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The Existence of Current-Vortex Sheets in Ideal Compressible Magnetohydrodynamics

Yuri Trakhinin

Abstract

We prove the local-in-time existence of solutions with a surface of current-vortex sheet (tangential discontinuity) of the equations of ideal compressible magnetohydrodynamics in three space dimensions provided that a stability condition is satisfied at each point of the initial discontinuity. This paper is a natural completion of our previous analysis in [43] where a sufficient condition for the weak stability of planar current-vortex sheets was found and a basic a priori estimate was proved for the linearized variable coefficients problem for nonplanar discontinuities. The original nonlinear problem is a free boundary hyperbolic problem. Since the free boundary is characteristic, the functional setting is provided by the anisotropic weighted Sobolev spaces H^m_*. The fact that the Kreiss-Lopatinski condition is satisfied only in a weak sense yields losses of derivatives in a priori estimates. Therefore, we prove our existence theorem by a suitable Nash-Moser-type iteration scheme.

1. Introduction

1.1. Free boundary value problem for compressible current-vortex sheets

The equations of ideal compressible magnetohydrodynamics (MHD) take the form of the system of conservation laws

$$\begin{gathered}\partial_t\rho + \operatorname{div}(\rho\mathbf{v}) = 0,\\ \partial_t(\rho\mathbf{v}) + \operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v} - \mathbf{H}\otimes\mathbf{H}) + \nabla q = 0,\\ \partial_t\mathbf{H} - \nabla\times(\mathbf{v}\times\mathbf{H}) = 0,\\ \partial_t\big(\rho e + \tfrac{1}{2}|\mathbf{H}|^2\big) + \operatorname{div}\big((\rho e + p)\mathbf{v} + \mathbf{H}\times(\mathbf{v}\times\mathbf{H})\big) = 0,\end{gathered} \tag{1}$$

where ρ, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{H} = (H_1, H_2, H_3)$, and p are the density, the fluid velocity, the magnetic field, and the pressure respectively, $q = p + \frac{1}{2}|\mathbf{H}|^2$ is the total pressure, $e = E + \frac{1}{2}|\mathbf{v}|^2$ is the total energy, $E = E(\rho, S)$ is the internal energy, and S is the entropy. The Gibbs relation

$$TdS = dE - \frac{p}{\rho^2}\,d\rho$$

implies that $p = \rho^2 E_\rho(\rho, S)$ and the temperature $T = E_S(\rho, S)$. Then, with a state equation of medium, $E = E(\rho, S)$, (1) is a closed system. As the unknown we can fix, for example, the vector $\mathbf{U} = \mathbf{U}(t, \mathbf{x}) = (p, \mathbf{v}, \mathbf{H}, S)$, where t is the time, $\mathbf{x} = (x_1, x_2, x_3)$ are space variables, $\partial_t := \partial/\partial t$, and $\partial_j := \partial/\partial x_j$.

System (1) is supplemented by the divergent constraint

$$\operatorname{div}\mathbf{H} = 0 \tag{2}$$

on the initial data $\mathbf{U}(0, \mathbf{x}) = \mathbf{U}_0(\mathbf{x})$. Taking into account (2), for smooth solutions we can rewrite (1) in the equivalent form

$$\begin{aligned} &\frac{1}{\rho c^2}\,\frac{dp}{dt} + \operatorname{div}\mathbf{v} = 0, \qquad \rho\,\frac{d\mathbf{v}}{dt} - (\mathbf{H}, \nabla)\mathbf{H} + \nabla q = 0, \\ &\frac{d\mathbf{H}}{dt} - (\mathbf{H}, \nabla)\mathbf{v} + \mathbf{H}\operatorname{div}\mathbf{v} = 0, \qquad \frac{dS}{dt} = 0, \end{aligned} \tag{3}$$

where c^2 is the square of the sound velocity and $d/dt = \partial_t + (\mathbf{v}, \nabla)$ (by $(\ ,\ )$ we denote the scalar product). Now $\rho = \rho(p, S)$ is considered as a state equation of medium and $1/c^2 = \rho_p(p, S)$. Equations (3) read as the symmetric quasilinear system

$$A_0(\mathbf{U})\partial_t\mathbf{U} + \sum_{j=1}^{n} A_j(\mathbf{U})\partial_j\mathbf{U} = 0, \tag{4}$$

