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Asymptotic model for free surface flow of an electrically conducting fluid in a high-frequency magnetic field

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Dedicated to Roderick S.C. Wong on the occasion of his 60th birthday

Abstract

We study the behaviour of a layer of an electrically conducting inviscid incompressible fluid in a high-frequency alternating magnetic field. We derive nonlinear asymptotic equations governing the evolution of the fluid layer in the high-frequency limit. As a test for the model, we consider the linearised stability problem for an infinite planar free surface of a layer of finite depth.

MSC: 76W05; 76E25; 76E17

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1. Introduction

Alternating electromagnetic fields are used in a number of metal processing techniques to levitate, stir and confine liquid metals. The effect of electromagnetic fields on the free surface of electrically conducting fluids has received considerable attention in the literature [2–5,7,8]. The simplest problem that has been studied by many authors (see, e.g., [2]) is the stability of a planar horizontal free surface of a conducting fluid in the presence of a horizontal alternating magnetic field. The field induces electric current in the

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fluid which, in turn, produces the Lorentz force acting on the fluid. At high frequency, the magnetic field penetrates only a shallow surface layer of the fluid. If we ignore finite thickness of this surface layer and regard it as a surface (of zero thickness), we arrive at the model where the fluid is perfectly conducting and where there is a current sheet at the free surface leading to an appropriate jump in tangent magnetic field. This idealized problem had been studied by Ladikov [4] who concluded that a high-frequency rotating magnetic field can stabilise a levitating layer of a liquid metal. Garnier and Moreau [3] had considered the linearised stability of a planar horizontal interface between a nonconducting fluid occupying half-space and carrying a uniform alternating magnetic field and an infinitely deep layer of a fluid of finite electric conductivity. They considered the limit of high frequency of the applied magnetic field and averaged the Lorentz force over the period. The result of their study demonstrated that the magnetic field is stabilising. Ramos and Castellanos [7] had taken into account viscosity of the fluid in a similar stability problem with a plane perfectly conducting cover at some distance from the interface. Parametric resonance at moderate frequency of the applied magnetic field has been studied by Cherepanov [1] in the case of perfectly conducting but viscous fluid and by Fautrelle and Sneyd [2] for a fluid with finite conductivity.

Most papers on the subject deal with the linearised stability problems in two limit cases: either the fluid is perfectly conducting or its conductivity is not only finite but large enough. Another approximation employed in many papers is that, while studying the linearised stability in high-frequency limit, an ad hoc procedure of averaging the Lorentz force is employed. Although the procedure seems to be physically reasonable, it is desirable to obtain the averaging procedure using some regular technique. It is the aim of the present paper to obtain the nonlinear equations that describe asymptotic behaviour of a layer of an inviscid fluid of finite electric conductivity in an arbitrary periodic magnetic field in the high-frequency limit.

The plan of the paper is as follows. In Section 2, we formulate the problem. In Section 3, we derive an asymptotic form of the governing equations using the method of multiple scales. Section 4 deals with the linearised stability of a planar free surface in the framework of the asymptotic model obtained in Section 3. Finally, in Section 5, we discuss the results.

2. Formulation of the problem

Consider a layer of a conducting inviscid fluid. The layer is bounded by a rigid perfectly conducting plane $z=-H$ and a free surface

$$z=\zeta(x,y,t) \tag{1}$$

and extends to infinity in x and y direction. In the space above the layer ($z>\zeta(x,y,t)$) there is vacuum (or a nonconducting gas of small density), and a periodic (in time) alternating magnetic field is applied at infinity (as $z\to\infty$). In the vacuum region, the magnetic field $\mathbf{B}(\mathbf{x},t)$ is irrotational

$$\mathbf{B}=\nabla\phi,\quad \nabla^2\phi=0\quad \text{for } z>\zeta(x,y,t), \tag{2}$$

$$\nabla\phi\to\mathbf{B}^{\infty}(\omega t)\quad \text{as } z\to\infty, \tag{3}$$

where $\mathbf{B}^{\infty}(\omega t)$ is a periodic function of t with period $2\pi/\omega$ with zero mean value,

$$\overline{\mathbf{B}}^{\infty}=\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\mathbf{B}^{\infty}(\omega t)\,\mathrm{d}t=0.$$

We assume that $\mathbf{B}^\infty$ is parallel to the horizontal plane, i.e. $\mathbf{B}^\infty \cdot \mathbf{e}_z = 0$. The evolution of the magnetic field and the velocity of the fluid $\mathbf{v}(\mathbf{x}, t)$ in the layer is governed by the standard equations of magnetohydrodynamics:

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - g\mathbf{e}_z + \frac{1}{\rho}(\mathbf{j} \times \mathbf{B}), \tag{4}$$

$$\mathbf{B}_t = \nabla \times (\mathbf{v} \times \mathbf{B}) - \eta\nabla \times (\nabla \times \mathbf{B}), \tag{5}$$

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla \cdot \mathbf{B} = 0. \tag{6}$$

Here $p(\mathbf{x}, t)$ is the pressure, ρ the density of the fluid, g the gravity constant, η the magnetic diffusivity. We employ the usual assumption that ρ and η are constants.

