On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part 4. Generalized isovorticity principle for three-dimensional flows

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The equations of magnetohydrodynamics (MHD) of an ideal fluid have two families of topological invariants: the magnetic helicity invariants and the cross-helicity invariants. It is first shown that these invariants define a natural foliation (described as isomagnetovortical, or imv for short) in the function space in which solutions $\{u(x,t), h(x,t)\}$ of the MHD equations reside. A relaxation process is constructed whereby total energy (magnetic plus kinetic) decreases on an imv folium (all magnetic and cross-helicity invariants being thus conserved). The energy has a positive lower bound determined by the global cross-helicity, and it is thus shown that a steady state exists having the (arbitrarily) prescribed families of magnetic and cross-helicity invariants.

The stability of such steady states is considered by an appropriate generalization of (Arnold) energy techniques. The first variation of energy on the imv folium is shown to vanish, and the second variation $\delta^2 E$ is constructed. It is shown that $\delta^2 E$ is a quadratic functional of the first-order variations $\delta^1 u, \delta^1 h$ of u and h (from a steady state $U(x), H(x)$), and that $\delta^2 E$ is an invariant of the linearized MHD equations. Linear stability is then assured provided $\delta^2 E$ is either positive-definite or negative-definite for all imv perturbations. It is shown that the results may be equivalently obtained through consideration of the frozen-in 'modified' vorticity field introduced in Part 1 of this series.

Finally, the general stability criterion is applied to a variety of classes of steady states $\{U(x), H(x)\}$, and new sufficient conditions for stability to three-dimensional imv perturbations are obtained.

1. Introduction

In Part 1 of this series (Vladimirov & Moffatt 1995), we have established two variational principles for steady three-dimensional solutions of the equations of magnetohydrodynamics (MHD) of an ideal incompressible fluid; in Parts 2 and 3 (Vladimirov, Moffatt & Ilin 1996, 1997), the techniques developed in Part 1 were used to obtain stability criteria for two-dimensional and axisymmetric flows. We now return to the general case of three-dimensional MHD flows. Our aim is first to extend the

concept of 'isovortical deformations' (Arnold 1965) to the MHD context, for which we shall find that a more general 'isomagnetovortical' (or imv, for short) deformation needs to be defined. This leads to the concept of an isomagnetovortical foliation of the function space in which solutions $\{u(x, t), h(x, t)\}$ of the MHD equations reside. The essential property of this foliation is that, under generalized deformations on an imv folium, all topological invariants associated with the MHD equations (and notably magnetic helicity and cross-helicity) are conserved. We then show that a relaxation process may be defined (a generalization of the relaxation to *magnetostatic* equilibrium described by Moffatt 1985) in which energy decreases to a minimum while the above topological invariants are conserved. This minimum corresponds to a steady solution of the MHD equations.

Our second aim is to extend Arnold's (1965) variational principle to the above MHD situation starting from first principles of fluid dynamics. This requires consideration of perturbations of an arbitrary steady state resulting from small imv deformations. We shall show that the first-order variation of energy $\delta^1 E$ vanishes at the equilibrium (i.e. at the steady state); and that the second-order variation of energy $\delta^2 E$ is an invariant of the MHD equations linearized about this steady state. These results are to be expected from general theory, but their explicit verification provides useful confirmation that the imv foliation has been correctly and completely identified.

It is known from general theory (Arnold 1965) that the steady state is stable if $\delta^2 E$ is either positive-definite or negative-definite for all admissible perturbations on the imv folium. We use this principle in §8 to obtain stability criteria for several classes of non-trivial steady MHD flows.

The abstract geometric approach to the equations of ideal MHD has been developed in a number of previous publications, notably Arnold (1966), Vishik & Dolzhanskii (1978), Marsden, Ratiu & Weinstein (1984), Holm *et al.* (1985), Khesin & Chekanov (1989), Zeitlin & Kambe (1993), Ono (1995), Friedlander & Vishik (1995). The background is extensively covered in the recent book by Arnold & Khesin (1998).

It is hoped that the treatment of the present paper, which deliberately avoids the abstraction of Lie algebras and general Hamiltonian dynamics, may be more readily accessible to readers of this Journal. We should point out however that the various results stated below as Propositions may in principle be obtained by appropriate specialization from corresponding results in the more general abstract theory; indeed, it was through such specialization (private communication) that Friedlander & Vishik (1995) obtained the results (5.9) and (5.10) which we obtain below by what we believe to be a simpler and more transparent technique.

The stability problem of magnetostatic equilibria has an extensive literature (see for example Bernstein 1983; Lifshitz 1989). We emphasize that the present paper is concerned with the more general (and more difficult) problem of stability of steady MHD flows.

2. Basic equations and their invariants

Consider an incompressible, inviscid and perfectly conducting fluid contained in a domain $\mathscr{D}$ with fixed boundary $\partial\mathscr{D}$. Let $u(x, t)$ be the velocity field, $\omega = \nabla \wedge u$ the vorticity field, $h(x, t)$ the magnetic field (in Alfvén velocity units), $j = \nabla \wedge h$ the current density, and $p(x, t)$ the pressure (divided by density). The governing equations are then

$$u_t = u \wedge \omega + j \wedge h - \nabla(p + \tfrac{1}{2}u^2), \tag{2.1}$$

$$h_t = \nabla \wedge (u \wedge h), \tag{2.2}$$

$$\nabla \cdot u = \nabla \cdot h = 0. \tag{2.3}$$

We shall use the 'commutator' notation for solenoidal fields u, h

$$[u, h] = \nabla \wedge (u \wedge h), \tag{2.4}$$

and we shall make frequent use of the property

$$[u, h] = -[h, u], \tag{2.5}$$

and the Jacobi identity for any three solenoidal fields

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0. \tag{2.6}$$

With this notation, (2.2) becomes

$$h_t = [u, h], \tag{2.7}$$

and the curl of (2.1) gives

$$\omega_t = [u, \omega] + [j, h]. \tag{2.8}$$

The boundary conditions on u and h are

$$n \cdot u = n \cdot h = 0 \quad \text{on } \partial\mathcal{D} \tag{2.9}$$

where n is the unit outward normal on $\partial\mathcal{D}$.

Equation (2.7) implies that the field h is frozen in the fluid, its flux through any surface S bounded by a material curve C being conserved; the conditions $\nabla \cdot h = 0$ and $n \cdot h = 0$ on $\partial\mathcal{D}$ are conserved under evolution governed by (2.7).

Let S_h be any closed material surface in $\mathcal{D}$ on which $n \cdot h = 0$ (again a condition conserved by (2.7)) and let V_h be the volume inside S_h. Then it is well known that there are two invariants for each such volume, namely the magnetic helicity

$$\mathscr{H}_M(V_h) = \int_{V_h} h \cdot \text{curl}^{-1} h \, dV \tag{2.10}$$

and the cross-helicity

$$\mathscr{H}_C(V_h) = \int_{V_h} h \cdot u \, dV. \tag{2.11}$$

Note that, if there is any closed h-line, γ_h say, then taking V_h to be a flux tube of vanishingly small cross-section centred on γ_h, we obtain the corresponding invariant

$$\Gamma_h = \oint_{\gamma_h} u \cdot dx = \int_{\Sigma_h} \omega \cdot n \, dS, \tag{2.12}$$

where Σ_h is any surface bounded by γ_h. Thus the flux of vorticity through any closed h-line is conserved. Note however that the vorticity field is not frozen in the fluid, since the Lorentz force $j \wedge h$ in (2.1) is in general rotational (i.e. $[j, h] \neq 0$).

The set of invariants (2.10) and (2.11) are topological in character (Moffatt 1969; see also Woltjer 1958; Moreau 1961), carrying information about the linkage of h-lines and the mutual linkage of ω-lines and h-lines. We shall describe them as the topological invariants of the system of equations (2.1)–(2.3). Note that a possible special choice of V_h in (2.10), (2.11) is $V_h = \mathcal{D}$, in which case we may talk of the global magnetic helicity and the global cross-helicity of the flow.

We wish to study the existence, structure and stability of steady solutions of (2.1)–(2.3), (2.9). The discussion below follows Moffatt (1989). Let $u = U(x)$, $h = H(x)$ be

one such solution, with $\boldsymbol{\Omega} = \nabla \wedge \boldsymbol{U}$ and $\boldsymbol{J} = \nabla \wedge \boldsymbol{H}$. Then, from (2.2),

$$\boldsymbol{U} \wedge \boldsymbol{H} = \nabla \Phi, \tag{2.13}$$

and, from (2.1),

$$\boldsymbol{U} \wedge \boldsymbol{\Omega} + \boldsymbol{J} \wedge \boldsymbol{H} = \nabla \Psi \tag{2.14}$$

where $\Psi = p + \frac{1}{2}\boldsymbol{u}^2$. From (2.13), we have

$$\boldsymbol{U} \cdot \nabla \Phi = \boldsymbol{H} \cdot \nabla \Phi = 0, \tag{2.15}$$

so that $\boldsymbol{U}$-lines (i.e. the streamlines of the flow) and $\boldsymbol{H}$-lines (i.e. the magnetic lines of force) lie on surfaces $\Phi = $ const. The surface $\partial \mathscr{D}$ is one such surface, i.e.

$$\Phi = \text{ const. } \text{ on } \partial \mathscr{D}. \tag{2.16}$$

In general therefore, there is a strong topological constraint on the structure of the fields $\boldsymbol{U}(\boldsymbol{x})$ and $\boldsymbol{H}(\boldsymbol{x})$. The only escape from this constraint arises if

$$\nabla \Phi \equiv 0 \quad \text{in some} \quad \mathscr{D}_1 \subseteq \mathscr{D}. \tag{2.17}$$

Then $\boldsymbol{U} \wedge \boldsymbol{H} = 0$ in $\mathscr{D}_1$, and so

$$\boldsymbol{U} = \alpha(\boldsymbol{x})\boldsymbol{H} \quad \text{with } (\boldsymbol{H} \cdot \nabla)\alpha = 0 \quad \text{in } \mathscr{D}_1. \tag{2.18}$$

Hence, again, the $\boldsymbol{U}$-lines and $\boldsymbol{H}$-lines are constrained to lie on surfaces $\alpha = $ const. in $\mathscr{D}_1$. Again the only escape from this constraint arises if

$$\nabla \alpha \equiv 0 \quad \text{in some} \quad \mathscr{D}_2 \subseteq \mathscr{D}_1. \tag{2.19}$$

In this case

$$\boldsymbol{U} = \alpha \boldsymbol{H} \quad \text{with } \alpha = \text{ const. in } \mathscr{D}_2. \tag{2.20}$$

It follows immediately that

$$\boldsymbol{\Omega} = \alpha \boldsymbol{J} \quad \text{in } \mathscr{D}_2 \tag{2.21}$$

and so (2.14) becomes

$$\left(1 - \alpha^2\right) \boldsymbol{U} \wedge \boldsymbol{\Omega} = \nabla \Psi. \tag{2.22}$$

Here, the possibility $\alpha = 1$ is very special: it corresponds to $\boldsymbol{U} \equiv \boldsymbol{H}$ in $\mathscr{D}_2$, in which case (2.13) and (2.14) are satisfied for any choice of $\boldsymbol{H}(\boldsymbol{x})$. If $\alpha \neq 1$, then (2.22) implies that

$$\boldsymbol{U} \cdot \nabla \Psi = \boldsymbol{\Omega} \cdot \nabla \Psi = 0 \quad \text{in } \mathscr{D}_2. \tag{2.23}$$

Hence the $\boldsymbol{U}$-lines (which now coincide with the $\boldsymbol{H}$-lines) still lie on surfaces $\Psi = $ const. – unless, that is,

$$\nabla \Psi \equiv 0 \quad \text{in some} \quad \mathscr{D}_3 \subseteq \mathscr{D}_2 \tag{2.24}$$

in which case

$$\boldsymbol{\Omega} = \beta(\boldsymbol{x})\boldsymbol{U} \quad \text{with} \quad \boldsymbol{U} \cdot \nabla \beta = 0 \quad \text{in } \mathscr{D}_3. \tag{2.25}$$

Now the $\boldsymbol{U}$-lines lie on surfaces $\beta = $ const. – it seems hard to escape the constraint that $\boldsymbol{U}$-lines lie on surfaces! However, and finally, it may happen that

$$\nabla \beta \equiv 0 \quad \text{in some} \quad \mathscr{D}_4 \subseteq \mathscr{D}_3, \tag{2.26}$$

in which case

$$\boldsymbol{\Omega} = \beta \boldsymbol{U} \quad \text{with} \quad \beta = \text{ const. in } \mathscr{D}_4, \tag{2.27}$$

i.e. $\boldsymbol{U}$ is a Beltrami field in $\mathscr{D}_4$. Now, at last, we are released from the constraint that

U-lines must lie on surfaces; under the condition (2.27), the U-lines may be chaotic in $\mathscr{D}_4$ (the prototype example is the ABC-flow studied by Hénon 1966 and Arnold 1966). Note that in $\mathscr{D}_4$,

$$U = \beta^{-1}\boldsymbol{\Omega} = \alpha H = \beta^{-1}\alpha J. \tag{2.28}$$

3. The isomagnetovortical (imv) foliation

It is known that, if we replace the field u in (2.7) by any other vector field $v(x, \tau)$†, so that

$$h_\tau = [v, h], \tag{3.1}$$

then the integrals $\mathscr{H}_M(V_h)$ given by (2.10) are still invariant; this is obvious because the field h is now frozen in the hypothetical flow $v(x, \tau)$ and so all its topological properties are conserved. It is natural to adopt the constraint

$$\nabla \cdot v = 0 \tag{3.2}$$

and the boundary condition $n \cdot v = 0$ on $\partial\mathscr{D}$.

