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Funded by: EPSRC; Grant Number: GR/S96609/02

Summary

We study Alfvén discontinuities for the equations of ideal compressible magnetohydrodynamics (MHD). By numerical testing of the Lopatinskii condition we find the parameter domains of violent instability of planar Alfvén discontinuities. We also show that Alfvén discontinuities can be only weakly (neutrally) stable. The similar situation takes place for Alfvén discontinuities in incompressible MHD flows whose linear stability was established long time ago. For planar incompressible Alfvén discontinuities we prove an energy a priori estimate that exhibits a big loss of regularity.

1 Introduction

We consider the MHD equations governing the motion of an ideal (inviscid and perfectly conducting) compressible fluid. They can be written as the quasilinear system of conservation laws (see, e.g. Reference [1]):

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{v}) = 0\,, \qquad \partial_t(\rho\mathbf{v}) + \operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v} - \mathbf{H}\otimes\mathbf{H}) + \nabla q = 0\,, \\ \partial_t\mathbf{H} - \nabla\times(\mathbf{v}\times\mathbf{H}) = 0\,, \\ \partial_t\left(\rho E + (1/2)(\rho|\mathbf{v}|^2 + |\mathbf{H}|^2)\right) + \operatorname{div}\left(\rho\mathbf{v}(E + (1/2)|\mathbf{v}|^2 + pV) + \mathbf{H}\times(\mathbf{v}\times\mathbf{H})\right) = 0\,, \end{cases} \tag{1}$$

where $\rho = \rho(t,\mathbf{x})$, $\mathbf{v} = \mathbf{v}(t,\mathbf{x}) = (v_1, v_2, v_3)$, $\mathbf{H} = \mathbf{H}(t,\mathbf{x}) = (H_1, H_2, H_3)$, $p = p(t,\mathbf{x})$ are the density, the fluid velocity, the magnetic field, and the pressure respectively, $q = p + (1/2)|\mathbf{H}|^2$ is the total pressure, $E = E(\rho, S)$ is the internal energy, $S = S(t,\mathbf{x})$ is the entropy, $V = 1/\rho$

is the specific volume, t is the time, and $\mathbf{x} = (x_1, x_2, x_3)$ are space variables. Equations (1) are supplemented by the equation of state of the medium, $E = E(\rho, S)$. Then we have a closed system of equations for the vector of unknowns $\mathbf{U} = \mathbf{U}(t, \mathbf{x}) = (p, \mathbf{v}, \mathbf{H}, S)$. The equation

$$\operatorname{div} \mathbf{H} = 0 \tag{2}$$

(that expresses the absence of magnetic charges) can be treated as an additional requirement for the initial data $\mathbf{U}(0, \mathbf{x}) = \mathbf{U}_0(\mathbf{x})$.

It is known that the MHD equations (1), written (taking into account (2)) in the form

$$\frac{1}{\rho c^2}\frac{dp}{dt} + \operatorname{div} \mathbf{v} = 0, \qquad \rho\frac{d\mathbf{v}}{dt} - (\mathbf{H}, \nabla)\mathbf{H} + \nabla q = 0,$$
$$\frac{d\mathbf{H}}{dt} - (\mathbf{H}, \nabla)\mathbf{v} + \mathbf{H}\operatorname{div} \mathbf{v} = 0, \qquad \frac{dS}{dt} = 0$$

can be put in the form of a symmetric t-hyperbolic system in the sense of Friedrichs,

$$A_0(\mathbf{U})\partial_t\mathbf{U} + \sum_{k=1}^{3} A_k(\mathbf{U})\partial_k\mathbf{U} = 0, \tag{3}$$

if the hyperbolicity condition $A_0 > 0$ holds:

$$\rho > 0, \quad c^2 > 0.$$

Here $c^2 = \partial_\rho(\rho^2\partial_\rho E)$ is the square of the sound speed, $d/dt = \partial_t + (\mathbf{v}, \nabla)$, $\partial_k = \partial/\partial x_k$, and the symmetric matrices A_α can be easily written down, in particular, $A_0 = \operatorname{diag}(1/(\rho c^2), \rho, \rho, \rho, 1, 1, 1, 1)$.

Let $\Gamma(t) = \{x_1 - f(t, \mathbf{y}) = 0\}$ be a smooth hypersurface in $\mathbb{R} \times \mathbb{R}^3$, where $\mathbf{y} = (x_2, x_3)$ are tangential coordinates. We assume that $\Gamma(t)$ is a surface of strong discontinuity for solutions of the MHD system. There are four types of MHD discontinuities (see, e.g. References [1, 2]). Namely, in addition to MHD shock waves ($j \neq 0$, $[\rho] \neq 0$) there are three types of characteristic discontinuities: tangential discontinuities or current-vortex sheets ($j = 0$, $H_{\mathrm{N}}^{+} = 0$), contact discontinuities ($j = 0$, $H_{\mathrm{N}}^{+} \neq 0$), and Alfvén or rotational discontinuities ($j \neq 0$, $[\rho] = 0$). Here $[g] = g^+ - g^-$ denotes the jump for every regularly discontinuous function g with corresponding values behind ($g^+ := g|_{x_1 - f(t,\mathbf{y}) \to +0}$) and ahead ($g^- := g|_{x_1 - f(t,\mathbf{y}) \to -0}$) of the discontinuity front, $j = j^{\pm} = \rho^{\pm}(v_{\mathrm{N}}^{\pm} - \partial_t f)$ is the mass flux across the discontinuity, $v_{\mathrm{N}} = (\mathbf{v}, \mathbf{N})$, $H_{\mathrm{N}} = (\mathbf{H}, \mathbf{N})$, and $\mathbf{N} = (1, -\partial_2 f, -\partial_3 f)$ is the vector normal to $\Gamma(t)$. Note that, in view of the MHD Rankine-Hugoniot conditions [1], $[j] = 0$ and $[H_{\mathrm{N}}] = 0$.

In this paper, we are interested in *Alfvén discontinuities.* Note that Alfvén discontinuities play an important role in solving the MHD Riemann problem [1, 2] and, respectively, in numerical simulations of compressible MHD flows (see, e.g. Reference [3]). For Alfvén discontinuities the general MHD Rankine-Hugoniot conditions reduce to the equations (see [1]):

$$\left\{\begin{array}{l} [p] = 0, \quad [S] = 0, \quad [H_{\mathrm{N}}] = 0, \\[2mm] \big[|\mathbf{H}|^2\big] = 0, \quad \left[\mathbf{v} - \dfrac{\mathbf{H}}{\sqrt{\rho}}\right] = 0, \quad j^+ = H_{\mathrm{N}}^{+}\sqrt{\rho^+}. \end{array}\right. \tag{4}$$

Observe that $H_{\mathrm{N}}^{\pm} \neq 0$ ($j \neq 0$), and the condition $[\rho] = 0$ automatically follows from the first two conditions in (4). That is, the density, the pressure, and the entropy are continuous. Moreover, the vector of magnetic field *rotates* on the Alfvén discontinuity whereas its absolute value has no jump. The Alfvén discontinuity is a *characteristic* discontinuity (see Section 2) but, as for shock waves, the plasma crosses its front. For this reason Alfvén (or rotational) discontinuities are sometimes called Alfvén shocks [2].

The initial boundary value problem for system (3) in the domains $\Omega^{\pm}(t) := \{x_1 \gtrless f(t, \mathbf{y})\}$ with the boundary conditions (4) on the hypersurface $\Gamma(t)$ is a free boundary problem. Indeed, the function $f(t, \mathbf{y})$ defining Γ is one of the unknowns of problem (3), (4) with the corresponding initial data

$$f(0, \mathbf{y}) = f_0(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^2; \quad \mathbf{U}(0, \mathbf{x}) = \mathbf{U}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega^{\pm}(0). \tag{5}$$

As for the Cauchy problem, for problem (3)–(5) constraint (2) in the domains $\Omega^{\pm}(t)$ can be regarded as the restriction only on the initial data (5) (see Section 2 for the proof).

Definition 1.1 A weak solution $\mathbf{U}(t, \mathbf{x})$ of the MHD equations (3) is called *solution with Alfvén discontinuity* if there exists a smooth hypersurface $\Gamma(t)$ such that $\mathbf{U}$ is a classical solution of (1) on either side of Γ and the jump conditions (4) (together with the requirement $j \neq 0$) hold at each point of Γ.

Piecewise constant solutions of (3) satisfying (4) on a *planar* discontinuity (e.g. with the equation $x_1 = 0$) are simplest solutions with Alfvén discontinuity. Such solutions clearly exist but if solutions that are close to them do not, then the flow with a planar Alfvén discontinuity is not physically realizable as well (in the framework of ideal MHD). From the physical point of view, it is also very important to know whether, under appropriate conditions, there exist solutions with an arbitrary curved Alfvén discontinuity. To prove the existence of such weak solutions of the MHD equations, one needs to answer the following question: does a solution $(\mathbf{U}, f)$ of problem (3)–(5) exist?

Before any attempt to prove the local-in-time existence of solutions of the nonlinear free boundary problem (3)–(5), a linearized problem associated with (3)–(5) should be considered. The linearized problem for a planar discontinuity is a constant coefficient problem. We prove that the *uniform* Kreiss-Lopatinskii condition [4, 5, 6, 7] is always violated for this problem, i.e. planar Alfvén discontinuities can be only weakly (neutrally) stable. This is a direct consequence of the fact that the symbol associated with the front of Alfvén discontinuity is not elliptic (see Section 3).

The present work is an extension of an earlier work by the present authors [8], where by testing the Lopatinskii condition numerically, we have found the domains of violent instability of planar

Alfvén discontinuities, i.e. the parameter domains of ill-posedness of the linearized problem. The numerical results indicate that there is no instability outside these domains, which corresponds to (weak) stability of the planar Alfvén discontinuity. The only previous work on the stability of planar compressible Alfvén discontinuities is Reference [9]. The results of our calculations show, in particular, that the conclusion in Reference [9] about the violent instability of planar Alfvén discontinuities for the case of an asymptotically strong magnetic field was not quite correct. That is, for this case Alfvén discontinuities are not always unstable and can be weakly stable for some angles determining the rotation of the magnetic field on the discontinuity front. The conclusion about instability in Reference [9] was caused by a difficulty associated with so-called *glancing modes* [4, 10] in the MHD system. As we will see, glancing modes are indeed the source of difficulties in the normal mode analysis for Alfvén discontinuities (see Section 3).

It should be underlined that the study of the stability of planar Alfvén discontinuities is of independent interest in connection with astrophysical and geophysical applications such as, for example, the model of *magnetopause* (see Reference [11] and references therein). The magnetopause is a boundary of complex structure between the Earth's magnetosphere and the solar wind. The magnetopause structure is locally classified as closed (or nightside) and open (or dayside) magnetopause. These types of magnetopause structure are usually treated as a current-vortex sheet and an Alfvén discontinuity respectively [11]. Besides, the linear stability of a planar Alfvén discontinuity is interpreted as the macroscopic stability of the open magnetopause. Note that a sufficient condition for the weak stability of a planar compressible current-vortex sheet was recently found in Reference [12] (see also Reference [13] and references therein for the case of incompressible fluid).

The question on the local-in-time existence of nonplanar weakly stable Alfvén discontinuities still remains open. Until recent times even the general question on the possibility of the existence of solutions of hyperbolic conservation laws with a surface of weakly stable discountinuity was open, but now we know at least three examples of the positive answer to this question. The first two examples are 2D supersonic vortex sheets and weakly stable shock waves in isentropic gas dynamics. The existence of these weakly stable discountinuties was recently proved by Coulombel and Secchi [14]. The third recent example is above mentioned compressible current-vortex sheets whose local-in-time existence was shown by one of the present authors [15] provided that the stability condition from Reference [12] is satisfied at each point of the initial discontinuity.

Since the symbol associated with the front of Alfvén discontinuity is not elliptic, weakly stable Alfvén discontinuities are in some sense less stable than discontinuties studied in References [14] and [15]. We show this on the example of Alfvén discontinuity in an *incompressible* fluid. The linear stability of planar Alfvén discontinuities in incompressible MHD was proved long time ago by Syrovatskii [16] (see also Section 4 and Appendix A). For incompressible Alfvén discontinuities the symbol associated with the front is also not elliptic, i.e. stability is of weak type. The energy a priori estimate that we prove for the constant coefficients problem for planar incompressible Alfvén discontinuities exhibits however a big loss of regularity (in particular, the loss of two derivatives from the initial data). It is still unclear whether such an estimate can be generalized to the problem with variable coefficients. It is quite possible that the "non-ellipticity of the front" is such a strong thing that it can even prevent the existence of slightly curved discontinuities. This question needs an additional investigation.

