Note on the three-dimensional stability of steady magnetohydrodynamic flows of an ideal fluid

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ABSTRACT

The stability of steady magnetohydrodynamic flows of an ideal incompressible fluid to small three-dimensional perturbations is studied. Two new sufficient conditions for linear stability of steady MHD flows are obtained by the energy method.

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I. Introduction

In this note, we consider the stability of steady magnetohydrodynamic flows of an ideal incompressible fluid to small three-dimensional perturbations. We exploit the approach first proposed by Bernstein *et al* (see Ref. 1) for analysis of the stability of magnetostatic equilibria and later generalized by Frieman and Rotenberg (see Ref. 2) to the case of steady MHD flows.

The idea of the method is to construct a quadratic in perturbations functional which is conserved by the linearized equations. This has been done by Frieman and Rotenberg who obtained the conserved energy functional for a general steady basic state. Though this result is known for almost forty years now, not many stability criteria seem to have been obtained with its help. So far, it is known only that the energy integral for linearized equations is non-negative definite for some magnetostatic equilibria (in this case it reduces to the second variation obtained in Ref. 1) and for a relatively trivial situation when in the basic state the magnetic field $\mathbf{H}(\mathbf{x})$ corresponds to a stable magnetostatic equilibrium and the velocity is given by $\mathbf{U}(\mathbf{x}) = \lambda\mathbf{H}(\mathbf{x})$ where λ is constant and $|\lambda| < 1$ (see e.g. Ref. 3), so that again the energy functional is effectively reduced to that corresponding to a magnetostatic equilibrium. It is therefore interesting to find out whether there are any non-trivial steady MHD flows that are stable to three-dimensional perturbations and that are *not* reducible to any magnetostatic equilibrium.

This question is addressed in the paper. Our analysis results in explicit stability criteria for two classes of non-trivial steady MHD flows. Namely, we obtain sufficient conditions for stability to small three-dimensional perturbations of (i) steady flows with $\mathbf{H} = H_0(x,y)\mathbf{e}_z$ and $\mathbf{U} = \lambda(x,y)\mathbf{H}$ where $H_0(x,y)$ and $\lambda(x,y)$ are arbitrary functions, and of (ii) general two-dimensional steady flows.

The plan of the paper is as follows. In section 2 we first formulate the linearized stability problem for an arbitrary steady state, and then, following the procedure of Ref. 2, we obtain the energy integral which is conserved by the linearized equations. In section 3 we analyse the properties of this integral invariant and formulate the stability criteria.

II. Basic equations

Consider an incompressible, inviscid and perfectly conducting fluid contained in a domain $\mathcal{D}$ with fixed boundary $\partial\mathcal{D}$. Let $\mathbf{u}(\mathbf{x},t)$ be the velocity field, $\mathbf{h}(\mathbf{x},t)$ the magnetic field (in Alfven velocity units), $p(\mathbf{x},t)$ the pressure (divided by density), and $\mathbf{j} = \nabla\times\mathbf{h}$ the current density. Then the governing equations are:

$$\partial\mathbf{u}/\partial t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \mathbf{j}\times\mathbf{h}\,, \quad (1)$$

$$\mathbf{h}_t = (\mathbf{h}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{h} \equiv [\mathbf{u},\mathbf{h}]\,, \quad (2)$$

$$\mathbf{j} = \nabla\times\mathbf{h}\,, \quad \nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{h} = 0\,. \quad (3)$$

Equation (2) implies that $\mathbf{h}$ is frozen in the fluid, its flux through any material surface being conserved. We suppose that the boundary $\partial\mathcal{D}$ is perfectly conducting and that the magnetic field $\mathbf{h}$ does not penetrate through $\partial\mathcal{D}$. The boundary conditions are then

$$\mathbf{n}\cdot\mathbf{u} = 0\,, \quad \mathbf{n}\cdot\mathbf{h} = 0 \quad \text{on} \quad \partial\mathcal{D}\,. \quad (4)$$

We suppose further that at $t = 0$, the fields $\mathbf{u}$ and $\mathbf{h}$ are smooth and satisfy (3) and (4), but are otherwise arbitrary. (Throughout the paper the term 'smooth' means smooth enough so as to justify all our mathematical manipulations.)