In similar spirit, we may enquire what simultaneous modification of (2.8) may be made that will still guarantee conservation of the set of cross-helicities (2.11). Simple manipulations reveal that we must replace u on the right-hand side of (2.8) by the same $v(x, \tau)$ as in (3.1); and at the same time, we may replace j by a vector field $c(x, \tau)$, so that (2.8) is replaced by

$$\omega_\tau = [v, \omega] + [c, h]. \tag{3.3}$$

It is again natural to impose the solenoidality condition

$$\nabla \cdot c = 0. \tag{3.4}$$

It will not be necessary at this stage to impose any boundary condition on c (just as there was no boundary condition on j in the parent problem (2.1)–(2.3)). We may refer to v and c as 'auxiliary' velocity and current fields. Here we have introduced the main imv equation for vorticity (3.3) in a heuristic way; more detailed discussion of how (3.3) can be obtained in a more systematic manner is given in §7.

In order to verify the above, we write (3.1) in the form

$$\frac{Dh}{D\tau} \equiv \frac{\partial h}{\partial \tau} + v \cdot \nabla h = h \cdot \nabla v, \tag{3.5}$$

and the 'uncurled' version of (3.3) in the form

$$\frac{Du}{D\tau} \equiv \frac{\partial u}{\partial \tau} + v \cdot \nabla u = v \cdot (\nabla u)^T + c \wedge h - \nabla P, \tag{3.6}$$

where $[v \cdot (\nabla u)^T]_i = v_j \partial u_j / \partial x_i$ and P is a function which can be found from the equation $\nabla \cdot u = 0$ and the boundary condition $n \cdot u = 0$. Then

$$\frac{d}{d\tau} \int_{V_h} (u \cdot h) dV = \int_{V_h} \left(u \cdot \frac{Dh}{D\tau} + h \cdot \frac{Du}{D\tau} \right) dV$$

$$= \int_{S_h} (h \cdot n)(u \cdot v - P) dS = 0, \tag{3.7}$$

using standard manipulations and the condition $n \cdot h = 0$ on S_h.

† Throughout §§ 3–5, we replace t by a 'virtual time' τ, in order to distinguish the virtual processes considered from real time evolution.

The pair of equations (3.1), (3.3) is wider in scope than the pair (2.7), (2.8) because in the latter

$$\boldsymbol{u} = \text{curl}^{-1}\boldsymbol{\omega}, \quad \boldsymbol{j} = \text{curl }\boldsymbol{h}, \tag{3.8}$$

whereas $\boldsymbol{v}$ and $\boldsymbol{c}$ suffer no such restrictions.

Note that, if $\boldsymbol{c} = 0$ in (3.3), then the vortex lines are frozen (like the $\boldsymbol{h}$-lines) in the flow $\boldsymbol{v}$. When $\boldsymbol{c} \neq 0$, the vortex lines are *not* frozen in this flow; but the equation is such that the quantities Γ_h defined by (2.12), which represent flux of vorticity through any closed $\boldsymbol{h}$-line, do remain invariant. The term $[\boldsymbol{c}, \boldsymbol{h}]$ in (3.3) redistributes vorticity flux within any such closed $\boldsymbol{h}$-line but conserves the integrated flux.

Equations (3.1), (3.3) thus conserve all known topological invariants of the parent equations, but allow a much wider range of behaviour due to the freedom of choice of the auxiliary fields $\{\boldsymbol{v}, \boldsymbol{c}\}$. Let $\mathscr{A}(\boldsymbol{x})$ represent the pair of fields $\{\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{h}(\boldsymbol{x})\}$ (where $\boldsymbol{u} = \text{curl}^{-1}\boldsymbol{\omega}$). Under the action of a pair of auxiliary fields $\{\boldsymbol{v}(\boldsymbol{x}, \tau), \boldsymbol{c}(\boldsymbol{x}, \tau)\}$ during a time interval $[\tau_1, \tau_2]$, evolution governed by (3.1) and (3.3) will convert a pair $\mathscr{A}_1(\boldsymbol{x})$ to a pair $\mathscr{A}_2(\boldsymbol{x})$ having the same set of topological invariants. Conversely, we may say that two pairs $\mathscr{A}_1(\boldsymbol{x})$ and $\mathscr{A}_2(\boldsymbol{x})$ lie on the same isomagnetovortical (or imv) folium of the function space of such pairs if and only if there exist fields $\{\boldsymbol{v}(\boldsymbol{x}, \tau), \boldsymbol{c}(\boldsymbol{x}, \tau)\}$ which effect the conversion $\mathscr{A}_1(\boldsymbol{x}) \to \mathscr{A}_2(\boldsymbol{x})$ in a time interval $[\tau_1, \tau_2]$. This requirement clearly defines an *imv foliation* of the function space, two pairs being on the same folium if and only if they are 'accessible' one from another via (3.1) and (3.3), and therefore certainly only if they have the same set of topological invariants. This foliation provides the required generalization of the 'isovortical' foliation introduced by Arnold (1965). For further explanations see §7 below.

It is of course obvious that the function space trajectory of a solution $\{\boldsymbol{u}(\boldsymbol{x}, t), \boldsymbol{h}(\boldsymbol{x}, t)\}$ of the parent equations (2.1)–(2.3) lies on an imv folium, since this evolution corresponds to the particular choice of auxiliary fields $\boldsymbol{v}(\boldsymbol{x}, \tau) = \boldsymbol{u}(\boldsymbol{x}, \tau)$ and $\boldsymbol{c}(\boldsymbol{x}, \tau) = \boldsymbol{j}(\boldsymbol{x}, \tau)$ and the restoration of real time $\tau \to t$. This may be stated as:

PROPOSITION 3.1. *A trajectory $\{\boldsymbol{u}(\boldsymbol{x}, t), \boldsymbol{h}(\boldsymbol{x}, t)\}$ of the ideal MHD equations lies on an isomagnetovortical folium.*

4. Relaxation to minimum energy states

Let us now consider the energy functional

$$E = \tfrac{1}{2} \int_{\mathscr{D}} \left(\boldsymbol{h}^2 + \boldsymbol{u}^2\right) \mathrm{d}V, \tag{4.1}$$

which is known to be an invariant of equations (2.1)–(2.3). Let us calculate the rate of change of energy under (3.1), (3.3), i.e. under a more general evolution on an imv folium. We have

$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = \int_{\mathscr{D}} \{\boldsymbol{h} \cdot [\boldsymbol{v}, \boldsymbol{h}] + \boldsymbol{u} \cdot (\boldsymbol{v} \wedge \boldsymbol{\omega} + \boldsymbol{c} \wedge \boldsymbol{h} - \nabla P\}\mathrm{d}V$$

$$= -\int_{\mathscr{D}} \{\boldsymbol{v} \cdot (\boldsymbol{u} \wedge \boldsymbol{\omega} + \boldsymbol{j} \wedge \boldsymbol{h}) + \boldsymbol{c} \cdot (\boldsymbol{u} \wedge \boldsymbol{h})\}\mathrm{d}V. \tag{4.2}$$

Here, we have used $\boldsymbol{n} \cdot \boldsymbol{u} = \boldsymbol{n} \cdot \boldsymbol{v} = 0$ on $\partial \mathscr{D}$ and have discarded surface terms. Note that if we put $\boldsymbol{v} = \boldsymbol{u}$ and $\boldsymbol{c} = \boldsymbol{j}$ in (4.2), then the integrand vanishes and $\mathrm{d}E/\mathrm{d}\tau = 0$ as expected.

We now exploit our freedom of choice of $\boldsymbol{v}$ and $\boldsymbol{c}$ in order to ensure monotonic

decrease of E, the idea being to drive the system towards a minimum energy state. The obvious choice is

$$v = u \wedge \omega + j \wedge h - \nabla\alpha,$$ (4.3)

$$c = u \wedge h - \nabla\beta,$$ (4.4)

where α and β are chosen so that

$$\nabla \cdot v = \nabla \cdot c = 0 \,, \quad n \cdot v = \beta = 0 \text{ on } \partial\mathscr{D}.$$ (4.5)

These requirements define a Neumann problems for α and a Dirichlet problem for β which provide unique solutions for $\nabla\alpha$ and $\nabla\beta$. Equation (4.2) now gives

$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = - \int_{\mathscr{D}} \left(v^2 + c^2\right) \mathrm{d}V \,,$$ (4.6)

where we have used (4.5) to discard surface contributions. Hence E does decrease monotonically as required.

In order for this result to be useful, we require a positive lower bound on E. This is provided by the cross-helicity invariant $\mathscr{H}_C$ which is conserved on the imv folium, for

$$E = \tfrac{1}{2} \int_{\mathscr{D}} \left(u^2 + h^2\right) \mathrm{d}V \geqslant \left| \int_{\mathscr{D}} u \cdot h \, \mathrm{d}V \right| = |\mathscr{H}_C|.$$ (4.7)

Hence, provided $\mathscr{H}_C \neq 0$, we certainly have a lower bound. (The technique of Freedman (1988) may presumably be adapted to demonstrate the existence of a positive lower bound on E even when $\mathscr{H}_C = 0$ provided there is a non-trivial linkage between the vorticity and magnetic fields.) It then follows that E tends to a limit as $\tau \to \infty$, and hence, from (4.6), that

$$\int_{\mathscr{D}} \left(v^2 + c^2\right) \mathrm{d}V \to 0 \quad \text{as} \quad \tau \to \infty.$$ (4.8)

On the reasonable assumption that v and c remain smooth for all τ, it follows that $v \to 0$ and $c \to 0$ as $\tau \to \infty$, and hence, from (4.3) and (4.4), that u and h must tend to steady (equilibrium) fields $U^E(x), H^E(x)$, say, satisfying the steady-state equations

$$[U^E, H^E] = 0 \,,$$ (4.9)

$$[U^E, \Omega^E] + [J^E, H^E] = 0 \,,$$ (4.10)

where $\Omega^E = \text{curl } U^E$, $J^E = \text{curl } H^E$. Moreover, since the energy E 'goes downhill all the way' during the relaxation process, it attains either a minimum in the asymptotic state or possibly a saddle point; in the former case, the asymptotic state is stable.

The process described above is very similar to the magnetic relaxation process described by Moffatt (1985). In that case, there was a lower bound on the magnetic energy obtained through a combination of Schwarz and Poincaré inequalities:

$$\tfrac{1}{2} \int_{\mathscr{D}} h^2 \mathrm{d}V \geqslant q^{-1} |\mathscr{H}_M| \,,$$ (4.11)

where q^{-1} is a positive constant (with dimensions of length) dependent on the geometry of the domain $\mathscr{D}$. The inequality (4.11) still holds in the present context and implies, in conjunction with (4.7), that

$$E \geqslant \max \left\{ |\mathscr{H}_C|, q^{-1} |\mathscr{H}_M| \right\} \,.$$ (4.12)

There is no lower bound on the kinetic energy $\tfrac{1}{2} \int u^2 \mathrm{d}V$.