Boundary conditions at the rigid perfectly conducting boundary $z = -H$ are

$$\mathbf{v} \cdot \mathbf{n} = 0, \tag{7}$$

$$B_3 = 0, \tag{8}$$

$$B_{1z} = 0, \quad B_{2z} = 0 \quad \text{at } z = -H. \tag{9}$$

Conditions (9) follow from the requirement that the tangent components of the electric field vanish at the boundary.

Boundary conditions at the free surface of the fluid include the continuity of the magnetic field

$$\mathbf{B} = \nabla\phi \quad \text{at } z = \zeta(x, y, t), \tag{10}$$

the kinematic condition

$$\zeta_t + \mathbf{v} \cdot \nabla\zeta = v_3 \quad \text{at } z = \zeta(x, y, t) \tag{11}$$

and the condition of continuity of normal stress, which, in view of (10), can be written as

$$p = -\gamma\kappa, \tag{12}$$

where γ is the surface tension and κ is the mean curvature of the free surface, given by

$$\kappa = \frac{(1 + \zeta_x^2)\zeta_{yy} + (1 + \zeta_y^2)\zeta_{xx} - 2\zeta_x\zeta_y\zeta_{xy}}{(1 + \zeta_x^2 + \zeta_y^2)^{3/2}}. \tag{13}$$

We consider fluid flows which are either periodic in x and y with periods L_1 and L_2 or decaying at infinity. In the former case, we assume that the mean depth of the layer is H, so that

$$\langle\zeta\rangle = \frac{1}{L_1 L_2}\int_0^{L_1}\int_0^{L_2} \zeta(x, y, t)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{14}$$

In the latter case, H is the layer depth at infinity and, hence, $\zeta(x, y, t) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$.

Now it is convenient to rewrite the problem (2)–(12) in dimensionless variables. Let

$$t = T\tilde{t}, \quad \mathbf{x} = L\tilde{\mathbf{x}}, \quad \mathbf{v} = \frac{L}{T}\tilde{\mathbf{v}}, \quad \mathbf{B} = B_0\tilde{\mathbf{B}}, \quad p = \rho\frac{L^2}{T^2}\tilde{p}, \tag{15}$$

where T and L are some characteristic scales for time and length, $B_0 = (\overline{|\mathbf{B}^\infty|^2})^{1/2}$ is the characteristic scale for magnetic field, $\tilde{t}$, $\tilde{\mathbf{x}}$, $\tilde{\mathbf{v}}$, $\tilde{\mathbf{B}}$ and $\tilde{p}$ are the corresponding dimensionless quantities. Now, we substitute formulas (15) into (2)–(12) and drop ‘tildes’ to simplify the notation. As a result, we have

$$\nabla^2\phi = 0 \quad \text{for } z > \zeta(x, y, t), \tag{16}$$

$$\nabla\phi \to \mathbf{B}^\infty(\tilde{\omega}t) \quad \text{as } z \to \infty, \tag{17}$$

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p - \tilde{g}\mathbf{e}_z + \alpha(\mathbf{j}\times\mathbf{B}), \tag{18}$$

$$\mathbf{B}_t = \nabla\times(\mathbf{v}\times\mathbf{B}) - \tilde{\eta}\nabla\times(\nabla\times\mathbf{B}), \tag{19}$$

$$\nabla\cdot\mathbf{v} = 0, \quad \nabla\cdot\mathbf{B} = 0. \tag{20}$$

Here $\tilde{\omega}$ is the dimensionless frequency, $\tilde{g} = gT^2/L$ the dimensionless gravity constant, $\alpha = B_0^2T^2/\rho L^2$ the squared ratio of the Alfven velocity to the characteristic velocity of the fluid, $\tilde{\eta} = \eta T/L^2$ the dimensionless magnetic diffusivity.

Boundary conditions (7)–(11) remain the same, while the dynamic condition (12) takes the form

$$p = -\tilde{\gamma}\kappa, \tag{21}$$

where $\tilde{\gamma} = \gamma T^2/\rho L^3$ is the dimensionless surface tension and κ is given by Eq. (13).