2 The linearized problem associated with (3)–(5)

2.1 The reduction to fixed domains

To deal with fixed domains rather that the time-dependent domains $\Omega^{\pm}(t)$, we make the usual change of variables [5]:

$$\widetilde{t} = t\,, \quad \widetilde{x}_1 = x_1 - f(t,\mathbf{y})\,, \quad \widetilde{\mathbf{y}} = \mathbf{y}\,. \tag{6}$$

Then, $\widetilde{\mathbf{U}}(\widetilde{t},\widetilde{\mathbf{x}}) := \mathbf{U}(t,\mathbf{x})$ is a smooth vector-function for $\widetilde{\mathbf{x}} \in \mathbb{R}^3_{\pm}$, and the initial boundary value problem (3)–(5) is reduced to the following problem (we omit tildes to simplify the notation):

$$L(\mathbf{U},\mathbf{F})\mathbf{U} = 0 \quad \text{in } [0,T]\times(\mathbb{R}^3_+ \cup \mathbb{R}^3_-)\,, \tag{7}$$

$$\begin{cases} [p] = 0\,, \quad [S] = 0\,, \quad [H_{\rm N}] = 0\,, \quad \big[|\mathbf{H}|^2\big] = 0\,, \\ \left[\mathbf{v} - \dfrac{\mathbf{H}}{\sqrt{\rho}}\right] = 0\,, \quad \partial_t f = v_{\rm N}^+ - \dfrac{H_{\rm N}^+}{\sqrt{\rho^+}} \end{cases} \quad \text{on } [0,T]\times\{x_1=0\}\times\mathbb{R}^2, \tag{8}$$

$$\mathbf{U}|_{t=0} = \mathbf{U}_0 \quad \text{in } \mathbb{R}^3_+ \cup \mathbb{R}^3_-, \qquad f|_{t=0} = f_0 \quad \text{in } \mathbb{R}^2. \tag{9}$$

Here

$$L = L(\mathbf{U},\mathbf{F}) = A_0(\mathbf{U})\partial_t + A_\nu(\mathbf{U},\mathbf{F})\partial_1 + \sum_{k=2}^{3} A_k(\mathbf{U})\partial_k\,, \quad \mathbf{F} = (\partial_t f, \partial_2 f, \partial_3 f)\,,$$

$$A_\nu = A_\nu(\mathbf{U},\mathbf{F}) = \sum_{\alpha=0}^{3} \nu_\alpha A_\alpha = A_1(\mathbf{U}) - A_0(\mathbf{U})\partial_t f - \sum_{k=2}^{3} A_k(\mathbf{U})\partial_k f,$$

$\boldsymbol{\nu} = (\nu_0,\ldots,\nu_3) = (-\partial_t f, \mathbf{N})$ is the space-time normal vector to $\Gamma(t)$, and A_ν is the so-called *boundary matrix*.

The matrix $A_0^{-1}A_\nu$ has the following eigenvalues, $\lambda_1 \leq \ldots \leq \lambda_8$ (see, e.g. Reference [7] for the planar discontinuity, $f \equiv 0$):

$$\lambda_{1,8} = (v_{\rm N} - \partial_t f) \mp c_{\rm f}\,, \quad \lambda_{2,7} = (v_{\rm N} - \partial_t f) \mp c_{\rm a}\,, \quad \lambda_{3,6} = (v_{\rm N} - \partial_t f) \mp c_{\rm s}\,, \quad \lambda_{4,5} = v_{\rm N} - \partial_t f\,.$$

Here $c_{\rm a} = H_{\rm N}/\sqrt{\rho}$ is the Alfvén velocity in the direction normal to the discontinuity front,

$$c_{\rm f} = \frac{1}{\sqrt{2}}\sqrt{b^2 + c^2 + \sqrt{(b^2+c^2)^2 - 4c_{\rm a}^2c^2}}\,, \quad c_{\rm s} = \frac{1}{\sqrt{2}}\sqrt{b^2 + c^2 - \sqrt{(b^2+c^2)^2 - 4c_{\rm a}^2c^2}}$$

are the fast and slow magnetosonic velocities, $b^2 = |\mathbf{H}|^2/\rho$. In view of the boundary conditions (8), the Alfvén discontinuity moves with the Alfvén velocity $c_{\rm a}$. This implies $\lambda_2|_{x_1=\pm 0} = 0$. Hence, the boundary matrix A_ν is singular on the discontinuity front ($\det A_\nu|_{x_1=\pm 0} = 0$), and the Alfvén discontinuity is a *characteristic* discontinuity.

As is known, in general $c_{\rm s} \leq c_{\rm a} \leq c_{\rm f}$. For the Alfvén discontinuity, taking into account the condition $j \neq 0$, one has the strict inequalities $c_{\rm s} < c_{\rm a} < c_{\rm f}$. Therefore, the matrix $A_\nu|_{x_1=-0}$ has only one negative eigenvalue, and the matrix $A_\nu|_{x_1=+0}$ has exactly six positive eigenvalues. In other words, the hyperbolic system (7) in a neighborhood of the boundary $x_1 = 0$ has one outgoing characteric direction for $x_1 < 0$ and six outgoing characteristic directions for $x_1 > 0$. This means that this system requires seven boundary conditions on the boundary $x_1 = 0$. Moreover, one more boundary condition is needed for finding the front f. Thus, system (7) has the correct number of boundary conditions in (8), i.e. the Alfvén discontinuity is *evolutionary* [1, 7].

Proposition 2.1 Let $\mathbf{U}$ is a solution with Alfvén discontinuity (see Definition 1.1) on the time interval $[0,T]$. If condition (2) holds for the initial data (5), then $\mathbf{U}$ also satisfies (2) in $\Omega^\pm(t)$ for all $t \in [0,T]$.

Proof. Clearly, it is enough to prove the proposition for $\mathbf{U}$ written in the new variables (6). After the change of variables (6) constraint (2) takes the form $g = 0$, where $g = \operatorname{div}\mathbf{B}$, $\mathbf{B} = (H_{\rm N}, H_2, H_3)$. The equation for $\mathbf{H}$ contained in (7) reads

$$\partial_t \mathbf{H} + (\mathbf{u}, \nabla)\mathbf{H} - (\mathbf{B}, \nabla)\mathbf{v} + \mathbf{H}\operatorname{div}\mathbf{u} = 0\,, \tag{10}$$

where $\mathbf{u} = (v_{\rm N} - \partial_t f, v_2, v_3)$. After some algebra it follows from (10) that

$$\partial_t \mathbf{B} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u}\operatorname{div}\mathbf{B} = 0\,. \tag{11}$$

Acting on (11) by div, we obtain

$$\partial_t g + \operatorname{div}(g\,\mathbf{u}) = 0\,, \tag{12}$$

where the function $\mathbf{u}$ is piecewise smooth. Moreover, using the boundary conditions (8), from the first equation in system (11) one gets (we omit calculations)

$$[g]|_{x_1=0} = 0\,.$$

Now if $g|_{t=0} = 0$, then, by the standard method of characteristic curves, we obtain from equation (12) that $g = \operatorname{div}\mathbf{B} = 0$ for all $t \in [0,T]$. $\square$

2.2 The variable coefficients problem

Let $(\bar{\mathbf{U}}(t,\mathbf{x}), \bar{f}(t,\mathbf{y}))$ be a given vector-function (*basic state*), where $\bar{\mathbf{U}} = (\bar{p}, \bar{\mathbf{v}}, \bar{\mathbf{H}}, \bar{S})$ is supposed to be smooth for $\mathbf{x} \in \mathbb{R}^3_\pm$. We also suppose that the basic state $(\bar{\mathbf{U}}, \bar{f})$ satisfies (8) with a given function $\bar{\rho}(t,\mathbf{x})$ subject to the equation of state $E=E(\rho,S)$. In particular, it follows from the Gibbs relation that $\bar{p}=\bar{\rho}^2\partial_\rho E(\bar{\rho},\bar{S})$. Moreover, we assume that $\bar{\rho}>0$, $\bar{c}^2>0$, and $\bar{H}_{\rm N}^{\pm}\neq 0$, where $\bar{c}^2=\partial_\rho(\rho^2\partial_\rho E)(\bar{\rho},\bar{S})$, $\bar{H}_{\rm N}=(\bar{\mathbf{H}},\bar{\mathbf{N}})$, $\bar{\mathbf{N}}=(1,-\partial_2\bar{f},-\partial_2\bar{f})$, etc.

Then the linearization of (7)–(9) results in the following variable coefficients problem for perturbations $(\delta\mathbf{U},\delta f)$ (below we drop δ):

$$L(\bar{\mathbf{U}},\bar{\mathbf{F}})\mathbf{U}+\bar{C}\mathbf{U}=\big\{L(\bar{\mathbf{U}},\bar{\mathbf{F}})f\big\}\partial_1\bar{\mathbf{U}}\quad\text{in } [0,T]\times(\mathbb{R}^3_+\cup\mathbb{R}^3_-)\,, \tag{13}$$

$$\begin{cases} [p]=0\,,\quad [S]=0\,,\quad \big[(\bar{\mathbf{H}},\mathbf{H})\big]=0\,,\\[2mm] [H_{\rm N}]=[\bar{H}_2]\partial_2 f+[\bar{H}_3]\partial_3 f\,,\\[2mm] \left[\mathbf{v}-\dfrac{\mathbf{H}}{\sqrt{\bar{\rho}}}+\dfrac{\bar{\mathbf{H}}}{2\bar{\rho}\sqrt{\bar{\rho}}}\,\rho\right]=0\,,\\[2mm] \partial_t f=v_{\rm N}^+-\dfrac{H_{\rm N}^+}{\sqrt{\bar{\rho}^+}}-\bar{w}_2^+\partial_2 f-\bar{w}_3^+\partial_3 f+\dfrac{\bar{H}_{\rm N}^+}{2\bar{\rho}^+\sqrt{\bar{\rho}^+}}\,\rho^+ \end{cases}\quad \text{on } [0,T]\times\{x_1=0\}\times\mathbb{R}^2, \tag{14}$$

and the initial data for the perturbation $(\mathbf{U},f)$ coincide with (9). Here

$$\bar{\mathbf{F}}=(\partial_t\bar{f},\partial_2\bar{f},\partial_3\bar{f})\,,\quad v_{\rm N}=(\mathbf{v},\bar{\mathbf{N}})\,,\quad H_{\rm N}=(\mathbf{H},\bar{\mathbf{N}})\,,$$

$$\rho=\frac{1}{\bar{c}^2}\,p-\frac{\bar{\rho}^2\partial^2_{\rho S}E(\bar{\rho},\bar{S})}{\bar{c}^2}\,S\,,\quad \bar{w}_k=\bar{v}_k-\frac{\bar{H}_k}{\sqrt{\bar{\rho}}}\,,\quad k=2,3\,.$$

The matrix $\bar{C}=\bar{C}(\bar{\mathbf{U}},\partial_t\bar{\mathbf{U}},\nabla\bar{\mathbf{U}},\bar{\mathbf{F}})$ is determined as follows:

$$\bar{C}\mathbf{U}=(\mathbf{U},\nabla_u A_0(\bar{\mathbf{U}}))\partial_t\bar{\mathbf{U}}+(\mathbf{U},\nabla_u A_\nu(\bar{\mathbf{U}},\bar{\mathbf{F}}))\partial_1\bar{\mathbf{U}}+\sum_{k=2}^{3}(\mathbf{U},\nabla_u A_k(\bar{\mathbf{U}}))\partial_k\bar{\mathbf{U}}\,,$$

$(\mathbf{U},\nabla_u):=\sum_{i=1}^8 u_i\partial/\partial u_i$, $(u_1,\dots,u_8):=(p,\mathbf{v},\mathbf{H},S)$.

The boundary conditions (14) can be put in the form

$$\bar{B}\mathbf{F}+\bar{M}^+\mathbf{U}^++\bar{M}^-\mathbf{U}^-=0\,,$$

where $\mathbf{F}=(\partial_t f,\partial_2 f,\partial_3 f)$, and the matrices $\bar{B}(\bar{\mathbf{U}}^+,\bar{\mathbf{U}}^-)$ and $\bar{M}^\pm(\bar{\mathbf{U}}^\pm,\bar{\mathbf{F}})$ can be easily written out if necessary. Problem (13), (14) is the *genuine linearization* of (7), (8) in the sense that we keep all the lower order terms in (13).

It should be noted that the differential operator in system (13) is a first order operator in f. To avoid this inconvenience and to simplify system (13) we make the change of unknowns [17]

$$\dot{\mathbf{U}}=\mathbf{U}-f\partial_1\bar{\mathbf{U}}\,.$$

In terms of this "good unknown" problem (13), (14) takes the form (we omit dots to simplify the notation):

$$L(\bar{\mathbf{U}},\bar{\mathbf{F}})\mathbf{U}+\bar{C}\mathbf{U}=-f\partial_1\{L(\bar{\mathbf{U}},\bar{\mathbf{F}})\bar{\mathbf{U}}\}\qquad\text{in } [0,T]\times(\mathbb{R}^3_+\cup\mathbb{R}^3_-), \tag{15}$$

$$\bar{B}\mathbf{F}+\bar{M}^+\mathbf{U}^++\bar{M}^-\mathbf{U}^-+f\bar{Q}=0\qquad\text{on } [0,T]\times\{x_1=0\}\times\mathbb{R}^2\,, \tag{16}$$

where $\bar{Q} = \bar{M}^+(\partial_1\bar{\mathbf{U}})^+ + \bar{M}^-(\partial_1\bar{\mathbf{U}})^-$. If we assume that a solution to problem (7)–(9) exists and that the basic state $(\bar{\mathbf{U}}, \bar{f})$ is this solution, then the right hand side of equation (15) vanishes.

Using arguments similar to those in the proof of Proposition 2.1 we come to the following conclusion.

Proposition 2.2 Suppose that the basic state obeys the equation $\operatorname{div}\bar{\mathbf{B}} = 0$ for $\mathbf{x} \in \mathbb{R}^3_\pm$, where $\bar{\mathbf{B}} = (\bar{H}_{\rm N}, \bar{H}_2, \bar{H}_3)$. If the condition

$$\operatorname{div}\mathbf{B} = 0\,, \quad \mathbf{x} \in \mathbb{R}^3_\pm\,,$$

where $\mathbf{B} = (H_{\rm N}, H_2, H_3)$, holds for the initial data of problem (15), (16), then this condition is also satisfied for all $t \in [0, T]$.