It may well be asked why the above procedure does not work if the fluid is non-conducting, and $h \equiv 0$. The answer is that then both $\mathcal{H}_C$ and $\mathcal{H}_M$ are zero, and no positive lower bound is available for E. Thus although a process may be constructed as above which monotonically reduces E, there is no guarantee of a limit other than $E = 0$. Thus the procedure is not guaranteed to yield any three-dimensional stable steady solutions of the Euler equations of ideal hydrodynamics, consistent with the conclusion of Rouchon (1991) that no such steady solutions exist satisfying Arnold's (1965) sufficient condition for stability.

5. Variational principle

Suppose now that $\{U(x), H(x)\}$ is a steady solution of equations (2.1)–(2.3); we may think of $\{U(x), H(x)\}$ as a fixed point in the function space. Following Moffatt (1986), we consider a virtual displacement $x \to \zeta(x, \tau_0)$ on the imv folium containing this fixed point, considered as the displacement associated with a *steady* auxiliary velocity field $v(x)$ acting over a short time interval $[0, \tau_0]$ (it is possible to represent displacements in this way because the domain $\mathcal{D}$ is fixed). Thus

$$\zeta_\tau = v(\zeta) \quad (0 \leqslant \tau \leqslant \tau_0) \tag{5.1}$$

and, for small τ_0,

$$\zeta(x, \tau) = x + \tau_0 v(x) + \tfrac{1}{2}\tau_0^2 v \cdot \nabla v + O(\tau_0^3). \tag{5.2}$$

We define the first-order displacement field†

$$\xi(x) = \tau_0 v(x), \tag{5.3}$$

which evidently satisfies

$$\nabla \cdot \xi = 0, \quad n \cdot \xi = 0 \quad \text{on} \quad \partial \mathcal{D}. \tag{5.4}$$

Under the frozen-field distortion of $H(x)$ associated with the virtual displacement $\zeta(x, \tau_0)$, it is known (Moffatt 1986) that the first- and second-order variations of H are

$$\delta^1 h = [\xi, H], \quad \delta^2 h = \tfrac{1}{2}[\xi, \delta^1 h]. \tag{5.5}$$

In similar spirit, we may now calculate from (3.3) the first- and second-order variations of $\Omega(x) = \operatorname{curl} U(x)$ consequent upon the 'application' of steady auxiliary fields $v(x)$ and $c(x)$ over the time interval $[0, \tau_0]$. Defining

$$\eta(x) = \tau_0 c(x), \quad \nabla \cdot \eta = 0, \tag{5.6}$$

these are

$$\delta^1 \omega = [\xi, \Omega] + [\eta, H], \tag{5.7}$$

and

$$\delta^2 \omega = \tfrac{1}{2}[\xi, \delta^1 \omega] + \tfrac{1}{2}[\eta, \delta^1 h]. \tag{5.8}$$

The corresponding perturbations of U are evidently

$$\delta^1 u = \xi \wedge \Omega + \eta \wedge H - \nabla \alpha, \tag{5.9}$$

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† The same expressions (5.5), (5.9), (5.10) can be obtained from (3.1), (3.3) with the use of usual definitions: $\delta^1 \omega \equiv \tau(\partial \omega / \partial \tau)$, $\delta^2 \omega \equiv \tfrac{1}{2}\tau^2(\partial^2 \omega / \partial \tau^2)$ etc. where all τ-derivatives are taken at $\tau = 0$. If we use this procedure we do not need to restrict attention to 'steady' fields v and c and fixed domain $\mathcal{D}$.

$$\delta^2 \boldsymbol{u} = \tfrac{1}{2}\boldsymbol{\xi} \wedge \delta^1 \boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\eta} \wedge \delta^1 \boldsymbol{h} - \nabla \beta \,. \tag{5.10}$$

By virtue of their construction, the variations (5.5), (5.9) and (5.10) are first- and second-order perturbations on the imv folium containing $\{\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{H}(\boldsymbol{x})\}$. The expressions (5.9) and (5.10) have been obtained previously by Friedlander & Vishik (1995) following the abstract prescription for general Hamiltonian systems of Khesin & Chekanov (1989). We believe that the above derivation is easier to understand, and that it sheds some light on the physical meaning of the displacement fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

The first-order variation of energy is

$$\begin{aligned}
\delta^1 E &= \int_{\mathscr{D}} \left(\boldsymbol{H} \cdot \delta^1 \boldsymbol{h} + \boldsymbol{U} \cdot \delta^1 \boldsymbol{u} \right) \mathrm{d}V \\
&= \int_{\mathscr{D}} \{ \boldsymbol{H} \cdot [\boldsymbol{\xi}, \boldsymbol{H}] + \boldsymbol{U} \cdot (\boldsymbol{\xi} \wedge \boldsymbol{\Omega} + \boldsymbol{\eta} \wedge \boldsymbol{H}) \} \, \mathrm{d}V \\
&= -\int_{\mathscr{D}} \{ \boldsymbol{\xi} \cdot (\boldsymbol{U} \wedge \boldsymbol{\Omega} + \boldsymbol{J} \wedge \boldsymbol{H}) + \boldsymbol{\eta} \cdot (\boldsymbol{U} \wedge \boldsymbol{H}) \} \, \mathrm{d}V \\
&= -\int_{\mathscr{D}} (\boldsymbol{\xi} \cdot \nabla \Psi + \boldsymbol{\eta} \cdot \nabla \Phi) \mathrm{d}V \\
&= 0 \,.
\end{aligned} \tag{5.11}$$

Here we have used $\nabla \cdot \boldsymbol{\xi} = \nabla \cdot \boldsymbol{\eta} = 0$, $\boldsymbol{n} \cdot \boldsymbol{\xi} = 0$ on $\partial \mathscr{D}$ and $\Phi = \text{const.}$ on $\partial \mathscr{D}$. We have thus proved the generalization of Arnold's (1965) energy variational principle to three-dimensional steady MHD flows:

PROPOSITION 5.1. *The energy functional has a stationary point at the steady state (or fixed point) $\{\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{H}(\boldsymbol{x})\}$ with respect to all perturbations on the imv folium containing the fixed point.*

Similarly, the second-order variation of energy on the imv folium is given by

$$\delta^2 E = \tfrac{1}{2} \int \left[(\delta^1 \boldsymbol{u})^2 + (\delta^1 \boldsymbol{h})^2 + 2 \boldsymbol{U} \cdot \delta^2 \boldsymbol{u} + 2 \boldsymbol{H} \cdot \delta^2 \boldsymbol{h} \right] \mathrm{d}V \,. \tag{5.12}$$

Substituting for $\delta^2 \boldsymbol{h}$, $\delta^2 \boldsymbol{u}$ from (5.5) and (5.10) and rearranging (using integration by parts) gives

$$\delta^2 E = \tfrac{1}{2} \int \left[(\delta^1 \boldsymbol{u})^2 + (\delta^1 \boldsymbol{h})^2 - \boldsymbol{\xi} \cdot \left(\boldsymbol{U} \wedge \delta^1 \boldsymbol{\omega} + \boldsymbol{J} \wedge \delta^1 \boldsymbol{h} \right) - \boldsymbol{\eta} \cdot \left(\boldsymbol{U} \wedge \delta^1 \boldsymbol{h} \right) \right] \mathrm{d}V \,. \tag{5.13}$$

This result is as previously stated (allowing for change of notation) by Friedlander & Vishik (1995).

The expression (5.13) apparently depends on the choice of fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. However $\delta^2 E$ in fact depends only on the perturbations $\delta^1 \boldsymbol{u}$ and $\delta^1 \boldsymbol{h}$ and is a quadratic functional of these fields. This follows from:

PROPOSITION 5.2. *Let $\{\boldsymbol{\xi}_1, \boldsymbol{\eta}_1\}$, $\{\boldsymbol{\xi}_2, \boldsymbol{\eta}_2\}$ be two distinct pairs of displacement fields giving the same $\delta^1 \boldsymbol{u}$, $\delta^1 \boldsymbol{h}$, i.e.*

$$\delta^1 \boldsymbol{h} = [\boldsymbol{\xi}_1, \boldsymbol{H}] = [\boldsymbol{\xi}_2, \boldsymbol{H}] \,, \tag{5.14}$$

$$\delta^1 \boldsymbol{u} = \boldsymbol{\xi}_1 \wedge \boldsymbol{\Omega} + \boldsymbol{\eta}_1 \wedge \boldsymbol{H} - \nabla \alpha_1 = \boldsymbol{\xi}_2 \wedge \boldsymbol{\Omega} + \boldsymbol{\eta}_2 \wedge \boldsymbol{H} - \nabla \alpha_2 \,. \tag{5.15}$$

Then

$$\delta^2 E(\boldsymbol{\xi}_1, \boldsymbol{\eta}_1) = \delta^2 E(\boldsymbol{\xi}_2, \boldsymbol{\eta}_2) \,. \tag{5.16}$$

This result is a natural generalization of that of Arnold (1966). Its proof involves long vector manipulation and is relegated to Appendix A.

The general stability principle is now applicable: *the flow $\{U(x), H(x)\}$ is stable provided $\delta^2 E$ is either positive definite or negative definite for all admissible $\{\xi(x), \eta(x)\}$.*

Before exploiting this principle, we shall first establish the relationship between $\delta^2 E$ and the linear stability problem, and then we shall provide an alternative formulation of the variational principle in terms of the frozen-in 'modified vorticity field' introduced in Part 1 of this series.

6. Linearized stability problem

Consider now an infinitesimal perturbation of the state $\{U(x), H(x), P(x)\}$ in the form

$$\{U + u(x, t), H + h(x, t), P + p(x, t)\}. \tag{6.1}$$

We shall identify the perturbation fields u, h with the first-order variations $\delta^1 u, \delta^1 h$ introduced in § 5; now however we are concerned with the actual time (t) evolution of the system. Substituting (6.1) in (2.1)–(2.3) and linearizing in the perturbations, we obtain the linear evolution equations

$$u_t + (U \cdot \nabla)u + (u \cdot \nabla)U = -\nabla p + J \wedge h + j \wedge H, \tag{6.2}$$

$$h_t = [u, H] + [U, h], \tag{6.3}$$

$$\nabla \cdot u = \nabla \cdot h = 0. \tag{6.4}$$

We may assume that at time $t = 0$, the exact solution lies on the imv folium containing $\{U, H\}$; then by Proposition 3.1, it remains on this imv folium for all $t > 0$. We may then infer the existence of time-dependent displacement fields $\xi(x, t), \eta(x, t)$ such that

$$h(x, t) = [\xi(x, t), H(x)], \tag{6.5}$$

$$u(x, t) = \xi(x, t) \wedge \Omega(x) + \eta(x, t) \wedge H(x) - \nabla \alpha. \tag{6.6}$$

which mimic the form of the first variations (5.5), (5.9). The corresponding vorticity perturbation is

$$\omega(x, t) = [\xi, \Omega] + [\eta, H]. \tag{6.7}$$

The evolution equations (6.2)–(6.4) imply corresponding evolution equations for ξ and η. These may be found as follows. First, from (6.5),

$$h_t = [\xi_t, H] \tag{6.8}$$

so that, substituting in (6.3) and using the Jacobi identity,

$$[\xi_t, H] = [u, H] + [U, [\xi, H]]$$

$$= [u, H] - [\xi, [H, U]] - [H, [U, \xi]]. \tag{6.9}$$

Hence, using $[H, U] = 0$ (from (2.13)), we have

$$[\xi_t, H] = [(u + [U, \xi]), H], \tag{6.10}$$

and so (up to an arbitrary additive vector field which commutes with H and therefore does not contribute to the h given by (6.5))

$$\xi_t = u + [U, \xi]. \tag{6.11}$$

This is the required evolution equation for ξ. Note that substituting u from (6.11) into (6.3) gives

$$(h - [\xi, H])_t = [U, h - [\xi, H]] , \tag{6.12}$$

thus verifying that, if (6.5) is satisfied at $t = 0$, then it is also satisfied for all $t > 0$.

Equation (6.11) implies the following interpretation for $\xi(x, t)$ (Chandrasekhar 1987): let x be the position of a particle in the undisturbed flow $U(x)$; then $x + \xi(x, t)$ is the position of the same particle in the (linearly) disturbed flow $U(x) + u(x, t)$. The correspondent field $\xi(x, t)$ is known as the Lagrangian displacement field.