Let

$$\mathbf{u} = \mathbf{U}(\mathbf{x})\,, \quad \mathbf{h} = \mathbf{H}(\mathbf{x}) \tag{5}$$

be a steady solution of the problem (1)-(4) whose stability will be studied. We shall refer to this solution as the basic state. Let $\mathbf{u}'(\mathbf{x}, t)$ and $\mathbf{h}'(\mathbf{x}, t)$ be infinitesimal perturbation to the basic state (5). Linearized equations governing the evolution of $\mathbf{u}'(\mathbf{x}, t)$ and $\mathbf{h}'(\mathbf{x}, t)$ are

$$\mathbf{u}'_t + (\mathbf{u}' \cdot \nabla)\mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{u}' = -\nabla p' + \mathbf{j}' \times \mathbf{H} + \mathbf{J} \times \mathbf{h}'\,, \quad \nabla \cdot \mathbf{u}' = 0\,, \tag{6a}$$

$$\mathbf{h}'_t = [\mathbf{u}', \mathbf{H}] + [\mathbf{U}, \mathbf{h}']\,, \quad \nabla \cdot \mathbf{h}' = 0 \quad \text{in} \quad \mathcal{D}\,; \tag{6b}$$

$$\mathbf{u}' \cdot \mathbf{n} = \mathbf{h}' \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{D}\,. \tag{6c}$$

From here on, 'primes' will be omitted to simplify the notations. Following Ref. 2, we introduce the Lagrangian displacement $\boldsymbol{\xi}(\mathbf{x}, t)$ of a fluid particle (i.e. the displacement at the time t of a fluid particle in perturbed flow relative to its position $\mathbf{x}$ (at the time t) in unperturbed flow) satisfying the equation

$$\mathbf{u} = \boldsymbol{\xi}_t + [\boldsymbol{\xi}, \mathbf{U}]\,. \tag{7}$$

Equations (6b) and (7) have a consequence that

$$(\mathbf{h} - [\boldsymbol{\xi}, \mathbf{H}])_t = [\mathbf{U}, (\mathbf{h} - [\boldsymbol{\xi}, \mathbf{H}])]\,.$$

It follows that if the relation

$$\mathbf{h} = [\boldsymbol{\xi}, \mathbf{H}] \tag{8}$$

is satisfied at $t = 0$ then it holds for any $t > 0$. This allows us to introduce a special class of *isomagnetic perturbations* as such perurbations that satisfy the relation (8). And only such perturbations will be considered.

Substitution of (7), (8) in (6a) yields the following equation for the Lagrangian displacement $\boldsymbol{\xi}$:

$$\boldsymbol{\xi}_{tt} + 2(\mathbf{U} \cdot \nabla)\boldsymbol{\xi}_t = \hat{K}\boldsymbol{\xi} - \nabla\alpha\,, \tag{9}$$

where $\hat{K}$ is symmetric operator defined by the formula

$$\hat{K}\boldsymbol{\xi} \equiv \mathbf{U} \times \mathrm{curl}[\boldsymbol{\xi}, \mathbf{U}] + [\boldsymbol{\xi}, \mathbf{U}] \times \boldsymbol{\Omega} - \mathbf{H} \times \mathrm{curl}[\boldsymbol{\xi}, \mathbf{H}] - [\boldsymbol{\xi}, \mathbf{H}] \times \mathbf{J}\,, \tag{10}$$

and where

$$\alpha \equiv p' + \mathbf{U} \cdot [\boldsymbol{\xi}, \mathbf{U}]$$

is a function which is determined from obvious conditions

$$\nabla \cdot \boldsymbol{\xi} = 0 \quad \text{in} \quad \mathcal{D}\,, \quad \boldsymbol{\xi} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{D}\,. \tag{11}$$

Eq. (9) represents the 'incompressible version' of the equation obtained by Frieman and Rotenberg (Ref. 2).

III. Sufficient conditions for stability

Taking a dot-product of Eq. (9) with $\boldsymbol{\xi}_t$ and integrating over $\mathcal{D}$, we obtain

$$\frac{d}{dt}\int_{\mathcal{D}}\Big(\tfrac{1}{2}\boldsymbol{\xi}_t^2-\tfrac{1}{2}\boldsymbol{\xi}\cdot\hat{K}\boldsymbol{\xi}\Big)\,dV=0\,,$$

i.e. the quadratic integral

$$E\equiv\int_{\mathcal{D}}\Big(\tfrac{1}{2}\boldsymbol{\xi}_t^2-\tfrac{1}{2}\boldsymbol{\xi}\cdot\hat{K}\boldsymbol{\xi}\Big)\,dV \tag{12}$$

is conserved by linearized equations and may be interpreted as the energy of the linearized problem.

Evidently, E as a quadratic functional of $\boldsymbol{\xi}_t$ and $\boldsymbol{\xi}$ is positive definite if the 'potential energy'

$$W\equiv-\tfrac{1}{2}\int_{\mathcal{D}}\boldsymbol{\xi}\cdot\hat{K}\boldsymbol{\xi}\,dV=\tfrac{1}{2}\int_{\mathcal{D}}\Big([\boldsymbol{\xi},\mathbf{H}]^2+[\boldsymbol{\xi},\mathbf{H}]\cdot(\mathbf{J}\times\boldsymbol{\xi})-[\boldsymbol{\xi},\mathbf{U}]^2-[\boldsymbol{\xi},\mathbf{U}]\cdot(\boldsymbol{\Omega}\times\boldsymbol{\xi})\Big)\,dV \tag{13}$$

is positive definite. Positive definiteness of E, in turn, means that E can be taken as a norm to measure the deviation of perturbed flow from unperturbed one, and the conservation of E by (9), (11) implies the stability of the basic state to small perturbations. However, as we shall show, the functional W is never (strictly) positive definite.