2.3 The problem with constant coefficients for a planar discontinuity

For planar discontinuities $\bar{f}(t,\mathbf{y})$ is a linear function: $\bar{f}(t,\mathbf{y}) = \sigma t + (\boldsymbol{\gamma},\mathbf{y})$, $\boldsymbol{\sigma} = (\sigma, \boldsymbol{\gamma}) \in \mathbb{R}^3$. Without loss of generality we suppose that $\boldsymbol{\sigma} = 0$. Consider a piecewise constant solution of (3)–(5) for the planar Alfvén discontinuity with the equation $x_1 = 0$:

$$\bar{\mathbf{U}} = \begin{cases} \bar{\mathbf{U}}^+ = (\bar{p}, \bar{\mathbf{v}}^+, \bar{\mathbf{H}}^+, \bar{S})\,, & x_1 > 0\,, \\ \bar{\mathbf{U}}^- = (\bar{p}, \bar{\mathbf{v}}^-, \bar{\mathbf{H}}^-, \bar{S})\,, & x_1 < 0\,, \end{cases} \tag{17}$$

where, in view of (4), $\bar{\mathbf{H}}^\pm = (\bar{H}_1, \bar{H}_2^\pm, \bar{H}_3^\pm)$, $\bar{\mathbf{v}}^\pm = (\bar{v}_1, \bar{v}_2^\pm, \bar{v}_3^\pm)$, and the constants are related by

$$\left[\bar{H}_2^2 + \bar{H}_3^2\right] = 0\,, \quad \bar{v}_1 = \frac{\bar{H}_1}{\sqrt{\bar{\rho}}}\,, \quad [\bar{\mathbf{v}}] = \frac{[\bar{\mathbf{H}}]}{\sqrt{\bar{\rho}}}\,. \tag{18}$$

Constant $\bar{\rho}$ is positive. Without loss of generality we suppose also that $\bar{H}_1 > 0$.

The problem obtained by the linearization of (3), (4) on solution (17) coincides with problem (13), (14) if the coefficients of (13), (14) are "frozen", $\bar{C} = 0$, and the right-hand side of equation (13) vanishes. Moreover, $A_\nu = A_1$ and $\bar{\mathbf{N}} = (1, 0, 0)$, i.e. $H_{\rm N} = H_1$, $\bar{H}_{\rm N}^\pm = \bar{H}_1 > 0$, etc.

Following [9], we now make several useful simplifications of the linearized problem with constant coefficients. First, we perform the Galilean transformation

$$\tilde{t} = t\,, \quad \tilde{x}_1 = x_1\,, \quad \tilde{x}_k = x_k - \left(\bar{v}_k^+ - \frac{\bar{\mathbf{H}}_k^+}{\sqrt{\bar{\rho}}}\right)t\,, \quad k = 2, 3\,,$$

where, in view of (18), $\bar{v}_k^- - (\bar{\mathbf{H}}_k^-/\sqrt{\bar{\rho}}) = \bar{v}_k^+ - (\bar{\mathbf{H}}_k^+/\sqrt{\bar{\rho}})$. Second, we reduce the problem to a dimensionless form by introducing the following scaled values:

$$\tilde{t}' = \frac{\tilde{t}\bar{c}}{l}\,, \quad \tilde{\mathbf{x}}' = \frac{\mathbf{x}}{l}\,, \quad p' = \frac{p}{\bar{\rho}\bar{c}^2}\,, \quad \mathbf{v}' = \frac{\mathbf{v}}{\bar{c}}\,, \quad \mathbf{H}' = \frac{\mathbf{H}}{\bar{c}\sqrt{\bar{\rho}}}\,,$$

where l is a typical length. After performing the Galilean transformation and dropping the tildes and primes the linearized interior equations have the form

$$\begin{cases} \dfrac{d^\pm p}{dt} + \operatorname{div}\mathbf{v} = 0\,, & \dfrac{d^\pm \mathbf{v}}{dt} - (\mathbf{h}^\pm, \nabla)\mathbf{H} + \nabla\left(p + (\mathbf{h}^\pm, \mathbf{H})\right) = 0\,, \\ \dfrac{d^\pm \mathbf{H}}{dt} - (\mathbf{h}^\pm, \nabla)\mathbf{v} + \mathbf{h}^\pm \operatorname{div}\mathbf{v} = 0\,, & \dfrac{d^\pm S}{dt} = 0 \qquad \text{for} \quad \mathbf{x} \in \mathbb{R}^3_\pm\,, \end{cases} \tag{19}$$

where $d^{\pm}/dt = \partial_t + (\mathbf{h}^{\pm}, \nabla)$, $\mathbf{h}^{\pm} = (h_1, h_2^{\pm}, h_3^{\pm}) = \bar{\mathbf{H}}^{\pm}/(\bar{c}\sqrt{\bar{\rho}})$. Third, since the function $S(t, \mathbf{x})$ is a solution of the separate problem

$$\frac{d^{\pm}S}{dt} = 0 \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3_{\pm}\,, \qquad [S]|_{x_1=0} = 0\,, \qquad S|_{t=0} = S_0(\mathbf{x})\,,$$

the assumption that $S_0 \equiv 0$ leads to $S \equiv 0$. Even if $S_0 \neq 0$, the presence of S does not play any role in the normal mode analysis for the linearized problem with constant coefficients because it just creates a separate "stable block" in the Lopatinskii determinant (see the next section). So, without loss of generality we assume that $S \equiv 0$.

After all the above simplifications, the linearized problem for planar Alfvén discontinuities becomes:

$$\partial_t \mathbf{U} + \sum_{k=1}^{3} \bar{A}_k^{\pm} \partial_k \mathbf{U} = 0 \qquad \text{if} \quad \mathbf{x} \in \mathbb{R}^3_{\pm}\,, \tag{20}$$

$$\begin{cases} [p] = 0\,, \quad [(\mathbf{h}, \mathbf{H})] = 0\,, \quad [H_1] = [h_2]\partial_2 f + [h_3]\partial_3 f\,, \\ [\mathbf{v} - \mathbf{H} + (1/2)p\,\mathbf{h}] = 0\,, \quad \partial_t f = v_1^+ - H_1^+ + (1/2)p^+\mathbf{h}^+ \qquad \text{if} \quad x_1 = 0\,. \end{cases} \tag{21}$$

With a little abuse of notation, $\mathbf{U}$ now denotes the vector of (scaled) perturbations $(p, \mathbf{v}, \mathbf{H})$, and the symmetric matrices $\bar{A}_k^{\pm}$ are given by

$$\bar{A}_1^{\pm} = \begin{pmatrix} h_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & h_1 & 0 & 0 & 0 & h_2^{\pm} & h_3^{\pm} \\ 0 & 0 & h_1 & 0 & 0 & -h_1 & 0 \\ 0 & 0 & 0 & h_1 & 0 & 0 & -h_1 \\ 0 & 0 & 0 & 0 & h_1 & 0 & 0 \\ 0 & h_2^{\pm} & -h_1 & 0 & 0 & h_1 & 0 \\ 0 & h_3^{\pm} & 0 & -h_1 & 0 & 0 & h_1 \end{pmatrix},$$

$$\bar{A}_i^{\pm} = \begin{pmatrix} h_i^{\pm} & 0 & \delta_{2i} & \delta_{3i} & 0 & 0 & 0 \\ 0 & h_i^{\pm} & 0 & 0 & -h_i^{\pm} & 0 & 0 \\ \delta_{2i} & 0 & h_i^{\pm} & 0 & \delta_{2i}h_1 & -\delta_{3i}h_3^{\pm} & \delta_{2i}h_3^{\pm} \\ \delta_{3i} & 0 & 0 & h_i^{\pm} & \delta_{3i}h_1 & \delta_{3i}h_2^{\pm} & -\delta_{2i}h_2^{\pm} \\ 0 & -h_i^{\pm} & \delta_{2i}h_1 & \delta_{3i}h_1 & h_i^{\pm} & 0 & 0 \\ 0 & 0 & -\delta_{3i}h_3^{\pm} & \delta_{3i}h_2^{\pm} & 0 & h_i^{\pm} & 0 \\ 0 & 0 & \delta_{2i}h_3^{\pm} & -\delta_{2i}h_2^{\pm} & 0 & 0 & h_i^{\pm} \end{pmatrix}, \quad i = 2, 3\,.$$

System (20) represents equations (19) without the last equation for S . The matrices $\bar{A}_1^{+}$ and $\bar{A}_1^{-}$ have five positive eigenvalues, one negative and one zero eigenvalue (cf., Subsection 2.1). It is easy to show that the constraint $\operatorname{div}\mathbf{H} = 0$ is again only an additional requirement on the initial data for problem (20), (21) (cf. Propositions 2.1, 2.2).

3 Stability of planar discontinuities

3.1 The reduced problem with noncharacteristic boundary

Problem (20), (21) is a problem for a linear symmetric hyperbolic system with characteristic boundary. If by means of unitary transformations $U_\pm$ we reduce systems (20) to the form

$$\partial_t \mathbf{R} + \sum_{k=1}^{3} B_k^\pm \partial_k \mathbf{R} = 0 \,, \qquad \mathbf{x} \in \mathbb{R}^3_\pm \,,$$

then the Kreiss-Lopatinskii condition for (20), (21) could be introduced by a standard way proposed in Reference [18] for the case of characteristic boundary, where $\mathbf{U} = U_\pm \mathbf{R}$ for $\mathbf{x} \in \mathbb{R}^3_\pm$, $U_\pm U_\pm^* = I$, $B_k^\pm = U_\pm^* \bar{A}_k^\pm U_\pm$, and $B_1^\pm = \operatorname{diag}(0, b^\pm, B^\pm)$ are block diagonal matrices with the constants (cf., the eigenvalue λ_1 from Subsection 2.1)

$$b^\pm = h_1 - \frac{1}{\sqrt{2}} \sqrt{1 + h^2 + \sqrt{(1+h^2)^2 - 4h_1^2}} < 0 \qquad (h = |\mathbf{h}^+| = |\mathbf{h}^-|)$$

and symmetric positive definite 5×5 matrices $B^\pm$.

We prefer to proceed in a different way and adopt an idea from Reference [9]. This idea enables us, in some sense, to reduce (20), (21) to a hyperbolic problem with noncharacteristic boundary in which boundary conditions do not contain the front perturbation f at all. It should be noted however that the price for using such a trick isthat matrices involved are no longer symmetric. Namely, following Reference [9], from equation (19) we deduce the system

$$\begin{cases} \partial_t p + \operatorname{div} \mathbf{w} = 0 \,, \qquad \partial_t r + h^2 \operatorname{div} \mathbf{w} - (\mathbf{h}^\pm, \nabla(\mathbf{h}^\pm, \mathbf{w})) = 0 \,, \\ \partial_t \mathbf{w} + 2(\mathbf{h}^\pm, \nabla)\mathbf{w} - \mathbf{h}^\pm(\mathbf{h}^\pm, \nabla p) + \nabla(p + r) = 0 \qquad \text{for} \quad \mathbf{x} \in \mathbb{R}^3_\pm \,, \end{cases} \tag{22}$$

where

$$r = (\mathbf{h}^\pm, \mathbf{H}) \,, \quad \mathbf{w} = (w_1, w_2, w_3) = \mathbf{v} - \mathbf{H} + p\,\mathbf{h}^\pm \qquad \text{for} \quad \mathbf{x} \in \mathbb{R}^3_\pm \,.$$

In the derivation of (22), we used the constraint $\operatorname{div} \mathbf{H} = 0$. System (22) is supplemented with the boundary conditions

$$[p] = 0 \,, \quad [r] = 0 \,, \quad [\mathbf{w} - (1/2)p\,\mathbf{h}] = 0 \qquad \text{at} \quad x_1 = 0 \tag{23}$$

following from (21). Problem (22), (23) can be put in the form

$$\partial_t \mathbf{W} + \sum_{k=1}^{3} \mathcal{B}_k^\pm \partial_k \mathbf{W} = 0 \quad \text{if} \quad \mathbf{x} \in \mathbb{R}^3_\pm \,, \qquad G^+ \mathbf{W}^+ + G^- \mathbf{W}^- = 0 \quad \text{if} \quad x_1 = 0 \,,$$

with $\mathbf{W} = (p, r, \mathbf{w})$ and corresponding matrices $\mathcal{B}_k^\pm$ and $G^\pm$.

In Reference [9] the equivalence of problems (20), (21) and (22), (23) was proved in the sense that knowing a solution $\mathbf{W}$ of (22), (23) we can define the functions v_k , H_k , and f (actually, using integration [9]) in such a way that $(\mathbf{U}, f)$ is the solution of (20), (21). However, problem (22), (23) containing no front f looses some important information about the original problem (20), (21). In particular, an exact relation between the Lopatinskii conditions for problems (20), (21) and (22), (23) is unclear.

Therefore, unlike Reference [9], we proceed as follows. We consider the systems

$$\partial_t \mathbf{V} + \sum_{k=1}^{3} \mathcal{A}_k^{\pm} \partial_k \mathbf{V} = 0 \qquad \text{for} \quad \mathbf{x} \in \mathbb{R}^3_{\pm} \tag{24}$$

formed by the equations

$$\partial_t z + (h_1 - h_2^{\pm})\partial_2 w_2 + \partial_3 (h_1 w_3 - h_3^{\pm} w_2) = 0\,, \tag{25}$$

$$\begin{aligned} &\partial_t H_1 + \partial_1 H_1 + \partial_2 \left(h_1 w_2 - h_2^{\pm} w_1 + z + (h_2^{\pm} - h_1)p\right) \\ &\quad + \partial_3 (h_1 w_3 - h_3^{\pm} w_1) + \frac{1}{h_3^{\pm}}\,\partial_3 \left(r - h_1 H_1 - h_2^{\pm}(z + (h_2^{\pm} - h_1)p)\right) = 0\,, \end{aligned} \tag{26}$$

and (22), where

$$\mathbf{V} = (z, H_1, \mathbf{W})\,, \quad z = H_2 + (h_1 - h_2^{\pm})p \qquad \text{for} \quad \mathbf{x} \in \mathbb{R}^3_{\pm}\,.$$

We assume that $(h_3^+)^2 + (h_3^-)^2 \neq 0$. Otherwise, since $h^2 - h_1^2 \neq 0$ (for $h^2 - h_1^2 = 0$ solution (17) is continuous), with corresponding changes in (25) and (26) we choose $z = H_2 + (h_1 - h_3^{\pm})p$ (if $(h_2^+)^2 + (h_2^-)^2 \neq 0$) or $z = H_l + (h_1 - h_l^+)p$ for $x_1 > 0$ and $z = H_m + (h_1 - h_m^-)p$ for $x_1 < 0$ (if $(h_m^+)^2 + (h_l^-)^2 \neq 0$), where $l = 2$ and $m = 3$ or $l = 3$ and $m = 2$. The boundary conditions for (24) read:

$$\begin{cases} [H_1] = [h_2]\partial_2 f + [h_3]\partial_3 f\,, \quad \partial_t f = w_1^+ - (h_1/2)p^+\,, \\ [p] = 0\,, \quad [r] = 0\,, \quad [\mathbf{w} - (1/2)p\,\mathbf{h}] = 0\,, \end{cases} \tag{27}$$

or in the matrix form

$$B\mathbf{F} + M^+\mathbf{Y}^+ + M^-\mathbf{Y}^- = 0\,,$$

with $\mathbf{F} = (\partial_t f, \partial_2 f, \partial_3 f)$, $\mathbf{Y} = (H_1, \mathbf{W})$, and corresponding matrices $M^{\pm}$.