Secondly, from (6.7) we have

$$\omega_t = [\xi_t, \Omega] + [\eta_t, H] , \tag{6.13}$$

while from the curl of (6.2) we have

$$\omega_t = [U, \omega] + [u, \Omega] + [j, H] + [J, h]. \tag{6.14}$$

Hence

$$[\xi_t, \Omega] + [\eta_t, H] = [U, [\xi, \Omega] + [\eta, H]] + [u, \Omega] + [j, H] + [J, [\xi, H]] . \tag{6.15}$$

This may be rearranged in the form

$$[\xi_t - u - [U, \xi], \Omega] + [\eta_t - j - [U, \eta] - [J, \xi], H] = 0. \tag{6.16}$$

The first term vanishes by (6.11); and we then have (again up to an arbitrary additive term commuting with H which does not contribute to ω in (6.7))

$$\eta_t = j + [U, \eta] + [J, \xi]. \tag{6.17}$$

This is the required evolution equation for η. Equation (6.17) implies that

$$(\omega - [\xi, \Omega] - [\eta, H])_t = [U, \omega - [\xi, \Omega] - [\eta, H]] \tag{6.18}$$

(provable by now standard manipulations), thus verifying that if (6.7) is satisfied at $t = 0$, then it is also satisfied for all $t > 0$.

Invariance of $\delta^2 E$ under linearized evolution

We know that the energy E is an exact invariant of the equations (2.1)–(2.3). Moreover, we know that under an imv perturbation from the state (U, H), $\delta^1 E = 0$, and so

$$E = E_0 + \delta^2 E + \dots , \tag{6.19}$$

where $\delta^2 E$ is a quadratic functional of the perturbation fields (u, h). It is to be expected therefore that $\delta^2 E$ should be an invariant of the linearized evolution equations (6.2)–(6.4), since if it were not, then E could not be an invariant of the exact equations. We thus have the following MHD counterpart of Arnold's (1966) result for ideal hydrodynamics:

PROPOSITION 6.1. *The second variation of energy* (5.13) *is an invariant of the linearized equations* (6.2)−(6.4).

The direct verification of this result involves long manipulations. Since not all readers may be prepared to accept the statement without proof, the direct verification is given in Appendix B.

7. Relation of variational principle to generalized Weber transformation

The theory developed in §§ 3–6 above is based on the need to consider perturbations which conserve the cross-helicity invariants (2.11), and the associated circulation invariants (2.12) when closed *h*-lines exist. These integrals are evidently invariant under the replacement

$$u \longrightarrow \tilde{u} = u + h \wedge m + \nabla \alpha, \tag{7.1}$$

where *m* is an arbitrary vector field and α is chosen so that $\nabla \cdot \tilde{u} = 0$ and $n \cdot \tilde{u} = 0$ on $\partial \mathscr{D}$.

It is natural to ask whether *m* may be chosen in such a way that the circulation of $\tilde{u}$ round *any* contour moving with the velocity field *u* (and not only those which coincide with closed *h*-lines) is conserved. If this is so, then the 'modified vorticity field'

$$\tilde{\omega} = \text{curl } \tilde{u} = \omega + [h, m] \tag{7.2}$$

must satisfy the frozen-field equation

$$\tilde{\omega}_t = [u, \tilde{\omega}]. \tag{7.3}$$

It follows from (7.2), (7.3) that

$$\omega_t + [h_t, m] + [h, m_t] = [u, \omega] + [u, [h, m]]. \tag{7.4}$$

Substituting for h_t and ω_t from (2.2), (2.8) and using the Jacobi identity, we obtain

$$[h, m_t - j - [u, m]] = 0 \tag{7.5}$$

and so, up to a field commuting with *h*, *m* must satisfy the evolution equation

$$m_t = [u, m] + j. \tag{7.6}$$

This equation is compatible with imposition of the subsidiary condition

$$\nabla \cdot m = 0. \tag{7.7}$$

Equations (7.2), (7.3), (7.6) represent a slightly modified form of the generalized Weber transformation for MHD equations, which was obtained by the direct procedure of integration of the basic MHD equations (2.1)–(2.3) in Part 1. The auxiliary field *m* appeared there naturally during the integration. Here, we provided an alternative interpretation of *m* as the generator of transformations (7.1) that leave cross-helicities invariant; equation (7.6) is then a consequence of the requirement (7.3) that the circulation of the modified velocity $\tilde{u}$ round any material circuit (in the flow *u*) be conserved.

The imv folium introduced in § 3 by equations (3.1), (3.3) now admits reformulation, putting the emphasis on the frozen-in fields *h* and $\tilde{\omega}$. We keep the same equation (3.1), but replace (3.3) with the 'frozen-in' equation for $\tilde{\omega}$:

$$h_\tau = [v, h], \tag{7.8}$$

$$\tilde{\omega}_\tau = [v, \tilde{\omega}]; \tag{7.9}$$

then now both *h* and $\tilde{\omega}$ are frozen-in in the flow *v* (with conservation of fluxes through all material circuits). One can show that equations (7.8), (7.9) are equivavalent to (3.1), (3.3) if we accept the following connection between fields *m* and *c*:

$$m_\tau = [v, m] + c, \tag{7.10}$$

where *c* is the 'modified current field' of § 3. Since $c(x, \tau)$ is an arbitrary solenoidal

field, we may equally regard $\boldsymbol{m}(\boldsymbol{x}, \tau)$ as the arbitrary field (with $\boldsymbol{c}$ then given by (7.10)), if this proves convenient.

Condition (7.9) for the 'modified vorticity field' is the same as the 'isovorticity condition' originally introduced by Arnold (1965) for the vorticity field (or for the velocity circulation). So *our definition of imv folium is logically identical to the original Arnold definition of isovortical folium: we just replace his frozen-in condition for vorticity by the frozen-in conditions for both magnetic field and modified vorticity.*

Consider now the basic state $\boldsymbol{U}(\boldsymbol{x})$, $\boldsymbol{H}(\boldsymbol{x})$ satisfying (2.13), (2.14), or equivalently

$$[\boldsymbol{U}, \boldsymbol{H}] = 0, \quad [\boldsymbol{U}, \boldsymbol{\Omega}] + [\boldsymbol{J}, \boldsymbol{H}] = 0. \tag{7.11}$$

Let $\boldsymbol{M}(\boldsymbol{x})$ be the associated steady field satisfying

$$[\boldsymbol{U}, \boldsymbol{M}] + \boldsymbol{J} = 0, \tag{7.12}$$

and let

$$\tilde{\boldsymbol{U}} = \boldsymbol{U} + \boldsymbol{H} \wedge \boldsymbol{M} + \nabla\alpha, \tag{7.13}$$

$$\tilde{\boldsymbol{\Omega}} = \boldsymbol{\Omega} + [\boldsymbol{H}, \boldsymbol{M}]. \tag{7.14}$$

Then, from (7.3), we have

$$[\boldsymbol{U}, \tilde{\boldsymbol{\Omega}}] = 0. \tag{7.15}$$

As in §5, we may now consider imv perturbations described by displacement fields $\{\boldsymbol{\xi}(\boldsymbol{x}), \boldsymbol{\eta}(\boldsymbol{x})\}$ where $\boldsymbol{\xi} = \tau\boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{\eta} = \tau\boldsymbol{c}(\boldsymbol{x})$ with τ small. The corresponding perturbation of the field $\boldsymbol{M}$ is given from (7.10) by $\boldsymbol{M} \to \boldsymbol{M} + \boldsymbol{\mu}(\boldsymbol{x})$ where

$$\boldsymbol{\mu}(\boldsymbol{x}) = [\boldsymbol{\xi}, \boldsymbol{M}] + \boldsymbol{\eta}, \tag{7.16}$$

and we may use the pair $\{\boldsymbol{\xi}, \boldsymbol{\mu}\}$ instead of $\{\boldsymbol{\xi}, \boldsymbol{\eta}\}$ to describe imv perturbations.

In terms of these fields the first- and second-order variations of $\tilde{\omega}$ are given by the frozen field relations

$$\delta^1\tilde{\omega} = [\boldsymbol{\xi}, \tilde{\boldsymbol{\Omega}}] \tag{7.17}$$

$$\delta^2\tilde{\omega} = \tfrac{1}{2}[\boldsymbol{\xi}, \delta^1\tilde{\omega}]. \tag{7.18}$$

Hence, from (7.2),

$$\delta^1\omega = [\boldsymbol{\xi}, \tilde{\boldsymbol{\Omega}}] - [\delta^1\boldsymbol{h}, \boldsymbol{M}] - [\boldsymbol{H}, \boldsymbol{\mu}], \tag{7.19}$$

$$\delta^2\omega = \tfrac{1}{2}[\boldsymbol{\xi}, \delta^1\tilde{\omega}] - [\delta^2\boldsymbol{h}, \boldsymbol{M}] - [\delta^1\boldsymbol{h}, \boldsymbol{\mu}]. \tag{7.20}$$

The corresponding first variation of energy is

$$\delta^1 E = \int_{\mathscr{D}} \{\boldsymbol{U} \cdot (\boldsymbol{M} \wedge [\boldsymbol{\xi}, \boldsymbol{H}] + \boldsymbol{\mu} \wedge \boldsymbol{H} + \boldsymbol{\xi} \wedge \tilde{\boldsymbol{\Omega}} - \nabla\alpha) + \boldsymbol{H} \cdot [\boldsymbol{\xi}, \boldsymbol{H}]\} \mathrm{d}V$$

$$= \int_{\mathscr{D}} \{(\boldsymbol{\xi} \wedge \boldsymbol{H}) \cdot ([\boldsymbol{U}, \boldsymbol{M}] + \boldsymbol{J}) + \boldsymbol{\mu} \cdot (\boldsymbol{H} \wedge \boldsymbol{U}) + \boldsymbol{\xi} \cdot (\tilde{\boldsymbol{\Omega}} \wedge \boldsymbol{U})\} \mathrm{d}V$$

$$= 0 \tag{7.21}$$

using (7.11)–(7.15). Thus we have re-established Proposition 5.1 from the alternative point of view.

Similarly the second variation of energy is now given by

$$\delta^2 E = \tfrac{1}{2} \int_{\mathscr{D}} \{(\delta^1\boldsymbol{u})^2 + (\delta^1\boldsymbol{h})^2 + \delta^1\omega \cdot (\boldsymbol{U} \wedge \boldsymbol{\xi}) + \delta^1\boldsymbol{h} \cdot (\boldsymbol{U} \wedge (\boldsymbol{\mu} - [\boldsymbol{\xi}, \boldsymbol{M}]) + \boldsymbol{J} \wedge \boldsymbol{\xi})\} \mathrm{d}V. \tag{7.22}$$

The relationship (7.16) guarantees that this expression is the same as (5.13); and Proposition 5.2 may be reformulated in an obvious way in terms of the fields $\{\boldsymbol{\xi}, \boldsymbol{\mu}\}$. We shall not labour the details; it is evident that we have merely reformulated the results of §5 from the alternative viewpoint.

8. Sufficient conditions for stability

Let us now consider in more detail the expression (5.13) for $\delta^2 E$; with the change of notation (as in §6) $\delta^1 \boldsymbol{u} \to \boldsymbol{u}$, $\delta^1 \boldsymbol{h} \to \boldsymbol{h}$, this may be written

$$\delta^2 E = \tfrac{1}{2} \int_{\mathscr{D}} \left\{ \boldsymbol{u}^2 + \boldsymbol{h}^2 - \boldsymbol{u} \cdot [\boldsymbol{\xi}, \boldsymbol{U}] - \boldsymbol{h} \cdot (\boldsymbol{\eta} \wedge \boldsymbol{U} + \boldsymbol{\xi} \wedge \boldsymbol{J}) \right\} \mathrm{d}V. \tag{8.1}$$

We restrict attention to the class of steady MHD flows for which

$$\boldsymbol{U}(\boldsymbol{x}) = \lambda(\boldsymbol{x})\boldsymbol{H}(\boldsymbol{x}), \quad \boldsymbol{H} \cdot \nabla\lambda = 0. \tag{8.2}$$

For this class of flows, the expression (8.1) may be reduced (see Appendix C) to the form

$$\delta^2 E = \tfrac{1}{2} \int_{\mathscr{D}} (\boldsymbol{u} - \lambda\boldsymbol{h} + \boldsymbol{H}(\boldsymbol{\xi} \cdot \nabla\lambda))^2 \, \mathrm{d}V + W, \tag{8.3}$$

where

$$W = \tfrac{1}{2} \int_{\mathscr{D}} \left\{ (1 - \lambda^2)\left(\boldsymbol{h}^2 + \boldsymbol{h} \cdot (\boldsymbol{J} \wedge \boldsymbol{\xi})\right) - 2\lambda(\boldsymbol{\xi} \cdot \nabla\lambda)(\boldsymbol{\xi} \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{H}) \right\} \mathrm{d}V, \tag{8.4}$$

with, as usual, $\boldsymbol{h} = [\boldsymbol{\xi}, \boldsymbol{H}]$. Note that W is a quadratic functional of $\boldsymbol{\xi}$ alone; the dependence of $\delta^2 E$ on $\boldsymbol{\eta}$ in (8.3) is contained in the first integral (via the dependence of $\boldsymbol{u}$ on $\boldsymbol{\eta}$).

