Existence and Stability of Compressible and Incompressible Current-Vortex Sheets

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Abstract

Recent author's results in the investigation of current-vortex sheets (MHD tangential discontinuities) are surveyed. A sufficient condition for the neutral stability of planar compressible current-vortex sheets is first found for a general case of the unperturbed flow. In astrophysical applications, this condition can be considered as the sufficient condition for the stability of the heliopause, which is modelled by an ideal compressible current-vortex sheet and caused by the interaction of the supersonic solar wind plasma with the local interstellar medium (in some sense, the heliopause is the boundary of the solar system). The linear variable coefficients problem for nonplanar compressible current-vortex sheets is studied as well. Since the tangential discontinuity is characteristic, the functional setting is provided by the anisotropic weighted Sobolev spaces. The a priori estimate deduced for this problem is a necessary step to prove the local-in-time existence of current-vortex sheet solutions of the nonlinear equations of ideal compressible MHD. Analogous results are obtained for incompressible current-vortex sheets. In the incompressibility limit the sufficient stability condition found for compressible current-vortex sheets describes exactly the half of the whole parameter domain of linear stability of planar discontinuities in ideal incompressible MHD.

1 Introduction

We consider the equations of magnetohydrodynamics (MHD) governing the motion of an ideal (inviscid and perfectly conducting) compressible fluid. In the nonconservative form the MHD equations read (see e.g. [11, 13]):

$$\begin{aligned} &\frac{1}{\rho c^2}\frac{dp}{dt} + \operatorname{div}\mathbf{v} = 0\,, \qquad \rho\frac{d\mathbf{v}}{dt} - (\mathbf{H},\nabla\mathbf{H}) + \nabla q = 0\,, \\ &\frac{d\mathbf{H}}{dt} - (\mathbf{H},\nabla)\mathbf{v} + \mathbf{H}\operatorname{div}\mathbf{v} = 0\,, \qquad \frac{dS}{dt} = 0\,. \end{aligned} \tag{1}$$

Here $\rho=\rho(t,\mathbf{x})$, $\mathbf{v}=\mathbf{v}(t,\mathbf{x})=(v_1,v_2,v_3)$, $\mathbf{H}=\mathbf{H}(t,\mathbf{x})=(H_1,H_2,H_3)$, $p=p(t,\mathbf{x})$, $S=S(t,\mathbf{x})$ are the density, the fluid velocity, the magnetic field, the pressure, and the entropy respectively, $q=p+|\mathbf{H}|^2/2$ is the total pressure, $c^2=\partial p/\partial\rho$ is the square of the sound velocity, t is the time, $\mathbf{x}=(x_1,x_2,x_3)$ are space variables, and $d/dt=\partial_t+(\mathbf{v},\nabla)$, $\partial_t=\partial/\partial t$, $\partial_j=\partial/\partial x_j$. With a state equation of medium, $p=p(\rho,S)$, we can consider (1) as a closed system for the vector of unknowns $\mathbf{U}=\mathbf{U}(t,\mathbf{x})=(p,\mathbf{v},\mathbf{H},S)$. Moreover, system (1) should be supplemented by the divergent constraint

$$\operatorname{div}\mathbf{H} = 0\,, \tag{2}$$

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that is just an additional requirement on the initial data $\mathbf{U}(0,\mathbf{x})=\mathbf{U}_0(\mathbf{x})$. As is known, system (1) is written in the form

$$A_0(\mathbf{U})\mathbf{U}_t+\sum_{k=1}^{3}A_k(\mathbf{U})\mathbf{U}_{x_k}=0\,, \tag{3}$$

with symmetric matrices A_α, and system (3) is symmetric hyperbolic if the following natural assumptions (hyperbolicity conditions) hold:

$$\rho>0\,,\quad c^2>0\,. \tag{4}$$

Let $\Gamma(t)=\{x_1-f(t,\mathbf{x}')=0\}$ be a smooth hypersurface in $\mathbb{R}\times\mathbb{R}^3$ ($\mathbf{x}'=(x_2,x_3)$ are tangential coordinates). We assume that $\Gamma(t)$ is a surface of tangential discontinuity [13] (*current-vortex sheet*) for solutions of the MHD system. This is the type of contact discontinuities for which the normal component of the magnetic field is zero on $\Gamma(t)$, and the tangential components of the velocity and the magnetic field may undergo any jump on $\Gamma(t)$:

$$H^{\pm}_{\rm N}=0\,,\quad [\mathbf{v}_\tau]\neq 0\,,\quad [\mathbf{H}_\tau]\neq 0$$

Here

$$H_{\rm N}=(\mathbf{H},\mathbf{N})\,,\quad v_{\rm N}=(\mathbf{v},\mathbf{N})\,,\quad \mathbf{v}_\tau=(v_{\tau_1},v_{\tau_2})\,,\quad \mathbf{H}_\tau=(H_{\tau_1},H_{\tau_2})\,,$$
$$v_{\tau_i}=(\mathbf{v},\boldsymbol{\tau}_i)\,,\quad H_{\tau_i}=(\mathbf{H},\boldsymbol{\tau}_i)\,,\quad \boldsymbol{\tau}_1=(f_{x_2},1,0)\,,\quad \boldsymbol{\tau}_2=(f_{x_3},0,1)\,;$$

$\mathbf{N}=(1,-f_{x_2},-f_{x_3})$ is a space normal vector to $\Gamma(t)$, $[g]=g^+-g^-$ denotes the jump for every regularly discontinuous function g with corresponding values behind ($g^+:=g|_{x_1-f(t,\mathbf{x}')\to+0}$) and ahead ($g^-:=g|_{x_1-f(t,\mathbf{x}')\to-0}$) of the discontinuity front Γ. For current-vortex sheets, the general MHD Rankine-Hugoniot conditions (see e.g. [11, 13, 4, 23]) are satisfied in the following way:

$$f_t=v^{\pm}_{\rm N}\,,\quad H^{\pm}_{\rm N}=0\,,\quad [q]=0\,. \tag{5}$$

The initial boundary value problem for system (1) in the domains $\Omega^\pm(t):=\{x_1\gtrless f(t,\mathbf{x}')\}$ with the boundary conditions (5) on the hypersurface $\Gamma(t)$ is a free boundary value problem. Indeed, the function $f(t,\mathbf{x}')$ defining Γ is one of the unknowns of problem (1), (5) with the corresponding initial data

$$f(0,\mathbf{x}')=f_0(\mathbf{x}')\,,\quad \mathbf{x}'\in\mathbb{R}^2\,;\quad \mathbf{U}(0,\mathbf{x})=\mathbf{U}_0(\mathbf{x})\,,\quad \mathbf{x}\in\Omega^\pm(0)\,. \tag{6}$$

It is worth to note that for problem (1), (5), (6) the divergent constraint (2) as well as the boundary conditions

$$H^{+}_{\rm N}=0\,,\quad H^{-}_{\rm N}=0 \tag{7}$$

can be regarded as the restrictions only on the initial data (6). This fact was not formally proved in [23] but this can be done by analogy with the proof in [24] for the case of incompressible MHD (see below).

Definition 1.1 *A piecewise smooth vector-function* $\mathbf{U}(t,\mathbf{x})$ *is called current-vortex sheet solution of the MHD equations (1) if there exists a smooth hypersurface* Γ *such that* $\mathbf{U}$ *is a classical solution of (1) on either side of* Γ *and conditions (5) hold at each point of* Γ.

To prove the existence of current-vortex sheet solutions for the MHD equations it needs to reply on the following question: does there exist a solution $(\mathbf{U},f)$ of problem (1), (5), (6)? Because of the general properties of hyperbolic conservation laws it is natural to expect only the local-in-time existence of current-vortex sheet solutions. In this connection, the question on the nonlinear Lyapunov's stability of current-vortex sheet has no sense.