First we note that for a particular class of perturbations (satisfying generalized isovorticity condition [4]) the corresponding integral invariant (of the linearized problem) is indefinite in sign provided that there is a region in the flow domain where $\mathbf{U}$ and $\boldsymbol{\Omega}$ are both non-zero and non-parallel to $\mathbf{H}$ (see Ref. 3). Moreover, in a somewhat different from Ref. 3 and Ref. 4 variational approach of Hameiri (see Ref. 5), the corresponding Lyapunov functional is indefinite in sign if $\mathbf{U}$ is not parallel to $\mathbf{H}$ or $|\mathbf{U}|>|\mathbf{H}|$ somewhere in $\mathcal{D}$. In our case the same arguments as in Refs. 3 and 5 show that W is indefinite in sign if there is a region in the flow domain where $\mathbf{U}$ is non-zero and non-parallel to $\mathbf{H}$. Indeed, in this region one can choose a function $\boldsymbol{\xi}(\mathbf{x})$ which rapidly oscillates along $\mathbf{U}$ and slowly varies in $\mathbf{H}$-direction, and which vanishes outside the region. For such a $\boldsymbol{\xi}$, the leading term in W has the form

$$W\sim-\tfrac{1}{2}\int_{\mathcal{D}}\Big((\mathbf{U}\cdot\nabla)\boldsymbol{\xi}\Big)^2\mathrm{d}V$$

and is obviously negative, so that W can take negative values.

Consider now the situation when $\mathbf{U}$ is parallel to $\mathbf{H}$ everywhere in $\mathcal{D}$. In this case, W is never strictly positive definite: for any $\boldsymbol{\xi}=g(\mathbf{x},t)\mathbf{H}(\mathbf{x})$ with arbitrary function $g(\mathbf{x},t)$ the functional W vanishes. Thus, W may be at most a positive semi-definite functional. Suppose now that W is positive semi-definite. The solutions of Eqs. (9), (11) such that $W=0$ may, in principle, grow with time. Nevertheless, it follows from the conservation of E and from the non-negativeness of W that $|\boldsymbol{\xi}|$ cannot grow faster than linearly with time. It is also known that linear with time growth of $\boldsymbol{\xi}$ does not always mean physical instability because the corresponding Eulerian velocity perturbation may remain bounded (see e.g. Ref. 6). We shall not discuss this subtle question here. In what follows we shall assume that *the non-negativeness of W is sufficient for linear stability*[7].

Consider now the functional W given by Eq. (13). Obviously, there are situations when W is positive semi-definite. For instance, it is well-known that it is so for certain magnetostatic equilibria ($\mathbf{U} \equiv 0$) (see e.g. Ref. 8). On the other hand, the results of Refs. 3, 5 and our simple arguments above indicate that for steady MHD flows with non-parallel $\mathbf{U}$ and $\mathbf{H}$ the 'potential energy' (13) is never of definite sign. Therefore, in what follows, we restrict ourselves to the case of flow and magnetic field everywhere parallel, i.e. we suppose that in the steady state (5)

$$\mathbf{U}(\mathbf{x}) = \lambda(\mathbf{x})\mathbf{H}(\mathbf{x}) \quad \text{in} \ \ \mathcal{D} \tag{14}$$

with some smooth function $\lambda(\mathbf{x})$. From the incompressibility condition, we obtain

$$\mathbf{H} \cdot \nabla\lambda = 0\,. \tag{15}$$

Eqs. (1), (14) and (15) have a consequence that

$$\nabla\Big(P + \tfrac{1}{2}\mathbf{H}^2\Big) = (1-\lambda^2)(\mathbf{H}\cdot\nabla)\mathbf{H}\,. \tag{16}$$

Also, we have

$$[\boldsymbol{\xi}, \mathbf{U}] = \lambda[\boldsymbol{\xi}, \mathbf{H}] + \nabla\lambda \times (\boldsymbol{\xi} \times \mathbf{H}) = \lambda[\boldsymbol{\xi}, \mathbf{H}] - \mathbf{H}(\boldsymbol{\xi}\cdot\nabla\lambda)\,, \tag{17}$$

$$\boldsymbol{\Omega} \times \boldsymbol{\xi} = \lambda\mathbf{J} \times \boldsymbol{\xi} - \boldsymbol{\xi} \times (\nabla\lambda \times \mathbf{H}) = \lambda\mathbf{J}\times\boldsymbol{\xi} - \nabla\lambda(\mathbf{H}\cdot\boldsymbol{\xi}) + \mathbf{H}(\boldsymbol{\xi}\cdot\nabla\lambda)\,. \tag{18}$$