The characteristic scales T and L can be chosen in a number of different ways. We choose $L = \delta$ and $T = \sqrt{\rho}\delta/B_0$, where $\delta = \sqrt{\eta/\omega}$ is the electromagnetic skin depth. It follows that

$$\alpha = 1, \quad \tilde{\omega} = \frac{\sqrt{\rho}\delta\omega}{B_0}, \quad \tilde{\eta} = \tilde{\omega}, \quad \tilde{g} = \frac{\rho g\delta}{B_0^2}, \quad \tilde{\gamma} = \frac{\gamma}{B_0^2\delta}.$$

3. Asymptotic model for $\tilde{\omega} \to \infty$

Our aim is to obtain an asymptotic form of the above problem for large $\tilde{\omega}$. We assume that $\tilde{g}$, $\tilde{\gamma}$ and the dimensionless depth of the layer $h = H/\delta$ are fixed. Note that since $\delta \to 0$ as $\omega \to \infty$, the last assumption implies that finite values of h correspond to $H \sim \delta$ and $H \to 0$ as $\omega \to \infty$, while $h = \infty$ corresponds to a fixed value of H. Both cases will be covered by the same asymptotic model.

To derive the asymptotic form of the governing equations, we employ the method of multiple scales. Let $\varepsilon = 1/\tilde{\omega}$, $\tau = \tilde{\omega}t$. We seek a solution of (16)–(21), (7)–(12) in the form

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t, \tau), \quad \mathbf{B} = \mathbf{B}(\mathbf{x}, t, \tau), \quad p = p(\mathbf{x}, t, \tau), \quad \phi = \phi(\mathbf{x}, t, \tau), \quad \zeta = \zeta(x, y, t, \tau).$$

Substitution of these equations into (11), (18), (19) yields

$$\mathbf{v}_\tau = -\varepsilon[\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p + \tilde{g}\mathbf{e}_z - \mathbf{j}\times\mathbf{B}], \tag{22}$$

$$\mathbf{B}_\tau + \nabla\times(\nabla\times\mathbf{B}) = -\varepsilon[\mathbf{B}_t - \nabla\times(\mathbf{v}\times\mathbf{B})] \tag{23}$$

for $-h < z < \zeta(x, y, t, \tau)$, and

$$\zeta_\tau + \varepsilon[\zeta_t + \mathbf{v}\cdot\nabla\zeta - v_3] = 0 \tag{24}$$

at $z=\zeta(x,y,t,\tau)$. We assume that for small ε the solution has the form

$$\begin{aligned}
\mathbf{v} &= \mathbf{v}_0(\mathbf{x},t,\tau)+\varepsilon\mathbf{v}_1(\mathbf{x},t,\tau)+O(\varepsilon^2),\\
\mathbf{B} &= \mathbf{B}_0(\mathbf{x},t,\tau)+\varepsilon\mathbf{B}_1(\mathbf{x},t,\tau)+O(\varepsilon^2),\\
p &= p_0(\mathbf{x},t,\tau)+\varepsilon p_1(\mathbf{x},t,\tau)+O(\varepsilon^2),\\
\zeta &= \zeta_0(\mathbf{x},t,\tau)+\varepsilon\zeta_1(\mathbf{x},t,\tau)+O(\varepsilon^2),\\
\phi &= \phi_0(\mathbf{x},t,\tau)+\varepsilon\phi_1(\mathbf{x},t,\tau)+O(\varepsilon^2).
\end{aligned} \tag{25}$$

Substituting these expansions into Eqs. (22)–(24), we obtain in the leading order in ε (terms proportional to ε^0)

$$\mathbf{v}_{0\tau}=0, \tag{26}$$

$$\mathbf{B}_{0\tau}+\nabla\times(\nabla\times\mathbf{B}_0)=0 \quad \text{for } -h<z<\zeta_0(x,y,t,\tau); \tag{27}$$

$$\zeta_{0\tau}=0. \tag{28}$$

In the leading order, Eqs. (16) and (20) yield

$$\nabla^2\phi_0=0 \quad \text{for } z>\zeta_0(x,y,t,\tau), \tag{29}$$

$$\nabla\cdot\mathbf{v}_0=0, \quad \nabla\cdot\mathbf{B}_0=0 \quad \text{for } -h<z<\zeta_0(x,y,t,\tau). \tag{30}$$

It follows from (26) and (28) that $\mathbf{v}_0$ and ζ_0 do not depend on fast time τ, i.e.