where the symmetric matrices A_α can be easily written down, in particular, $A_0 = \operatorname{diag}(1/(\rho c^2), \rho, \rho, \rho, 1, 1, 1, 1)$. System (4) is symmetric hyperbolic if the state equation $\rho = \rho(p, S)$ satisfies the hyperbolicity condition $A_0 > 0$:

$$\rho(p, S) > 0, \quad \rho_p(p, S) > 0. \tag{5}$$

We consider the MHD equations for $t \in [0, T]$ in the unbounded space domain $\mathbb{R}^3$ and suppose that $\varGamma(t) = \{x_1 - f(t, \mathbf{x}') = 0\}$ is a smooth hypersurface in $[0, T] \times \mathbb{R}^3$, where $\mathbf{x}' = (x_2, x_3)$ are tangential coordinates. We assume that $\varGamma(t)$ is a surface of strong discontinuity for the conservation laws (1), i.e., we are interested in solutions of (1) that are smooth on either side of $\varGamma(t)$. To be weak solutions of (1) such piecewise smooth solutions should satisfy the MHD Rankine-Hugoniot conditions

$$\begin{aligned} &[\mathrm{j}] = 0, \quad [H_{\mathrm{N}}] = 0, \quad \mathrm{j}\,[v_{\mathrm{N}}] + [q] = 0, \quad \mathrm{j}\,[\mathbf{v}_\tau] = H_{\mathrm{N}}[\mathbf{H}_\tau], \\ &H_{\mathrm{N}}[\mathbf{v}_\tau] = \mathrm{j}\,[\mathbf{H}_\tau/\rho], \quad \mathrm{j}\left[e + \tfrac{1}{2}(|\mathbf{H}|^2/\rho)\right] + [q v_{\mathrm{N}} - H_{\mathrm{N}}(\mathbf{H}, \mathbf{v})] = 0 \end{aligned} \tag{6}$$

at each point of Γ (see, e.g., [18,19]), where $[g] = g^+|_\Gamma - g^-|_\Gamma$ denotes the jump of g, with $g^\pm := g$ in $\Omega^\pm(t) = \{x_1 \gtrless f(t, \mathbf{x}')\}$,

$$\mathfrak{j} = \rho(v_{\rm N} - \partial_t f), \quad v_{\rm N} = (\mathbf{v}, \mathbf{N}), \quad H_{\rm N} = (\mathbf{H}, \mathbf{N}), \quad \mathbf{N} = (1, -\partial_2 f, -\partial_3 f),$$

$$\mathbf{v}_\tau = (v_{\tau_1}, v_{\tau_2}), \quad \mathbf{H}_\tau = (H_{\tau_1}, H_{\tau_2}), \quad v_{\tau_i} = (\mathbf{v}, \boldsymbol{\tau}_i),$$
$$H_{\tau_i} = (\mathbf{H}, \boldsymbol{\tau}_i), \quad \boldsymbol{\tau}_1 = (\partial_2 f, 1, 0), \quad \boldsymbol{\tau}_2 = (\partial_3 f, 0, 1),$$

$H_{\rm N}|_\Gamma := H_{\rm N}^\pm|_\Gamma$, and $\mathfrak{j} := \mathfrak{j}^\pm|_\Gamma$ is the mass transfer flux across the discontinuity surface.

From the mathematical point of view, there are two types of strong discontinuities: shock waves and characteristic discontinuities. Following Lax [20], characteristic discontinuities, which are characteristic free boundaries, are called contact discontinuities. For the Euler equations contact discontinuities are contact also from the physical point of view, i.e., there is no mass transfer across the discontinuity. However, in MHD the situation with characteristic discontinuities is richer than in gas dynamics. Namely, besides MHD shock waves ($\mathfrak{j} \neq 0$, $[\rho] \neq 0$) there are three types of characteristic discontinuities [4,5,18,19]: tangential discontinuities or current-vortex sheets ($\mathfrak{j} = 0$, $H_{\rm N}|_\Gamma = 0$), contact discontinuities ($\mathfrak{j} = 0$, $H_{\rm N}|_\Gamma \neq 0$), and Alfvén or rotational discontinuities ($\mathfrak{j} \neq 0$, $[\rho] = 0$). Current-vortex sheets and contact MHD discontinuities are contact from the physical point of view ($\mathfrak{j} = 0$), but Alfvén discontinuities are not.