Substitution of Eqs. (17), (18) in (13) yields

$$W = \tfrac{1}{2}\int_{\mathcal{D}} \Big\{(1-\lambda^2)\Big([\boldsymbol{\xi},\mathbf{H}]^2 + [\boldsymbol{\xi},\mathbf{H}]\cdot(\mathbf{J}\times\boldsymbol{\xi})\Big) - \lambda(\boldsymbol{\xi}\cdot\nabla\lambda)\,\boldsymbol{\xi}\cdot(\mathbf{J}\times\mathbf{H}) + \lambda(\boldsymbol{\xi}\cdot\nabla\lambda)\,\mathbf{H}\cdot[\boldsymbol{\xi},\mathbf{H}] + \lambda(\mathbf{H}\cdot\boldsymbol{\xi})\,\nabla\lambda\cdot[\boldsymbol{\xi},\mathbf{H}]\Big\}\,dV\,. \tag{19}$$

It may be shown that

$$X \equiv \lambda(\boldsymbol{\xi}\cdot\nabla\lambda)\,\mathbf{H}\cdot[\boldsymbol{\xi},\mathbf{H}] + \lambda(\mathbf{H}\cdot\boldsymbol{\xi})\,\nabla\lambda\cdot[\boldsymbol{\xi},\mathbf{H}]$$
$$= (\mathbf{H}\cdot\nabla)\Big(\lambda(\boldsymbol{\xi}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot\mathbf{H})\Big) - \lambda(\boldsymbol{\xi}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot\nabla)\Big(\tfrac{1}{2}\mathbf{H}^2\Big) - \lambda(\boldsymbol{\xi}\cdot\nabla\lambda)\Big(\boldsymbol{\xi}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}\Big)\,.$$

With help of this identity, Eq. (19) may be written in the form

$$W = \tfrac{1}{2}\int_{\mathcal{D}} \Big\{(1-\lambda^2)\Big([\boldsymbol{\xi},\mathbf{H}]^2 + [\boldsymbol{\xi},\mathbf{H}]\cdot(\mathbf{J}\times\boldsymbol{\xi})\Big) - 2\lambda(\boldsymbol{\xi}\cdot\nabla\lambda)\Big(\boldsymbol{\xi}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}\Big)\Big\}d\tau\,. \tag{20}$$

Now it is clear that the following stability criterion is valid.

Proposition 1. *Steady MHD flow satisfying (14) is stable to small three-dimensional perturbations provided that the quadratic functional W, given by eqn. (20), is non-negative definite.*

Below we consider three particular classes of steady MHD flows for which we formulate sufficient conditions for stability in explicit form.

A. *Flows reducible to magnetostatic equilibria.* The simplest special class of flows (14) for which W is non-negative definite is well-known (see e.g. Ref. 3). It comprises flows with $\lambda = const$, $|\lambda| < 1$. Indeed, for such flows the quadratic functional (20) simplifies to

$$W = (1-\lambda^2)W_0\,, \quad W_0 \equiv \tfrac{1}{2}\int_{\mathcal{D}}\Big([\boldsymbol{\xi},\mathbf{H}]^2 + [\boldsymbol{\xi},\mathbf{H}]\cdot(\mathbf{J}\times\boldsymbol{\xi})\Big)d\tau\,. \tag{21}$$

The sign of W is thus determined by the value of λ and by the sign of W_0.

Note that the integral W_0 coincides with the well-known potential energy of Ref. 1 which is related to the stability of magnetostatic equilibria and which has been studied by numerous authors (e.g. Ref. 8). Moreover, for steady flows (14) with constant λ Eq. (16) reduces to the equation

$$\mathbf{J}\times\mathbf{H} = \nabla\tilde{P}\,, \quad \tilde{P} \equiv \frac{1}{1-\lambda^2}(P + \tfrac{1}{2}\mathbf{U}^2)\,,$$

which evidently coincides with the equation describing a magnetoststic equlibrium with the magnetic field $\mathbf{H}$ and the pressure $\tilde{P}$. In other words, there is one-to-one correspondence between steady flows (14) (with constant λ) and magnetostatic equilibria with the same magnetic field and the modified pressure $\tilde{P}$. Therefore, we conclude that *for any stable (in the sense of non-negative definite W_0) magnetostatic equilibrium the corresponding steady flow of the form (14) with constant λ is also linearly stable provided that $|\lambda| < 1|$.*

B. *Parallel flow and field.* Let the flow domain $\mathcal{D}$ be an infinite cylinder (of arbitrary cross-section) parallel to the z-axis. We suppose that in the basic state both the velocity and the magnetic field are along the axis of the cylinder and depend only upon transverse coordinates x, y, i.e.

$$\mathbf{H} = H_0(x,y)\mathbf{e}_z\,, \quad \mathbf{U} = \lambda(x,y)\mathbf{H}\,. \tag{22}$$