At the same time, the study of the linearized stability of current-vortex sheet solutions is not only a necessary step to prove local-in-time existence but also is of independent interest in connection with various astrophysical applications (see e.g. [18, 3, 19]). Piecewise constant solutions of (1) satisfying (5) on a *planar* discontinuity are a simplest case of current-vortex sheet solutions.

In astrophysics and geophysics the linear stability of a planar compressible current-vortex sheet is usually interpreted as the macroscopic stability of the *heliopause* (see e.g. [19] and references therein). The model of heliopause was suggested in [3], and the heliopause is in fact a current-vortex sheet separating the interstellar plasma compressed at the bow shock from the solar wind plasma compressed at the termination shock wave. That is, the heliopause is the model for the boundary of the solar system.

We can show that for the constant coefficients linearized problem for planar current-vortex sheets (see Sect. 3) the uniform Kreiss-Lopatinski condition [12, 16] is never satisfied [23]. That is, planar current-vortex sheets are never uniformly stable and can be only neutrally (*weakly*) stable or violently unstable. In the 1960–90's, in a number of works (see [19] and references therein) motivated by astrophysical applications (in particular, by applications to the heliopause) the linear stability of planar compressible current-vortex sheets was examined by the normal modes analysis. But, before the recent result in [23] neither stability nor instability conditions were found for a general case of the unperturbed flow. The main difficulty in the normal modes analysis is connected with the fact that the Lopatinski determinant is generically reduced to an algebraic equation of the tenth degree depending on seven dimensionless parameters and one more inner parameter determining the wave vector (see [23]). Moreover, the squaring was applied under the reduction of the Lopatinski determinant to this algebraic equation and, therefore, it can introduce spurious roots. For all these reasons both the analytical analysis and the *full* numerical study of the Lopatinski determinant are unacceptable for finding the Lopatinski condition for compressible current-vortex sheets. The alternative energy method suggested in [23] has first enabled to find sufficient conditions for their weak stability.

Unlike the case of compressible current-vortex sheets, for planar current-sheets in *incompressible* MHD the linear stability conditions can be straightforwardly found [22, 2]. At the same time, the question on the local-in-time existence of incompressible current-vortex sheets remains open. First results in this direction were recently obtained in [24].

A current-vortex sheet solution of the system of ideal incompressible MHD

$$\frac{d\mathbf{v}}{dt} - (\mathbf{H}, \nabla\mathbf{H}) + \nabla q = 0\,, \quad \frac{d\mathbf{H}}{dt} - (\mathbf{H}, \nabla)\mathbf{v} = 0\,, \quad \operatorname{div}\mathbf{v} = 0 \tag{8}$$

is determined as a piecewise smooth solution $\mathbf{U} = (\mathbf{v}, \mathbf{H})$ of (8) being a classical solution of (8) on either side of a smooth hypersurface Γ and satisfying the jump conditions (5) at each point of Γ. Here the magnetic field is measured in Alfvén velocity units and the pressure p was divided by the density ρ ($\rho \equiv \text{const} > 0$). Other notations are the same as in (1). Generically, for current-vortex sheets the density can be piecewise constant. But, since this gives no trouble, we suppose for simplicity that it is the same constant ($\rho^+ = \rho^- = \rho$) on either side of Γ. We can show (see [24]) that for the free boundary value problem (8), (5), (6) the divergent constraint (2) and the boundary conditions (7) can be regarded as the restrictions only on the initial data (6).

2 A "secondary" symmetrization of the compressible MHD equations

The crucial role in obtaining the a priori estimate for the linearized problem associated to (1), (5), (6) (see Sect. 3) is played by a new symmetric form [23] of the MHD equations that is a kind of "secondary" symmetrization of the symmetric system (3). Using the linear analog of this symmetrization one can get a conserved energy integral for the constant coefficients linearized problem, $dI(t)/dt = 0$ (see Sect. 3), that implies the desired a priori estimate for planar compressible current-vortex sheets, provided that $I(t) > 0$. The last inequality gives a sufficient condition of the linear stability of a planar compressible current-vortex sheet (the macroscopic stability of the heliopause).

The mentioned "secondary" symmetrization is performed as follows. In view of the divergent constraint (2), system (3) implies

$$PA_0\mathbf{U}_t + \sum_{k=1}^{3} PA_k\mathbf{U}_{x_k} + \mathbf{R}\,\mathrm{div}\mathbf{H} = 0\,, \tag{9}$$

where the matrix $P = P(\mathbf{U})$ and the vector $\mathbf{R} = \mathbf{R}(\mathbf{U})$ are yet arbitrary. If we choose

$$
P = \begin{pmatrix}
1 & \dfrac{\lambda H_1}{\rho c^2} & \dfrac{\lambda H_2}{\rho c^2} & \dfrac{\lambda H_3}{\rho c^2} & 0 & 0 & 0 & 0 \\
\lambda H_1 \rho & 1 & 0 & 0 & -\rho\lambda & 0 & 0 & 0 \\
\lambda H_2 \rho & 0 & 1 & 0 & 0 & -\rho\lambda & 0 & 0 \\
\lambda H_3 \rho & 0 & 0 & 1 & 0 & 0 & -\rho\lambda & 0 \\
0 & -\lambda & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -\lambda & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -\lambda & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad \mathbf{R} = -\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ H_1 \\ H_2 \\ H_3 \\ 0 \end{pmatrix}
$$

(the function $\lambda = \lambda(\mathbf{U})$ is arbitrary), then system (9) is again symmetric:

$$
\mathcal{A}_0(\mathbf{U})\mathbf{U}_t + \sum_{k=1}^{3} \mathcal{A}_k(\mathbf{U})\mathbf{U}_{x_k} = 0\,, \tag{10}
$$

where

$$
\mathcal{A}_0 = PA_0 = \begin{pmatrix}
\dfrac{1}{\rho c^2} & \dfrac{\lambda H_1}{c^2} & \dfrac{\lambda H_2}{c^2} & \dfrac{\lambda H_3}{c^2} & 0 & 0 & 0 & 0 \\
\dfrac{\lambda H_1}{c^2} & \rho & 0 & 0 & -\rho\lambda & 0 & 0 & 0 \\
\dfrac{\lambda H_2}{c^2} & 0 & \rho & 0 & 0 & -\rho\lambda & 0 & 0 \\
\dfrac{\lambda H_3}{c^2} & 0 & 0 & \rho & 0 & 0 & -\rho\lambda & 0 \\
0 & -\rho\lambda & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -\rho\lambda & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -\rho\lambda & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix};
$$

for the concrete form of the matrices $\mathcal{A}_k$ we refer to [23]. Note that $\lambda\sqrt{\rho}$ is a dimensionless value, and for $\lambda = 0$ system (10) coincides with (3).

The symmetric system (10) is hyperbolic if $\mathcal{A}_0 > 0$ (this also guarantees that $\det P \neq 0$). Direct calculations show that the last condition is satisfied if inequalities (4) hold together with the additional requirement

$$
\rho\lambda^2 < \frac{1}{1 + c_{\mathrm{A}}^2/c^2}, \tag{11}
$$

where $c_{\mathrm{A}} = |\mathbf{H}|/\sqrt{\rho}$. Of course, the hyperbolicity conditions for system (10) are much more restrictive than the usual natural assumptions (4). It should be also noted that condition (11) guarantees the equivalence of systems (1) and (10) on smooth solutions provided that $\lambda(\mathbf{U})$ is a smooth function of its variables (components of the vector $\mathbf{U}$).

Proposition 2.1 *Let the hyperbolicity conditions (4) and (11) hold for systems (1) and (10) respectively, and the initial data for these systems satisfy the divergent constraint (2). Let* $\lambda = \lambda(\mathbf{U}) : \mathbb{R}^8 \to \mathbb{R}$ *is a smooth enough function of their arguments. Assume* $[0, T]$ *is a time interval on which both hyperbolic systems (1) and (10) have a unique classical solution. Then classical solutions of the Cauchy problems for systems (1) and (10) coincide on the interval* $[0, T]$ *.*

For the proof of Proposition 2.1 we refer to [23]. In principle, analogous assertion could be proved for current-vortex sheet solutions of the MHD system.

