psi.$$

It can be shown using Hersh's lemma [14] that if $\mathrm{Re}(\sigma) > 0$, both polynomials have 4 roots with $\mathrm{Re}(\lambda^\pm) < 0$ and 1 root with $\mathrm{Re}(\lambda^\pm) > 0$ (see Appendix). If these roots are simple, we have four linearly independent solutions of the equation $\hat{A}^+ \mathbf{U}_0^+ = 0$ and one solution of the equation $\hat{A}^- \mathbf{U}_0^- = 0$, satisfying condition (23). Let λ_k^+ ($k = 1, 2, 3, 4$) and λ^- be the roots corresponding to these solutions and let

$$\mathbf{U} = \begin{cases} \sum\limits_{k=1}^{4} C_k \hat{\mathbf{U}}_k^+ e^{k(\sigma t + \lambda_k^+ x_1 + i(\mathbf{q}\cdot\mathbf{x}))}, & x_1 > 0 \\ C_5 \hat{\mathbf{U}}^- e^{k(\sigma t + \lambda^- x_1 + i(\mathbf{q}\cdot\mathbf{x}))}, & x_1 < 0 \end{cases},$$

where $C_1, \ldots, C_5$ are arbitrary constants. Substituting this in the boundary condition, we obtain the linear system for $C_1, \ldots, C_5$

$$Q\mathbf{C} = 0,$$

where

$$Q = \left(\hat{\mathbf{U}}_1^+ \;\; \hat{\mathbf{U}}_2^+ \;\; \hat{\mathbf{U}}_3^+ \;\; \hat{\mathbf{U}}_4^+ \;\; -B\hat{\mathbf{U}}^- \right),$$

and $\mathbf{C} = (C_1, C_2, C_3, C_4, C_5)^t$. (Here the superscript t denotes transposition.)

The existence of nontrivial solutions requires that

$$L(\sigma, \beta, \theta, \phi, \psi) = \det Q = 0. \tag{25}$$

$L(\sigma, \beta, \theta, \phi, \psi)$ is called the Lopatinskii determinant [15]. For given β, θ, ϕ and ψ, equation (25) is an algebraic equation for σ. If for given values of β, θ and ϕ which characterize the Alfvén discontinuity, there is ψ ($\psi \in (-\pi, \pi]$) such that

$$\mathrm{Re}\,\{\sigma(\beta, \theta, \phi, \psi)\} > 0,$$

then *the corresponding Alfvén discontinuity is unstable, moreover, the growth rate can be arbitrarily large (*recall that the growth rate is $k\sigma$ with arbitrary positive k*), i.e. the linearised problem is ill-posed in the sense of Hadamard.*

In general, it is impossible to find zeros of the Lopatiskii determinant analytically. Therefore, we did this numerically using MATLAB.

D. Numerical results. To find zeros of $L(\sigma)$ for fixed β, θ and ϕ, we employed the secant method. It turned out that unstable modes ($\mathrm{Re}\sigma > 0$) exist for certain values of θ and ϕ and all finite values of β. Typically, an unstable mode corresponds to a narrow interval in ψ (recall that ψ is the angle between the positive direction of the x_2 axis and the direction of the wave vector in the x_2-x_3 plane). Typical curves of $\eta \equiv \mathrm{Re}(\sigma)$ versus ψ are shown in Fig. 2.

Once we have found unstable modes, we wanted to compute the instability domain in the plane of parameters θ and ϕ. To do this, we first fixed some values of β and θ and computed

$$\eta_m(\beta, \theta, \phi) = \max_{0 \le \psi \le \pi} \mathrm{Re}\,\{\sigma(\beta, \theta, \phi, \psi)\}.$$

for various values of $\psi \in [0, \pi]$. Typical curves of η_m versus ϕ are shown in Figure 3.

We then used linear interpolation to find ϕ^* at which η_m vanishes, this gave us the boundary of the interval of instability in ϕ. Then we varied β and θ and repeated the whole procedure again, and so on. The resulting instability domains in the ϕ-θ plane for some values of β are shown in Figure 4. For each curve in Figure 4, the unstable region is above the curve. One can see that the Alfvén discontinuity is unstable in a wide range of values of ψ as $\theta \to \pi/2$. When

θ decreases from $\pi/2$, the interval of values of ψ for which the Alfvén discontinuity is unstable shrinks. At certain $\theta = \theta^*$, this interval degenerates to a point, and there is no instability for $0 < \theta \le \theta^*$. For each fixed value of β, the maximum growth rate is attained when $\theta \to \pi/2$.