The local-in-time existence of shock front solutions of the Euler equations was proved by Blokhin [3,4] and Majda [23,24], provided that the uniform Kreiss-Lopatinski condition [17,22,4] is satisfied at each point of the initial shock discontinuity (i.e., shock waves are uniformly stable). For the 3D Euler equations contact discontinuities with a nonzero jump in the velocity (vortex sheets) are always violently unstable [41,10]. For the 2D isentropic Euler equations, Coulombel & Secchi [8] have recently proved the existence of supersonic vortex sheets. In [8], using the result of [6], they have also shown the existence of weakly stable shock waves in isentropic gas dynamics.[1]

In MHD there are two types of Lax shocks: slow and fast shock waves (see, e.g., [18,19]). A complete 2D stability analysis of fast MHD shock waves was carried out in [42] for an ideal gas equation of state. Taking into account the recent work of Métivier & Zumbrun [26] extending the Kreiss–Majda theory [17,22] to a class of hyperbolic symmetrizable systems with characteristics of variable multiplicities (this class contains the MHD system), uniformly stable fast MHD shock waves found in [42] exist locally in time. In this paper, we are interested in current-vortex sheets and continue our analysis in [43] where a sufficient condition for the weak

[1] The existence theorems in [8] were proved for a finite (not necessarily short) time but under the condition that the initial discontinuity is close to a weakly stable rectilinear (for 2D) or planar (for 3D) discontinuity. It seems that the assumption that the initial data are close to a piecewise constant solution is technical and could be removed, provided that the time of existence is sufficiently short.

stability of compressible current-vortex sheets was first found for a general case of the unperturbed flow (see a discussion below). Concerning other types of characteristic MHD discontinuities, contact MHD discontinuities ($H_{\rm N}|_{\Gamma} \neq 0$) are known to be always weakly stable [4] whereas Alfvén discontinuities are violently unstable for a wide range of flow parameters [15].

For *current-vortex sheets*, with the requirements $\mathrm{j} = 0$ and $H_{\rm N}|_{\Gamma} = 0$ the Rankine-Hugoniot conditions (6) give the boundary conditions

$$\partial_t f = v_{\rm N}^{\pm}, \quad H_{\rm N}^{\pm} = 0, \quad [q] = 0 \qquad \text{on} \quad \Gamma(t). \tag{7}$$

Observe that the tangential components of the velocity and the magnetic field may undergo any jump: $[\mathbf{v}_{\tau}] \neq 0$, $[\mathbf{H}_{\tau}] \neq 0$.

Our final goal is to find conditions on the initial data

$$\mathbf{U}^{\pm}(0,\mathbf{x}) = \mathbf{U}_0^{\pm}(\mathbf{x}), \quad \mathbf{x} \in \Omega^{\pm}(0), \qquad f(0,\mathbf{x}') = f_0(\mathbf{x}'), \quad \mathbf{x}' \in \mathbb{R}^2, \tag{8}$$

providing the existence of current-vortex sheet solutions to the MHD system, i.e., the existence of a solution $(\mathbf{U}^{\pm}, f)$ of the free boundary value problem (1), (7), (8), where $\mathbf{U}^{\pm} := \mathbf{U}$ in $\Omega^{\pm}(t)$, and $\mathbf{U}^{\pm}$ is smooth in $\Omega^{\pm}(t)$. Because of the general properties of hyperbolic conservation laws it is natural to expect only the local-in-time existence of current-vortex sheet solutions. Therefore, the question on the nonlinear Lyapunov's stability of an ideal current-vortex sheet has no sense.

At the same time, the study of the linearized stability of current-vortex sheets is not only a necessary step to prove local-in-time existence but also is of independent interest in connection with various astrophysical applications. In particular, the ideal compressible current-vortex sheet is used for modeling the *heliopause*, which is caused by the interaction of the supersonic solar wind plasma with the local interstellar medium (in some sense, the heliopause is the *boundary of the solar system*). The generally accepted model of heliopause was suggested by BARANOV, KRASNOBAEV & KULIKOVSKY [2], and the heliopause is in fact a current-vortex sheet separating the interstellar plasma compressed at the bow shock from the solar wind plasma compressed at the termination shock wave. Note that in December 2004 the spacecraft Voyager 1 has crossed the termination shock at the distance of 93 AU from the Sun, and astrophysicists predict that it should reach the heliopause in the next ten years. So, the mathematical modeling of the heliopause in which current-vortex sheets play the key role becomes very urgent.