The change of unknowns $\mathcal{T}:\ \mathbf{U} \to \mathbf{V}$ is, of course, invertible. But, since we used the constraint $\operatorname{div}\mathbf{H} = 0$ while obtaining equations (22) from (20), we need to show the equivalence of problems (20), (21) and (24), (27) (in the usual sense). For this purpose, we should prove the following statement.

Proposition 3.1 If the equation

$$\partial_1 H_1 + \partial_2 \left(z + (h_2^{\pm} - h_1)p\right) + \frac{1}{h_3^{\pm}}\,\partial_3 \left(r - h_1 H_1 - h_2^{\pm}(z + (h_2^{\pm} - h_1)p)\right) = \operatorname{div}\mathbf{H} = 0 \tag{28}$$

for $\mathbf{x} \in \mathbb{R}^3_{\pm}$ holds for the initial data for problem (24), (27), then $\mathbf{V}$ also satisfies (28) for all $t > 0$.

Proof. It follows from the relation $\mathbf{U}=\mathcal{T}^{-1}\mathbf{V}$ that

$$H_2=z+(h_2^{\pm}-h_1)p\,,\quad H_3=\frac{r-h_1H_1-h_2^{\pm}H_2}{h_3^{\pm}}\,,\quad \mathbf{v}=\mathbf{w}+\mathbf{H}-p\,\mathbf{h}^{\pm}\qquad\text{for}\quad \mathbf{x}\in\mathbb{R}^3_{\pm}\,.$$

Then equation (26) implies

$$\partial_t H_1-(\mathbf{h}^{\pm},\nabla\Omega_1)+h_1\mathrm{div}\,\mathbf{\Omega}+\mathrm{div}\,\mathbf{H}=0\,,\quad \mathbf{x}\in\mathbb{R}^3_{\pm}\,,\tag{29}$$

where $\mathbf{\Omega}=(\Omega_1,\Omega_2,\Omega_3)=\mathbf{w}-(1/2)p\,\mathbf{h}^{\pm}$ for $\mathbf{x}\in\mathbb{R}^3_{\pm}$. From (29) we obtain

$$\partial_t[H_1]-[h_2]\partial_2\Omega_1-[h_3]\partial_3\Omega_1+[\mathrm{div}\,\mathbf{H}]=0\qquad\text{at}\quad x_1=0\,,$$

where $\Omega_1|_{x_1=0}=\Omega_1^+=\Omega_1^-$. In view of the first two conditions in (27), the last equation yields

$$[g]|_{x_1=0}=0\,,\tag{30}$$

where $g=\mathrm{div}\,\mathbf{H}$. From (25), (26), and (22) we have

$$\partial_t\mathbf{H}-(\mathbf{h}^{\pm},\nabla)\mathbf{w}+\mathbf{h}^{\pm}\mathrm{div}\,\mathbf{w}+\mathbf{a}^{\pm}\mathrm{div}\,\mathbf{H}=0\,,\quad \mathbf{x}\in\mathbb{R}^3_{\pm}\,,\tag{31}$$

where $\mathbf{a}^{\pm}=(1,0,-h_1/h_3^{\pm})$. Acting on (31) by div , we get

$$\partial_t g+\mathrm{div}\,(g\,\mathbf{a}^{\pm})=0\,,\quad \mathbf{x}\in\mathbb{R}^3_{\pm}\,,\tag{32}$$

Finally, (28), (30), and (32) imply $g=0$ for all $t>0$. □

Thus, instead of problem (20), (21) we can consider the equivalent problem (24), (27). Clearly, system (24) is still hyperbolic but no longer symmetric. The boundary matrices $\mathcal{A}_1^{\pm}$ have the form:

$$\mathcal{A}_1^{\pm}=\begin{pmatrix}0&0&\mathbf{0}\\0&1&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathcal{B}_1^{\pm}\end{pmatrix}\,,$$

where $\mathbf{0}$ is the row (or column) null vector, and the 5×5 matrices $\mathcal{B}_1^{\pm}$ have one negative and four positive eigenvalues (this follows from the properties of the matrices $\bar{A}_1^{\pm}$ in (20) and can be also checked directly). That is, the function z is a "characteristic unknown" of problem (24), (27). Moreover, the separate subproblem (22), (23) is a hyperbolic problem with noncharacteristic boundary. Note also that the boundary conditions in (23) are standard (like those in [4, 18]) in the sense that they do not contain the unknown front f.

3.2 The Lopatinskii condition for problem (24), (27)

We apply the Fourier-Laplace transform to problem (24), (27). Namely, we apply a Laplace transform in t with the dual variable $s=\eta+i\xi$, where $\eta>0$, $\xi\in\mathbb{R}$, and a Fourier transform in $\mathbf{y}$ with the dual variable $\boldsymbol{\omega} = (\omega_2, \omega_3) \in \mathbb{R}^2$. The transforms of $\mathbf{V}$ and f are denoted by $\hat{\mathbf{V}} = (\hat{z}, \hat{H}_1, \hat{\mathbf{W}}) = (\hat{z}, \hat{\mathbf{Y}})$ and $\hat{f}$ respectively.

We arrive at

$$s\hat{z} = i\mathbf{a}^{\pm}(\boldsymbol{\omega}) \cdot \hat{\mathbf{W}}\,, \qquad x_1 \in \mathbb{R}_{\pm}\,, \tag{33}$$

$$\frac{d\hat{H}_1}{dx_1} = i\left(\frac{h_2^{\pm}}{h_3^{\pm}}\,\omega_3 - \omega_2\right)\hat{z} + \left(i\frac{h_1}{h_3^{\pm}}\,\omega_3 - s\right)\hat{H}_1 + i\mathbf{b}^{\pm}(\boldsymbol{\omega}) \cdot \hat{\mathbf{W}}\,, \qquad x_1 \in \mathbb{R}_{\pm}\,, \tag{34}$$

$$\frac{d\hat{\mathbf{W}}}{dx_1} = \mathcal{M}^{\pm}(s, i\boldsymbol{\omega})\hat{\mathbf{W}}\,, \qquad x_1 \in \mathbb{R}_{\pm} \tag{35}$$

(without loss of generality we consider the case $(h_3^+)^2 + (h_3^-)^2 \neq 0$), where

$$\mathcal{M}^{\pm}(s, i\boldsymbol{\omega}) = -(\mathcal{B}_1^{\pm})^{-1}(sI + i\omega_2\mathcal{B}_2^{\pm} + i\omega_3\mathcal{B}_3^{\pm})\,,$$

and the vectors $\mathbf{a}^{\pm}(\boldsymbol{\omega})$ and $\mathbf{b}^{\pm}(\boldsymbol{\omega})$ are not used in what follows. Substituting (33) into (34), we obtain the following system of ordinary differential equations:

$$\frac{d\hat{\mathbf{Y}}}{dx_1} = \mathcal{P}^{\pm}(s, i\boldsymbol{\omega})\hat{\mathbf{Y}}\,, \qquad x_1 \in \mathbb{R}_{\pm}\,, \tag{36}$$

where

$$\mathcal{P}^{\pm}(s, i\boldsymbol{\omega}) = \begin{pmatrix} -s + i(h_1/h_3^{\pm})\omega_3 & \dfrac{\mathbf{c}^{\pm}(\boldsymbol{\omega})}{s} + i\mathbf{b}^{\pm}(\boldsymbol{\omega}) \\ \mathbf{0} & \mathcal{M}^{\pm}(s, i\boldsymbol{\omega}) \end{pmatrix},$$

$\mathbf{c}^{\pm}(\boldsymbol{\omega}) = (\omega_2 - (h_2^{\pm}/h_3^{\pm})\omega_3)\,\mathbf{a}^{\pm}(\boldsymbol{\omega})$. The matrices $\mathcal{P}^{\pm}$ have a simple pole at $s = 0$. In principle, for hyperbolic problems with characteristic boundary [18] fractional roots of $|\boldsymbol{\omega}|/s$ can occur as eigenvalues of $\mathcal{P}^{\pm}$ but, as we can see, this is not the case for Alfvén discontinuities. Applying a Fourier-Laplace transform to (27) results in the boundary conditions for (36):

$$\mathbf{b}(s, i\boldsymbol{\omega})\hat{f} + M^+\hat{\mathbf{Y}}^+ + M^-\hat{\mathbf{Y}}^- = 0\,, \tag{37}$$

where $\mathbf{b}(s, i\boldsymbol{\omega}) = (i[h_2]\omega_2 + i[h_3]\omega_3, s, \mathbf{0})$.

First of all, it should be noted that in the boundary conditions (27) the symbol associated with the front f is *not elliptic*. Roughly speaking, it means that the vector $\mathbf{F}$ cannot be expressed through $\mathbf{V}^+$ and $\mathbf{V}^-$. More precisely, there is a point $(\eta_0, \xi_0, \boldsymbol{\omega}_0) \in \mathcal{R}_+$ such that $\mathbf{b}(s_0, i\boldsymbol{\omega}_0) = 0$, where $\mathcal{R}_+ = \overline{\mathbb{R}^4_+} \setminus \{0\}$ (i.e. $\Re s_0 = \eta_0 \geq 0$, $|s_0|^2 + |\boldsymbol{\omega}_0|^2 \neq 0$) . In our case,

$$s_0 = 0\,, \quad \boldsymbol{\omega}_0 = \alpha\,(-[h_3], [h_2])\,, \quad \alpha \neq 0\,. \tag{38}$$

Since $\mathbf{b}$ is not elliptic, the front cannot be eliminated from (37), i.e. there is no matrix $N(s, i\boldsymbol{\omega})$ such that

$$N(s, i\boldsymbol{\omega})\mathbf{b}(s, i\boldsymbol{\omega}) = \begin{pmatrix} g(s, i\boldsymbol{\omega}) \\ \mathbf{0} \end{pmatrix},$$

with

$$\min_{(\eta,\xi,\boldsymbol{\omega})\in\mathcal{R}_+} |g(s,i\boldsymbol{\omega})| > 0\,.$$

Therefore, we have to introduce the Lopatinskii condition for the original boundary conditions (37) containing the front. The Lopatinskii condition is satisfied for problem (24), (27) if for all fixed $(s,\boldsymbol{\omega})$ with $\eta>0$ the only bounded solution of problem (36), (37) is the trivial solution $\hat{\mathbf{Y}}=0$, $\hat{f}=0$.

To write down the Lopatinskii condition in a precise (algebraic) form [4, 5, 6, 7] we now consider the characteristic equations for the eigenvalues $\lambda^{\pm}$ $(\lambda_0^{\pm},\lambda_1^{\pm},\ldots,\lambda_5^{\pm})$ of the matrices $\mathcal{P}^{\pm}$:

$$\det(\mathcal{P}^{\pm}(s,\boldsymbol{\omega})-\lambda^{\pm}I)=0\,.$$

These equations imply the dispersion relations

$$\big(s+\lambda^{\pm}-i(h_1/h_3^{\pm})\,\omega_3\big)\ \det(sI+\lambda^{\pm}\mathcal{B}_1^{\pm}+i\omega_2\mathcal{B}_2^{\pm}+i\omega_3\mathcal{B}_3^{\pm})=0\,. \tag{39}$$

The first multiplier in (39) gives

$$\lambda_0^{\pm}=-s+i(h_1/h_3^{\pm})\,\omega_3\,.$$

Concerning the second multiplier in (39), for the hyperbolic systems (22) with noncharacteristic boundary we apply Hersh's lemma [19] to arrive at the following proposition (for the case of characteristic boundary see Reference [18]).

Proposition 3.2 For all fixed $(s,\boldsymbol{\omega})$ with $\eta>0$ the matrix $\mathcal{M}^+(s,i\boldsymbol{\omega})$ has four eigenvalues λ^+ with $\Re\lambda^+<0$ and one eigenvalue with $\Re\lambda^+>0$. The same property takes place for the eigenvalues λ^- of the matrix $\mathcal{M}^-(s,i\boldsymbol{\omega})$.

Hence, we can reduce the matrices $\mathcal{P}^{\pm}$ to the form

$$\mathcal{P}^{\pm}=\Lambda^{\pm}\begin{pmatrix}\lambda_0^{\pm} & \mathbf{0} & 0\\ \mathbf{0} & \mathcal{D}^{\pm} & \mathbf{0}\\ 0 & \mathbf{0} & \lambda_5^{\pm}\end{pmatrix}(\Lambda^{\pm})^{-1}$$

with nonsingular matrices $\Lambda^{\pm}(s,i\boldsymbol{\omega})$, where for $\eta>0$ all the eigenvalues of the 4×4 matrices $\mathcal{D}^{\pm}$ lie in the left half-plane ($\Re\lambda_k^{\pm}<0$, $k=\overline{1,4}$) and $\Re\lambda_5^{\pm}>0$.