When for given θ and ϕ we vary β from 0 to ∞, the growth rate increases form zero, attains its maximum value at some $\beta = \beta^*$ and then decreases monotonically to zero as $\beta \to \infty$. Typical curves for some fixed θ, ϕ and ψ are shown in Fig. 5. Note that $\beta = \infty$ corresponds to the incompressible fluid. It is well-known [9] and can be shown independently by the energy method that incompressible Alfvén discontinuities are always stable, which agrees with our numerical results. In Reference 12, the Lopatinskii determinant for the reduced problem had been analyzed in the limit case of magnetically dominated flows ($\beta \to 0$) and its zero had been found. It can be shown however that the root of the Lopatinskii determinant found in Reference 12 corresponds to a double root of the dispersion relation for λ^+ and is therefore a fictitious zero of the Lopatinskii determinant.

As was mentioned above, the maximum growth rate corresponds to the limit as $\theta \to \pi/2$. Note that the limit case $\theta = \pi/2$ corresponds to a particular case of a tangential discontinuity. Our results suggest that this tangential discontinuity (that can be treated as the degenerate case of Alfvén discontinuity) is unstable in the class of flows with Alfvén discontinuities.

III Conclusion

We have studied the stability of the planar Alfvén discontinuity to small perturbations and have shown that, for certain values of the angle θ that the magnetic field makes with the normal to the front and the angle ϕ of rotation of the magnetic field across the front, the planar Alfvén discontinuity is unstable. Moreover, the growth rate can be made arbitrarily large, and this means that the corresponding linearized initial-boundary-value problem is ill-posed in the sense of Hadamard. This means that the corresponding Alfvén discontinuity cannot exist for any finite, however short, time interval. We have determined numerically the instability domain in the space of the parameters of the Alfvén discontinuity. We should point out here that, although we did not find unstable modes outside the instability domain described above (and we believe that the Alfvén discontinuity is stable there), we did not formally prove the stability of the Alfvén discontinuity outside this instability domain.

We have found that the maximum growth rate corresponds to the limit case $\theta \to \pi/2$. In this limit, the Alfvén discontinuity degenerates into a tangential discontinuity. Our results show that this tangential discontinuity is unstable to perturbations which belong to the class of flows with Alfvén discontinuities.

It is unclear at the moment whether such a strong instability as described in this paper can be suppressed by the effects of viscosity and/or resistivity or by the Hall effect. Although there are examples which show the existence of viscous, resistive analogues of Alfvén discontinuities [16] and the possibility of regularization of Alfvén discontinuities in the Hall magnetohydrodynamics [17], nothing is known about the stability of these regularized solutions. One can speculate that for sufficiently small viscosity and resistivity or sufficiently weak Hall effect, the instability found in this paper would persist, but this is a problem for a further investigation.

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Appendix: Hersh's lemma and the roots of polynomials $F^{\pm}$

Here we show that polynomials $F^{\pm}$ have exactly 4 roots with $\mathrm{Re}(\lambda^{\pm}) < 0$ and 1 root with $\mathrm{Re}(\lambda^{\pm}) > 0$ provided that

$$\mathrm{Re}(\sigma) > 0. \tag{26}$$

First we note that matrices $A_1^{\pm}$ are non-singular. Then we define matrices $M^{\pm}$ by the formulas:

$$M^{\pm} = \left(A_1^{\pm}\right)^{-1} \left(\sigma I + i \left(\cos\psi A_2^{\pm} + \sin\psi A_3^{\pm}\right)\right).$$

It follows from Eq. (24) and the definition of $F^{\pm}$ ($F^{\pm} = \det \hat{A}^{\pm}$) that if $\mu^{\pm}$ are eigenvalues of $M^{\pm}$, then $\lambda^{\pm} = -\mu^{\pm}$ are roots of $F^{\pm}$. Hence, all we need to show is that, under the condition (26), both M^{+} and M^{-} have four eigenvalues with positive real part and one eigenvalue with negative real part.

Hersh's lemma [14] can be formulated as follows: *let B_1, B_2 and B_3 be real $(n \times n)$ matrices and let B_1 be non-singular with k positive and $(n-k)$ negative eigenvalues. If for arbitrary real ξ_1, ξ_2 and ξ_3 the matrix $\xi_1 B_1 + \xi_2 B_2 + \xi_3 B_3$ has only real eigenvalues, then for all σ satifying (26) and all real η_2, η_3 the matrix*

$$M(\sigma, \eta_2, \eta_3) = B_1^{-1} \left(\sigma I + i(\eta_2 B_2 + \eta_3 B_3)\right)$$

has k eigenvalues with positive real part and $(n-k)$ with negative real part.