Piecewise constant solutions of (1) satisfying (8) on a planar discontinuity are a simplest case of current-vortex sheet solutions. In astrophysics the linear stability of a planar compressible current-vortex sheet is usually interpreted as the macroscopic stability of the heliopause [32]. One can show that for the constant coefficients linearized problem for planar current-vortex sheets the uniform Kreiss-Lopatinski condition is never satisfied [43]. That is, planar current-vortex sheets can be only *weakly* stable or violently unstable. In the 1960–90's, in a number of works motivated by astrophysical applications (see [32] and references therein) the linear stability of planar compressible current-vortex sheets was examined by the normal modes analysis. But, because of insuperable technical difficulties neither stability nor instability conditions were found for a general case of the unperturbed flow.

In [43] we have proposed an alternative energy method that has first enabled one to find a *sufficient* condition for the weak stability of planar compressible current-vortex sheets. For the most general case when for the unperturbed flow the tangential magnetic fields $\mathbf{H}'^{\pm} = (H_2^{\pm}, H_3^{\pm})$ are nonzero and noncollinear this condition reads

$$G(\mathbf{U}^+, \mathbf{U}^-) > 0, \tag{9}$$

where

$$G(\mathbf{U}^+, \mathbf{U}^-) = |\sin(\varphi^+ - \varphi^-)| \min\left\{\frac{\gamma^+}{|\sin\varphi^-|}, \frac{\gamma^-}{|\sin\varphi^+|}\right\} - |[\mathbf{v}']|,$$

$$\mathbf{v}'^{\pm} = (v_2^{\pm}, v_3^{\pm}), \quad \gamma^{\pm} = \frac{c^{\pm}c_{\mathrm{A}}^{\pm}}{\sqrt{(c^{\pm})^2 + (c_{\mathrm{A}}^{\pm})^2}}, \quad \cos\varphi^{\pm} = \frac{([\mathbf{v}'], \mathbf{H}'^{\pm})}{|[\mathbf{v}']|\,|\mathbf{H}'^{\pm}|},$$

and $c_{\mathrm{A}} = |\mathbf{H}|/\sqrt{\rho}$ is the Alfvén velocity.[2] Later on we will refer to (9) for nonplanar current-vortex sheets and (9) will be assumed to be satisfied at each point of $\Gamma(t)$, but now, when we speak about planar discontinuities, all the values in (9) are constants describing a piecewise constant solution of (1), (8) (unperturbed flow). Without loss of generality the planar discontinuity is supposed to be given by the equation $x_1 = 0$, i.e., it follows from (8) that for the piecewise constant solution $v_1^{\pm} = H_1^{\pm} = 0$.

As was shown in [43], the case when the vectors $\mathbf{H}'^+$ and $\mathbf{H}'^-$ are collinear or one of them is zero corresponds to the transition to violent instability. We exclude these critical cases from the consideration and without loss of generality suppose that

$$H_2^+H_3^- - H_3^+H_2^- \geq \epsilon > 0, \tag{10}$$

where ϵ is a fixed constant. Recall that if $\mathbf{H}'^+ = \mathbf{H}'^- = 0$ we have a planar vortex sheet, which is always violently unstable in 3D.

1.2. Reduction to a fixed domain

The function $f(t, \mathbf{x}')$ determining the discontinuity surface Γ is one of the unknowns of the free boundary value problem (1), (7), (8). To reduce this problem to that in a fixed domain we straighten, as usual (see, e.g.,

[2] It is known that current sheets ($[\mathbf{v}'] = 0$) are always weakly stable. To include this particular case into (9) we set that, for example, $\varphi^+ := \pi/2$ and $\varphi^- := 0$ for $[\mathbf{v}'] = 0$. That is, (9) is automatically satisfied for the case $[\mathbf{v}'] = 0$.