Since we are seeking bounded solutions, $\hat{\mathbf{Y}}(x_1)$ is as follows:

$$\begin{aligned}\hat{\mathbf{Y}}(x_1)&=\Lambda^+\begin{pmatrix}e^{\lambda_0^+x_1}c_0\\ e^{\mathcal{D}^+x_1}\mathbf{C}_+\\ 0\end{pmatrix}\qquad\text{for}\quad x_1>0\,,\\ \hat{\mathbf{Y}}(x_1)&=\Lambda^-\begin{pmatrix}\mathbf{0}\\ e^{\lambda_5^-x_1}c_5\end{pmatrix}\qquad\text{for}\quad x_1<0\,,\end{aligned}\tag{40}$$

where $\mathbf{C}_+ = (c_1, \dots, c_4)$, c_i $(i = \overline{0,5})$ are constants. Clearly, the first columns in the matrices $\Lambda^\pm$ are the eigenvectors $(1, \mathbf{0})$ for the eigenvalues $\lambda_0^\pm$. More precisely, the matrices $\Lambda^\pm$ have the following structure:

$$\Lambda^\pm = \begin{pmatrix} 1 & \mathbf{g}^\pm/s \\ \mathbf{0} & T^\pm \end{pmatrix}, \qquad \mathcal{M}^\pm = T^\pm \begin{pmatrix} \mathcal{D}^\pm & \mathbf{0} \\ \mathbf{0} & \lambda_5^\pm \end{pmatrix} (T^\pm)^{-1},$$

for some $\mathbf{g}^\pm(s, i\boldsymbol{\omega})$ (we do not need explicit formulae for $\mathbf{g}^\pm(s, i\boldsymbol{\omega})$).

The constants $c_0, \dots, c_5$, $\hat{f}$ are linked by the relations

$$c_0 + \frac{1}{s}\mathbf{g}^+ \begin{pmatrix} \mathbf{C}_+ \\ 0 \end{pmatrix} - \frac{1}{s}\mathbf{g}^- \begin{pmatrix} \mathbf{0} \\ c_5 \end{pmatrix} = i([h_2]\omega_2 + [h_3]\omega_3)\hat{f}, \qquad s\hat{f} = \begin{pmatrix} -h_1/2 & 0 & 1 & 0 & 0 \end{pmatrix} \widetilde{T}^+ \mathbf{C}_+, \tag{41}$$

$$G^+ \widetilde{T}^+ \mathbf{C}_+ + G^- \boldsymbol{\gamma}_5 c_5 = 0 \tag{42}$$

following from (37). Here $\boldsymbol{\gamma}_5$ is the eigenvector for λ_5^- (the last column in T^-), and the matrix $\widetilde{T}^+$ is formed by the first four columns, $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_4$, of the matrix T^+ which are the basis of the eigenspace for $\lambda_1^+, \dots, \lambda_4^+$. Relations (41) and (42) can be put in the form

$$L(s, i\boldsymbol{\omega}) \begin{pmatrix} c_0 \\ \hat{f} \\ \mathbf{C} \end{pmatrix} = 0,$$

where $\mathbf{C} = (\mathbf{C}_+, c_5)$,

$$L(s, i\boldsymbol{\omega}) = \begin{pmatrix} 1 & -i([h_2]\omega_2 + [h_3]\omega_3) & \dfrac{\mathbf{a}_0(s, i\boldsymbol{\omega})}{s} \\ 0 & s & \mathbf{a}_1(s, i\boldsymbol{\omega}) \\ \mathbf{0} & \mathbf{0} & \mathcal{L}(s, i\boldsymbol{\omega}) \end{pmatrix}, \tag{43}$$

$\det L$ is the so-called *Lopatinskii determinant*, $\mathbf{a}_0(s, i\boldsymbol{\omega})$ makes no contribution to $\det L$, and the column vector $\mathbf{a}_1(s, i\boldsymbol{\omega})$ depends on the first and third components of the vectors $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_4$. The Lopatinskii determinant $\det L$ has no singularities at $s = 0$. It is clear that $\det \mathcal{L}$ is the Lopatinskii determinant for the reduced problem (22), (23). We are now in a position to introduce the Lopatinskii condition for the whole problem (24), (27).

Definition 3.1 Problem (24), (27) satisfies the *Lopatinskii condition* if

$$\det L(s, i\boldsymbol{\omega}) \neq 0 \tag{44}$$

for all $\eta > 0$, $(\xi, \boldsymbol{\omega}) \in \mathbb{R}^3$. Problem (24), (27) satisfies the *uniform Lopatinskii condition* if requirement (44) is fulfilled for all $\eta \geq 0$, $(\xi, \boldsymbol{\omega}) \in \mathbb{R}^3$, with $|s|^2 + |\boldsymbol{\omega}|^2 \neq 0$.

It follows from the structure of the matrix L that $\det L(0, i\boldsymbol{\omega}) = 0$. That is, the uniform Lopatinskii condition is always violated. Actually, this is a direct consequence of the fact that $\mathbf{b}(s, i\boldsymbol{\omega})$ is not elliptic. Indeed, problem (24), (27) has the solution $\mathbf{V} = 0$, $f = \hat{f}e^{i(\boldsymbol{\omega}_0, \mathbf{y})}$ associated with the neutral mode (38) (see also Remark 3.2 below). Thus, we have proved the following proposition.

Proposition 3.3 Planar Alfvén discontinuities are never uniformly stable. A planar Alfvén discontinuity is violently unstable if and only if the Lopatinskii condition for the reduced problem (22), (23) is not satisfied, i.e. $\det \mathcal{L}(s, i\boldsymbol{\omega})$ vanishes for some $\eta > 0$, $(\xi, \boldsymbol{\omega}) \in \mathbb{R}^3$.

Remark 3.1 In Reference [6] it was shown that if the uniform Lopatinskii condition is satisfied for the linearized problem for Lax shock waves, then the symbol associated with the front of shock wave is elliptic. The same is, of course, true for characteristic discontinuities and a corresponding general proposition could be formulated as well.

Remark 3.2 The most general normal mode solution of problem (24), (27) corresponding to the root $s = 0$ of the Lopatinskii determinant is

$$H_1 = c_0 e^{i\left((h_1/h_3^{+})\omega_3 + (\boldsymbol{\omega}, \mathbf{y})\right)}, \quad \mathbf{W} = 0 \quad \text{for} \quad x_1 > 0, \quad \mathbf{Y} = 0 \quad \text{for} \quad x_1 < 0, \quad f = \hat{f}e^{i(\boldsymbol{\omega}, \mathbf{y})},$$

where $c_0 = i([h_2]\omega_2 + [h_3]\omega_3)\hat{f}$, with surface waves of finite energy developing for the "characteristic unknown" z:

$$z = \text{const}\ e^{\gamma^{\pm} x_1 + i(\boldsymbol{\omega}^{\pm}, \mathbf{y})} \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3_{\pm},$$

where $\Re\gamma^{+} < 0$, $\Re\gamma^{-} > 0$, $\boldsymbol{\omega}^{\pm} = (\omega_2^{\pm}, \omega_3^{\pm}) \in \mathbb{R}^2$, and $h_2^{\pm}\omega_3^{\pm} = h_3^{\pm}\omega_2^{\pm}$.

3.3 Normal mode analysis for the reduced problem (22), (23)

The test of the Lopatinskii condition (especially, the uniform Lopatinskii condition) is in general a *nontrivial* linear algebraic problem even when we treat it numerically (see, e.g. Reference [20]). Our present goal is to test the Lopatinskii condition for the reduced problem (22), (23). Since even the roots $\lambda^{\pm} = \lambda^{\pm}(s, i\boldsymbol{\omega})$ $(\lambda_1^{\pm}, \ldots, \lambda_5^{\pm})$ of the dispersion relations (cf. (39))

$$\det(sI + \lambda^{\pm}\mathcal{B}_1^{\pm} + i\omega_2\mathcal{B}_2^{\pm} + i\omega_3\mathcal{B}_3^{\pm}) = 0 \tag{45}$$

cannot be found analytically in the general case, we are not able to calculate the Lopatinskii determinant $\det \mathcal{L}$ in an explicit form. So, the only way to proceed is to find (for fixed s and $\boldsymbol{\omega}$) and analyze the function $\mathcal{G}(s) = \det \mathcal{L}(s, i\boldsymbol{\omega})$ numerically.

To prepare problem (22), (23) for numerical testing of the Lopatinskii condition we make some useful modifications of this problem proposed in Reference [9]. First, we introduce the new dependent and independent variables

$$t' = ht, \quad r' = (1/h)r, \quad \mathbf{w}' = (1/h)\mathbf{w}.$$

Second, without loss of generality we suppose that

$$\mathbf{h}^{\pm} = h\mathbf{b}_1^{\pm}\,, \quad \mathbf{b}_1^{-} = (\cos\theta, \sin\theta, 0)\,, \quad \mathbf{b}_1^{+} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)\,,$$

where $0 < \theta < \pi/2$ and $0 < \phi < 2\pi$. Introducing the orthonormal basis $\{\mathbf{b}_1^{\pm}, \mathbf{b}_2^{\pm}, \mathbf{b}_3^{\pm}\}$ associated with the vectors $\mathbf{h}^{\pm}$, we have

$$\mathbf{w}' = \sum_{k=1}^{3} \widetilde{w}_k \mathbf{b}_k^{\pm}\,, \quad \widetilde{w}_k = (\mathbf{w}', \mathbf{b}_k^{\pm}) \qquad \text{for} \quad \mathbf{x} \in \mathbb{R}^3_{\pm}\,,$$

$$\mathbf{b}_2^{+} = (-\sin\theta, \cos\theta\cos\phi, \cos\theta\sin\phi)\,, \quad \mathbf{b}_3^{+} = (0, -\sin\phi, \cos\phi)\,,$$

$$\mathbf{b}_2^{-} = (-\sin\theta, \cos\theta, 0)\,, \quad \mathbf{b}_3^{-} = (0, 0, 1)\,.$$

In particular, the last three boundary conditions in (23) are now written as

$$\left[\frac{p}{2}\mathbf{b}_1 - \sum_{k=1}^{3} \widetilde{w}_k \mathbf{b}_k\right] = 0\,. \tag{46}$$

Now, to simplify the notation, we drop the primes and tildes and denote by $\mathbf{W} = (p, r, \mathbf{w})$ the vector $(p, r', \widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)$. Moreover, the matrices in the interior equations and the boundary conditions are again denoted by $\mathcal{B}_k^{\pm}$ and $G^{\pm}$ respectively. Their explicit form is as follows:

$$\mathcal{B}_k^{\pm} = \begin{pmatrix} 0 & 0 & b_{1k}^{\pm} & b_{2k}^{\pm} & b_{3k}^{\pm} \\ 0 & 0 & 0 & b_{2k}^{\pm} & b_{3k}^{\pm} \\ (\beta-1)b_{1k}^{\pm} & b_{1k}^{\pm} & 2b_{1k}^{\pm} & 0 & 0 \\ \beta b_{2k}^{\pm} & b_{2k}^{\pm} & 0 & 2b_{1k}^{\pm} & 0 \\ \beta b_{3k}^{\pm} & b_{3k}^{\pm} & 0 & 0 & 2b_{1k}^{\pm} \end{pmatrix}, \quad G^{-} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ d_1 & 0 & d_2 & 2d_3 & 2d_4 \\ -d_3 & 0 & 2d_3 & d_5 & d_6 \\ d_4 & 0 & -d_4 & -d_6 & d_7 \end{pmatrix},$$

$G^{+} = I$, where $\beta = 1/h^2 = \bar{\rho}\bar{c}^2/|\bar{\mathbf{H}}^{\pm}|^2$ is the so-called *value of plasma*, b_{ik}^{+} and b_{ik}^{-} are the kth components of the vectors $\mathbf{b}_i^{+}$ and $\mathbf{b}_i^{-}$ respectively ($b_{11}^{\pm} = \cos\theta$, $b_{12}^{-} = \sin\theta$, $b_{12}^{+} = \sin\theta\cos\phi$, etc.),

$$d_1 = \frac{1}{2}\sin^2\theta(\cos\phi - 1)\,, \quad d_2 = -(\cos^2\theta + \sin^2\theta\cos\phi)\,, \quad d_3 = \frac{1}{2}\cos\theta\sin\theta(1 - \cos\phi)\,,$$

$$d_4 = -\frac{1}{2}\sin\theta\sin\phi\,, \quad d_5 = -(\sin^2\theta + \cos^2\theta\cos\phi)\,, \quad d_6 = -\cos\theta\sin\phi\,, \quad d_7 = \cos\phi\,.$$

Multiplying boundary conditions (46) by the vectors $\mathbf{b}_1^{+}$, $\mathbf{b}_2^{+}$, and $\mathbf{b}_3^{+}$, we rewrite them in the form

$$G^{+}\mathbf{W}^{+} + G^{-}\mathbf{W}^{-} = 0. \tag{47}$$

After some algebra we find the explicit form of the dispersion relations (45):

$$K_2^{\pm}\left\{\beta s K_2^{\pm}\Big(|\boldsymbol{\omega}|^2 - (\lambda^{\pm})^2\Big) + (K_1^{\pm})^2\Big(|\boldsymbol{\omega}|^2 - (\lambda^{\pm})^2 + (K_1^{\pm})^2\Big)\right\} = 0\,, \tag{48}$$

where

$$K_m^{\pm} = K_m^{\pm}(s, i\boldsymbol{\omega}, \lambda^{\pm}) = s + mN^{\pm}\,, \quad m = 1, 2\,, \quad N^{\pm} = N^{\pm}(\lambda^{\pm}, i\boldsymbol{\omega}) = \lambda^{\pm}\cos\theta + ig_1^{\pm}(\boldsymbol{\omega})\sin\theta\,,$$

$$g_1^+(\boldsymbol{\omega}) = \omega_2\cos\phi + \omega_3\sin\phi\,, \quad g_1^-(\boldsymbol{\omega}) = \omega_2\,, \quad N^\pm = \sqrt{\beta}\,\left(\mathbf{h}^\pm,\,\boldsymbol{\xi}^\pm\right)\,, \quad \boldsymbol{\xi}^\pm = (\lambda^\pm,\,i\boldsymbol{\omega})\,.$$