The flow (8.2) is stable if $\delta^2 E$ is sign-definite; hence from (8.3) we have immediately:

PROPOSITION 8.1. *Steady MHD flows satisfying* (8.2) *are linearly stable to three-dimensional perturbations provided*

$$W \geqslant 0 \text{ for all admissible } \boldsymbol{\xi}. \tag{8.5}$$

(The possibility $W = 0$ is included to cover those displacements $\boldsymbol{\xi}$ for which $\boldsymbol{h} = [\boldsymbol{\xi}, \boldsymbol{H}] = 0$.) Let us consider some particular cases.

(*i*) *Flows with constant* λ

If $\lambda = $ const. in $\mathscr{D}$, (8.4) reduces to

$$W = (1 - \lambda^2)W_0, \quad W_0 = \tfrac{1}{2} \int_{\mathscr{D}} \left\{ \boldsymbol{h}^2 + \boldsymbol{h} \cdot (\boldsymbol{J} \wedge \boldsymbol{\xi}) \right\} \mathrm{d}V. \tag{8.6}$$

If $|\lambda| < 1$ (i.e. the flow is *sub-Alfvénic*) then it is stable provided

$$W_0 \geqslant 0 \quad \text{(all admissible } \boldsymbol{\xi}\text{)}. \tag{8.7}$$

This is as found previously by Friedlander & Vishik (1995). The criterion (8.7) applies equally to the case of magnetostatic equilibrium ($\lambda = 0$) and is the well-known criterion of Bernstein *et al.* (1958). Sub-Alfvénic flows with constant λ have the same structure as 'equivalent' magnetostatic equilibria, and are evidently governed by the same stability criterion.

If $|\lambda| > 1$, then the flow would be stable provided $W_0 \leqslant 0$ (all admissible $\boldsymbol{\xi}$). However, as shown by Friedlander & Vishik (1995), this condition is never satisfied:

perturbations $\boldsymbol{\xi}$ of sufficiently small length scale can always be found such that $\boldsymbol{h}^2 > |\boldsymbol{h} \cdot (\boldsymbol{J} \wedge \boldsymbol{\xi})|$ in (8.6).

(*ii*) *Parallel flow and field*

Let $\mathscr{D}$ be an infinite cylinder of arbitrary cross-section with axis parallel to Oz, and suppose that

$$\boldsymbol{H} = H_0(x, y)\boldsymbol{e}_z, \quad \boldsymbol{U} = \lambda(x, y)\boldsymbol{H}, \tag{8.8}$$

a particular case of (8.2). Then $(\boldsymbol{H} \cdot \nabla)\boldsymbol{H} = 0$, and

$$\boldsymbol{h} = \nabla \wedge (\boldsymbol{\xi} \wedge \boldsymbol{H}) = H_0(\boldsymbol{e}_z \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla H_0)\boldsymbol{e}_z. \tag{8.9}$$

After some algebra, (8.4) reduces to

$$W = \tfrac{1}{2} \int_{\mathscr{D}} (1 - \lambda^2) H_0^2 ((\boldsymbol{e}_z \cdot \nabla)\boldsymbol{\xi})^2 \, \mathrm{d}V. \tag{8.10}$$

We assume that the fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ decay sufficiently rapidly as $|z| \to \infty$, so that the integrals in (8.3), (8.4) converge. (Alternatively, we may consider perturbations periodic in z, and take $\mathscr{D}$ to cover one period.) Thus we obtain:

PROPOSITION 8.2. *The state* (8.8) *is linearly stable to imv perturbations provided* $|\lambda(x, y)| \leqslant 1$ *in* $\mathscr{D}$.

(*iii*) *Annular basic state*

Let $\mathscr{D}$ be an annular region between two cylinders C_1, C_2 of arbitrary cross-sections, and let

$$\boldsymbol{H} = -\boldsymbol{e}_z \wedge \nabla A, \quad \boldsymbol{U} = \lambda \boldsymbol{H}, \quad \lambda = \lambda(A), \tag{8.11}$$

where $A(x, y)$ is the flux-function of $\boldsymbol{H}$. We shall suppose that $|\nabla A| \neq 0$ in $\mathscr{D}$, i.e. $\boldsymbol{H}$ has no neutral points in $\mathscr{D}$. The function A satisfies the Grad–Shafranov equation, which may be written

$$(1 - \lambda^2)\nabla^2 A - \lambda\lambda'\boldsymbol{H}^2 = G(A) \tag{8.12}$$

where $\lambda' = \mathrm{d}\lambda/\mathrm{d}A$. Also, in the state (8.11),

$$\boldsymbol{J} = -\nabla^2 A\boldsymbol{e}_z, \quad (\boldsymbol{H} \cdot \nabla)\boldsymbol{H} = \nabla(\tfrac{1}{2}\boldsymbol{H}^2) - (\nabla^2 A)\nabla A. \tag{8.13}$$

Defining the unit vector

$$\boldsymbol{v} = \nabla A/|\nabla A|, \tag{8.14}$$

the expression for W may be reduced (see Appendix D) to the form

$$W = W_1 + W_2, \tag{8.15}$$

where

$$W_1 = \tfrac{1}{2} \int_{\mathscr{D}} (1 - \lambda^2)(\boldsymbol{h} + (\boldsymbol{\xi} \cdot \boldsymbol{v})\boldsymbol{J} \wedge \boldsymbol{v})^2 \mathrm{d}V, \tag{8.16}$$

$$W_2 = -\int_{\mathscr{D}} \left\{ (1 - \lambda^2)(\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{v} + \lambda\lambda'|\boldsymbol{H}|(\boldsymbol{v} \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{H})(\boldsymbol{\xi} \cdot \boldsymbol{v})^2 \right\} \mathrm{d}V. \tag{8.17}$$

Further transformation of W_2, using (8.13), (8.14) yields

$$W_2 = \tfrac{1}{2} \int_{\mathscr{D}} (1 - \lambda^2) \left(\nabla^2 A - \frac{\lambda\lambda'}{1 - \lambda^2}\boldsymbol{H}^2 \right) \left(-\nabla^2 A + \frac{\nabla A \cdot \nabla(\boldsymbol{H}^2)}{2\boldsymbol{H}^2} \right) (\boldsymbol{\xi} \cdot \boldsymbol{v})^2 \mathrm{d}V. \tag{8.18}$$

From (8.15)–(8.18) we conclude:

PROPOSITION 8.3. *The state* (8.11) *is linearly stable to imv perturbations provided* $|\lambda| < 1$ (*i.e. the flow is sub-Alfvénic*) *and*
either

$$\frac{\lambda\lambda'}{1-\lambda^2}\boldsymbol{H}^2 \leqslant \nabla^2 A \leqslant \frac{\nabla A \cdot \nabla(\boldsymbol{H}^2)}{2\boldsymbol{H}^2} \tag{8.19}$$

or

$$\frac{\nabla A \cdot \nabla(\boldsymbol{H}^2)}{2\boldsymbol{H}^2} \leqslant \nabla^2 A \leqslant \frac{\lambda\lambda'}{1-\lambda^2}\boldsymbol{H}^2 \tag{8.20}$$

throughout $\mathscr{D}$.

Note that for the limiting case in which $\boldsymbol{H} = H_0(y)\boldsymbol{e}_x$, $\boldsymbol{U} = \lambda(y)\boldsymbol{H}$, the integral W_2 given by (8.18) vanishes, and the stability criterion of Proposition 8.3 reduces simply to $|\lambda| < 1$, in conformity with Proposition 8.2.

To clarify the conditions (8.19), (8.20), consider the simple case for which $\mathscr{D}$ is the annular domain $a < r < b$, and

$$\boldsymbol{H} = H_0(r)\boldsymbol{e}_\theta, \quad H_0(r) = -A'(r), \quad \boldsymbol{U} = \lambda(r)\boldsymbol{H}. \tag{8.21}$$

Then, defining

$$\chi(r) = r\lambda(r)\lambda'(r)/(1-\lambda^2), \tag{8.22}$$

(8.19), (8.20) are equivalent to the inequalities
either

$$H_0(r) \geqslant 0 \quad \text{and} \quad H_0'(r) + \frac{1-\chi(r)}{r}H_0(r) \leqslant 0 \tag{8.23}$$

or

$$H_0(r) \leqslant 0 \quad \text{and} \quad H_0(r) + \frac{1-\chi(r)}{r}H_0(r) \geqslant 0. \tag{8.24}$$

Both (8.23), (8.24) may be combined in the single inequality

$$H_0(r)\left(H_0'(r) + \frac{1-\chi(r)}{r}H_0(r)\right) \leqslant 0 \quad \text{for } a < r < b \tag{8.25}$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[(1-\lambda^2)(rH_0)^2\right] \leqslant 0. \tag{8.26}$$

We conclude that if (8.26) is satisfied, then the state (8.21) is linearly stable to arbitrary imv perturbations.

Note that, for the case of magnetostatic equilibrium ($\lambda = 0$), (8.26) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}r}(rH_0)^2 \leqslant 0. \tag{8.27}$$

This criterion for stability to arbitrary three-dimensional imv perturbations is, as might be expected, more restrictive than the criterion

$$\frac{\mathrm{d}}{\mathrm{d}r}(H_0/r)^2 \leqslant 0 \tag{8.28}$$

obtained by Moffatt (1986) for stability to axisymmetric perturbations.

Appendix A. Proof of Proposition 5.2

For simplicity we consider here only a simply connected domain $\mathscr{D}$; more complicated geometry can be treated similarly. Let $\hat{\boldsymbol{\xi}} \equiv \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1$, $\hat{\boldsymbol{\eta}} \equiv \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1$. Then, from (5.14) and (5.15),

$$[\hat{\boldsymbol{\xi}}, \boldsymbol{H}] = 0, \quad [\hat{\boldsymbol{\xi}}, \boldsymbol{\Omega}] + [\hat{\boldsymbol{\eta}}, \boldsymbol{H}] = 0. \tag{A1}$$

Hence

$$\Delta E \equiv \delta^2 E(\boldsymbol{\xi}_2, \boldsymbol{\eta}_2) - \delta^2 E(\boldsymbol{\xi}_1, \boldsymbol{\eta}_1) = \tfrac{1}{2} \int_{\mathscr{D}} (\delta^1 \boldsymbol{\omega} \cdot (\boldsymbol{U} \wedge \hat{\boldsymbol{\xi}}) + \delta^1 \boldsymbol{h} \cdot (\boldsymbol{U} \wedge \hat{\boldsymbol{\eta}} + \boldsymbol{J} \wedge \hat{\boldsymbol{\xi}})) \mathrm{d}V. \tag{A2}$$