[22,25]), the unknown front Γ. That is, the unknowns $\mathbf{U}^{\pm}$ being smooth in $\Omega^{\pm}(t)$ are replaced by the vector-functions

$$\widetilde{\mathbf{U}}^{\pm}(t,\mathbf{x}) := \mathbf{U}^{\pm}(t,\Phi(t,\pm x_1,\mathbf{x}'),\mathbf{x}'),$$

that are smooth in the fixed domain $\mathbb{R}^3_+ = \{x_1 > 0,\ \mathbf{x}' \in \mathbb{R}^2\}$, where $\Phi(t,0,\mathbf{x}') = f(t,\mathbf{x}')$ and $\partial_1\Phi > 0$. In [43], as in [22], we used the simple choice $\Phi(t,\mathbf{x}) := x_1 + f(t,\mathbf{x}')$. Such a choice was suitable for our aims in [43], where we have proved a basic a priori estimate for the variable coefficients linearized problem, provided that the stable (in the sense of (9)) state about which we linearize problem (1), (7) belongs to $W^3_\infty([0,T]\times\mathbb{R}^3_+)$.

In this paper, to avoid assumptions about compact support of the initial data in the nonlinear existence theorem and work globally in $\mathbb{R}^3_+$ we use the choice similar to that suggested by Métivier [25]:

$$\Phi(t,\mathbf{x}) := x_1 + \chi(x_1)f(t,\mathbf{x}'),$$

where $\chi \in C_0^\infty(\mathbb{R})$ equals to 1 on $[-1,1]$, and $\|\chi'\|_{L_\infty(\mathbb{R})} < 1/2$. Then, the fulfillment of the requirement $\partial_1\Phi > 0$ is guaranteed if we consider solutions for which $\|f\|_{L_\infty([0,T]\times\mathbb{R}^2)} \leq 1$. The last is fulfilled if, without loss of generality, we consider the initial data satisfying $\|f_0\|_{L_\infty(\mathbb{R}^2)} \leq 1/2$, and the time T in our existence theorem is sufficiently small.

Dropping for convenience tildes in $\widetilde{\mathbf{U}}^{\pm}$ and introducing the functions

$$\Phi^{\pm}(t,\mathbf{x}) := \Phi(t,\pm x_1,\mathbf{x}') = \pm x_1 + \Psi^{\pm}(t,\mathbf{x}), \quad \Psi^{\pm}(t,\mathbf{x}) := \chi(\pm x_1)f(t,\mathbf{x}'),$$

we reduce (1), (7), (8) to the initial boundary value problem

$$\mathbb{L}(\mathbf{U}^+,\Psi^+) = 0, \quad \mathbb{L}(\mathbf{U}^-,\Psi^-) = 0 \quad \text{in } [0,T]\times\mathbb{R}^3_+, \tag{11}$$

$$\mathbb{B}(\mathbf{U}^+,\mathbf{U}^-,f) = 0 \quad \text{on } [0,T]\times\{x_1=0\}\times\mathbb{R}^2, \tag{12}$$

$$\mathbf{U}^+|_{t=0} = \mathbf{U}^+_0, \quad \mathbf{U}^-|_{t=0} = \mathbf{U}^-_0 \quad \text{in } \mathbb{R}^3_+, \qquad f|_{t=0} = f_0 \quad \text{in } \mathbb{R}^2, \tag{13}$$

where $\mathbb{L}(\mathbf{U},\Psi) = L(\mathbf{U},\Psi)\mathbf{U}$,

$$L(\mathbf{U},\Psi) = A_0(\mathbf{U})\partial_t + \widetilde{A}_1(\mathbf{U},\Psi)\partial_1 + A_2(\mathbf{U})\partial_2 + A_3(\mathbf{U})\partial_3,$$

$$\widetilde{A}_1(\mathbf{U}^{\pm},\Psi^{\pm}) = \frac{1}{\partial_1\Phi^{\pm}}\Big(A_1(\mathbf{U}^{\pm}) - A_0(\mathbf{U}^{\pm})\partial_t\Psi^{\pm} - \sum_{k=2}^{3} A_k(\mathbf{U}^{\pm})\partial_k\Psi^{\pm}\Big)$$

($\partial_1\Phi^{\pm} = \pm 1 + \partial_1\Psi^{\pm}$), and (12) is the compact form of the boundary conditions

$$\partial_t f - v_{\rm N}^+ = 0, \quad \partial_t f - v_{\rm N}^- = 0, \quad [q] = 0 \qquad \text{on } [0,T]\times\{x_1=0\}\times\mathbb{R}^2,$$

with $v_{\rm N}^{\pm} = v_1^{\pm} - v_2^{\pm}\partial_2 f - v_3^{\pm}\partial_3 f$, $\quad [q] = q^+|_{x_1=0} - q^-|_{x_1=0}$.