The "incompressible" counterpart of symmetrization (10) reads [24]:

$$
\mathcal{A}_0(\mathbf{U})\mathbf{U}_t + \sum_{k=1}^{3} \mathcal{A}_k(\mathbf{U})\mathbf{U}_{x_k} + \mathbf{b} \otimes \nabla q = 0\,, \tag{12}
$$

where

$$\mathcal{A}_0 = \begin{pmatrix} 1 & -\lambda \\ -\lambda & 1 \end{pmatrix} \otimes I_3\,, \quad \mathcal{A}_k = \begin{pmatrix} v_k + \lambda H_k & -H_k - \lambda v_k \\ -H_k - \lambda v_k & v_k + \lambda H_k \end{pmatrix} \otimes I_3\,, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -\lambda \end{pmatrix};$$

$\lambda = \lambda(\mathbf{U})$ is an arbitrary function, I_j is the unit matrix of order j. System (12) is equivalent to (8) if $\det A_0 \neq 0$ and coincides with (8) if $\lambda = 0$. Moreover, the matrix $\mathcal{A}_0$ is positive definite if $|\lambda| < 1$. In [24] the "symmetric" form (12) of system (8) plays the crucial role for obtaining "linearized" a priori estimates for problem (8), (5), (6). The condition $|\lambda| < 1$ being satisfied on either side of a planar current-vortex sheet defines exactly the half of the whole parameter domain [22, 2] of linear stability of incompressible current-vortex sheets (see Sect. 3).

3 Linear stability of planar current-vortex sheets

To work in fixed domains instead of the domains $\Omega^{\pm}(t)$ we make, as usual (see e.g. [16]), the following change of variables in $\mathbb{R} \times \mathbb{R}^3$:

$$\widetilde{t} = t\,, \quad \widetilde{x}_1 = x_1 - f(t, \mathbf{x}')\,, \quad \widetilde{\mathbf{x}}' = \mathbf{x}'\,. \tag{13}$$

Then, $\widetilde{\mathbf{U}}(\widetilde{t}, \widetilde{\mathbf{x}}) := \mathbf{U}(t, \mathbf{x})$ is a smooth vector-function for $\widetilde{\mathbf{x}} \in \mathbb{R}^3_{\pm}$, and problem (1), (5), (6) is reduced to the following problem (we omit tildes to simplify the notation):

$$L(\mathbf{U}, \mathbf{F})\mathbf{U} = 0 \quad \text{in } [0, T] \times (\mathbb{R}^3_+ \cup \mathbb{R}^3_-), \tag{14}$$

$$v_{\mathrm{N}}^{\pm} = f_t\,, \quad H_{\mathrm{N}}^{\pm} = 0\,, \quad [q] = 0 \quad \text{on } [0, T] \times \{x_1 = 0\} \times \mathbb{R}^2, \tag{15}$$

$$\mathbf{U}|_{t=0} = \mathbf{U}_0 \quad \text{in } \mathbb{R}^3_+ \cup \mathbb{R}^3_-, \qquad f|_{t=0} = f_0 \quad \text{in } \mathbb{R}^2. \tag{16}$$

Here

$$L = L(\mathbf{U}, \mathbf{F}) = A_0(\mathbf{U})\partial_t + A_\nu(\mathbf{U}, \mathbf{F})\partial_1 + A_2(\mathbf{U})\partial_2 + A_3(\mathbf{U})\partial_3, \quad \mathbf{F} = \mathbf{F}(t, \mathbf{x}') = (f_t, f_{x_2}, f_{x_3});$$

$$A_\nu = A_\nu(\mathbf{U}, \mathbf{F}) = \sum_{\alpha=0}^{3} \nu_\alpha A_\alpha = A_1(\mathbf{U}) - f_t A_0(\mathbf{U}) - \sum_{k=2}^{3} f_{x_k} A_k(\mathbf{U}) \quad \text{is the } \textit{boundary matrix},$$

$\boldsymbol{\nu} = (\nu_0, \ldots, \nu_n) = (-f_t, \mathbf{N})$ is the space-time normal vector to $\Gamma(t)$.

Since the boundary matrix A_ν is singular at $x_1 = 0$ (see [23]), compressible current-vortex sheets are *characteristic* discontinuities. Note also that the boundary matrix is of constant rank 2 on the boundary $x_1 = 0$. Unlike [8], where 2D compressible vortex sheets were analyzed, we make here the straightening of variables that is standard for shock waves [16]. The change of variables (13) is quite suitable for studying problem (14)–(16) by the *energy method*. However, for applying, as in [8], Kreiss' symmetrizer technique it needs to make another change of variables [10, 8]. For the change of variable used in [10, 8] the boundary matrix will have constant rank in the whole spaces $\mathbb{R}^3_{\pm}$, but not only on the boundary $x_1 = 0$.

Let $(\widehat{\mathbf{U}}(t, \mathbf{x}), \hat{f}(t, \mathbf{x}'))$ be a given vector-function, where $\widehat{\mathbf{U}} = (\hat{p}, \hat{\mathbf{v}}, \widehat{\mathbf{H}}, \widehat{S})$ is supposed to be smooth for $\mathbf{x} \in \mathbb{R}^3_{\pm}$. Then the linearization of (14)–(16) results in the following variable coefficients problem for determining small perturbations $(\delta\mathbf{U}, \delta f)$ (below we drop δ):

$$L(\widehat{\mathbf{U}}, \widehat{\mathbf{F}})\mathbf{U} + \widehat{C}\mathbf{U} = \big\{L(\widehat{\mathbf{U}}, \widehat{\mathbf{F}})f\big\}\widehat{\mathbf{U}}_{x_1} \quad \text{in } [0, T] \times (\mathbb{R}^3_+ \cup \mathbb{R}^3_-), \tag{17}$$

$$v_{\mathrm{N}}^{\pm} = f_t + \hat{v}_2^{\pm} f_{x_2} + \hat{v}_3^{\pm} f_{x_3}\,, \quad H_{\mathrm{N}}^{\pm} = \widehat{H}_2^{\pm} f_{x_2} + \widehat{H}_3^{\pm} f_{x_3}\,, \quad [q] = 0 \qquad \text{if } x_1 = 0, \tag{18}$$

and the initial data for the perturbation $(\mathbf{U}, f)$ coincide with (16). Here, $\widehat{\mathbf{F}} = (\hat{f}_t, \hat{f}_{x_2}, \hat{f}_{x_2})$, $v_{\mathrm{N}} = (\mathbf{v}, \widehat{\mathbf{N}})$, $H_{\mathrm{N}} = (\mathbf{H}, \widehat{\mathbf{N}})$, $q = p + (\widehat{\mathbf{H}}, \mathbf{H})$, $\widehat{\mathbf{N}} = (1, -\hat{f}_{x_2}, -\hat{f}_{x_2})$, etc. The matrix $\widehat{C} = \widehat{C}(\widehat{\mathbf{U}}, \widehat{\mathbf{U}}_t, \nabla\widehat{\mathbf{U}}, \widehat{\mathbf{F}})$ is determined as follows:

$$\widehat{C}\mathbf{U} = (\mathbf{U}, \nabla_u A_0(\widehat{\mathbf{U}}))\widehat{\mathbf{U}}_t + (\mathbf{U}, \nabla_u A_\nu(\widehat{\mathbf{U}}, \widehat{\mathbf{F}}))\widehat{\mathbf{U}}_{x_1} + \sum_{k=2}^{3} (\mathbf{U}, \nabla_u A_k(\widehat{\mathbf{U}}))\widehat{\mathbf{U}}_{x_k},$$

$(\mathbf{U},\nabla_u) := \sum_{i=1}^{8} u_i\partial/\partial u_i$, $(u_1,\dots,u_8) := (p,\mathbf{v},\mathbf{H},S)$. Problem (17), (18), (16) is the *genuine linearization* of (14)–(16) in the sense that we keep all the lower order terms in (17).