Using integration by parts and the Jacobi identity, we obtain

$$
\begin{aligned}
I_1 &\equiv \int_{\mathscr{D}} \delta^1 \boldsymbol{\omega} \cdot (\boldsymbol{U} \wedge \hat{\boldsymbol{\xi}}) \mathrm{d}V = \int_{\mathscr{D}} \delta^1 \boldsymbol{u} \cdot [\boldsymbol{U}, \hat{\boldsymbol{\xi}}] \mathrm{d}V \\
&= \int_{\mathscr{D}} (\boldsymbol{\xi}_1 \wedge \boldsymbol{\Omega} + \boldsymbol{\eta}_1 \wedge \boldsymbol{H}) \cdot [\boldsymbol{U}, \hat{\boldsymbol{\xi}}] \mathrm{d}V \\
&= \int_{\mathscr{D}} (\boldsymbol{\Omega} \cdot (\boldsymbol{\xi}_1 \wedge [\hat{\boldsymbol{\xi}}, \boldsymbol{U}]) + (\boldsymbol{\eta}_1 \wedge \boldsymbol{H}) \cdot [\boldsymbol{U}, \hat{\boldsymbol{\xi}}]) \mathrm{d}V \\
&= \int_{\mathscr{D}} (\boldsymbol{U} \cdot [\boldsymbol{\xi}_1, [\hat{\boldsymbol{\xi}}, \boldsymbol{U}]] + (\boldsymbol{\eta}_1 \wedge \boldsymbol{H}) \cdot [\boldsymbol{U}, \hat{\boldsymbol{\xi}}]) \mathrm{d}V \\
&= \int_{\mathscr{D}} (-\boldsymbol{U} \cdot [\hat{\boldsymbol{\xi}}, [\boldsymbol{U}, \boldsymbol{\xi}_1]] - \boldsymbol{U} \cdot [\boldsymbol{U}, [\boldsymbol{\xi}_1, \hat{\boldsymbol{\xi}}]] + (\boldsymbol{\eta}_1 \wedge \boldsymbol{H}) \cdot [\boldsymbol{U}, \hat{\boldsymbol{\xi}}]) \mathrm{d}V. \tag{A3}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &\equiv \int_{\mathscr{D}} \delta^1 \boldsymbol{h} \cdot (\boldsymbol{J} \wedge \hat{\boldsymbol{\xi}}) \mathrm{d}V = \int_{\mathscr{D}} \boldsymbol{J} \cdot (\hat{\boldsymbol{\xi}} \wedge \delta^1 \boldsymbol{h}) \mathrm{d}V = \int_{\mathscr{D}} \boldsymbol{H} \cdot [\hat{\boldsymbol{\xi}}, \boldsymbol{h}] \mathrm{d}V \\
&= \int_{\mathscr{D}} \boldsymbol{H} \cdot [\hat{\boldsymbol{\xi}}, [\boldsymbol{\xi}_1, \boldsymbol{H}]] \mathrm{d}V = \int_{\mathscr{D}} \boldsymbol{H} \cdot (-[\boldsymbol{\xi}_1, [\boldsymbol{H}, \hat{\boldsymbol{\xi}}]] - [\boldsymbol{H}, [\hat{\boldsymbol{\xi}}, \boldsymbol{\xi}_1]]) \mathrm{d}V \\
&= -\int_{\mathscr{D}} \boldsymbol{H} \cdot [\boldsymbol{H}, [\hat{\boldsymbol{\xi}}, \boldsymbol{\xi}_1]] \mathrm{d}V = -\int_{\mathscr{D}} (\boldsymbol{J} \wedge \boldsymbol{H}) \cdot [\hat{\boldsymbol{\xi}}, \boldsymbol{\xi}_1] \mathrm{d}V, \tag{A4}
\end{aligned}
$$

where we have used (A1). Substituting (A3) and (A4) in (A2) and taking account of (2.14), we obtain

$$2\Delta E = \int_{\mathscr{D}} (\delta^1 \boldsymbol{h} \cdot (\boldsymbol{U} \wedge \hat{\boldsymbol{\eta}}) + (\hat{\boldsymbol{\xi}} \wedge \boldsymbol{\Omega}) \cdot [\boldsymbol{U}, \boldsymbol{\xi}_1]) \mathrm{d}V. \tag{A5}$$

Hence, in view of (A1),

$$
\begin{aligned}
2\Delta E &= \int_{\mathscr{D}} \left(\delta^1 \boldsymbol{h} \cdot (\boldsymbol{U} \wedge \hat{\boldsymbol{\eta}}) - (\hat{\boldsymbol{\eta}} \wedge \boldsymbol{H}) \cdot [\boldsymbol{U}, \boldsymbol{\xi}_1] \right) \mathrm{d}V \\
&= \int_{\mathscr{D}} \hat{\boldsymbol{\eta}} \cdot (\boldsymbol{H} \wedge [\boldsymbol{\xi}_1, \boldsymbol{U}] - \boldsymbol{U} \wedge [\boldsymbol{\xi}_1, \boldsymbol{H}]) \mathrm{d}V. \tag{A6}
\end{aligned}
$$

Since $\nabla \cdot \hat{\boldsymbol{\eta}} = 0$ in $\mathscr{D}$, it may be expressed in the form $\hat{\boldsymbol{\eta}} = \nabla \wedge \boldsymbol{g}$ for some vector field $\boldsymbol{g}$.

From (A6), we then obtain

$$
\begin{aligned}
2\Delta E &= \int_{\mathscr{D}} (\nabla \wedge \boldsymbol{g}) \cdot (\boldsymbol{H} \wedge [\boldsymbol{\xi}_1, \boldsymbol{U}] - \boldsymbol{U} \wedge [\boldsymbol{\xi}_1, \boldsymbol{H}]) \mathrm{d}V \\
&= \int_{\mathscr{D}} \boldsymbol{g} \cdot ([\boldsymbol{H}, [\boldsymbol{\xi}_1, \boldsymbol{U}]] + [\boldsymbol{U}, [\boldsymbol{H}, \boldsymbol{\xi}_1]]) \mathrm{d}V = -\int_{\mathscr{D}} \boldsymbol{g} \cdot [\boldsymbol{\xi}_1, [\boldsymbol{U}, \boldsymbol{H}]] \mathrm{d}V = 0, \tag{A7}
\end{aligned}
$$

and this completes the proof.

Appendix B. Proof of Proposition 6.1

The proof is a direct verification that $\delta^2 E$ is conserved by the linearized equations (6.2)–(6.4). First, we identify $\delta^1 u$ and $\delta^1 u$ with infinitesimal perturbations $\boldsymbol{u}$ and $\boldsymbol{h}$ whose evolution is governed by (6.2)–(6.4), so that

$$\delta^2 E = \tfrac{1}{2}\int_{\mathscr{D}} \left\{ u^2 + h^2 - \boldsymbol{u}\cdot[\xi, U] - \boldsymbol{h}\cdot(\boldsymbol{\eta}\wedge U + \xi\wedge J) \right\}\mathrm{d}V . \tag{B 1}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta^2 E\right) = \int_{\mathscr{D}}\{\boldsymbol{u}\cdot\boldsymbol{u}_t + \boldsymbol{h}\cdot\boldsymbol{h}_t + \tfrac{1}{2}\boldsymbol{\omega}_t\cdot(U\wedge\xi)$$
$$+\tfrac{1}{2}\boldsymbol{h}_t\cdot(U\wedge\boldsymbol{\eta} + J\wedge\xi) + \tfrac{1}{2}\boldsymbol{\omega}\cdot(U\wedge\xi_t) + \tfrac{1}{2}\boldsymbol{h}\cdot(U\wedge\boldsymbol{\eta}_t + J\wedge\xi_t)\}\mathrm{d}V .$$

Now we show that

$$I \equiv \int_{\mathscr{D}}\{\boldsymbol{\omega}\cdot(U\wedge\xi_t) + \boldsymbol{h}\cdot(U\wedge\boldsymbol{\eta}_t + J\wedge\xi_t)\}\mathrm{d}V$$
$$= \int_{\mathscr{D}}\{\boldsymbol{\omega}_t\cdot(U\wedge\xi) + \boldsymbol{h}_t\cdot(U\wedge\boldsymbol{\eta} + J\wedge\xi)\}\mathrm{d}V . \tag{B 2}$$

Let

$$I_1 \equiv \int_{\mathscr{D}}\boldsymbol{\omega}\cdot(U\wedge\xi_t)\mathrm{d}V , \quad I_2 \equiv \int_{\mathscr{D}}\boldsymbol{h}\cdot(U\wedge\boldsymbol{\eta}_t + J\wedge\xi_t)\mathrm{d}V . \tag{B 3}$$

We have

$$I_1 = \int_{\mathscr{D}}([\xi, \boldsymbol{\Omega}] + [\boldsymbol{\eta}, H])\cdot(U\wedge\xi_t)\mathrm{d}V = \int_{\mathscr{D}}(\xi\wedge\boldsymbol{\Omega} + \boldsymbol{\eta}\wedge H)\cdot[U, \xi_t]\mathrm{d}V$$
$$= \int_{\mathscr{D}}\{\boldsymbol{\Omega}\cdot(\xi\wedge[\xi_t, U]) + \boldsymbol{\eta}\cdot(H\wedge[U, \xi_t])\}\mathrm{d}V = \int_{\mathscr{D}}\{U\cdot[\xi,[\xi_t, U]] + \boldsymbol{g}\cdot[H,[U, \xi_t]]\}\mathrm{d}V$$
$$= -\int_{\mathscr{D}}\{U\cdot([\xi_t,[U, \xi]] + [U,[\xi, \xi_t]]) + \boldsymbol{g}\cdot([U,[\xi_t, H]] + [\xi_t,[H, U]])\}\mathrm{d}V$$
$$= \int_{\mathscr{D}}\{(\xi_t\wedge\boldsymbol{\Omega})\cdot[U, \xi] - (\boldsymbol{\Omega}\wedge U)\cdot[\xi, \xi_t] + (U\wedge\boldsymbol{\eta})\cdot[\xi_t, H]\}\mathrm{d}V . \tag{B 4}$$

Here the property that $\nabla\cdot\boldsymbol{\eta} = 0$ has been used: we have introduced the vector field $\boldsymbol{g}$ such that $\boldsymbol{\eta} = \nabla\wedge\boldsymbol{g}$.

Similar manipulations result in

$$I_2 = \int_{\mathscr{D}}\{(\boldsymbol{\eta}_t\wedge H)\cdot[U, \xi] + (J\wedge\xi)\cdot[\xi_t, H] + (J\wedge H)\cdot[\xi, \xi_t]\}\mathrm{d}V . \tag{B 5}$$

It follows from (B 4), (B 5) that

$$I = I_1 + I_2 = \int_{\mathscr{D}}\{[U, \xi]\cdot(\xi_t\times\boldsymbol{\Omega} + \boldsymbol{\eta}_t\wedge H)$$
$$+[\xi, \xi_t]\cdot(U\wedge\boldsymbol{\Omega} + J\wedge H) + (J\wedge\xi)\cdot[\xi_t, H]\}\mathrm{d}V . \tag{B 6}$$

Finally, using (6.5), (6.6) and (2.14), we arrive at the formula (B 2).

It follows from (B 2) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta^2 E\right) = \int_{\mathscr{D}}\{\boldsymbol{u}\cdot\boldsymbol{u}_t + \boldsymbol{h}\cdot\boldsymbol{h}_t + \boldsymbol{\omega}_t\cdot(U\wedge\xi) + \boldsymbol{h}_t\cdot(U\wedge\boldsymbol{\eta} + J\wedge\xi)\}\mathrm{d}V . \tag{B 7}$$

After substitution of $\boldsymbol{u}_t$, $\boldsymbol{h}_t$ from (6.2)–(6.3) and some manipulations, this may be

written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta^2 E) = I_3 + I_4 + I_5 \tag{B 8}$$

where

$$I_3 = \int_{\mathscr{D}} \{\boldsymbol{u} \cdot (\boldsymbol{U} \wedge \boldsymbol{\omega}) + (\boldsymbol{U} \wedge \boldsymbol{\xi}) \cdot ([\boldsymbol{u}, \boldsymbol{\Omega}] + [\boldsymbol{U}, \boldsymbol{\omega}])\} \mathrm{d}V, \tag{B 9}$$

$$I_4 = \int_{\mathscr{D}} \{\boldsymbol{u} \cdot \left(\boldsymbol{j} \wedge \boldsymbol{H} + \boldsymbol{J} \wedge \boldsymbol{h}\right) + \boldsymbol{h} \cdot [\boldsymbol{u}, \boldsymbol{H}] + [\boldsymbol{u}, \boldsymbol{H}] \cdot (\boldsymbol{J} \wedge \boldsymbol{\xi})\} \mathrm{d}V, \tag{B 10}$$

$$I_5 = \int_{\mathscr{D}} \{(\boldsymbol{U} \wedge \boldsymbol{\xi}) \cdot ([\boldsymbol{j}, \boldsymbol{H}] + [\boldsymbol{J}, \boldsymbol{h}]) + \boldsymbol{h} \cdot [\boldsymbol{U}, \boldsymbol{h}] \tag{B 11}$$

$$+ [\boldsymbol{u}, \boldsymbol{H}] \cdot (\boldsymbol{U} \wedge \boldsymbol{\eta}) + [\boldsymbol{U}, \boldsymbol{h}] \cdot (\boldsymbol{U} \wedge \boldsymbol{\eta} + \boldsymbol{J} \wedge \boldsymbol{\xi})\} \mathrm{d}V. \tag{B 12}$$