$$\mathbf{v}_0=\mathbf{v}_0(\mathbf{x},t), \quad \zeta_0=\zeta_0(x,y,t).$$

At any fixed (slow) time t, Eqs. (27), (29) and (30) together with boundary conditions

$$\nabla\phi_0\to\mathbf{B}^{\infty}(\tau) \quad \text{as } z\to\infty, \tag{31}$$

$$\mathbf{B}_0=\nabla\phi_0 \quad \text{at } z=\zeta_0(x,y,t), \tag{32}$$

$$B_3=0, \quad B_{1z}=0, \quad B_{2z}=0 \quad \text{at } z=-h \tag{33}$$

represent a closed problem for $\phi_0(\mathbf{x},t,\tau)$ and $\mathbf{B}_0(\mathbf{x},t,\tau)$. It can be shown that any periodic solution of this problem has zero mean value, i.e.

$$\overline{\mathbf{B}}_0(\mathbf{x},t)=\frac{1}{2\pi}\int_0^{2\pi}\mathbf{B}_0(\mathbf{x},t,\tau)\,\mathrm{d}\tau=0.$$

Thus, in the leading order we have high-frequency oscillating magnetic field and slowly varying velocity and free surface.

To determine $\mathbf{v}_0(\mathbf{x},t)$ and $\zeta_0(\mathbf{x},t)$, we must employ the first-order approximation (terms proportional to ε). We have the following equations for $\mathbf{v}_1$ and ζ_1:

$$\mathbf{v}_{1\tau}=-[\mathbf{v}_{0t}+(\mathbf{v}_0\cdot\nabla)\mathbf{v}_0+\nabla p_0+\tilde{g}\mathbf{e}_z-\mathbf{j}_0\times\mathbf{B}_0], \tag{34}$$

$$\zeta_{1\tau}=-[\zeta_{0t}+\mathbf{v}_0\cdot\nabla\zeta_0-v_{03}] \quad \text{at } z=\zeta_0(x,y,t,\tau). \tag{35}$$

We seek periodic (in τ) solutions. On averaging Eqs. (34) and (35), we obtain

$$\mathbf{v}_{0t} + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0 + \nabla \overline{p}_0 + \tilde{g}\mathbf{e}_z - \overline{\mathbf{j}_0 \times \mathbf{B}_0} = 0, \tag{36}$$

$$\zeta_{0t} + \mathbf{v}_0 \cdot \nabla\zeta_0 - v_{03} = 0 \quad \text{at } z = \zeta_0(x, y, t), \tag{37}$$

where

$$\overline{\mathbf{j}_0 \times \mathbf{B}_0} = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{j}_0(\mathbf{x}, t, \tau) \times \mathbf{B}_0(\mathbf{x}, t, \tau)\, \mathrm{d}\tau, \quad \overline{p}_0 = \frac{1}{2\pi}\int_0^{2\pi} p_0(\mathbf{x}, t, \tau)\, \mathrm{d}\tau.$$

Eq. (36) must be supplemented with the incompressibility condition $\nabla \cdot \mathbf{v}_0 = 0$ and boundary conditions

$$\overline{p}_0 = -\tilde{\gamma}\kappa_0 \quad \text{at } z = \zeta_0(x, y, t), \tag{38}$$

$$v_{03} = 0 \quad \text{at } z = -h. \tag{39}$$

From here on, we forget about the derivation of the model and work only with the asymptotic equations. It is convenient to change the notation

$$\mathbf{v}_0 \to \mathbf{v}, \quad \mathbf{B}_0 \to \mathbf{B}, \quad \zeta_0 \to \zeta, \quad p_0 \to p, \quad \phi_0 \to \phi.$$

With this notation the complete asymptotic problem has the form

$$\nabla^2\phi = 0 \quad \text{for } z > \zeta(x, y, t), \tag{40}$$

$$\nabla\phi \to \mathbf{B}^{\infty}(\tau) \quad \text{as } z \to \infty; \tag{41}$$

$$\mathbf{B}_\tau + \nabla \times (\nabla \times \mathbf{B}) = 0, \tag{42}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{for } -h < z < \zeta(x, y, t); \tag{43}$$

$$\mathbf{B} = \nabla\phi \quad \text{at } z = \zeta(x, y, t), \tag{44}$$

$$B_3 = 0, \quad B_{1z} = 0, \quad B_{2z} = 0 \quad \text{at } z = -h; \tag{45}$$

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p - \tilde{g}\mathbf{e}_z + \overline{\mathbf{j} \times \mathbf{B}}, \tag{46}$$

$$\nabla \cdot \mathbf{v} = 0 \quad \text{for } -h < z < \zeta(x, y, t); \tag{47}$$

$$\zeta_t + \mathbf{v} \cdot \nabla\zeta = v_3, \tag{48}$$

$$p = -\tilde{\gamma}\kappa \quad \text{at } z = \zeta(x, y, t), \tag{49}$$

$$v_3 = 0 \quad \text{at } z = -h. \tag{50}$$

In Eq. (49), κ is given by (13).