It should be noted that the differential operator in system (17) is a first order operator in f. This fact can give some trouble in the application of the energy method to (17), (18), (16). To avoid this difficulty we make the change of unknowns (see [1])

$$\bar{\mathbf{U}} = \mathbf{U} - f\widehat{\mathbf{U}}_{x_1}. \tag{19}$$

In terms of the "good unknown" (19) problem (17), (18) takes the form

$$L(\widehat{\mathbf{U}},\widehat{\mathbf{F}})\bar{\mathbf{U}} + \widehat{C}\bar{\mathbf{U}} + f\partial_1\{L(\widehat{\mathbf{U}},\widehat{\mathbf{F}})\widehat{\mathbf{U}}\} = 0 \quad \text{in } [0,T]\times(\mathbb{R}^3_+\cup\mathbb{R}^3_-), \tag{20}$$

$$\left\{\begin{array}{ll} \bar{v}_{\rm N}^{\pm} = f_t + \hat{v}_2^{\pm} f_{x_2} + \hat{v}_3^{\pm} f_{x_3} - (\hat{v}_{\rm N})^{\pm}_{x_1} f\,, & \\ \bar{H}_{\rm N}^{\pm} = \widehat{H}_2^{\pm} f_{x_2} + \widehat{H}_3^{\pm} f_{x_3} - (\widehat{H}_{\rm N})^{\pm}_{x_1} f\,, & \\ [\bar{q}] = -f[\hat{q}_{x_1}] & \text{if} \quad x_1 = 0\,, \end{array}\right. \tag{21}$$

where $\bar{v}_{\rm N} = (\bar{\mathbf{v}},\widehat{\mathbf{N}})$, $(\hat{v}_{\rm N})^{\pm}_{x_1} = (\hat{v}_{\rm N})_{x_1}|_{x_1\to\pm 0}$, $\bar{q} = \bar{p} + (\widehat{\mathbf{H}},\bar{\mathbf{H}})$, etc.

For the successful application of the energy method to (20), (21) it would be enough if the operator in (20) had not involved first order terms in f (zero order terms in f give no trouble while applying the energy method). Therefore, without loss of generality we can drop the term $f\partial_1\{L(\widehat{\mathbf{U}},\widehat{\mathbf{F}})\widehat{\mathbf{U}}\}$ as well as the term $\widehat{C}\bar{\mathbf{U}}$ appearing in (20). As the result, the linearized equations associated to (14), (15) and obtained by dropping the lower order terms in (20) read:

$$L(\widehat{\mathbf{U}},\widehat{\mathbf{F}})\mathbf{U} = \mathbf{f} \quad \text{in } [0,T]\times(\mathbb{R}^n_+\cup\mathbb{R}^n_-), \tag{22}$$

$$\begin{pmatrix} f_t + \hat{v}_2^{+} f_{x_2} + \hat{v}_3^{+} f_{x_3} - (\hat{v}_{\rm N})^{+}_{x_1} f - v_{\rm N}^{+} \\ f_t + \hat{v}_2^{-} f_{x_2} + \hat{v}_3^{-} f_{x_3} - (\hat{v}_{\rm N})^{-}_{x_1} f - v_{\rm N}^{-} \\ \widehat{H}_2^{+} f_{x_2} + \widehat{H}_3^{+} f_{x_3} - (\widehat{H}_{\rm N})^{+}_{x_1} f - H_{\rm N}^{+} \\ \widehat{H}_2^{-} f_{x_2} + \widehat{H}_3^{-} f_{x_3} - (\widehat{H}_{\rm N})^{-}_{x_1} f - H_{\rm N}^{-} \\ [q] + f[\hat{q}_{x_1}] \end{pmatrix} = \mathbf{g} \qquad \text{if} \quad x_1 = 0\,. \tag{23}$$

Here we introduce the source terms $\mathbf{f}(t,\mathbf{x}) = \mathbf{f}^{\pm}(t,\mathbf{x})$ for $\mathbf{x}\in\mathbb{R}^n_{\pm}$ and $\mathbf{g}(t,\mathbf{x}')$ to make the interior equations and the boundary conditions inhomogeneous (this is needed to attack the nonlinear problem).

For planar discontinuities $\hat{f}(t,\mathbf{x}')$ is a linear function:

$$\hat{f}(t,\mathbf{x}') = \sigma t + (\boldsymbol{\sigma}',\mathbf{x}'), \quad \boldsymbol{\sigma} = (\sigma,\boldsymbol{\sigma}')\in\mathbb{R}^3. \tag{24}$$

Without loss of generality we can suppose that $\boldsymbol{\sigma} = 0$. For the case of a piecewise constant solution,

$$\widehat{\mathbf{U}} = \left\{\begin{array}{ll} \widehat{\mathbf{U}}^{+}, & x_1 > 0, \\ \widehat{\mathbf{U}}^{-}, & x_1 < 0, \end{array}\right.$$

equations (22), (23) have constant ("frozen") coefficients:

$$\widehat{A}_0^{\pm}\mathbf{U}_t + \sum_{k=1}^{3}\widehat{A}_k^{\pm}\mathbf{U}_{x_k} = 0 \quad \text{if} \quad \mathbf{x}\in\mathbb{R}^3_{\pm}, \tag{25}$$

$$\left\{\begin{array}{ll} f_t = v_1^{\pm} - \hat{v}_2^{\pm} f_{x_2} - \hat{v}_3^{\pm} f_{x_3}\,, & \\ H_1^{\pm} = \widehat{H}_2^{\pm} f_{x_2} + \widehat{H}_3^{\pm} f_{x_3}\,, & \\ [q] = 0 & \text{if} \quad x_1 = 0\,, \end{array}\right. \tag{26}$$

where $\widehat{A}^{\pm}_{\alpha} := A_{\alpha}(\widehat{\mathbf{U}}^{\pm})$; $q = p + (\widehat{\mathbf{H}}^{\pm}, \mathbf{H})$ for $\mathbf{x} \in \mathbb{R}^3_{\pm}$. Since the constant coefficients linearized problem for compressible current-vortex sheets is of independent interest in connection with astrophysical applications mentioned in Sect.1, we do not introduce in (25), (26) artificial source terms. At the same time, the a priori estimates proved in [23] for problem (25), (26) (see below) can be easily generalized to the case of inhomogeneous problem (see also the next section, where the variable coefficients inhomogeneous problem (22), (23) is considered).

We can show that compressible current-vortex sheets cannot be *uniformly* stable, i.e. the following important proposition is true [23].