Then,

$$(\mathbf{H}\cdot\nabla)\mathbf{H} = 0$$

and

$$[\boldsymbol{\xi},\mathbf{H}] = H_0(\mathbf{e}_z\cdot\nabla)\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\nabla H_0)\mathbf{e}_z\,,$$

whence, after some algebra, we obtain

$$W = \tfrac{1}{2}\int_{\mathcal{D}}(1-\lambda^2)H_0^2\Big((\mathbf{e}_z\cdot\nabla)\boldsymbol{\xi}\Big)^2 d\tau\,. \tag{23}$$

Here, we suppose that the vector fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ decay sufficiently rapidly as $|z|\to\infty$, so that the integral in (23) exists[8]. Thus, we have obtained the following:

Proposition 2. *The steady state (22) is linearly stable provided that $|\lambda(x,y)| \le 1$ in $\mathcal{D}$.*

C. *General two-dimensional basic state.* Let the flow domain be the same as in the previous example, and, in addition to the condition (14), suppose that both the velocity and the magnetic field are independent of z and parallel to the x, y-plane. Then

$$\mathbf{H} = \nabla A \times \mathbf{e}_z\,, \quad \mathbf{U} = \Psi'(A)\mathbf{H}\,, \quad A = A(x,y) \tag{24}$$

where A is the flux function for the magnetic field and Ψ is the stream function for the velocity. Note that, according to (14) and (24), $\lambda = \Psi'(A)$.

In the basic state,

$$\nabla^2 A - \Psi'(A)\nabla^2\Psi = G(A)$$

for some function $G(A)$. After some manipulations, this equation may be rewritten in the following equivalent form

$$(1-\lambda^2)\nabla^2 A - \lambda\lambda'\mathbf{H}^2 = G(A)\,, \tag{25}$$

where $\lambda' \equiv \mathrm{d}\lambda/\mathrm{d}A = \Psi''$. Also, in the basic state (24),

$$\mathbf{J} = -\nabla^2 A\mathbf{e}_z\,, \quad (\mathbf{H}\cdot\nabla)\mathbf{H} = -\nabla^2 A\nabla A + \nabla(\mathbf{H}^2/2)\,. \tag{26}$$

We assume that $|\nabla A| \neq 0$ in $\mathcal{D}$ and define a unit vector $\boldsymbol{\nu}$

$$\boldsymbol{\nu} \equiv \nabla A/|\nabla A|\,. \tag{27}$$

It may be shown by standard but tedious manipulations (see Appendix) that the integral (20) for the basic state (24) takes the form

$$W = W_1 + W_2\,, \quad W_1 = \tfrac{1}{2}\int_{\mathcal{D}}(1-\lambda^2)\Big([\boldsymbol{\xi},\mathbf{H}] + (\boldsymbol{\xi}\cdot\boldsymbol{\nu})\mathbf{J}\times\boldsymbol{\nu}\Big)^2 d\tau$$
$$W_2 = -\int_{\mathcal{D}}\Big((1-\lambda^2)(\mathbf{J}\times\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} + \lambda\lambda'|\mathbf{H}|(\boldsymbol{\nu}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H})\Big)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2 d\tau\,. \tag{28}$$

Further transformation of W_2 with the help of (25) and (26) results in

$$W_2 = \tfrac{1}{2}\int_{\mathcal{D}}(1-\lambda^2)\Big(\nabla^2 A - \frac{\lambda\lambda'}{1-\lambda^2}\mathbf{H}^2\Big)\Big(-\nabla^2 A + \frac{\nabla A\cdot\nabla(\mathbf{H}^2)}{2\mathbf{H}^2}\Big)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2 d\tau\,. \tag{29}$$

The following statement is a direct consequence of (28), (29).

Proposition 3. *The steady state (24) is stable to small three-dimensional perturbations provided that the flow is sub-alfvenic, i.e.* $|\lambda| = |\Psi'(A)| < 1$, *and either of the inequalities*

$$\Phi(A)\mathbf{H}^2 \leq \nabla^2 A \leq \frac{\nabla A\cdot\nabla(\mathbf{H}^2)}{2\mathbf{H}^2}\,, \tag{30}$$

$$\frac{\nabla A\cdot\nabla(\mathbf{H}^2)}{2\mathbf{H}^2} \leq \nabla^2 A \leq \Phi(A)\mathbf{H}^2 \tag{31}$$

is satisfied, where

$$\Phi(A) \equiv \frac{\lambda\lambda'}{1-\lambda^2}\,.$$

Note that for plane parallel flow and magnetic field when $\mathbf{H} = H_0(y)\mathbf{e}_x$, $\mathbf{U} = \lambda(y)\mathbf{H}$ the integral W_2 given by (29) is identically equal to zero, so that, in agreement with Proposition 1, the stability condition reduces to the single condition $|\lambda| < 1$.