As was mentioned above, the fast (or ‘magnetic’) part of the problem, given by Eqs. (40)–(45), depends on ‘slow time’ t only via the equation of free boundary and is otherwise completely separated from the slow (or 'fluid') part of the problem. Therefore, the 'magnetic' problem (40)–(45) can be solved independently for each $\zeta(x, y, t)$. Then, on substituting $\mathbf{B}[\zeta(x, y, t)]$ in Eq. (46), we obtain the closed problem for $\mathbf{v}$, p and ζ.

4. Stability of planar free-surface

Here we consider the stability of the exact solution of the asymptotic equations (40)–(50) that corresponds to a steady planar free-surface.

4.1. Basic state

The basic state whose stability is studied is produced by the alternating magnetic field at infinity, given by

$$\mathbf{B}^{\infty}(\tau) = (q_1 \cos\tau, q_2 \sin\tau, 0),$$

where q_1 and q_2 are real constants such that $(q_1^2 + q_2^2)/2 = 1$. In the basic state,

$$\zeta = 0, \quad \mathbf{v} = 0, \quad p = P(z), \quad \mathbf{B} = \mathbf{B}^0(z, \tau), \quad \phi = \mathbf{B}^{\infty} \cdot \mathbf{x}. \tag{51}$$

The magnetic field $\mathbf{B}^0(z, \tau)$ is the periodic (in τ) solution of the following problem

$$\mathbf{B}^0_{\tau} + \nabla \times (\nabla \times \mathbf{B}^0) = 0, \quad \nabla \cdot \mathbf{B}^0 = 0 \quad \text{for } -h < z < 0;$$
$$\mathbf{B}^0 = \mathbf{B}^{\infty}(\tau) \quad \text{at } z = 0, \quad B^0_3 = B^0_{1z} = B^0_{2z} = 0 \quad \text{at } z = -h.$$

It can be written in the form

$$\mathbf{B}^0 = Re\left[\frac{\mathrm{ch}[\lambda(z+h)]}{\mathrm{ch}[\lambda h]} \mathrm{e}^{\mathrm{i}\tau} \mathbf{a}\right], \tag{52}$$

where

$$\lambda = \frac{1+\mathrm{i}}{\sqrt{2}}, \quad \mathbf{a} = q_1 \mathbf{e}_x - \mathrm{i} q_2 \mathbf{e}_y. \tag{53}$$

The pressure in the basic state is given by

$$P(z) = -\tilde{g}z - \tfrac{1}{2}\overline{|\mathbf{B}^0(z, \tau)|^2} + \tfrac{1}{2}\overline{|\mathbf{B}^0(0, \tau)|^2}. \tag{54}$$

4.2. Linearised stability problem

Let now $\mathbf{b}(\mathbf{x}, t, \tau)$ and $\nabla\phi(\mathbf{x}, t, \tau)$ be the perturbation magnetic field in the fluid and in the vacuum region, respectively, $\mathbf{v}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ the perturbation velocity and the perturbation pressure, and let $z = \zeta(x, y, t)$ represent the perturbed free surface. We assume that $\mathbf{b}(\mathbf{x}, t, \tau)$, $\nabla\phi(\mathbf{x}, t, \tau)$, $\mathbf{v}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ and $\zeta(x, y, t)$ are small, so that their evolution is governed by the following linearised equations and boundary conditions:

$$\nabla^2\phi = 0 \quad \text{for } z > 0, \tag{55}$$

$$\nabla\phi \to 0 \quad \text{as } z \to \infty; \tag{56}$$

$$\mathbf{b}_\tau + \nabla\times(\nabla\times\mathbf{b}) = 0, \tag{57}$$

$$\nabla\cdot\mathbf{b} = 0 \quad \text{for } -h < z < 0; \tag{58}$$

$$\mathbf{b} + \mathbf{B}^0_z\zeta = \nabla\phi \quad \text{at } z = 0, \tag{59}$$

$$b_3 = 0, \quad b_{1z} = 0, \quad b_{2z} = 0 \quad \text{at } z = -h; \tag{60}$$

$$\mathbf{v}_t = -\nabla p + \overline{\mathbf{j}\times\mathbf{B}^0} + \overline{\mathbf{J}^0\times\mathbf{b}}, \tag{61}$$

$$\nabla\cdot\mathbf{v} = 0 \quad \text{for } -h < z < 0; \tag{62}$$

$$\zeta_t = v_3, \tag{63}$$

$$p + P_z\zeta = -\tilde{\gamma}(\zeta_{xx} + \zeta_{yy}) \quad \text{at } z = 0; \tag{64}$$

$$v_3 = 0 \quad \text{at } z = h. \tag{65}$$

We seek solutions of (55)–(65) in the form

$$\mathbf{b}(\mathbf{x},\tau,t) = Re(\hat{\mathbf{b}}(z,\tau,t)e^{i(k_1x+k_2y)}), \quad \mathbf{v}(\mathbf{x},t) = Re(\hat{\mathbf{v}}(z,t)e^{i(k_1x+k_2y)}), \quad \text{etc.}$$