Proposition 3.2 *For the initial boundary value problem (25), (26), (16), the uniform Kreiss-Lopatinski condition is never satisfied.*

Since problem (25), (26), (16) is a hyperbolic problem with characteristic boundary, there appears a loss of control on derivatives in the normal (x_1-)direction. Therefore, in the theorem below we use the following "nonsymmetric" Sobolev norms for solutions of (25), (26), (16):

$$\|\mathbf{U}(t)\|^2_{\widetilde{W}^1_2(\mathbb{R}^3_{\pm})} = \|\mathbf{U}_{\rm n}(t)\|^2_{W^1_2(\mathbb{R}^3_{\pm})} + \|\mathbf{U}_{\rm tan}(t)\|^2_{W^{1,\tan}_2(\mathbb{R}^3_{\pm})} ,$$

where $\mathbf{U}_{\rm n} = (q, v_1, H_1)$, $\mathbf{U}_{\rm tan} = (v_2, v_3, H_2, H_3, S)$,

$$\|(\cdot)(t)\|^2_{W^{1,\tan}_2(\mathbb{R}^3_{\pm})} = \|(\cdot)(t)\|^2_{L_2(\mathbb{R}^3_{\pm})} + \|(\cdot)_{x_2}(t)\|^2_{L_2(\mathbb{R}^3_{\pm})} + \|(\cdot)_{x_3}(t)\|^2_{L_2(\mathbb{R}^3_{\pm})} .$$

Theorem 3.1 *If* $\widehat{\mathbf{H}}^{+} \times \widehat{\mathbf{H}}^{-} \neq 0$, $[\hat{\mathbf{v}}] \neq 0$ *, and*

$$|[\hat{\mathbf{v}}]| < |\sin(\varphi^+ - \varphi^-)| \min\left\{ \frac{\gamma^+}{|\sin\varphi^-|} , \frac{\gamma^-}{|\sin\varphi^+|} \right\} , \tag{27}$$

where

$$\gamma^{\pm} = \frac{\hat{c}^{\pm}\hat{c}^{\pm}_{\rm A}}{\sqrt{(\hat{c}^{\pm})^2 + (\hat{c}^{\pm}_{\rm A})^2}} , \quad \cos\varphi^{\pm} = \frac{([\hat{\mathbf{v}}], \widehat{\mathbf{H}}^{\pm})}{|[\hat{\mathbf{v}}]|\,|\widehat{\mathbf{H}}^{\pm}|} ,$$

then, for Problem (25), (26), (16), the Lopatinski condition is satisfied and the a priori estimates

$$\begin{aligned} &\|\mathbf{U}(t)\|_{\widetilde{W}^1_2(\mathbb{R}^3_+)} + \|\mathbf{U}(t)\|_{\widetilde{W}^1_2(\mathbb{R}^3_-)} \\ &\qquad \leq C_1 \left\{ \|\mathbf{U}_0\|_{\widetilde{W}^1_2(\mathbb{R}^3_+)} + \|\mathbf{U}_0\|_{\widetilde{W}^1_2(\mathbb{R}^3_-)} \right\} , \end{aligned} \tag{28}$$

$$\|f(t)\|_{W^1_2(\mathbb{R}^2)} \leq \|f_0\|_{L_2(\mathbb{R}^2)} + C_2 \left\{ \|\mathbf{U}_0\|_{\widetilde{W}^1_2(\mathbb{R}^3_+)} + \|\mathbf{U}_0\|_{\widetilde{W}^1_2(\mathbb{R}^3_-)} \right\} \tag{29}$$

hold for any $t \in (0,T)$ *. Here* T *is a positive constant;* C_1 *and* $C_2(T)$ *are positive constants independent of the initial data (16).*

If $\widehat{\mathbf{H}}^{+} \times \widehat{\mathbf{H}}^{-} = 0$, $\widehat{\mathbf{H}}^{\pm} \times [\hat{\mathbf{v}}] = 0$, $[\hat{\mathbf{v}}] \neq 0$ *, and*

$$|[\hat{\mathbf{v}}]| < \max\left\{ \max\{\gamma^+, \gamma^-\} , 2\min\{\gamma^+, \gamma^-\} \right\} \tag{30}$$

the a priori estimate (28) holds as well, but for the function $f(t, \mathbf{x}')$ *we have the weaker estimate*

$$\|f(t)\|_{L_2(\mathbb{R}^2)} \leq \|f_0\|_{L_2(\mathbb{R}^2)} + C_3 \left\{ \|\mathbf{U}_0\|_{\widetilde{W}^1_2(\mathbb{R}^3_+)} + \|\mathbf{U}_0\|_{\widetilde{W}^1_2(\mathbb{R}^3_-)} \right\} , \tag{31}$$

where $C_3(T)$ *is a positive constant independent of (16).*

For current sheets, i.e. for the case $[\hat{\mathbf{v}}] = 0$ *the Lopatinski condition is always satisfied and estimate (28) takes place. Furthermore, estimate (29) holds if* $\widehat{\mathbf{H}}^{+} \times \widehat{\mathbf{H}}^{-} \neq 0$ *, otherwise we have the weaker estimate (31). For current sheets the case* $\widehat{\mathbf{H}}^{+} \times \widehat{\mathbf{H}}^{-} = 0$ *corresponds to the transition to violent instability (the Lopatinski condition is violated if* $\widehat{\mathbf{H}}^{+} \times \widehat{\mathbf{H}}^{-} = 0$ *and* $|[\hat{\mathbf{v}}]|/\hat{c}^{+} \ll 1$*).*

For the detailed proof of Theorem 3.1 we refer to [23]. The crucial point in the proof is that (25) implies a symmetric system which is the linearization of system (10) about the piecewise constant solution. This symmetric system accompanied with the boundary conditions (26) has the conserved integral

$$I(t)=\int_{\mathbb{R}^3_+}(\widehat{\mathcal{A}}_0^+\mathbf{U},\mathbf{U})\,d\mathbf{x}+\int_{\mathbb{R}^3_-}(\widehat{\mathcal{A}}_0^-\mathbf{U},\mathbf{U})\,d\mathbf{x}$$

for a certain choice of the constants $\lambda^{\pm}=\lambda(\widehat{\mathbf{U}}^{\pm})$, where $\widehat{\mathcal{A}}_0^{\pm}=\mathcal{A}_0(\widehat{\mathbf{U}}^{\pm})$ (see (10)). In view of (11), for the most general case $\widehat{\mathbf{H}}^+\times\widehat{\mathbf{H}}^-\neq 0$, $[\hat{\mathbf{v}}]\neq 0$, the hyperbolicity conditions $\widehat{\mathcal{A}}_0^{\pm}>0$ give the stability condition (27), that can be interpreted as the sufficient condition of the macroscopic stability of the *heliopause*. Note also that the process of deducing the a priori estimates (28), (29) can be formalized by introducing the notations of *dissipative p-symmetrizers* [25]. In fact, for problem (25), (26) the dissipative (but *not* strictly dissipative [25]) 0-symmetrizer

$$\mathbb{S}=\left\{P(\widehat{\mathbf{U}}^+),P(\widehat{\mathbf{U}}^-),\mathbf{R}(\widehat{\mathbf{U}}^+),\mathbf{R}(\widehat{\mathbf{U}}^-)\right\}$$

(cf. (9)) has been constructed [23].

For *incompressible* current-vortex sheets, the variable coefficients linearized problem is the initial boundary value problem for the system

$$L(\widehat{\mathbf{U}},\widehat{\mathbf{F}})\mathbf{U}+\mathbf{e}\otimes\nabla_f q=\mathbf{f},\quad \operatorname{div}\mathbf{u}=0\qquad \text{in } [0,T]\times(\mathbb{R}^n_+\cup\mathbb{R}^n_-), \tag{32}$$

with the boundary conditions (23) and the initial data (16), where

$$\mathbf{e}=(1,0)\,,\quad \nabla_f=\widehat{\mathbf{N}}\partial_1+\mathbf{e}_2\partial_2+\mathbf{e}_3\partial_3\,,\quad \mathbf{e}_k=(0,\delta_{2k},\delta_{3k})\,,\quad \mathbf{u}=(v_{\rm N},v_2,v_3)\,,$$

$A_\alpha:=\mathcal{A}_\alpha|_{\lambda=0}$ (see (12)), and other notations are the same as for compressible fluid.