Consider first I_3. We have

$$I_3 = \int_{\mathscr{D}} \{\boldsymbol{\omega} \cdot (\boldsymbol{u} \wedge \boldsymbol{U}) + [\boldsymbol{U}, \boldsymbol{\xi}] \cdot (\boldsymbol{u} \wedge \boldsymbol{\Omega}) + (\boldsymbol{U} \wedge \boldsymbol{\xi})([\boldsymbol{U}, [\boldsymbol{\xi}, \boldsymbol{\Omega}]] + [\boldsymbol{U}, [\boldsymbol{\eta}, \boldsymbol{H}]])\} \mathrm{d}V$$

$$= \int_{\mathscr{D}} \{([\boldsymbol{\xi}, \boldsymbol{\Omega}] + [\boldsymbol{\eta}, \boldsymbol{H}]) \cdot (\boldsymbol{u} \wedge \boldsymbol{U}) + \boldsymbol{\Omega} \cdot (\boldsymbol{u} \wedge [\boldsymbol{\xi}, \boldsymbol{U}])$$
$$- (\boldsymbol{U} \wedge \boldsymbol{\xi}) \cdot ([\boldsymbol{\xi}, [\boldsymbol{\Omega}, \boldsymbol{U}]] + [\boldsymbol{\Omega}, [\boldsymbol{U}, \boldsymbol{\xi}]] + [\boldsymbol{\eta}, [\boldsymbol{H}, \boldsymbol{U}]] + [\boldsymbol{H}, [\boldsymbol{U}, \boldsymbol{\eta}]])\} \mathrm{d}V$$

$$= \int_{\mathscr{D}} \{(\boldsymbol{\xi} \wedge \boldsymbol{\Omega}) \cdot [\boldsymbol{u}, \boldsymbol{U}] + [\boldsymbol{\eta}, \boldsymbol{H}] \cdot (\boldsymbol{u} \wedge \boldsymbol{U}) + \boldsymbol{U} \cdot [\boldsymbol{u}, [\boldsymbol{\xi}, \boldsymbol{U}]]$$
$$- [\boldsymbol{U}, \boldsymbol{\xi}] \cdot (\boldsymbol{\xi} \wedge [\boldsymbol{\Omega}, \boldsymbol{U}]) - [\boldsymbol{U}, \boldsymbol{\xi}] \cdot (\boldsymbol{H} \wedge [\boldsymbol{U}, \boldsymbol{\eta}])\} \mathrm{d}V$$

$$= \int_{\mathscr{D}} \{(\boldsymbol{\xi} \wedge \boldsymbol{\Omega}) \cdot [\boldsymbol{u}, \boldsymbol{U}] + [\boldsymbol{\eta}, \boldsymbol{H}] \cdot (\boldsymbol{u} \wedge \boldsymbol{U}) - \boldsymbol{U} \cdot [\boldsymbol{\xi}, [\boldsymbol{U}, \boldsymbol{u}]] - \boldsymbol{U} \cdot [\boldsymbol{U}, [\boldsymbol{u}, \boldsymbol{\xi}]]$$
$$- (\boldsymbol{\Omega} \wedge \boldsymbol{U}) \cdot [\boldsymbol{\xi}, [\boldsymbol{\xi}, \boldsymbol{U}]] + (\boldsymbol{U} \wedge \boldsymbol{\eta}) \cdot [\boldsymbol{H}, [\boldsymbol{U}, \boldsymbol{\xi}]]\} \mathrm{d}V$$

$$= \int_{\mathscr{D}} \{[\boldsymbol{\eta}, \boldsymbol{H}] \cdot (\boldsymbol{u} \wedge \boldsymbol{U}) - (\boldsymbol{U} \wedge \boldsymbol{\eta}) \cdot [\boldsymbol{U}, \boldsymbol{h}] - (\boldsymbol{\Omega} \wedge \boldsymbol{U})([\boldsymbol{u}, \boldsymbol{\xi}] + [\boldsymbol{\xi}, [\boldsymbol{\xi}, \boldsymbol{U}]])\} \mathrm{d}V. \tag{B 13}$$

It can be shown by similar calculations that

$$I_4 = -\int_{\mathscr{D}} (\boldsymbol{J} \wedge \boldsymbol{H}) \cdot [\boldsymbol{\xi}, \boldsymbol{u}] \mathrm{d}V, \tag{B 14}$$

$$I_5 = \int_{\mathscr{D}} \{(\boldsymbol{J} \wedge \boldsymbol{H}) \cdot [\boldsymbol{\xi}, [\boldsymbol{\xi}, \boldsymbol{U}]] + ([\boldsymbol{u}, \boldsymbol{H}] + [\boldsymbol{U}, \boldsymbol{h}]) \cdot (\boldsymbol{U} \wedge \boldsymbol{\eta})\} \mathrm{d}V. \tag{B 15}$$

From (B 13)–(B 15) and (2.14), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta^2 E) = \int_{\mathscr{D}} \{[\boldsymbol{\eta}, \boldsymbol{H}] \cdot (\boldsymbol{u} \wedge \boldsymbol{U}) + [\boldsymbol{u}, \boldsymbol{H}] \cdot (\boldsymbol{U} \wedge \boldsymbol{\eta})\} \mathrm{d}V. \tag{B 16}$$

Finally, since

$$\int_{\mathscr{D}} [\boldsymbol{u}, \boldsymbol{H}] \cdot (\boldsymbol{U} \times \boldsymbol{\eta}) \mathrm{d}V = -\int_{\mathscr{D}} \boldsymbol{\eta} \cdot (\boldsymbol{U} \wedge [\boldsymbol{u}, \boldsymbol{H}]) \mathrm{d}V$$

$$= -\int_{\mathscr{D}} \boldsymbol{g} \cdot [\boldsymbol{U}, [\boldsymbol{u}, \boldsymbol{H}]] \mathrm{d}V$$

$$= \int_{\mathscr{D}} \boldsymbol{g} \cdot ([\boldsymbol{u}, [\boldsymbol{H}, \boldsymbol{U}]] + [\boldsymbol{H}, [\boldsymbol{U}, \boldsymbol{u}]]) \mathrm{d}V$$

$$= \int_{\mathscr{D}} \boldsymbol{\eta} \cdot (\boldsymbol{H} \wedge [\boldsymbol{U}, \boldsymbol{u}]) = \int_{\mathscr{D}} [\boldsymbol{\eta}, \boldsymbol{H}] \cdot (\boldsymbol{U} \wedge \boldsymbol{u}) \mathrm{d}V,$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta^2 E) = 0. \tag{B 17}$$

Note that while obtaining this formula we have only used the linearized equations (6.2)–(6.4) and the relations (6.5), (6.6) between fields $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and infinitesimal perturbations of the velocity and the magnetic field (we have not explicitly used the evolution equations for $\boldsymbol{\xi}$, $\boldsymbol{\eta}$).

Appendix C. Derivation of equation (8.3)

First we note that for the state (8.2) the relationship (6.6) takes the form

$$\boldsymbol{u} = \boldsymbol{\gamma} + \boldsymbol{\eta} \wedge \boldsymbol{H} - \nabla\alpha, \quad \boldsymbol{\gamma} = \lambda\,\boldsymbol{\xi} \wedge \boldsymbol{J} + \nabla\lambda(\boldsymbol{\xi}\cdot\boldsymbol{H}) - \boldsymbol{H}(\boldsymbol{\xi}\cdot\nabla\lambda). \tag{C 1}$$

From (8.1), (8.2), we have

$$2\delta^2 E = \int_{\mathscr{D}} \{\boldsymbol{u}^2 + \boldsymbol{h}^2 - \boldsymbol{u}\cdot[\boldsymbol{\xi}, \boldsymbol{U}] - \boldsymbol{h}\cdot(\lambda\,\boldsymbol{\eta}\wedge\boldsymbol{H} + \boldsymbol{\xi}\wedge\boldsymbol{J})\}\mathrm{d}V. \tag{C 2}$$

Since

$$[\boldsymbol{\xi}, \boldsymbol{U}] = \lambda[\boldsymbol{\xi}, \boldsymbol{H}] + \nabla\lambda\wedge(\boldsymbol{\xi}\wedge\boldsymbol{H}) = \lambda\boldsymbol{h} - \boldsymbol{H}(\boldsymbol{\xi}\cdot\nabla\lambda), \tag{C 3}$$

we obtain

$$2\delta^2 E = \int_{\mathscr{D}} \{\boldsymbol{u}^2 + \boldsymbol{h}^2 - \lambda\boldsymbol{u}\cdot\boldsymbol{h} + (\boldsymbol{u}\cdot\boldsymbol{H})(\boldsymbol{\xi}\cdot\nabla\lambda) + \boldsymbol{h}\cdot(\boldsymbol{J}\wedge\boldsymbol{\xi}) - \lambda\boldsymbol{h}\cdot(\boldsymbol{\eta}\wedge\boldsymbol{H})\}\mathrm{d}V. \tag{C 4}$$

Further, using (C 1), we get

$$2\delta^2 E = \int_{\mathscr{D}} \{\boldsymbol{u}^2 + \boldsymbol{h}^2 - 2\lambda\boldsymbol{u}\cdot\boldsymbol{h} + (\boldsymbol{u}\cdot\boldsymbol{H})(\boldsymbol{\xi}\cdot\nabla\lambda) + \boldsymbol{h}\cdot(\boldsymbol{J}\wedge\boldsymbol{\xi}) + \lambda\,\boldsymbol{h}\cdot\boldsymbol{\gamma} - \lambda\nabla\alpha\cdot\boldsymbol{h}\}\mathrm{d}V. \tag{C 5}$$

Now

$$\int_{\mathscr{D}} \lambda\nabla\alpha\cdot\boldsymbol{h}\mathrm{d}V = \int_{\mathscr{D}} \lambda\nabla\alpha\cdot[\boldsymbol{\xi}, \boldsymbol{H}]\mathrm{d}V = \int_{\mathscr{D}} (\nabla\lambda\wedge\nabla\alpha)\cdot(\boldsymbol{\xi}\wedge\boldsymbol{H})\mathrm{d}V$$

$$= \int_{\mathscr{D}} \boldsymbol{\xi}\cdot(\boldsymbol{H}\wedge(\nabla\lambda\wedge\nabla\alpha))\mathrm{d}V = \int_{\mathscr{D}} (\boldsymbol{\xi}\cdot\nabla\lambda)(\boldsymbol{H}\cdot\nabla\alpha))\mathrm{d}V.$$

Substitution of this in (C 5) and some manipulations yield

$$2\delta^2 E = \int_{\mathscr{D}} \{(\boldsymbol{u} + \boldsymbol{H}(\boldsymbol{\xi}\cdot\nabla\lambda) - \lambda\boldsymbol{h})^2 + (1 - \lambda^2)(\boldsymbol{h}^2 + \boldsymbol{h}\cdot(\boldsymbol{J}\wedge\boldsymbol{\xi}))$$

$$+ \lambda(\boldsymbol{h}\cdot\boldsymbol{H})(\boldsymbol{\xi}\cdot\nabla\lambda) + \lambda(\boldsymbol{h}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot\boldsymbol{H}) - \lambda(\boldsymbol{\xi}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot(\boldsymbol{J}\wedge\boldsymbol{H}))\}\mathrm{d}V. \tag{C 6}$$

It may be shown that

$$X \equiv \lambda(\boldsymbol{h}\cdot\boldsymbol{H})(\boldsymbol{\xi}\cdot\nabla\lambda) + \lambda(\boldsymbol{h}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot\boldsymbol{H})$$

$$= (\boldsymbol{H}\cdot\nabla)(\lambda(\boldsymbol{\xi}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot\boldsymbol{H})) - \lambda(\boldsymbol{\xi}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot\nabla)(\tfrac{1}{2}H^2) - \lambda(\boldsymbol{\xi}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot(\boldsymbol{H}\cdot\nabla)\boldsymbol{H}).$$

With help of this identity (C 6) may be transformed to the equation

$$2\delta^2 E = \int_{\mathscr{D}} \{(\boldsymbol{u} + \boldsymbol{H}(\boldsymbol{\xi}\cdot\nabla\lambda) - \lambda\boldsymbol{h})^2$$

$$+ (1 - \lambda^2)(\boldsymbol{h}^2 + \boldsymbol{h}\cdot(\boldsymbol{J}\wedge\boldsymbol{\xi})) - 2\lambda(\boldsymbol{\xi}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot(\boldsymbol{H}\cdot\nabla)\boldsymbol{H})\}\mathrm{d}V, \tag{C 7}$$

which evidently coincides with (8.3).