Substitution of these into (55)–(65) yields

$$\hat{\phi}_{zz} - k^2\hat{\phi} = 0 \quad \text{for } z > 0, \tag{66}$$

$$\hat{\phi} \to 0 \quad \text{as } z \to \infty; \tag{67}$$

$$\hat{\mathbf{b}}_\tau = \hat{\mathbf{b}}_{zz} - k^2\hat{\mathbf{b}}, \tag{68}$$

$$i\mathbf{k}\cdot\hat{\mathbf{b}} + \hat{b}_{3z} = 0 \quad \text{for } -h < z < 0; \tag{69}$$

$$\hat{\mathbf{b}} = -\mathbf{B}^0_z\hat{\zeta} - \nabla\hat{\phi} \quad \text{at } z = 0, \tag{70}$$

$$\hat{b}_3 = 0, \quad \hat{b}_{1z} = 0, \quad \hat{b}_{2z} = 0 \quad \text{at } z = -h; \tag{71}$$

$$\hat{\mathbf{v}}_t = -\nabla\Pi + \overline{(\mathbf{B}^0\cdot\nabla)\hat{\mathbf{b}}} + \overline{(\hat{\mathbf{b}}\cdot\nabla)\mathbf{B}^0}, \tag{72}$$

$$i\mathbf{k}\cdot\hat{\mathbf{v}} + \hat{v}_{3z} = 0 \quad \text{for } -h < z < 0; \tag{73}$$

$$\hat{\zeta}_t = \hat{v}_3, \tag{74}$$

$$\hat{\Pi} = -P_z\hat{\zeta} + k^2\hat{\zeta} + \overline{\mathbf{B}^0\cdot\mathbf{b}} \quad \text{at } z = 0, \tag{75}$$

$$\hat{v}_3 = 0 \quad \text{at } z = -h. \tag{76}$$

Here $\mathbf{k} = (k_1, k_2, 0)$, $k = |\mathbf{k}|$, $\Pi = \hat{p} + \overline{\mathbf{B}^0\cdot\mathbf{b}}$.

4.3. ‘Magnetic’ part of the linearised problem

General solution of Eq. (66) satisfying condition (67) has the form

$$\hat{\phi}(z,t,\tau)=A(t,\tau)\mathrm{e}^{-kz}.$$

Substituting this into the boundary condition (70), we find that

$$\mathbf{k}\cdot\hat{\mathbf{b}}=-\mathbf{k}\cdot\mathbf{B}_z^0\hat{\zeta}-\mathrm{i}k^2A,\quad \hat{b}_3=-kA\quad \text{at } z=0.$$

These conditions and Eq. (69) allow us to eliminate $\mathbf{k}\cdot\hat{\mathbf{b}}$ and A and obtain the boundary condition for $\hat{b}_3$ alone:

$$\hat{b}_{3z}+k\hat{b}_3=\mathrm{i}\mathbf{k}\cdot\mathbf{B}_z^0\,\hat{\zeta}\quad \text{at } z=0. \tag{77}$$

Also, we have

$$\hat{b}_3=0\quad \text{at } z=-h \tag{78}$$

and

$$\hat{b}_{3\tau}=\hat{b}_{3zz}-k^2\hat{b}_3\quad \text{for } -h<z<0. \tag{79}$$

The periodic solution of the problem (77)–(79) is given by

$$\hat{b}_3=\mathrm{i}\hat{\zeta}(t)\,Re\left(\frac{(\mathbf{k}\cdot\mathbf{a})\lambda\,\mathrm{th}[\lambda h]\mathrm{sh}[\varkappa(z+h)]}{\mathrm{ch}(\varkappa h)[\varkappa+k\,\mathrm{th}(\varkappa h)]}\mathrm{e}^{\mathrm{i}\tau}\right), \tag{80}$$

where $\varkappa=\sqrt{k^2+\mathrm{i}}$, and the square root branch is chosen in such a way that $Re\,\varkappa>0$. In what follows we do not need $\hat{b}_1$ and $\hat{b}_2$ and, therefore, we do not give the corresponding expressions here.