The constant coefficients linearized problem for planar incompressible current-vortex sheets is the problem for the system

$$\mathbf{U}_t+\sum_{k=2}^{3}\widehat{A}_k^{\pm}\mathbf{U}_{x_k}+\mathbf{e}\otimes\nabla q=0\,,\quad \operatorname{div}\mathbf{v}=0\,,\qquad \mathbf{x}\in\mathbb{R}^3_{\pm}, \tag{33}$$

with the boundary conditions (26) and the initial data (16). The necessary and sufficient conditions for the nonexistence of Hadamard-type ill-posedness examples for problem (33), (26) (*linear stability* of a planar current-vortex sheet) were found in [22, 2] (see also [17] for the 2D case):

$$|[\hat{\mathbf{v}}]|^2<2\left\{|\widehat{\mathbf{H}}^+|^2+|\widehat{\mathbf{H}}^-|^2\right\},\quad \left\{\left|\widehat{\mathbf{H}}^+\times[\hat{\mathbf{v}}]\right|^2+\left|\widehat{\mathbf{H}}^-\times[\hat{\mathbf{v}}]\right|^2\right\}\leq 2\left|\widehat{\mathbf{H}}^+\times\widehat{\mathbf{H}}^-\right|^2. \tag{34}$$

Using the "symmetric" form (12), a priori estimates for problem (33), (26), (16) were obtained in [24] for a part of the parameter domain (34).

Theorem 3.2 *If* $\widehat{\mathbf{H}}^+\times\widehat{\mathbf{H}}^-\neq 0$, $[\hat{\mathbf{v}}]\neq 0$, *and*

$$|[\hat{\mathbf{v}}]|<|\sin(\varphi^+-\varphi^-)|\min\left\{\frac{|\mathbf{H}^+|}{|\sin\varphi^-|},\,\frac{|\mathbf{H}^-|}{|\sin\varphi^+|}\right\}, \tag{35}$$

where $\varphi^{\pm}$ *are the same as in Theorem 3.1, then the a priori estimates (28), (29), and*

$$\|\nabla q(t)\|_{L_2(\mathbb{R}^3_+)}+\|\nabla q(t)\|_{L_2(\mathbb{R}^3_-)}\leq C_4\left\{\|\mathbf{U}_0\|_{\widetilde{W}^1_2(\mathbb{R}^3_+)}+\|\mathbf{U}_0\|_{\widetilde{W}^1_2(\mathbb{R}^3_-)}\right\} \tag{36}$$

hold for any $t\in(0,T)$. *If* $\widehat{\mathbf{H}}^+\times\widehat{\mathbf{H}}^-=0$, $\widehat{\mathbf{H}}^{\pm}\times[\hat{\mathbf{v}}]=0$, $[\hat{\mathbf{v}}]\neq 0$, *and*

$$|[\hat{\mathbf{v}}]|<\max\left\{\max\{|\mathbf{H}^+|,|\mathbf{H}^-|\},\,2\min\{|\mathbf{H}^+|,|\mathbf{H}^-|\}\right\}, \tag{37}$$

then the a priori estimates (28), (36) hold as well, but for the function $f(t,\mathbf{x}')$ *we have the weaker estimate (31).*

For current sheets, i.e., for the case $[\hat{\mathbf{v}}]=0$, *the a priori estimates (28), (36) always take place. Furthermore, estimate (29) holds if* $\widehat{\mathbf{H}}^+\times\widehat{\mathbf{H}}^-\neq 0$, *otherwise we have the weaker estimate (31).*

For the detailed proof of Theorem 3.2 we refer to [24]. Note that, in view of (34), the particular case $\widehat{\mathbf{H}}^+ \times \widehat{\mathbf{H}}^- = 0$ corresponds to transition to violent instability (ill-posedness). For a general case when $\widehat{\mathbf{H}}^+ \times \widehat{\mathbf{H}}^- \neq 0$, we can show that the first inequality in (34) is redundant, and the second inequality in (34) can be rewritten as follows:

$$|[\hat{\mathbf{v}}]| \leq \frac{\sqrt{2}\, |\widehat{\mathbf{H}}^+|\, |\widehat{\mathbf{H}}^-|\, |\sin(\varphi^+ - \varphi^-)|}{\sqrt{|\widehat{\mathbf{H}}^+|^2 \sin^2\varphi^+ + |\widehat{\mathbf{H}}^-|^2 \sin^2\varphi^-}}\,. \tag{38}$$

In the *incompressibility limit*, $\hat{c}^\pm = \infty$, the sufficient stability condition (27) is reduced to inequality (35) (for simplicity we consider the case $\hat{\rho}^+ = \hat{\rho}^-$). Moreover, if we introduce the dimensionless parameters

$$x = \frac{|[\hat{\mathbf{v}}]|^2 \sin^2\varphi^+}{|\widehat{\mathbf{H}}^-|^2 \sin^2(\varphi^+ - \varphi^-)}\,, \quad y = \frac{|[\hat{\mathbf{v}}]|^2 \sin^2\varphi^-}{|\widehat{\mathbf{H}}^+|^2 \sin^2(\varphi^+ - \varphi^-)}\,,$$

then in the xy-plane inequalities (35) and (38) determine the domains

$$D_1 = \bigl\{x > 0\,,\ y > 0\,,\ \max\{x, y\} < 1\bigr\} \quad \text{and} \quad D_2 = \{x > 0\,,\ y > 0\,,\ x + y \leq 2\}$$

respectively. It is clear that $\operatorname{mes} D_1 = (1/2) \operatorname{mes} D_2$, and the domain $D_2 \backslash D_1$ is an "interlayer" between the domain D_1 of well-posedness (cf. Theorem 3.2) and the domain of ill-posedness: $\{x > 0\,,\ y > 0\,,\ x + y > 2\}$.

As regards a possibility to prove a priori estimates for the "interlayer" domain $D_2 \backslash D_1$, it seems that this could be done by adapting Kreiss's symmetrizer technique [12] to the *nonhyperbolic* problem (33), (26), (16). In this connection, there appears an interesting problem on obtaining "exponentially weighted" a priori estimates for nonhyperbolic initial boundary value problems like (33), (26), (16).

4 The variable coefficients analysis

Since the original nonlinear problem (14)–(16) is a *free boundary value problem*, to prove local-in-time existence by standard fixed-point argument we should gain the "additional derivative" for the front perturbation f (see [4, 16]). As we can see from (28), (29), we do not have an estimate for the second derivatives of f provided that $\mathbf{U} \in \widetilde{W}_2^1(\mathbb{R}^3_+) \cap \widetilde{W}_2^1(\mathbb{R}^3_-)$. This is because the uniform Lopatinski condition is violated and, therefore, it is principally impossible that the boundary conditions were *strictly* dissipative. And so, we have the loss of the trace $(\mathbf{U}_{\mathrm{n}}^+, \mathbf{U}_{\mathrm{n}}^-)$ in a high norm, i.e., we are not able to include the norm $\|\mathbf{U}_{\mathrm{n}}^+\|_{W_2^1(\partial\mathbb{R}^3_+)} + \|\mathbf{U}_{\mathrm{n}}^-\|_{W_2^1(\partial\mathbb{R}^3_-)}$ in estimate (28) (to estimate f in a high norm we need only the "noncharacteristic part" of the trace of $(\mathbf{U}^+, \mathbf{U}^-)$ whereas the loss of the trace for the "characteristic" unknown $\mathbf{U}_{\mathrm{tan}}$ even in a lower norm does not give any trouble).

Since even for constant coefficients we have a loss of one derivative for f, the standard scheme of proving the local-in-time existence theorem for the original nonlinear problem does not work for current-vortex sheets. In this connection, another possibility to attack the nonlinear problem is to use the so-called Nash-Moser technique (for hyperbolic problems see [1, 15, 10] and references therein). To apply the Nash-Moser method for compressible current-vortex sheets it needs to carry out an accurate analysis of the corresponding variable coefficients linearized problem. At the same time, after performing the change of unknowns (19) the lower order terms in the interior equations (20) can be neglected because they give no trouble while deducing a priori estimates. Thus, it is enough to study problem (22), (23). The main difficulties in the variable coefficients analysis are connected with lower order terms in the boundary conditions (23). Analogous remarks take place for the nonhyperbolic problem (32), (23) for incompressible fluid.