It follows from Hersh's lemma that the required property holds provided that (i) for arbitrary real ξ_1, ξ_2 and ξ_3 the matrices $G^{\pm} = \xi_1 A_1^{\pm} + \xi_2 A_2^{\pm} + \xi_3 A_3^{\pm}$ have only real eigenvalues and (ii) both A_1^+ and A_1^- have four eigenvalues with positive real part and one eigenvalue with negative real part. Note that condition (i) is exactly the hyperbolicity condition for system (20). Below we drop the superscripts "+" and "-" to simplify the notation. We have

$$G = \begin{pmatrix} 0 & 0 & \xi_a & \xi_b & \xi_c \\ 0 & 0 & 0 & \xi_b & \xi_c \\ (\beta-1)\xi_a & \xi_a & 2\xi_a & 0 & 0 \\ \beta\xi_b & \xi_b & 0 & 2\xi_a & 0 \\ \beta\xi_c & \xi_c & 0 & 0 & 2\xi_a \end{pmatrix},$$

where $\xi_a = \boldsymbol{\xi}\cdot\mathbf{a}$, $\xi_b = \boldsymbol{\xi}\cdot\mathbf{b}$, $\xi_c = \boldsymbol{\xi}\cdot\mathbf{c}$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$. Eigenvalues of G are

$$2\xi_a, \quad \xi_a \pm \frac{1}{2}\sqrt{2(1+\beta)(\xi_a^2+\xi_b^2+\xi_c^2)+2\sqrt{D}}, \quad \xi_a \pm \frac{1}{2}\sqrt{2(1+\beta)(\xi_a^2+\xi_b^2+\xi_c^2)-2\sqrt{D}},$$

where

$$D = (\beta^2+1)(\xi_a^2+\xi_b^2+\xi_c^2)^2 + 2\beta((\xi_b^2+\xi_c^2)^2 - \xi_a^4).$$

These eigenvalues are all real if $D \geq 0$ and $(1+\beta)(\xi_a^2+\xi_b^2+\xi_c^2) - \sqrt{D} \geq 0$, and it can be shown that the last two inequalities are satisfied for all real ξ_a, ξ_b and ξ_c and all positive β.

Eigenvalues of A_1^+ and A_1^- are the same and given by

$$2cos\theta, \quad cos\theta \pm \frac{1}{2}\sqrt{2(1+\beta)+2\sqrt{(1+\beta)^2-4\beta\cos^2\theta}}, \quad cos\theta \pm \frac{1}{2}\sqrt{2(1+\beta)-2\sqrt{(1+\beta)^2-4\beta\cos^2\theta}}.$$

It is not difficult to show that for arbitrary $\beta > 0$ and $\theta \in (0, \pi/2)$ all these eigenvalues are real, four of them being positive and one, namely, $cos\theta - \frac{1}{2}\sqrt{2(1+\beta)+2\sqrt{(1+\beta)^2-4\beta\cos^2\theta}}$, negative. This concludes the proof.

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Figure captions

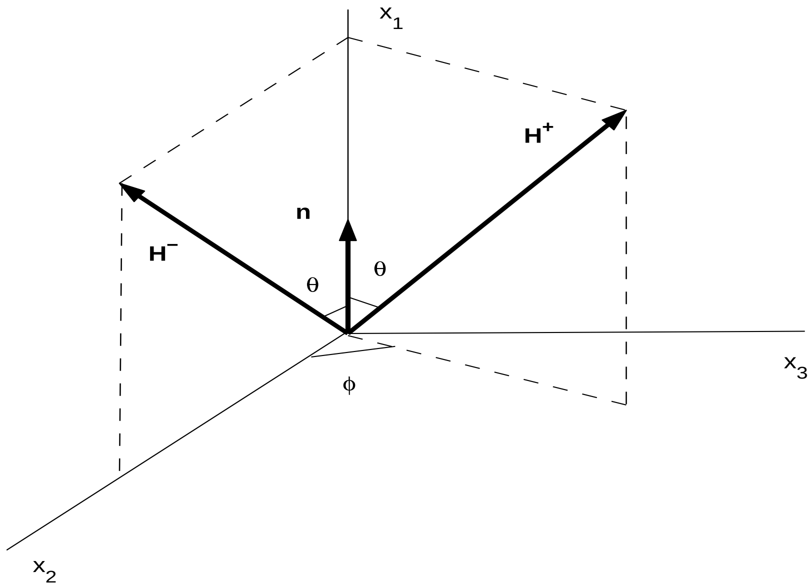
Figure 1. Schematic diagram of the planar Alfvén discontinuity.