In the sequel, we will also use the notations

$$g_2^+(\boldsymbol{\omega}) = -\omega_2\sin\phi + \omega_3\cos\phi\,, \quad P^\pm = P^\pm(\lambda^\pm, i\boldsymbol{\omega}) = -\lambda^\pm\sin\theta + ig_1^\pm(\boldsymbol{\omega})\cos\theta\,.$$

We are interested in the roots λ_k^+ ($k=\overline{1,4}$) and λ_5^- of (48) with $\Re\lambda_k^+ < 0$ and $\Re\lambda_5^- > 0$ for $\eta > 0$ (see the previous subsection). From the equation $K_2^+ = 0$, cf. (48), we find

$$\lambda_1^+ = -\,\frac{s + ig_1^+(\boldsymbol{\omega})\sin\theta}{\cos\theta}\,.$$

The eigenvector for this eigenvalue (the first column in $\widetilde{T}^+$) is

$$\boldsymbol{\gamma}_1 = (0\,,\,0\,,\,0\,,\,-ig_2^+(\boldsymbol{\omega})\,,\,P_1^+(s,i\boldsymbol{\omega}))\,, \tag{49}$$

where $P_1^+(s,i\boldsymbol{\omega}) = P^+(\lambda_1^+, i\boldsymbol{\omega})$. It is easily verified that $\boldsymbol{\gamma}_1 = 0$ if and only if $|\boldsymbol{\xi}^+\times\mathbf{h}^+| = 0$, with $\lambda^+ = \lambda_1^+$. This happens for the neutral mode $s = s_0 = -ig_1^+(\boldsymbol{\omega})/\sin\theta$ and the wave vector $\boldsymbol{\omega}$ such that $g_2^+(\boldsymbol{\omega}) = 0$.

The eigenvalues λ_2^+, λ_3^+, λ_4^+, and λ_5^- cannot be calculated analytically. But, assuming that λ_2^+, λ_3^+, and λ_4^+ are simple eigenvalues, we can analytically find the eigenvectors for them:

$$\begin{aligned} &\boldsymbol{\gamma}_k = (-h^2a_k^+N_k^+\,,\,b_k^+N_k^+\,,\,(b_k^+ + h^2a_k^+)s\,,\,sP_k^+N_k^+\,,\,ig_2^+sN_k^+)\,, \quad k=2,3,4\,,\\ &\boldsymbol{\gamma}_5 = (-h^2a_5^-N_5^-\,,\,b_5^-N_5^-\,,\,(b_5^- + h^2a_5^-)s\,,\,sP_5^-N_5^-\,,\,i\omega_3 sN_5^-)\,, \end{aligned} \tag{50}$$

where

$$a_m^\pm = |\boldsymbol{\omega}|^2 - (\lambda_m^\pm)^2 + (K_{1,m}^\pm)^2\,, \quad b_m^\pm = |\boldsymbol{\omega}|^2 - (\lambda_m^\pm)^2 + (N_m^\pm)^2\,, \quad N_m^\pm = N^\pm(\lambda_m^\pm, i\boldsymbol{\omega})\,,$$

$$P_m^\pm = P^\pm(\lambda_m^\pm, i\boldsymbol{\omega})\,, \quad K_{1,m}^\pm = K_1^\pm(s, i\boldsymbol{\omega}, \lambda_m^\pm)\,, \quad m=\overline{2,5}\,.$$

That is, if $\eta > 0$ and the eigenvalues λ_2^+, λ_3^+, and λ_4^+ are simple, then from (42) we can determine the matrix $\mathcal{L}$ and find the Lopatinskii determinant $\det\mathcal{L}$, that depends on the functions $\lambda_k^+(s,i\boldsymbol{\omega})$ ($k=2,3,4$), $\lambda_5^-(s,i\boldsymbol{\omega})$ whose explicit form is unknown.

At the same time, there exist so-called *glancing modes* s at which some roots λ^+ of (48) are not simple. Two glancing modes can be analytically found and correspond to the case when the "explicit" eigenvalue λ_1^+ becomes double or triple. Indeed, it happens when in (48) either simultaneously $K_2^+ = 0$ and $|\boldsymbol{\omega}|^2 - (\lambda^+)^2 + (K_1^+)^2 = 0$ or simultaneously $K_2^+ = 0$ and $K_1^+ = 0$.

The glancing mode for the first case is

$$s = s_1 = \frac{|g_2^+|\cos\theta - ig_1^+}{\sin\theta}\,.$$

If $g_2^+(\boldsymbol{\omega}) \neq 0$, then $\Re s_1 > 0$, the eigenvalue λ_1^+ is double, and

$$\lambda_1^+ = -\,\frac{|g_2^+| - ig_1^+\cos\theta}{\sin\theta}\,.$$

But, if $g_2^+(\boldsymbol{\omega}) = 0$, it becomes triple: $\lambda_1^+ = \lambda_2^+ = \lambda_5^+ = ig_1^+ \cos\theta / \sin\theta$. Moreover, $s_1 = s_0$ and the eigenvalue λ_5^+ comes from the right half-plane:

$$\lambda_5^+ = \lim_{\delta \to 0} \frac{\delta + ig_1^+ \cos\theta}{\sin\theta}\,, \quad \delta = |g_2^+(\boldsymbol{\omega})|\,.$$

Remark 3.3 The limit case when λ_1^+ is a triple eigenvalue for $s = s_0$ corresponds to $|\boldsymbol{\xi}^+ \times \mathbf{h}^+| = 0$ (see above). The so-called *nonglancing condition* [10] requires that any eigenvalue $\lambda^\pm$ should be simple as soon as $|\boldsymbol{\xi}^\pm \times \mathbf{h}^\pm| = 0$. This means that the nonglancing condition, which is satisfied for MHD shock waves (see Appendix A in Reference [10]), is violated for Alfvén discontinuities.

The glancing mode for the second case when both K_1^+ and K_2^+ vanish is $s = 0$. Clearly, the eigenvalue $\lambda_1^+ = -ig_1^+ \sin\theta / \cos\theta$ is triple. Other glancing modes cannot be found analytically, but the numerical analysis of the dispersion relation (48) (for λ^+) for the case $\eta > 0$ shows that the maximum number of glancing modes (together with the mode $s = s_1$) is five. However, for pressure dominated flows when β is large enough, in particular, for the incompressibility limit $\beta \to \infty$ there exists only one glancing mode with $\eta > 0$. This mode is $s = s_1$ calculated above.

The Lopatinskii determinant $\det \mathcal{L}$ computed for the eigenvectors (50), of course, vanishes at glancing modes. But these modes are, generally speaking, fictitious roots of the equation $\det \mathcal{L} = 0$ when $\mathcal{L}$ is properly determined. For glancing modes associated with the "explicit" eigenvalue λ_1^+ the basis of the eigenspace can be calculated analytically. In particular, for $s = s_1$ with $g_2^+ \neq 0$ the geometric multiplicity of the double eigenvalue λ_1^+ is one, and the corresponding eigenvector $\boldsymbol{\gamma}_1$ and the adjoint vector $\boldsymbol{\gamma}_2$ can be easily found (we do not present them here).

Remark 3.4 In Reference [9] the Lopatinskii condition for the reduced problem (22), (23) was analyzed for the limit case $\beta \to 0$ of magnetically dominated flows. For this case the eigenvalues λ_2^+ , λ_3^+ , λ_4^+ , and λ_5^- can be found as series in the small parameter $\sqrt{\beta}$, where $s = s' + s''\sqrt{\beta} + s'''\beta + \ldots$. The Lopatinskii determinant was calculated in Reference [9] in the zero limit in $\sqrt{\beta}$: $\det \mathcal{L} = \det \mathcal{L}_0 + O(\sqrt{\beta})$. But the root $s' = \cos\theta - ig_1^+(\boldsymbol{\omega}) \sin\theta$ of the equation $\det \mathcal{L}_0 = 0$ exhibited in Reference [9] as an unstable mode corresponds actually to a glancing mode and, therefore, s is a fictitious zero of the Lopatinskii determinant.

Matrix $\mathcal{L}(s, i\boldsymbol{\omega})$ has the form

$$\mathcal{L} = \begin{pmatrix} \boldsymbol{\gamma}_1 & \boldsymbol{\gamma}_2 & \boldsymbol{\gamma}_3 & \boldsymbol{\gamma}_4 & G^- \boldsymbol{\gamma}_5 \end{pmatrix},$$

where $\boldsymbol{\gamma}_k$ $(k = 1, \ldots, 5)$ are the column vectors given by (49), (50). In general, it is impossible to find zeros of the function $\mathcal{G}(s) = \det \mathcal{L}(s, i\boldsymbol{\omega})$ analytically. Therefore, we do this numerically with the help of MATLAB.

3.4 Numerical results

Without loss of generality, we restrict out analysis to the case

$$\boldsymbol{\omega} = (\omega_2, \omega_3) = (\cos\psi, \sin\psi)$$

for $\psi \in [0, \pi]$. This means that $|\boldsymbol{\omega}| = 1$ and that ψ is the angle between the unit vector $\boldsymbol{\omega}$ and the x_2 axis in the $x_2 - x_3$ plane.

We used the secant method for finding zeros of $\mathcal{G}(s)$ (in the upper half plane of complex variable s) for fixed β, θ, ϕ and ψ. It turned out that unstable modes ($\mathrm{Re}s > 0$) exist for certain values of θ and ϕ and all finite values of β. Unstable mode exist only within a narrow interval in ψ. Typical curves of $\eta \equiv \mathrm{Re}(s)$ versus ψ are shown in Fig. 1.

Once we have found unstable modes, we computed the instability domain in the plane of parameters θ and ϕ. To do this, we first fixed some values of β and θ and computed

$$\eta_m(\beta, \theta, \phi) = \max_{0 \le \psi \le \pi} \mathrm{Re}\,\{s\}.$$

for various values of $\psi \in [0, \pi]$. Typical curves of η_m versus ϕ are shown in Fig. 2.

We applied linear interpolation to find ϕ^* at which η_m vanishes, this gave us the boundary of the interval of instability in ϕ. Then we changed values of β and θ and repeated the whole procedure again, and so on. The resulting instability domains in the ϕ-θ plane for some values of β are shown in Fig. 3. For each curve in Fig. 3, the unstable domain is above the curve. Figure 3 shows that the Alfvén discontinuity is unstable in a wide range of ϕ as $\theta \to \pi/2$. When θ decreases from $\pi/2$, the interval of values of ϕ for which the Alfvén discontinuity is unstable shrinks. At certain $\theta = \theta^*$, this interval degenerates to a point, and there is no instability for $0 < \theta \le \theta^*$. For each fixed value of β, the maximum growth rate is attained when $\theta \to \pi/2$.

If β is increased from 0 to ∞ for given θ and ϕ, the growth rate increases from zero, attains its maximum value at some $\beta = \beta^*$ and then decreases monotonically to zero as $\beta \to \infty$. Typical curves for some fixed θ, ϕ and ψ are shown in Fig. 4. The case $\beta = \infty$ corresponds to the incompressible fluid. It is shown in the next section that incompressible Alfvén discontinuities are always stable, which agrees with our numerical results. In [9], the Lopatinskii determinant for the reduced problem had been analyzed in the limit ($\beta \to 0$) and its zero had been found. The numerical calculations however show that this root of the Lopatinskii determinant corresponds to a double root of the dispersion relation for λ^+ and is therefore a fictitious zero of the Lopatinskii determinant.

As was mentioned above, the maximum growth rate corresponds to the limit as $\theta \to \pi/2$. Note that the limit case $\theta = \pi/2$ corresponds to a particular case of a tangential discontinuity. Our results suggest that this tangential discontinuity (that can be treated as the degenerate case of Alfvén discontinuity) is unstable in the class of flows with Alfvén discontinuities.

4 Incompressible Alfvén discontinuities

A solution with Alfvén discontinuity of the system of ideal incompressible MHD

$$\frac{d\mathbf{v}}{dt} - (\mathbf{H}, \nabla\mathbf{H}) + \nabla q = 0\,, \quad \frac{d\mathbf{H}}{dt} - (\mathbf{H}, \nabla)\mathbf{v} = 0\,, \quad \operatorname{div}\mathbf{v} = 0 \tag{51}$$

is determined as a piecewise smooth solution $\mathbf{U} = (\mathbf{v}, \mathbf{H})$ of (51) being a classical solution of (51) on either side of a smooth hypersurface Γ and satisfying the jump conditions

$$[H_{\rm N}] = 0\,, \quad [\mathbf{v} - \mathbf{H}] = 0\,, \quad \partial_t f = v_{\rm N}^+ - H_{\rm N}^+\,, \quad [q] = 0 \tag{52}$$

at each point of Γ. Here the magnetic field is measured in Alfvén velocity units ($\mathbf{H} := \mathbf{H}/\sqrt{\bar{\rho}}$, $\bar{\rho} \equiv \text{const} > 0$) and the pressure p was divided by the density $\bar{\rho}$ ($p := p/\bar{\rho}$). Other notations are the same as in Section 1. Applying arguments similar to those in the proof of Proposition 2.1 we can show that for the free boundary value problem (51), (52), (5) the divergent constraint (2) can be regarded as the restriction only on the initial data (5).

For incompressible MHD there are two types of strong discontinuities: current-vortex sheets [13, 21] ($j = 0$, $H_{\rm N}^+ = 0$) and Alfvén discontinuities ($j \neq 0$, $[\bar{\rho}] = 0$). We omit the deduction of (52) from the general jump conditions for incompressible MHD (see, e.g. Reference [21]) and just refer to Reference [16]. Note that, unlike compressible Alfvén discontinuities, the pressure p can in principle have a jump on the surface of incompressible Alfvén discontinuity, but the total pressure q is, of course, continuous.