For the basic state $(\widehat{\mathbf{U}}, \hat{f})$ (it can be, in particular, an exact current-vortex sheet solution to the MHD equations), we assume that

$$\widehat{\mathbf{U}} \in X_4([0, T], \mathbb{R}^3_+) \cap X_4([0, T], \mathbb{R}^3_-)\,, \quad \hat{f} \in W_2^5([0, T] \times \mathbb{R}^2), \tag{39}$$

where

$$X_k([0,T],\mathbb{R}^3_\pm) := \bigcap_{j=0}^{k} C^j([0,T],W_2^{k-j}(\mathbb{R}^3_\pm))\,, \quad \|\cdot\|_{X_k} = \max_{t\in[0,T]}\sum_{j=0}^{k}\|\partial_t^j(\cdot)(t)\|^2_{W_2^{k-j}}\,.$$

Then, in view of Sobolev's imbedding, we have

$$\widehat{\mathbf{U}} \in W^2_\infty([0,T]\times\mathbb{R}^3_+)\cap W^2_\infty([0,T]\times\mathbb{R}^3_-)\,, \quad \widehat{\mathbf{F}} \in W^2_\infty([0,T]\times\mathbb{R}^2)\,.$$

In view of (39), there exists a constant $M>0$ such that

$$\|\hat{f}\|_{W_2^5([0,T]\times\mathbb{R}^2)} + \sum_{\pm}\|\widehat{\mathbf{U}}\|_{X_4([0,T],\mathbb{R}^3_\pm)} \le M\,.$$

For the constant coefficients linearized problem, in the a priori estimate (28) we have the so-called loss of derivatives in the normal direction to the boundary. But, for constant coefficients it was enough to use usual Sobolev norms. At the same time, in the variable coefficients analysis we have to require a little bit more regularity for solutions. The natural functional setting is provided by the anisotropic weighted Sobolev space $W_2^{m,\sigma}$ ($:= H^m_*$; see e.g. [21] and references therein). Following [21], we now give the definition of the spaces $W_2^{m,\sigma}(\mathbb{R}^3_\pm)$. Let $\sigma(x_1)\in C^\infty(\mathbb{R}_+)\cap C^\infty(\mathbb{R}_-)$ is a monotone increasing function for $x_1>0$ and monotone decreasing for $x_1<0$ such that $\sigma(x_1)=|x_1|$ in a neighborhood of the origin and $\sigma(x_1)=1$ for $|x_1|$ large enough. Let us introduce the so-called conormal derivative

$$\partial_*^\alpha = (\sigma(x_1)\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}\,.$$

Then, given $m\ge 1$, the function space $W_2^{m,\sigma}(\Omega)$ ($\Omega=\mathbb{R}^3_+$ or $\Omega=\mathbb{R}^3_-$) is defined as the set of functions $u\in L_2(\Omega)$ such that $\partial_*^\alpha\partial_1^k u\in L_2(\Omega)$ if $|\alpha|+2k\le m$. The space $W_2^{m,\sigma}(\Omega)$ is normed by

$$\|u\|^2_{W_2^{m,\sigma}(\Omega)} = \sum_{|\alpha|+2k\le m}\|\partial_*^\alpha\partial_1^k u\|^2_{L_2(\Omega)}\,.$$

For solutions of problem (22), (23) we use also the norms

$$|||\mathbf{U}(t)|||^2_{\widetilde{W}_2^{m,\sigma}(\mathbb{R}^3_\pm)} = |||\mathbf{U}(t)|||^2_{W_2^{m,\sigma}(\mathbb{R}^3_\pm)} + |||\partial_1\mathbf{U}_{\rm n}(t)|||^2_{W_2^{m-1,\sigma}(\mathbb{R}^3_\pm)}\,,$$

where $|||(\cdot)(t)|||^2_{W_2^{k,\sigma}} = \sum_{j=0}^k\|\partial_t^j(\cdot)(t)\|^2_{W_2^{k-j,\sigma}}$; $\mathbf{U}_{\rm n}=(q,v_{\rm N},H_{\rm N})$ is the "noncharacteristic part" of $\mathbf{U}$. We are now in a position to formulate the main result from [23] obtained for the variable coefficients linearized problem for compressible current-vortex sheets. Note that unlike [23] we formulate the theorem below for the inhomogeneous problem (22), (23) (arguments in [23] can be easily extended to the case when $\mathbf{f}\neq 0$ and $\mathbf{g}\neq 0$).

Theorem 4.3 *Let the basic state $(\widehat{\mathbf{U}},\hat{f})$ satisfies assumptions (39), the Rankine-Hugoniot conditions (5), and the hyperbolicity conditions (4). Let also there exists a positive constant δ such that*

$$|\hat{\mathbf{h}}^+(t,\mathbf{x}')\times\hat{\mathbf{h}}^-(t,\mathbf{x}')|\ge\delta>0 \tag{40}$$

for all $t\in[0,T]$, $\mathbf{x}'\in\mathbb{R}^2$ and the condition

$$r(t,\mathbf{x}) < b(t,\mathbf{x}) \tag{41}$$

holds for all $t\in[0,T]$ at each point $\mathbf{x}\in\overline{\mathbb{R}^3_\pm}$ such that $\hat{\mathbf{u}}^+(t,\mathbf{x}')\neq\hat{\mathbf{u}}^-(t,\mathbf{x}')$, where

$$\hat{\mathbf{h}} = (\widehat{H}_{\rm N},\widehat{H}_2,\widehat{H}_3)\,, \quad \hat{\mathbf{u}} = (\hat{v}_{\rm N}-\hat{f}_t,\hat{v}_2,\hat{v}_3)\,,$$

$$r(t,\mathbf{x}) = \sqrt{\hat{\rho}(1+\hat{c}_{\rm A}^2/\hat{c}^2)}\,, \quad b(t,\mathbf{x}) = \left\{ \begin{array}{lll} b^+(t,\mathbf{x}') & \textit{if} & x_1>0\,, \\ b^-(t,\mathbf{x}') & \textit{if} & x_1<0\,; \end{array}\right.$$

$$b^{\pm}(t,\mathbf{x}') = \frac{|\hat{\mathbf{h}}^{\pm}|\,|\sin(\varphi^+-\varphi^-)|}{|[\hat{\mathbf{u}}]|\,|\sin\varphi^{\mp}|}\,, \quad \cos\varphi^{\pm}(t,\mathbf{x}') = \frac{([\hat{\mathbf{u}}],\hat{\mathbf{h}}^{\pm})}{|[\hat{\mathbf{u}}]|\,|\hat{\mathbf{h}}^{\pm}|}\,.$$

Then, for problem (22), (23) the a priori estimate

$$\sum_{\pm} |||\mathbf{U}(t)|||_{\widetilde{W}_2^{1,\sigma}(\mathbb{R}^3_{\pm})} + \|f\|_{W_2^1([0,T]\times\mathbb{R}^2)}$$
$$\leq C\left\{ \sum_{\pm}\left\{ \|\mathbf{f}^{\pm}\|_{W_2^1([0,T]\times\mathbb{R}^3_{\pm})} + |||\mathbf{U}_0|||_{\widetilde{W}_2^{1,\sigma}(\mathbb{R}^3_{\pm})}\right\} + \|\mathbf{g}\|_{W_2^2([0,T]\times\mathbb{R}^2)} + \|f_0\|_{W_2^1(\mathbb{R}^2)}\right\} \tag{42}$$

holds for any $t\in[0,T]$. Here $C=C(T,M)$ is a positive constant independent of the initial data $(\mathbf{U}_0,f_0)$.

For the detailed proof of Theorem 4.3 we refer to [23]. Inequality (41) appearing in this theorem is the analogue of the stability condition (27) for variable coefficients. The a priori estimate (42) is, in some sense, a base estimate, and to attack the original nonlinear problem we should deduce estimates in higher order norms.