4.4. ‘Fluid’ part of the linearised problem

Consider now Eqs. (72)–(76). Taking scalar product of Eq. (72) with i$\mathbf{k}$ and using Eq. (73), we find that

$$\Pi=\frac{1}{k^2}[-\hat{v}_{3zt}+\mathrm{i}(\overline{(\mathbf{k}\cdot\mathbf{B}^0)\hat{b}_{3z}}-\overline{(\mathbf{k}\cdot\mathbf{B}_z^0)\hat{b}_3})]. \tag{81}$$

Substitution of this formula into the z-component of Eq. (72) results in the equation

$$\hat{v}_{3tzz}-k^2\hat{v}_{3t}=\mathrm{i}Q, \tag{82}$$

where

$$Q=(\overline{(\mathbf{k}\cdot\mathbf{B}^0)\hat{b}_{3z}}-\overline{(\mathbf{k}\cdot\mathbf{B}_z^0)\hat{b}_3})_z-k^2\overline{(\mathbf{k}\cdot\mathbf{B}^0)\hat{b}_3}.$$

Now we assume that $Q(z,t)$ is a known function and solve Eq. (82), subject to condition (76). This yields

$$\hat{v}_{3t}=\frac{\mathrm{i}}{k}\int_{-h}^{z}\mathrm{sh}[k(z-s)]Q(s,t)\,\mathrm{d}s+C\,\mathrm{sh}[k(z+h)] \tag{83}$$

with arbitrary constant C. Then, this constant is determined from the boundary condition (74):

$$C = \frac{1}{\operatorname{sh}(kh)} \left(\hat{\zeta}_{tt} + \frac{\mathrm{i}}{k} \int_{-h}^{0} \operatorname{sh}(kz) Q(z,t)\, \mathrm{d}s \right). \tag{84}$$

From (81) and (83), we can obtain an expression for $\Pi(z,t)$. Finally, substitution of $\Pi(z,t)$ into (75) and certain manipulations give us the following equation for $\hat{\zeta}(t)$:

$$\hat{\zeta}_{tt} + \Omega^2 \hat{\zeta} = \mathrm{i}(1 + \operatorname{th}(kh))\, \overline{(\mathbf{k} \cdot \mathbf{B}^0)\hat{b}_3}|_{z=0} - 2\mathrm{i} I(k),$$

where

$$I(k) = \frac{1}{\operatorname{ch}(kh)} \int_{-h}^{0} \overline{(\mathbf{k} \cdot \mathbf{B}^0_z)\hat{b}_3}\, \operatorname{ch}[k(z+h)]\, \mathrm{d}z$$

and $\Omega^2 = \tilde{\gamma} k^3 + \tilde{g} k$ is the frequency of capillary-gravity waves. Then, after taking account of formula (80) for $\hat{b}_3$ and some manipulations, we arrive at the following equation:

$$\hat{\zeta}_{tt} + \Omega^2 \hat{\zeta} + \frac{|\mathbf{k} \cdot \mathbf{a}|^2}{2} F(k) \hat{\zeta} = 0, \tag{85}$$

where

$$F(k,h) = Re \left\{ \frac{\lambda \operatorname{th}(\lambda h)(\operatorname{th}(kh)[k + \varkappa \operatorname{th}(\varkappa h)] - \lambda^* \operatorname{th}(\lambda^* h))}{\varkappa[\varkappa + k \operatorname{th}(\varkappa h)]} \right\}. \tag{86}$$

For any given h, function $F(k,h)$ is positive for any $k \geqslant 0$. Asymptotic behaviour of $F(k,h)$ for large k is $F = (1/k)\, Re\{\lambda \operatorname{th}(\lambda h)\} + O(k^{-2})$. For large h, $F(k,h)$ reduces to the formula of Garnier and Moreau [3]:

$$F(k,h) \to F_0(k) = \frac{\sqrt{\sqrt{k^4+1} + k^2} + \left(1 - \sqrt{2}k\right)\sqrt{\sqrt{k^4+1} - k^2}}{2\sqrt{k^4+1}}$$

as $h \to \infty$.