Consider a piecewise constant solution of (51), (52) for the planar Alfvén discontinuity with the equation $x_1 = 0$:

$$(\bar{q}, \bar{\mathbf{U}}^\pm) = (\bar{q}, \bar{\mathbf{v}}^\pm, \bar{\mathbf{H}}^\pm) \qquad \text{for} \quad x_1 \gtrless 0\,, \tag{53}$$

where, in view of (52),

$$\bar{\mathbf{H}}^\pm = (\bar{H}_1, \bar{H}_2^\pm, \bar{H}_3^\pm)\,, \quad \bar{\mathbf{v}}^\pm = (\bar{v}_1, \bar{v}_2^\pm, \bar{v}_3^\pm)\,, \quad \bar{v}_1 = \bar{H}_1\,, \quad [\bar{\mathbf{v}}_k] = [\bar{\mathbf{H}}_k]\,, \quad k = 2, 3\,.$$

Without loss of generality we suppose that $\bar{H}_1 > 0$. Linearization of equations (51), (52) about solution (53) yields the linear problem with constant coefficients:

$$\partial_t\mathbf{v} + (\bar{\mathbf{H}}^\pm, \nabla)\mathbf{w} + \nabla q = 0\,, \quad \partial_t\mathbf{H} - (\bar{\mathbf{H}}^\pm, \nabla)\mathbf{w} = 0\,, \quad \operatorname{div}\mathbf{v} = 0 \qquad \text{if} \quad \mathbf{x} \in \mathbb{R}^3_\pm\,, \tag{54}$$

$$[H_1] = [h_2]\partial_2 f + [h_3]\partial_3 f\,, \quad \partial_t f = w_1^+\,, \quad [\mathbf{w}] = 0\,, \quad [q] = 0 \qquad \text{if} \quad x_1 = 0\,. \tag{55}$$

Here $\mathbf{w} = (w_1, w_2, w_3) = \mathbf{v} - \mathbf{H}$, and in the derivation of (54), (55), we performed the Galilean transformation

$$\tilde{t} = t\,, \quad \tilde{x}_1 = x_1\,, \quad \tilde{x}_k = x_k - \left(\bar{v}_k^+ - \bar{H}_k^+\right)t\,, \quad k = 2, 3$$

(the tildes were removed).

Taking into account the constraint $\operatorname{div}\mathbf{H}=0$, from (54), (55) we easily obtain the following reduced problem in which boundary conditions do not contain the front f (cf. (22), (23)):

$$\partial_t\mathbf{w}+2(\bar{\mathbf{H}}^{\pm},\nabla)\mathbf{w}+\nabla q=0\,,\quad \operatorname{div}\mathbf{w}=0\qquad\text{if}\quad \mathbf{x}\in\mathbb{R}^3_{\pm}\,,\tag{56}$$

$$[q]=0\,,\quad [\mathbf{w}]=0\qquad\text{if}\quad x_1=0\,.\tag{57}$$

Let us now consider the problem which contains (56), (57) as a separate subproblem and which can be proved to be equivalent to the original problem (54), (55). The unknown for this problem is $\mathbf{V}=(z_2,z_3,H_1,\mathbf{W})$, where $z_k=v_k+H_k$ $(k=2,3)$, $\mathbf{W}=(q,\mathbf{w})$, and the interior equations include

$$\begin{aligned}&\partial_t z_k+\partial_k q=0\,,\quad k=2,3\,,\\ &\partial_t H_1+2\partial_1 H_1+\partial_2 z_2+\partial_3 z_3\\ &\qquad +(\bar{H}_1-1)(\partial_2 w_2+\partial_3 w_3)-\bar{H}_2^{\pm}\partial_2 w_1-\bar{H}_3^{\pm}\partial_3 w_1=0\qquad\text{for}\quad \mathbf{x}\in\mathbb{R}^3_{\pm}\end{aligned}\tag{58}$$

and equations (56). The boundary conditions for (58), (56) are relations (55). The functions z_2 and z_3 are, in some sense, “incompressible” analogues of “characteristic unknowns” for hyperbolic problems.

Since we used the constraint $\operatorname{div}\mathbf{H}=0$ while deducing (56), (58) from (54), we need the following proposition to show the equivalence of problems (54), (55) and (58), (56), (55).

Proposition 4.1 If the equations

$$\partial_1 H_1+\frac{1}{2}\partial_2(z_2-w_2)+\frac{1}{2}\partial_3(z_3-w_3)=\operatorname{div}\mathbf{H}=0\tag{59}$$

for $\mathbf{x}\in\mathbb{R}^3_{\pm}$ hold for the initial data for problem (58), (56), (55), then $\mathbf{V}$ also satisfies (59) for all $t>0$.

The proof of Proposition 4.1 is similar to that of Proposition 3.1 and we drop it.

Applying a Fourier-Laplace transform to system (58), (56) we obtain the system of ordinary differential equations in the form of (36) with $\hat{\mathbf{Y}}=(\hat{H}_1,\hat{\mathbf{W}})$ and

$$\mathcal{P}^{\pm}(s,i\boldsymbol{\omega})=\begin{pmatrix}-\dfrac{s}{2} & \dfrac{\mathbf{c}(\boldsymbol{\omega})}{s}+i\mathbf{b}^{\pm}(\boldsymbol{\omega})\\ \mathbf{0} & \mathcal{M}^{\pm}(s,i\boldsymbol{\omega})\end{pmatrix}.$$

Here

$$\mathbf{c}(\boldsymbol{\omega})=\big(-|\boldsymbol{\omega}|^2/2\,,\,\mathbf{0}\big)\,,\quad \mathbf{b}^{\pm}(\boldsymbol{\omega})=\frac{1}{2}\,\big(0\,,\,a^{\pm}(\boldsymbol{\omega})\,,\,(1-\bar{H}_1)\omega_2\,,\,(1-\bar{H}_1)\omega_3\big)\,,$$

$a^{\pm}(\boldsymbol{\omega})=\bar{H}_2^{\pm}\omega_2+\bar{H}_3^{\pm}\omega_3$, the matrices $\mathcal{M}^{\pm}$ have no singularities at $s=0$ and are written out in Appendix A.

Applying a Fourier-Laplace transform to the boundary conditions and omitting usual arguments (see Subsection 3.2), we finally get the Lopatinskii determinant $\det L$ with the matrix $L(s,i\boldsymbol{\omega})$ in the form of (43). The determinant of the matrix $\mathcal{L}(s, i\boldsymbol{\omega})$ is the Lopatinskii determinant for the reduced problem (56), (57) and for its explicit form we refer to Appendix A. That is, we arrive at the following proposition.

Proposition 4.2 Planar incompressible Alfvén discontinuities are never uniformly stable. They are violently unstable if and only if $\det \mathcal{L}(s, i\boldsymbol{\omega}) = 0$ for some $\eta > 0$, $(\xi, \boldsymbol{\omega}) \in \mathbb{R}^3$.

The linear stability of planar incompressible Alfvén discontinuities was shown in Reference [16] by normal modes analysis (see also Appendix A). Moreover, in Appendix A we prove that the reduced problem (56), (57) satisfies the *uniform* Lopatinskii condition, which is however violated for the full problem (54), (55) (cf. Proposition 4.2). At the same time, the stability of planar discontinuities is trivially proved because it follows from the conserved integral

$$I(t) = I(0) \tag{60}$$

which can be easily obtained for problem (56), (57), where $I(t) = \sum_{\pm} \|\mathbf{w}(t)\|^2_{L_2(\mathbb{R}^3_{\pm})}$.

For the original problem (54), (55) the a priori estimate (60) is an estimate in a *seminorm* of the solution $\mathbf{U}$. Nevertheless, we now show that it enables us to deduce an energy estimate for a norm of $\mathbf{U}$ as well as to estimate ∇q and the front f . This estimate is however very weak and we present its deduction just to demonstrate that there is not much hope to prove an estimate for the variable coefficients problem (see Remark 4.1 below).

Theorem 4.1 Solutions of (54), (55) obey the a priori estimates

$$\sum_{\pm} \|\mathbf{U}(t)\|_{L_2(\mathbb{R}^3_{\pm})} \leq C_1 \sum_{\pm} \left\{ \|\mathbf{U}(0)\|_{H^2_{\tan}(\mathbb{R}^3_{\pm})} + \|\partial_t \mathbf{U}(0)\|_{H^1_{\tan}(\mathbb{R}^3_{\pm})} \right\} , \tag{61}$$

$$\sum_{\pm} \|\nabla q\|_{L_2([0,T]\times\mathbb{R}^3_{\pm})} \leq C_2 \sum_{\pm} \left\{ \|\mathbf{U}(0)\|_{H^2_{\tan}(\mathbb{R}^3_{\pm})} + \|\partial_t \mathbf{U}(0)\|_{H^1_{\tan}(\mathbb{R}^3_{\pm})} \right\} , \tag{62}$$

$$\|f(t)\|_{H^1(\mathbb{R}^2)} \leq \|f(0)\|_{H^1(\mathbb{R}^2)} + C_3 \sum_{\pm} \left\{ \|\mathbf{U}(0)\|_{H^2_{\tan}(\mathbb{R}^3_{\pm})} + \|\partial_t \mathbf{U}(0)\|_{H^1_{\tan}(\mathbb{R}^3_{\pm})} \right\} \tag{63}$$

for any $t \in (0, T)$. Here $C_k = C_k(T)$ $(k = 1, 2, 3)$ are positive constants independent of the initial data;

$$\|(\cdot)(t)\|_{H^m_{\tan}(\mathbb{R}^3_{\pm})} = \sum_{|\alpha|\leq m} \|\partial^\alpha_y (\cdot)(t)\|^2_{L_2(\mathbb{R}^3_{\pm})} , \quad \partial^\alpha_y = \partial_2^{\alpha_1} \partial_3^{\alpha_2} , \quad \alpha = (\alpha_1, \alpha_2) .$$

Proof. By tangential differentiation (with respect to $\mathbf{y}$ and t) of the first equation in (56) we easily obtain the conserved integral

$$J(t) = J(0) , \tag{64}$$

where

$$J(t) = \sum_{\pm} \left\{ \|\mathbf{w}(t)\|^2_{H^2_{\tan}(\mathbb{R}^3_{\pm})} + \|\partial_t \mathbf{w}(t)\|^2_{H^1_{\tan}(\mathbb{R}^3_{\pm})} \right\} .$$

Multiplying the first equation in (56) by ∇q and taking into account the boundary conditions (57) and the equation $\operatorname{div}\mathbf{w}=0$, we have

$$Q(t)=\sum_{\pm}\|\nabla q(t)\|^2_{L_2(\mathbb{R}^3_\pm)}=-\sum_{\pm}\int_{\mathbb{R}^3_\pm}\left\{\operatorname{div}(q\partial_t\mathbf{w})+\operatorname{div}(2q(\bar{\mathbf{H}}^\pm,\nabla)\mathbf{w})\right\}d\mathbf{x}$$

$$=2\int_{\mathbb{R}^2}q^+\left[(\bar{\mathbf{H}},\nabla w_1)\right]d\mathbf{y}=2\int_{\mathbb{R}^2}q^+\left([\bar{H}_2]\partial_2w_1^++[\bar{H}_3]\partial_3w_1^+\right)d\mathbf{y}$$

$$=-2[\bar{H}_2]\int_{\mathbb{R}^2}w_1^+\partial_2q^+d\mathbf{y}-2[\bar{H}_3]\int_{\mathbb{R}^2}w_1^+\partial_3q^+d\mathbf{y}\,.$$

The boundary integral

$$\int_{\mathbb{R}^2}w_1^+\partial_2q^+d\mathbf{y}=-\sum_{\pm}\int_{\mathbb{R}^3_\pm}(\partial_1w_1\partial_2q+w_1\partial_1\partial_2q)d\mathbf{x}\,,$$

where from (56) one has:

$$-\partial_1\partial_2q=\partial_t\partial_2w_1-2\bar{H}_1(\partial_2^2w_2+\partial_2\partial_2w_3)+2\bar{H}_2^\pm\partial_2^2w_1+2\bar{H}_3^\pm\partial_2\partial_3w_1\,,\qquad \mathbf{x}\in\mathbb{R}^3_\pm\,.$$

Handling analogously the boundary integral $\int_{\mathbb{R}^2}w_1^+\partial_3q^+d\mathbf{y}$ and applying the Young inequality, we obtain

$$Q(t)\le c_1\left(\varepsilon Q(t)+\frac{1}{\varepsilon}J(t)\right)\,,$$

where $\varepsilon>0$ is a constant. Here and below $c_i=c_i(\bar{\mathbf{U}}^+,\bar{\mathbf{U}}^-,T)$ $(i=1,2,\ldots)$ are positive constants. Choosing ε small enough we get

$$Q(t)\le c_2J(t)=c_2J(0)$$

that yields (62).

It follows from (56) that

$$\sum_{\pm}\|\partial_1\mathbf{w}(t)\|^2_{L_2(\mathbb{R}^3_\pm)}\le c_3(Q(t)+J(t))\le c_4J(0)\,. \tag{65}$$

At last, multiplying the second equation in (54) by $\mathbf{H}$ and using then (64) and (65) we deduce the estimate

$$\sum_{\pm}\|\mathbf{H}(t)\|^2_{L_2(\mathbb{R}^3_\pm)}\le c_5J(0)\,. \tag{66}$$

Using the equality $\mathbf{v}=\mathbf{w}+\mathbf{H}$ and inequality (66) we estimate the velocity $\mathbf{v}$ that gives (61). Estimate (63) for the front f is trivially follows from the second boundary condition in (55) by applying trace's property for w_1 and taking into account the equation $\operatorname{div}\mathbf{w}=0$. $\square$

Remark 4.1 The a priori estimates (61)–(63) are very weak in many respects. Moreover, since q is an "elliptic" unknown, the differentiation of the interior equations and the boundary conditions with respect to t is an absolutely forbidden trick for the case of variable coefficients (see discussion in Reference [13]). At the same time, the main difficulty to carry over the a priori estimates to the case of variable coefficients is, of course, the "non-ellipticity of the front." We refer, for example, to References [12, 13, 14] where the fact that the symbol associated with the front of discontinuity was elliptic played the crucial role in the variable coefficient analysis for current-vortex sheets and for 2D vortex sheets.