Corollary 4.1 *Let all the assumptions of Theorem 4.3 are satisfied and*

$$(\widehat{\mathbf{U}},\hat{f}) \in \Big\{\bigcap_{\pm}\bigcap_{j=0}^{s} W^j_{\infty}([0,T],W_2^{s-j,\sigma}(\mathbb{R}^3_{\pm}))\Big\}\times W_2^{s+1}([0,T]\times\mathbb{R}^2)$$

for some $s\geq 8$. Let $1\leq m\leq s$. Then, for problem (22), (23) the a priori estimate

$$\sum_{\pm} |||\mathbf{U}(t)|||_{\widetilde{W}_2^{m,\sigma}(\mathbb{R}^3_{\pm})} + \|f\|_{W_2^m([0,T]\times\mathbb{R}^2)}$$
$$\leq C\left\{ \sum_{\pm}\left\{ \|\mathbf{f}^{\pm}\|_{W_2^{m,\sigma}([0,T]\times\mathbb{R}^3_{\pm})} + |||\mathbf{U}_0|||_{\widetilde{W}_2^{m,\sigma}(\mathbb{R}^3_{\pm})}\right\} + \|\mathbf{g}\|_{W_2^{m+1}([0,T]\times\mathbb{R}^2)} + \|f_0\|_{W_2^m(\mathbb{R}^2)}\right\}$$

holds for any $t\in[0,T]$, where $\|\mathbf{f}^{\pm}\|^2_{W_2^{m,\sigma}([0,T]\times\mathbb{R}^3_{\pm})} = \int_0^T |||\mathbf{f}^{\pm}(t)|||^2_{W_2^{m,\sigma}(\mathbb{R}^3_{\pm})}dt$.

To prove Corollary 4.1 it needs to take into account the imbeddings $W_2^{s,\sigma}\subset W_2^{[s/2]}\subset W_2^4$ (cf. (39)) and use the standard technique based on the application of the Gagliardo-Nirenberg inequalities.

For incompressible current-vortex sheets, the technique of obtaining a priori estimates for problem (32), (23) is similar to that used in [23] for compressible current-vortex sheets. However, there is, of course, an essential principal difference from the "hyperbolic" energy method utilized in [23]. As is known, for the system of incompressible MHD the total pressure q is an "elliptic" unknown. Hence, the differentiation of system (32) with respect to t cannot help to prove the energy a priori estimate. Instead of this, roughly speaking, we estimate ∇q through spatial derivatives of $\mathbf{U}$ and then, from system (32), we obtain an estimate for $\mathbf{U}_t$. Actually, since in our case the boundary is a strong discontinuity, for the function q we have an elliptic boundary value problem that is like *diffraction problems* [14]. It is true that the estimate for ∇q can be obtained directly from system (32) (see [24] for details).

Let us now formulate the corresponding theorem proved in [24]. Let

$$\widehat{\mathbf{U}} \in \bigcap_{k=0}^{1} C^k\Big([0,T],\bigcap_{\pm}W_2^{4-k}(\mathbb{R}^3_{\pm})\Big)\,, \quad \hat{f}\in C\big([0,T],W_2^4(\mathbb{R}^2)\big)\,. \tag{43}$$

Moreover, we assume also that

$$\nabla \hat{q} \in C\big([0,T], W^1_\infty(\mathbb{R}^3_+) \cap W^1_\infty(\mathbb{R}^3_-)\big) \tag{44}$$

and

$$\nabla \hat{q} \in \bigcap_{j=0}^{1} C\left([0,T];\, W_2^j\big(\mathbb{R}_+, W_\infty^{1-j}(\mathbb{R}^2)\big) \cap W_2^j\big(\mathbb{R}_-, W_\infty^{1-j}(\mathbb{R}^2)\big)\right)\,. \tag{45}$$

Theorem 4.4 *Let the basic state* $(\widehat{\mathbf{U}}, \hat{f})$ *satisfies assumptions (43)–(45). Let also conditions (40) and (41) are fulfilled with* $r(t,\mathbf{x}) \equiv 1$ *(other notations are the same as in Theorem 4.3). Then, for problem (32), (23) the a priori estimate*

$$\begin{gathered}\sum_{\pm}\Big\{|||\mathbf{U}(t)|||_{\widetilde{W}_2^{1,\sigma}(\mathbb{R}^3_\pm)} + \|f\|_{W_2^1([0,T]\times\mathbb{R}^2)} + \|\nabla q\|_{L_2([0,T]\times\mathbb{R}^3_\pm)}\Big\} \\ \leq C\left\{\sum_{\pm}\Big\{\|\mathbf{f}^\pm\|_{W_2^1([0,T]\times\mathbb{R}^3_\pm)} + |||\mathbf{U}_0|||_{\widetilde{W}_2^{1,\sigma}(\mathbb{R}^3_\pm)}\Big\} + \|\mathbf{g}\|_{W_2^2([0,T]\times\mathbb{R}^2)} + \|f_0\|_{W_2^1(\mathbb{R}^2)}\right\}\end{gathered} \tag{46}$$

holds for any $t \in [0,T]$ *.*

As for problem (22), (23) (see Corollary 4.1), we can obtain estimates in higher order norms for problem (32), (23). Note also that the a priori estimates (42) and (46) imply the *uniqueness* of solutions of the original nonlinear problems for compressible and incompressible current-vortex sheets respectively (see [24] for standard arguments).

5 Concluding remarks

In [23], the sufficient condition (27) for the neutral stability of planar compressible current-vortex sheets is first found for a general case of the unperturbed flow. This condition can be interpreted as the sufficient condition of the macroscopic stability of the heliopause [3, 19]. In the incompressibility limit this condition describes exactly the half of the whole parameter domain of stability. The variable coefficients linearized problem for nonplanar compressible current-vortex sheets has been studied as well. Since the current-vortex sheet is a characteristic discontinuity, there appears a loss of derivatives in the normal direction to the discontinuity front, and the natural functional setting is provided by the anisotropic weighted Sobolev spaces [21]. Furthermore, since the uniform Lopatinski condition is not satisfied, we have a loss of one derivative for the front perturbation as well as for the source term in the boundary conditions.

To prove the local-in-time existence of current-vortex sheets solutions for compressible MHD it needs, first, to prove the existence of solutions of the variable coefficients linearized problem (22), (23) in the functional spaces indicated in the estimate in Corollary 4.1. To this end, it is necessary to generalize recent results from [5] to the case of boundary conditions with variable coefficients, when the boundary conditions are not strictly dissipative. Note that in [5] earlier results from [20] for linear symmetric hyperbolic problems with characteristic boundary were extended to nonhomogeneous problems. Then, it seems that we can construct solutions of the nonlinear problem by the Nash-Moser method.

At the same time, one can suggest an alternative programme based on Kreiss' symmetrizer technique and paradifferential calculus (and the Nash-Moser method as well). Such a programme is now being realized for weakly stable shock waves and 2D vortex sheets (see [6, 7, 8]). Indeed, the energy method applied in [23] for the constant coefficients problem for compressible current-vortex sheets can be considered as an indirect test of the Kreiss-Lopatinski condition. Then, we should construct Kreiss' symmetrizer for this problem and follow the arguments from [6, 7, 8] for variable coefficients.

As regards incompressible current-vortex sheets [24], the nonhyperbolic problem for them has, of course, some peculiarities. But, in principle, with appropriate modifications the above remarks for compressible current-vortex sheets can be made for this problem as well.

At last, we note that for astrophysical applications it would be extremely important to generalize the sufficient stability condition (27) to improved heliopause models like, for example, the multifluid neutral MHD model newly developed by astrophysicists (see [9] and references therein). For possible future results, it is especially important that this MHD model is a system of hyperbolic balance laws.

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