Appendix D. Derivation of formula (8.15)

Consider the integral

$$I = \tfrac{1}{2} \int_{\mathscr{D}} (1 - \lambda^2)(h^2 + \boldsymbol{h} \cdot (\boldsymbol{J} \wedge \boldsymbol{\xi})) \mathrm{d}V, \tag{D 1}$$

which enters the expression (8.4). We start with some transformations of this integral similar to those of Bernstein *et al.* (1958). First, following Bernstein *et al.* (1958), we shall prove the identity

$$2(\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{v} = \boldsymbol{J}^2 + (\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{H})) + \boldsymbol{H} \cdot (\boldsymbol{J} \wedge \boldsymbol{v})\nabla \cdot \boldsymbol{v}, \tag{D 2}$$

where $\boldsymbol{v}$ is defined by (8.14).

It follows from (8.11), (8.14) that $\boldsymbol{H} \cdot \boldsymbol{v} = 0$. Therefore,

$$0 = \nabla(\boldsymbol{H} \cdot \boldsymbol{v}) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{H} + (\boldsymbol{H} \cdot \nabla)\boldsymbol{v} + \boldsymbol{H} \wedge (\nabla \wedge \boldsymbol{v}) + \boldsymbol{v} \wedge \boldsymbol{J}, \tag{D 3}$$

and so

$$\nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{H}) + \boldsymbol{H}\nabla \cdot \boldsymbol{v} = (\boldsymbol{H} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{H}$$

$$= 2(\boldsymbol{H} \cdot \nabla)\boldsymbol{v} + \boldsymbol{H} \wedge (\nabla \wedge \boldsymbol{v}) + \boldsymbol{v} \wedge \boldsymbol{J}.$$

Further, we have

$$Y \equiv \boldsymbol{J}^2 + (\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{H}) + \boldsymbol{H}\nabla \cdot \boldsymbol{v})$$

$$= \boldsymbol{J}^2 + (\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (2(\boldsymbol{H} \cdot \nabla)\boldsymbol{v} + \boldsymbol{H} \wedge (\nabla \wedge \boldsymbol{v}) - \boldsymbol{J} \wedge \boldsymbol{v})$$

$$= 2(\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{v} + \boldsymbol{H} \cdot ((\nabla \wedge \boldsymbol{v}) \wedge (\boldsymbol{J} \wedge \boldsymbol{v})) = 2(\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{v}, \tag{D 4}$$

whence the identity (D 2) immediately follows.

Now we rewrite the integral (D 1) in the form

$$2I = \int_{\mathscr{D}} (1 - \lambda^2)\{(\boldsymbol{h} + (\boldsymbol{\xi} \cdot \boldsymbol{v})\boldsymbol{J} \wedge \boldsymbol{v})^2 - 2(\boldsymbol{\xi} \cdot \boldsymbol{v})^2(\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{v}\}\mathrm{d}V + I_1, \tag{D 5}$$

where

$$I_1 = \int_{\mathscr{D}} (1 - \lambda^2)\{2(\boldsymbol{\xi} \cdot \boldsymbol{v})^2(\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{v} - 2\boldsymbol{h} \cdot (\boldsymbol{J} \wedge \boldsymbol{v})(\boldsymbol{\xi} \cdot \boldsymbol{v})$$

$$- (\boldsymbol{J} \wedge \boldsymbol{v})^2(\boldsymbol{\xi} \cdot \boldsymbol{v})^2 + \boldsymbol{h} \cdot (\boldsymbol{J} \wedge \boldsymbol{\xi})\}\mathrm{d}V. \tag{D 6}$$

Substitution of (D 2) in (D 6) yields

$$I_1 = \int_{\mathscr{D}} (1 - \lambda^2)\{((\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{H})) - J_0|\boldsymbol{H}|\nabla \cdot \boldsymbol{v})(\boldsymbol{\xi} \cdot \boldsymbol{v})^2$$

$$- 2\boldsymbol{h} \cdot (\boldsymbol{J} \wedge \boldsymbol{v})(\boldsymbol{\xi} \cdot \boldsymbol{v}) + \boldsymbol{h} \cdot (\boldsymbol{J} \wedge \boldsymbol{\xi})\}\mathrm{d}V. \tag{D 7}$$

Here $J_0 \equiv -\nabla^2 A$ and we have used (8.11), (8.13), (8.14).

Let

$$\boldsymbol{\xi} = (\boldsymbol{\xi} \cdot \boldsymbol{v})\boldsymbol{v} + \tilde{\boldsymbol{\xi}}, \quad \tilde{\boldsymbol{\xi}} = b\,\boldsymbol{e}_z + c\,\boldsymbol{H}, \tag{D 8}$$

i.e. $b = \boldsymbol{\xi} \cdot \boldsymbol{e}_z$, $c = (\boldsymbol{\xi} \cdot \boldsymbol{H})/H^2$. Substitution of (D8) in (D7) results in

$$I_1 = \int_{\mathscr{D}} (1 - \lambda^2)\{((\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{H})) - J_0|\boldsymbol{H}|\nabla \cdot \boldsymbol{v})(\boldsymbol{\xi} \cdot \boldsymbol{v})^2$$

$$+ \boldsymbol{h} \cdot (\boldsymbol{J} \wedge \tilde{\boldsymbol{\xi}}) - \boldsymbol{h} \cdot (\boldsymbol{J} \wedge \boldsymbol{v})(\boldsymbol{\xi} \cdot \boldsymbol{v})\}\mathrm{d}V. \tag{D 9}$$

Using (6.5) and (D 8), we obtain

$$Y_1 \equiv (\boldsymbol{\xi} \cdot \boldsymbol{v})^2 (\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{H})) - \boldsymbol{h} \cdot (\boldsymbol{J} \wedge \boldsymbol{v})(\boldsymbol{\xi} \cdot \boldsymbol{v})$$

$$= (\boldsymbol{\xi} \cdot \boldsymbol{v})(\boldsymbol{J} \wedge \boldsymbol{v}) \cdot ((\boldsymbol{\xi} \cdot \boldsymbol{v})\nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{H}) - \nabla \wedge (\boldsymbol{\xi} \wedge \boldsymbol{H}))$$

$$= (\boldsymbol{\xi} \cdot \boldsymbol{v})(\boldsymbol{J} \wedge \boldsymbol{v}) \cdot ((\boldsymbol{\xi} \cdot \boldsymbol{v})\nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{H}) - \nabla \wedge ((\boldsymbol{\xi} \cdot \boldsymbol{v})\boldsymbol{v} \wedge \boldsymbol{H} + b(\boldsymbol{e}_z \times \boldsymbol{H})))$$

$$= -J_0 |\boldsymbol{H}|((\boldsymbol{v} \cdot \nabla)\frac{(\boldsymbol{\xi} \cdot \boldsymbol{v})^2}{2} + (\boldsymbol{\xi} \cdot \boldsymbol{v})(\boldsymbol{e}_z \cdot \nabla b)). \tag{D 10}$$

From (D 8) and the condition $\nabla \cdot \boldsymbol{\xi} = 0$, we have

$$\boldsymbol{e}_z \cdot \nabla b = -\boldsymbol{v} \cdot \nabla(\boldsymbol{\xi} \cdot \boldsymbol{v}) - (\boldsymbol{\xi} \cdot \boldsymbol{v})\nabla \cdot \boldsymbol{v} - \boldsymbol{H} \cdot \nabla c, \tag{D 11}$$

whence

$$Y_1 = J_0 |\boldsymbol{H}|((\boldsymbol{\xi} \cdot \boldsymbol{v})^2 \nabla \cdot \boldsymbol{v} + (\boldsymbol{\xi} \cdot \boldsymbol{v})(\boldsymbol{H} \cdot \nabla c)). \tag{D 12}$$

Substituting this in (D 9), we get

$$I_1 = \int_{\mathscr{D}} (1 - \lambda^2)\{J_0 |\boldsymbol{H}|(\boldsymbol{\xi} \cdot \boldsymbol{v})(\boldsymbol{H} \cdot \nabla c) + \boldsymbol{h} \cdot (\boldsymbol{J} \wedge \tilde{\boldsymbol{\xi}})\}\mathrm{d}V. \tag{D 13}$$

Consider now the integral

$$I_2 \equiv \int_{\mathscr{D}} (1 - \lambda^2)\boldsymbol{h} \cdot (\boldsymbol{J} \wedge \tilde{\boldsymbol{\xi}})\mathrm{d}V. \tag{D 14}$$

Since $\boldsymbol{J} \wedge \tilde{\boldsymbol{\xi}} = J_0 c \nabla A$, we have

$$I_2 = \int_{\mathscr{D}} (1 - \lambda^2)J_0 c \nabla A \cdot (\nabla \wedge (\boldsymbol{\xi} \wedge \boldsymbol{H}))\mathrm{d}V$$

$$= \int_{\mathscr{D}} (\nabla((1 - \lambda^2)J_0 c) \wedge \nabla A) \cdot (\boldsymbol{\xi} \wedge \boldsymbol{H})\mathrm{d}V$$

$$= \int_{\mathscr{D}} (1 - \lambda^2)\boldsymbol{\xi} \cdot (\boldsymbol{H} \wedge (\nabla(J_0 c) \wedge \nabla A))\mathrm{d}V$$

$$= -\int_{\mathscr{D}} (1 - \lambda^2)(\boldsymbol{\xi} \cdot \boldsymbol{v})|\boldsymbol{H}|(\boldsymbol{H} \cdot (J_0 c))\mathrm{d}V,$$

whence,

$$I_1 = \int_{\mathscr{D}} (1 - \lambda^2)|\boldsymbol{H}|c(\boldsymbol{\xi} \cdot \boldsymbol{v})(\boldsymbol{H} \cdot \nabla(\nabla^2 A))\mathrm{d}V. \tag{D 15}$$

Now let us recall equation (8.12). Using (8.11) and (8.12), we obtain

$$(1 - \lambda^2)(\boldsymbol{H} \cdot \nabla)\nabla^2 A = \lambda\lambda'(\boldsymbol{H} \cdot \nabla(H^2)). \tag{D 16}$$

Equation (D 15) can therefore be written as

$$I_1 = \int_{\mathscr{D}} \lambda\lambda'|\boldsymbol{H}|(\boldsymbol{H} \cdot \nabla(H^2))c(\boldsymbol{\xi} \cdot \boldsymbol{v})\mathrm{d}V. \tag{D 17}$$

Now, taking account of (D 5) and (D 17), we can rewrite (8.4) in the form

$$W = W_1 + R, \tag{D 18}$$

where

$$W_1 = \tfrac{1}{2} \int_{\mathscr{D}} (1 - \lambda^2)(\boldsymbol{h} + (\boldsymbol{\xi} \cdot \boldsymbol{v})\boldsymbol{J} \wedge \boldsymbol{v})^2 \mathrm{d}V$$

$$R = \tfrac{1}{2} \int_{\mathscr{D}} \{\lambda\lambda'|\boldsymbol{H}|(\boldsymbol{H} \cdot \nabla(\boldsymbol{H}^2))c(\boldsymbol{\xi} \cdot \boldsymbol{v})$$
$$-2(1 - \lambda^2)(\boldsymbol{\xi} \cdot \boldsymbol{v})^2(\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{v} - 2\lambda\lambda'|\boldsymbol{H}|(\boldsymbol{\xi} \cdot \boldsymbol{v})\boldsymbol{\xi} \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{H}\}\mathrm{d}V . \qquad \text{(D 19)}$$

All that remains to be done is to show that R in (D 19) coincides with the integral W_2 given by (8.17). We have

$$Y_3 \equiv -\lambda\lambda'|\boldsymbol{H}|(\boldsymbol{\xi} \cdot \boldsymbol{v})\boldsymbol{\xi} \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{H}$$

$$= -\lambda\lambda'|\boldsymbol{H}|(\boldsymbol{\xi} \cdot \boldsymbol{v})((\boldsymbol{\xi} \cdot \boldsymbol{v})\boldsymbol{v} + c\,\boldsymbol{H}) \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{H} . \qquad \text{(D 20)}$$

Substitution of this in R yields

$$R = -\int_{\mathscr{D}} \{(1 - \lambda^2)(\boldsymbol{\xi} \cdot \boldsymbol{v})^2(\boldsymbol{J} \wedge \boldsymbol{v}) \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{v} + \lambda\lambda'|\boldsymbol{H}|(\boldsymbol{\xi} \cdot \boldsymbol{v})^2\boldsymbol{v} \cdot (\boldsymbol{H} \cdot \nabla)\boldsymbol{H}\}\mathrm{d}V . \qquad \text{(D 21)}$$

After comparison of this formula with (8.17) we conclude that R is indeed the same as W_2.

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