To clarify the conditions (30), (31), we consider the following simple example. Let in the basic state (24)

$$\mathbf{H} = H_0(r)\mathbf{e}_\theta\,, \quad H_0(r) = -A'(r)\,, \quad \mathbf{U} = \lambda(r)\mathbf{H} \quad \text{for} \;\; a \le r \le b\,. \tag{32}$$

Then the condition (30) is equivalent to the inequalities

$$H_0(r) \ge 0 \quad \text{and} \quad H_0'(r) + \frac{1-\chi(r)}{r}H_0(r) \le 0 \quad \text{for} \;\; a \le r \le b\,, \tag{33a}$$

while (31) simplifies to

$$H_0(r) \le 0 \quad \text{and} \quad H_0'(r) + \frac{1-\chi(r)}{r}H_0(r) \ge 0 \quad \text{for} \;\; a \le r \le b\,. \tag{33b}$$

In Eqs. (33a,b), $\chi(r) \equiv r\lambda(r)\lambda'(r)/(1-\lambda^2)$. Clearly, either (33a) or (33b) is satisfied provided that

$$H_0(r)\Big(H_0'(r) + \frac{1-\chi(r)}{r}H_0(r)\Big) \le 0 \quad \text{for} \;\; a \le r \le b\,. \tag{34}$$

We may conclude that *if inequality (34) is satisfied then the steady state (32) is linearly stable.*

IV. Conclusions

We studied the linear stability of steady magnetohydrodynamic flows of an incompressible fluid. We have shown that there are non-trivial steady flows for which the integral invariant (12) of the linearized problem (9), (11) is positive semi-definite. From this fact we have concluded that these flows are linearly stable and formulated the appropriate stability criteria (Propositions 2 and 3). Strictly speaking (see the discussion in the beginning of section III), our stability conditions ensure only that there are no perturbations which grow with time faster than linearly. An important open question is whether the linear growth of $|\boldsymbol{\xi}|$ implies the growth of physical fields such as the velocity and the magnetic field. This is a problem for further investigation.

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Appendix: derivation of the formula (28)

Consider the integral

$$I = \tfrac{1}{2}\int_{\mathcal{D}}(1-\lambda^2)\Big(\mathbf{h}^2 + \mathbf{h}\cdot(\mathbf{J}\times\boldsymbol{\xi})\Big)\mathrm{d}\tau\,, \qquad (A1)$$

which enters the expression (20) (recall that, according to (8), $\mathbf{h} = [\boldsymbol{\xi}, \mathbf{H}]$). We start with some transformations of this integral. First, following Ref. 1, we shall prove the identity

$$2(\mathbf{J}\times\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} = \mathbf{J}^2 + (\mathbf{J}\times\boldsymbol{\nu})\cdot(\nabla\times(\boldsymbol{\nu}\times\mathbf{H})) + \mathbf{H}\cdot(\mathbf{J}\times\boldsymbol{\nu})\mathrm{div}\boldsymbol{\nu}\,, \qquad (A2)$$

where $\boldsymbol{\nu}$ is defined by (27).

It follows from (24), (27) that $\mathbf{H}\cdot\boldsymbol{\nu} = 0$. Therefore,

$$0 = \nabla(\mathbf{H}\cdot\boldsymbol{\nu}) = (\boldsymbol{\nu}\cdot\nabla)\mathbf{H} + (\mathbf{H}\cdot\nabla)\boldsymbol{\nu} + \mathbf{H}\times(\nabla\times\boldsymbol{\nu}) + \boldsymbol{\nu}\times\mathbf{J}\,.$$

Hence,

$$\begin{aligned}\nabla\times(\boldsymbol{\nu}\times\mathbf{H}) + \mathbf{H}\mathrm{div}\boldsymbol{\nu} &= (\mathbf{H}\cdot\nabla)\boldsymbol{\nu} - (\boldsymbol{\nu}\cdot\nabla)\mathbf{H}\\ &= 2(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} + \mathbf{H}\times(\nabla\times\boldsymbol{\nu}) + \boldsymbol{\nu}\times\mathbf{J}\,.\end{aligned}$$

Further, we have

$$\begin{aligned}Y &\equiv \mathbf{J}^2 + (\mathbf{J}\times\boldsymbol{\nu})\cdot\Big(\nabla\times(\boldsymbol{\nu}\times\mathbf{H}) + \mathbf{H}\mathrm{div}\boldsymbol{\nu}\Big)\\ &= \mathbf{J}^2 + (\mathbf{J}\times\boldsymbol{\nu})\cdot\Big(2(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} + \mathbf{H}\times(\nabla\times\boldsymbol{\nu}) - \mathbf{J}\times\boldsymbol{\nu}\Big)\\ &= 2(\mathbf{J}\times\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} + \mathbf{H}\cdot\Big((\nabla\times\boldsymbol{\nu})\times(\mathbf{J}\times\boldsymbol{\nu})\Big) = 2(\mathbf{J}\times\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu}\,,\end{aligned}$$

whence the identity (A2) immediately follows.