Appendix A

In this Appendix we show that the reduced problem (56), (57) for planar incompressible Alfvén discontinuities satisfies the uniform Lopatinskii condition.

Rearranging the equations of system (56) and applying to them a Fourier-Laplace transform, we obtain the system of ordinary equations in the form of (35) with $\hat{\mathbf{W}} = (\hat{q}, \hat{\mathbf{w}})$ and

$$\mathcal{M}^{\pm}(s, i\boldsymbol{\omega}) = \begin{pmatrix} 0 & -s-2ia^{+} & 2i\bar{H}_1\omega_2 & 2i\bar{H}_1\omega_3 \\ 0 & 0 & -i\omega_2 & -i\omega_3 \\ -\dfrac{i\omega_2}{2\bar{H}_1} & 0 & -\dfrac{s+2ia^{\pm}}{2\bar{H}_1} & 0 \\ -\dfrac{i\omega_3}{2\bar{H}_1} & 0 & 0 & -\dfrac{s+2ia^{\pm}}{2\bar{H}_1} \end{pmatrix},$$

where $a^{\pm}(\boldsymbol{\omega}) = \bar{H}_2^{\pm}\omega_2 + \bar{H}_3^{\pm}\omega_3$. The eigenvalues $\lambda^{\pm}$ of the matrices $\mathcal{M}^{\pm}$ are easily calculated:

$$\lambda_1^{\pm} = \lambda_2^{\pm} = -\frac{s+2ia^{\pm}}{2\bar{H}_1}, \quad \lambda_3^{\pm} = -|\boldsymbol{\omega}|, \quad \lambda_4^{\pm} = |\boldsymbol{\omega}|.$$

We need to find the bases of eigenspaces for λ_1^{+}, λ_3^{+}, and λ_4^{-}. Supposing that $\eta > 0$ and $\lambda_1^{+} \neq \lambda_3^{+}$ we calculate two eigenvectors $\boldsymbol{\gamma}_{1,2}$ for the double eigenvalue λ_1^{+} and the eigenvectors $\boldsymbol{\gamma}_3$ and $\boldsymbol{\gamma}_4$ for the simple eigenvalues λ_3^{+} and λ_4^{-}:

$$\boldsymbol{\gamma}_1 = (0\,, i\omega_2\,, -\lambda_1^{+}\,, 0)\,, \quad \boldsymbol{\gamma}_2 = (0\,, i\omega_3\,, 0\,, -\lambda_1^{+})\,,$$

$$\boldsymbol{\gamma}_3 = (s+2ia^{+} - 2\bar{H}_1|\boldsymbol{\omega}|\,, |\boldsymbol{\omega}|\,, -i\omega_2\,, -i\omega_3)\,, \quad \boldsymbol{\gamma}_4 = (s+2ia^{-} + 2\bar{H}_1|\boldsymbol{\omega}|\,, -|\boldsymbol{\omega}|\,, -i\omega_2\,, -i\omega_3)\,.$$

Here we also suppose that $\boldsymbol{\omega} \neq 0$. This is natural assumption for the case of incompressible fluid. Indeed, the Lopatinskii determinant formally vanishes for $\boldsymbol{\omega} = 0$ but this is just because the total pressure q is an "elliptic" unknown and determined up to an arbitrary function of t. The fact that the Lopatinskii determinant is zero for $\boldsymbol{\omega} = 0$ does not imply the existence of a 1D Hadamard-type ill-posedness example for the "hyperbolic" unknown $\mathbf{w}$. Since the Lopatinskii determinant is a homogenous function with respect to $(s, \boldsymbol{\omega})$, without loss of generality we suppose that $|\boldsymbol{\omega}| = 1$.

In view of the boundary conditions (57), the matrix $\mathcal{L}$ whose determinant is the Lopatinskii determinant is formed by the column vectors $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\gamma}_3$, and $-\boldsymbol{\gamma}_4$. The Lopatinskii determinant is then easily computed:

$$\det \mathcal{L} = 2\lambda_1^+ \Big\{ \lambda_1^+ \left(s + i(a^+ + a^-) \right) + 2\bar{H}_1 + i(a^- - a^+) \Big\} .$$

The Lopatinskii determinant vanishes when either $\lambda_1^+ = 0$ or the expression in curly braces is zero. For the last case, omitting detailed calculations we find two roots:

$$s = s_1 = 2(\bar{H}_1 - ia^+) , \quad s = s_2 = -2\bar{H}_1 - i(a^+ + a^-) .$$

The "unstable" root s_1 is actually a "glancing mode" for which $\lambda_1^+ = \lambda_2^+ = \lambda_3^+ = -1$ (recall that $|\boldsymbol{\omega}| = 1$). Analogously, for the neutral mode $s = -2ia^+$ at which $\lambda_1^+ = 0$ the vectors $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ become linearly dependent. If the vectors $\boldsymbol{\gamma}_{1,2,3}$ for the modes $s = s_1$ and $s = -2ia^+$ are calculated properly (we drop calculations), it can be shown that in the first case $\det \mathcal{L} = -2i\bar{H}_1 \neq 0$ and in the second case $\det \mathcal{L} = 4\bar{H}_1 + 2i(a^- - a^+) \neq 0$. Thus, the only genuine zero of the Lopatinskii determinant is the stable root $s = s_2$. Hence, problem (56), (57) satisfies the uniform Lopatinskii condition.

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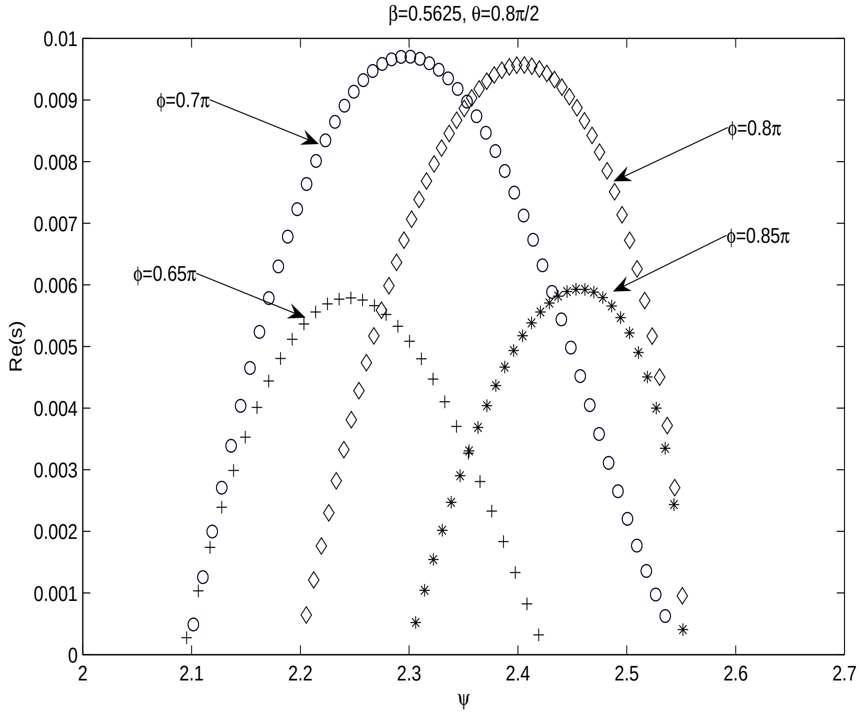
Figure captions

Figure 1. Typical graphs of $\eta = \mathrm{Re}(s)$ as function of ψ.

Figure 2. $\eta_m = \max_{\psi} \mathrm{Re}(s)$ as function of ϕ.

Figure 3. Instability domain in ϕ-θ plane.

Figure 4. $\mathrm{Re}(s)$ as function of $\beta^{1/2}$ for $\phi = \pi$, $\psi = \pi$.



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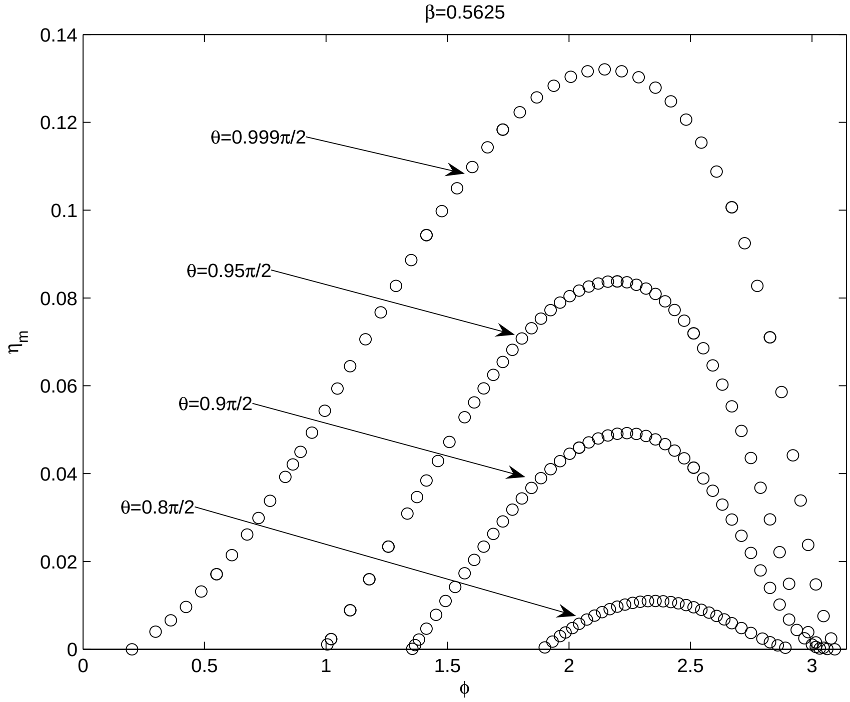

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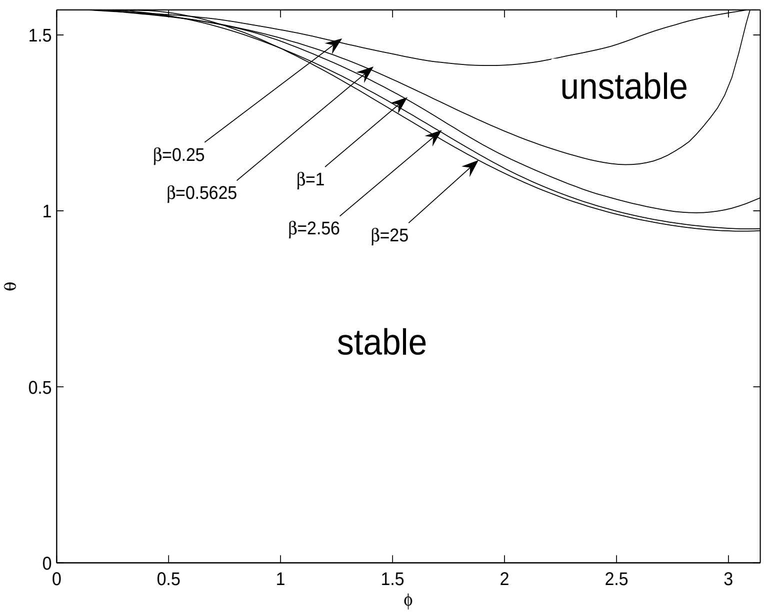
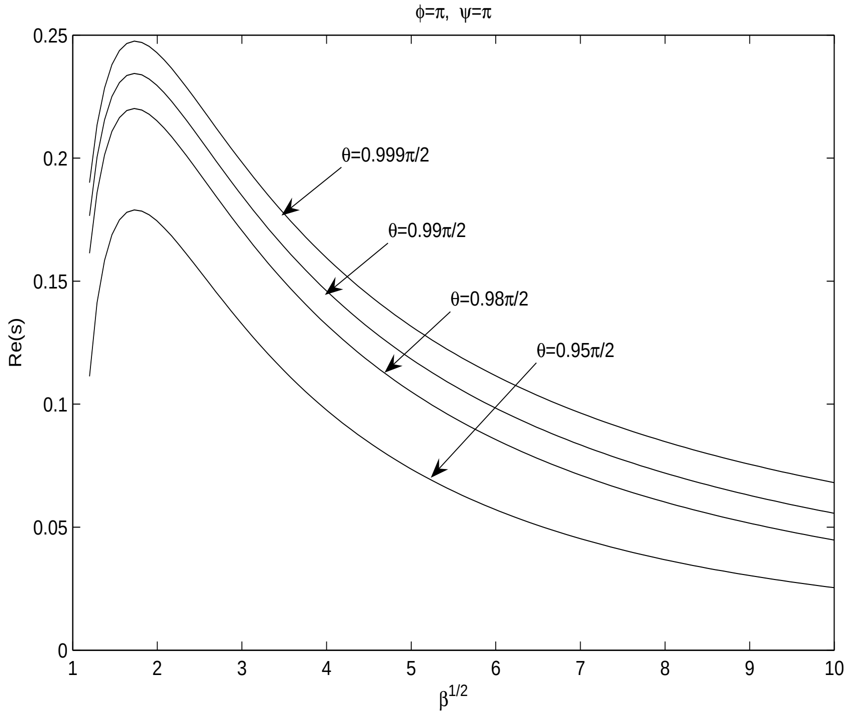


Figure 3: Instability domain in ϕ-θ plane.



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