It follows from (85) that the plane free surface is always stable provided that

$$\Omega^2_{\text{eff}} = \Omega^2 + \frac{|\mathbf{k} \cdot \mathbf{a}|^2}{2} F(k) > 0.$$

This is always true if $\tilde{g} > 0$, i.e. when the vacuum region is above the fluid layer. Moreover, the magnetic field always improves the stability and the most stable regime corresponds to a rotating magnetic field when $\mathbf{B}^\infty(\tau) = (\cos\tau, \sin\tau, 0)$ and $\mathbf{a} = \mathbf{e}_x - \mathrm{i}\mathbf{e}_y$. It turns out that the rotating magnetic field can also provide stable levitation of a fluid layer. Indeed, if $\tilde{g} < 0$, then

$$\Omega^2_{\text{eff}} = \tilde{\gamma} k^3 - |\tilde{g}| k + \frac{k^2}{2} F(k).$$

In this case, there always exists a critical value k_* such that $\Omega^2_{\text{eff}}(k) > 0$ for $k > k_*$ and $\Omega^2_{\text{eff}}(k) < 0$ for $k < k_*$. This means that only perturbations with sufficiently long wavelength can result in instability. Therefore, a layer which extends to infinity in x and y directions cannot be stabilised by the magnetic field, while a layer of finite size in both x and y directions can be stabilised provided that $L > 2\pi/k_*$, where L is the horizontal size of the layer.

Note that a nonrotating magnetic field cannot stabilise a levitating fluid layer. For example, if $\mathbf{B}^\infty(\tau) = (\cos\tau, 0)$, then $\mathbf{a} = \mathbf{e}_x$ and

$$\Omega_{\text{eff}}^2 = \tilde{\gamma}k^3 - |\tilde{g}|k + \frac{k_1^2}{2}F(k).$$

Evidently, there always are perturbations with $k_2 \neq 0$ which result in $\Omega_{\text{eff}}^2 < 0$. The physical mechanism of no stabilising effect in this case is that there exist perturbations which do not bend the magnetic field lines. If the magnetic field is rotating, i.e. both a_1 and a_2 nonzero, then no such perturbations exist, and the field is stabilising.

5. Conclusion

We have considered free surface flows of a layer of a conducting inviscid fluid in the presence of a periodic (in time) magnetic field and derived the asymptotic form of the governing equations in the limit of high frequency of the applied magnetic field. The 'magnetic' part of the model that describes the high-frequency periodic magnetic field is separated from the 'fluid' part that governs slow evolution of the free surface. Earlier, Garnier and Moreau [3] and others (see, e.g., [7]) have used similar arguments in linearised stability analysis of a planar interface between two fluids with different electric conductivities. In contrast with those papers, our asymptotic equations are *nonlinear* and obtained using a well-defined formal procedure. As a test for the asymptotic model, we have studied the linearised stability of the infinite planar free surface of a layer of finite depth and have found that the effect of a rotating magnetic field is stabilising and, moreover, that a layer of finite depth and finite horizontal size can, in principle, levitate in the presence of a rotating magnetic field. Our results are in agreement with the results by Garnier and Moreau [3], and this provides certain justification of the model.

The domain of applicability of the model follows from our key assumption that the dimensionless frequency of the alternating magnetic field $\tilde{\omega}$ goes to infinity. This requires that $\tilde{\omega} = (\rho\eta\omega)^{1/2}/B_0 \gg 1$. Another assumption was that in our model the viscosity of the fluid is neglected. This is justified if the magnetic Prandtl number $P_{\text{m}} = \nu/\eta$ is very small (ν is the kinematic viscosity of the fluid). Indeed, if we take account of viscosity, the right-hand side of Eq. (18) would contain the viscous term $\tilde{\nu}\nabla^2\mathbf{v}$, where $\tilde{\nu}$ is the dimensionless viscosity, given by

$$\tilde{\nu} = \nu\frac{\sqrt{\rho}}{B_0\delta} = \frac{\nu}{\eta}\tilde{\omega}.$$

It follows that the viscous term does not affect the leading terms of the asymptotic expansion of Section 3 and can therefore be ignored provided that $P_{\text{m}} = o(1/\tilde{\omega})$. As has been argued in [3], the above assumptions can, in principle, be satisfied in laboratory experiment.

Though the asymptotic equations obtained here are much simpler than the original equations, they still represent a very complicated nonlinear problem which, in the absence of the magnetic field, reduces to the initial-boundary-value problem describing nonlinear capillary-gravity waves on the free surface of an inviscid fluid. Possibly, somewhat simpler asymptotic equations can be obtained for flow regimes whose characteristic length scale is much larger than the electromagnetic skin depth δ. In this case, the magnetic field is confined to a shallow surface layer, and equations of boundary-layer type can be obtained. This is the subject of a continuing investigation. Another important open problem concerns the effect of viscosity on free surface flows of a conducting fluid generated by a high-frequency alternating magnetic field. Qualitative analysis of Moffatt [6] shows that for sufficiently large but finite Reynolds numbers the net effect of a rotating magnetic field is to induce an effective stress just inside the shallow layer on the free surface. Construction of an asymptotic model of this phenomenon is another topic for a further investigation.

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