Now we rewrite the integral (A1) in the form

$$2I = \int_{\mathcal{D}}(1-\lambda^2)\Big\{\Big(\mathbf{h} + (\boldsymbol{\xi}\cdot\boldsymbol{\nu})\mathbf{J}\times\boldsymbol{\nu}\Big)^2 - 2(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2(\mathbf{J}\times\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu}\Big\}\mathrm{d}\tau + I_1\,, \qquad (A3)$$

where

$$\begin{aligned}I_1 = \int_{\mathcal{D}}(1-\lambda^2)\Big\{&2(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2(\mathbf{J}\times\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} - 2\mathbf{h}\cdot(\mathbf{J}\times\boldsymbol{\nu})(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\\ &- (\mathbf{J}\times\boldsymbol{\nu})^2(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2 + \mathbf{h}\cdot(\mathbf{J}\times\boldsymbol{\xi})\Big\}\mathrm{d}\tau\,. \qquad (A4)\end{aligned}$$

Substituting (A2) in (A4), we obtain

$$\begin{aligned}I_1 = \int_{\mathcal{D}}(1-\lambda^2)\Big\{&\Big((\mathbf{J}\times\boldsymbol{\nu})\cdot(\nabla\times(\boldsymbol{\nu}\times\mathbf{H})) - J_0|\mathbf{H}|\mathrm{div}\boldsymbol{\nu}\Big)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2\\ &- 2\mathbf{h}\cdot(\mathbf{J}\times\boldsymbol{\nu})(\boldsymbol{\xi}\cdot\boldsymbol{\nu}) + \mathbf{h}\cdot(\mathbf{J}\times\boldsymbol{\xi})\Big\}\mathrm{d}\tau\,. \qquad (A5)\end{aligned}$$

Here $J_0 \equiv -\nabla^2 A$ and we have used eqns. (24), (26), (27).

Now let

$$\boldsymbol{\xi} = (\boldsymbol{\xi}\cdot\boldsymbol{\nu})\boldsymbol{\nu} + \tilde{\boldsymbol{\xi}}\,, \quad \tilde{\boldsymbol{\xi}} = b\,\mathbf{e}_z + c\,\mathbf{H}\,, \tag{A6}$$

i.e. $b = \boldsymbol{\xi}\cdot\mathbf{e}_z$, $c = (\boldsymbol{\xi}\cdot\mathbf{H})/\mathbf{H}^2$. Substitution of (A6) in (A5) results in

$$I_1 = \int_{\mathcal{D}} (1-\lambda^2)\Big\{\Big((\mathbf{J}\times\boldsymbol{\nu})\cdot(\nabla\times(\boldsymbol{\nu}\times\mathbf{H})) - J_0|\mathbf{H}|\mathrm{div}\boldsymbol{\nu}\Big)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2 \\ + \mathbf{h}\cdot(\mathbf{J}\times\tilde{\boldsymbol{\xi}}) - \mathbf{h}\cdot(\mathbf{J}\times\boldsymbol{\nu})(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\Big\}\mathrm{d}\tau\,. \tag{A7}$$

Using (8) and (A6), we obtain

$$\begin{aligned} Y_1 &\equiv (\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2(\mathbf{J}\times\boldsymbol{\nu})\cdot(\nabla\times(\boldsymbol{\nu}\times\mathbf{H})) - \mathbf{h}\cdot(\mathbf{J}\times\boldsymbol{\nu})(\boldsymbol{\xi}\cdot\boldsymbol{\nu}) \\ &= (\boldsymbol{\xi}\cdot\boldsymbol{\nu})(\mathbf{J}\times\boldsymbol{\nu})\cdot\Big((\boldsymbol{\xi}\cdot\boldsymbol{\nu})\nabla\times(\boldsymbol{\nu}\times\mathbf{H}) - \nabla\times(\boldsymbol{\xi}\times\mathbf{H})\Big) \\ &= (\boldsymbol{\xi}\cdot\boldsymbol{\nu})(\mathbf{J}\times\boldsymbol{\nu})\cdot\Big((\boldsymbol{\xi}\cdot\boldsymbol{\nu})\nabla\times(\boldsymbol{\nu}\times\mathbf{H}) - \nabla\times\Big((\boldsymbol{\xi}\cdot\boldsymbol{\nu})\boldsymbol{\nu}\times\mathbf{H} + b(\mathbf{e}_z\times\mathbf{H})\Big)\Big) \\ &= -J_0|\mathbf{H}|\Big((\boldsymbol{\nu}\cdot\nabla)\frac{(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2}{2} + (\boldsymbol{\xi}\cdot\boldsymbol{\nu})(\mathbf{e}_z\cdot\nabla b)\Big)\,. \end{aligned} \tag{A8}$$