Figure 2. Typical graphs of $\mathrm{Re}(\sigma)$ as function of ψ.

Figure 3. $\eta_m = \max\limits_{\psi} \mathrm{Re}(\sigma)$ as function of ϕ.

Figure 4. Instability domain in ϕ-θ plane.

Figure 5. $\mathrm{Re}(\sigma)$ as function of $\beta^{1/2}$ for $\phi = \pi$, $\psi = \pi$.



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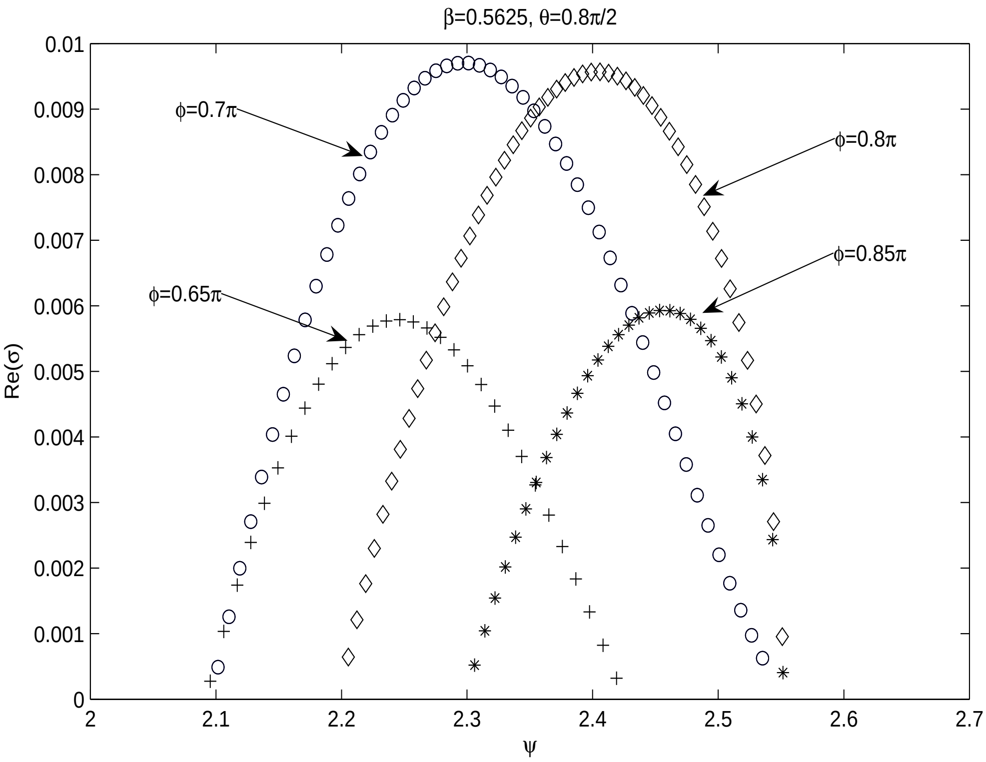

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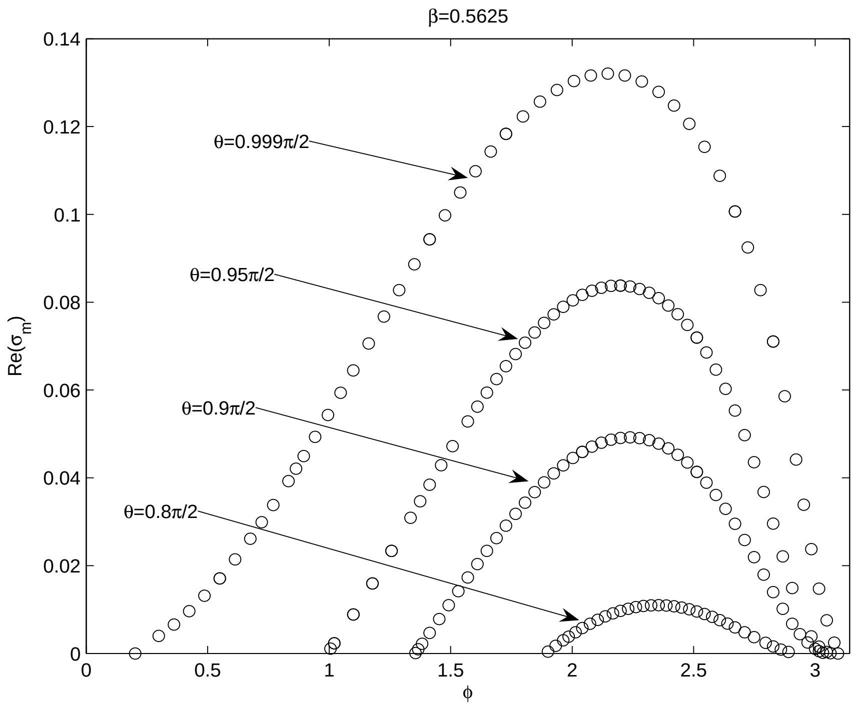