From (A6) and the condition $\mathrm{div}\boldsymbol{\xi} = 0$, we have

$$\mathbf{e}_z\cdot\nabla b = -\boldsymbol{\nu}\cdot\nabla(\boldsymbol{\xi}\cdot\boldsymbol{\nu}) - (\boldsymbol{\xi}\cdot\boldsymbol{\nu})\mathrm{div}\boldsymbol{\nu} - \mathbf{H}\cdot\nabla c\,,$$

whence

$$Y_1 = J_0|\mathbf{H}|\Big((\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2\mathrm{div}\boldsymbol{\nu} + (\boldsymbol{\xi}\cdot\boldsymbol{\nu})(\mathbf{H}\cdot\nabla c)\Big)\,. \tag{A9}$$

On substituting this in eqn. (A7), we get

$$I_1 = \int_{\mathcal{D}}(1-\lambda^2)\Big\{J_0|\mathbf{H}|(\boldsymbol{\xi}\cdot\boldsymbol{\nu})(\mathbf{H}\cdot\nabla c) + \mathbf{h}\cdot(\mathbf{J}\times\tilde{\boldsymbol{\xi}})\Big\}\mathrm{d}\tau\,. \tag{A10}$$

Consider now the integral

$$I_2 \equiv \int_{\mathcal{D}}(1-\lambda^2)\mathbf{h}\cdot(\mathbf{J}\times\tilde{\boldsymbol{\xi}})\mathrm{d}\tau\,.$$

Since $\mathbf{J}\times\tilde{\boldsymbol{\xi}} = J_0 c\nabla A$, we have

$$\begin{aligned} I_2 &= \int_{\mathcal{D}}(1-\lambda^2)J_0 c\nabla A\cdot(\nabla\times(\boldsymbol{\xi}\times\mathbf{H}))\mathrm{d}\tau = \int_{\mathcal{D}}\Big(\nabla((1-\lambda^2)J_0 c)\times\nabla A\Big)\cdot(\boldsymbol{\xi}\times\mathbf{H})\mathrm{d}\tau \\ &= \int_{\mathcal{D}}(1-\lambda^2)\boldsymbol{\xi}\cdot\Big(\mathbf{H}\times\Big(\nabla(J_0 c)\times\nabla A\Big)\Big)\mathrm{d}\tau = -\int_{\mathcal{D}}(1-\lambda^2)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})|\mathbf{H}|\Big(\mathbf{H}\cdot(J_0 c)\Big)\mathrm{d}\tau\,. \end{aligned}$$

Whence,

$$I_1 = \int_{\mathcal{D}}(1-\lambda^2)|\mathbf{H}|c(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\Big(\mathbf{H}\cdot\nabla(\nabla^2 A)\Big)\mathrm{d}\tau\,. \tag{A11}$$

From (25),

$$(1-\lambda^2)(\mathbf{H}\cdot\nabla)\nabla^2 A = \lambda\lambda'\Big(\mathbf{H}\cdot\nabla(\mathbf{H}^2)\Big)\,.$$

Eqn. (A11) can therefore be written as

$$I_1 = \int_{\mathcal{D}} \lambda\lambda'|\mathbf{H}|\Big(\mathbf{H}\cdot\nabla(\mathbf{H}^2)\Big)c(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\mathrm{d}\tau\,. \qquad (A12)$$

Now, taking account of eqns. (A1), (A3) and (A12), we can rewrite (20) in the form $W = W_1 + R$ where

$$\begin{aligned}
W_1 &= \tfrac{1}{2}\int_{\mathcal{D}}(1-\lambda^2)\Big(\mathbf{h}+(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\mathbf{J}\times\boldsymbol{\nu}\Big)^2 d\tau \\
R &= \tfrac{1}{2}\int_{\mathcal{D}}\Big\{\lambda\lambda'|\mathbf{H}|\Big(\mathbf{H}\cdot\nabla(\mathbf{H}^2)\Big)c(\boldsymbol{\xi}\cdot\boldsymbol{\nu}) \\
&\quad - 2(1-\lambda^2)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2(\mathbf{J}\times\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} - 2\lambda\lambda'|\mathbf{H}|(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\boldsymbol{\xi}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}\Big\}\mathrm{d}\tau\,. \qquad (A13)
\end{aligned}$$

All that remains to be done is to show that R in (A13) coincides with the integral W_2 given by (28). We have

$$-\lambda\lambda'|\mathbf{H}|(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\boldsymbol{\xi}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H} = -\lambda\lambda'|\mathbf{H}|(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\Big((\boldsymbol{\xi}\cdot\boldsymbol{\nu})\boldsymbol{\nu}+c\,\mathbf{H}\Big)\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}\,.$$

Substitution of this in the expression for R yields

$$R = -\int_{\mathcal{D}}\Big\{(1-\lambda^2)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2(\mathbf{J}\times\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} + \lambda\lambda'|\mathbf{H}|(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2\boldsymbol{\nu}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}\Big\}\mathrm{d}\tau\,.$$

After comparison of this formula with (28) we conclude that R is indeed the same as W_2.

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[9] Also, we may consider perturbations which are periodic in z, then the integral in (23) is taken over the period.