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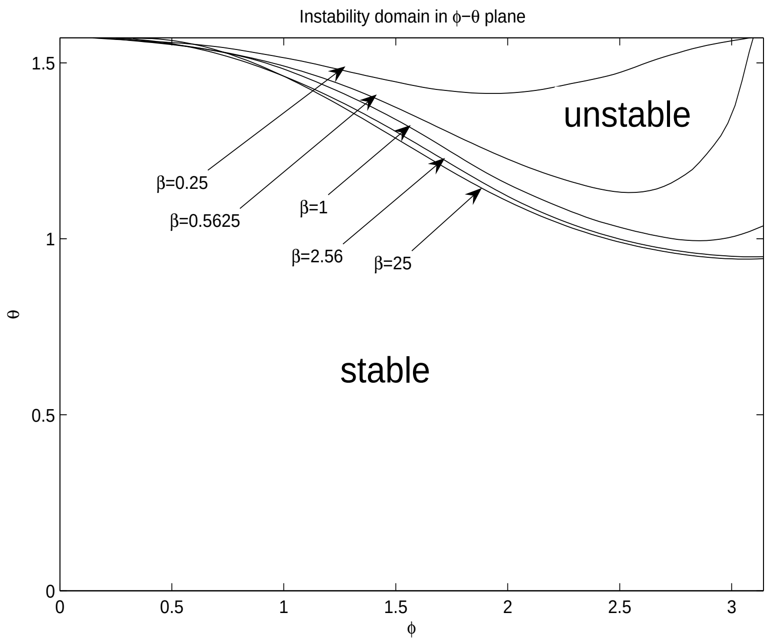

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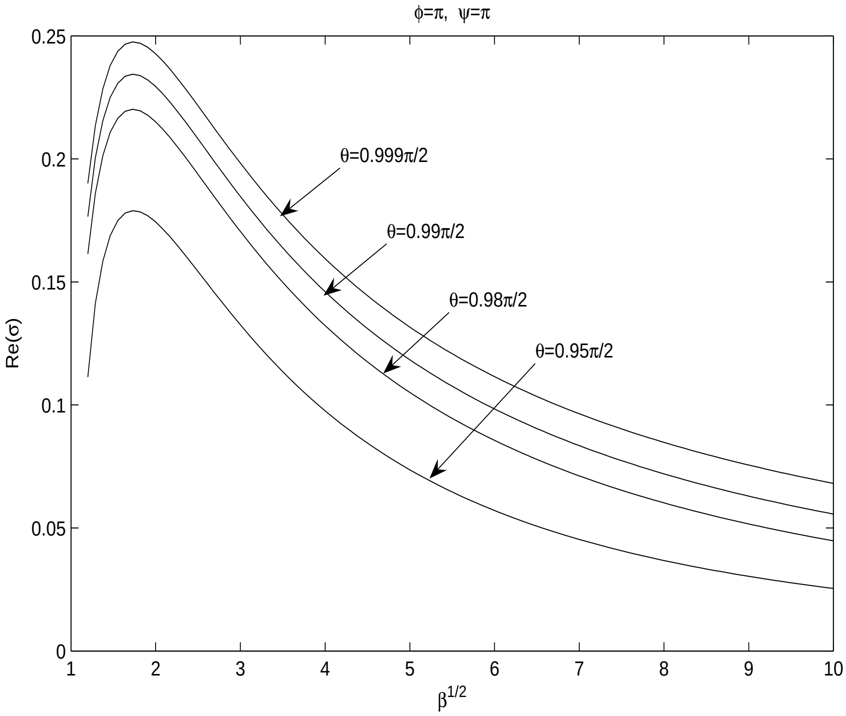

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