There appear two natural questions. The first one: Why the boundary conditions $H_{\rm N}^+|_{x_1=0} = 0$ and $H_{\rm N}^-|_{x_1=0} = 0$ have not been included in (12)? And the second question: Why systems (1) and (4) are equivalent on current-vortex sheet solutions, i.e., why system (1) in the straightened variables is equivalent to (11)? The answer to these questions is given by the following proposition.

Proposition 1. *Let the initial data (13) satisfy*

$$\operatorname{div}\mathbf{h}^+ = 0, \quad \operatorname{div}\mathbf{h}^- = 0 \tag{14}$$

and the boundary conditions

$$H_{\mathrm{N}}^+|_{x_1=0} = 0, \quad H_{\mathrm{N}}^-|_{x_1=0} = 0, \tag{15}$$

where

$$\mathbf{h}^{\pm} = (H_{\mathrm{n}}^{\pm}, H_2^{\pm}\partial_1\Phi^{\pm}, H_3^{\pm}\partial_1\Phi^{\pm}), \quad H_{\mathrm{n}}^{\pm} = H_1^{\pm} - H_2^{\pm}\partial_2\Psi^{\pm} - H_3^{\pm}\partial_3\Psi^{\pm}$$

($H_{\mathrm{n}}^{\pm}|_{x_1=0} = H_{\mathrm{N}}^{\pm}|_{x_1=0}$). If problem (11)–(13) has a solution $(\mathbf{U}^{\pm}, f)$, then this solution satisfies (14) and (15) for all $t \in [0,T]$. The same is true for current-vortex sheet solutions of system (1).

The proof of Proposition 1 is given in Appendix A. Equations (14) are just the divergent constraint (12) on either side of the straightened front. Using (14), we can easily prove that system (1) in the straightened variables is equivalent to (11). Concerning the boundary conditions $H_{\mathrm{N}}^+|_{x_1=0} = 0$ and $H_{\mathrm{N}}^-|_{x_1=0} = 0$, we must regard them as the restrictions on the initial data (13). Otherwise, the hyperbolic problem (11), (12), (15) does not have a correct number of boundary and, in particular, its linearization does not have the property of *maximality* [30]. Indeed, one can show (see [43] and Section 2) that the boundary matrix on the boundary $x_1 = 0$,

$$A_\nu|_{x_1=0} = \operatorname{diag}(\widetilde{A}_1(\mathbf{U}^+, \Psi^+), \widetilde{A}_1(\mathbf{U}^-, \Psi^-))|_{x_1=0},$$

has two positive ("outgoing") and two negative eigenvalues, and other eigenvalues are zeros. That is, the boundary $x_1 = 0$ is characteristic, and since one of the boundary conditions is needed for determining the function $f(t, \mathbf{x}')$, the correct number of boundary conditions is three (that is the case in (12)).

1.3. Main result and discussion

In this paper, our main goal is to prove the local-in-time existence of solutions to problem (11)–(13), provided that the initial data (13) satisfy the stability condition (9) at each point of the straightened front $x_1 = 0$ together with all the other necessary conditions, see Theorem 1 below.

As usual, we will construct solutions to the nonlinear problem (11)–(13) by considering a sequence of linearized problems. However, since for current-vortex sheets the Kreiss-Lopatinski condition is satisfied only in a weak sense, there appears a loss of derivatives phenomena and, therefore, standard Picard iterations, which convergence is usually proved by fixed-point argument, are inapplicable for our case. We overcome this principal difficulty by solving our nonlinear problem by a suitable Nash-Moser-type iteration scheme (see, e.g., [14]). For multidimensional hyperbolic conservations laws, the Nash-Moser method was earlier used by ALINHAC [1] to prove the existence of rarefaction waves and by FRANCHETEAU & MÉTIVIER

[11] to construct shock front solutions when the strength of the shock tends to zero. Recently the Nash-Moser method was successfully exploited by Coulombel & Secchi [8] for 2D compressible vortex sheets for the isentropic Euler equations.

As in [8], the Nash-Moser procedure we shall use in this paper is not completely standard. Exactly as in [8], some nonlinear constraints must be satisfied at each step of the iteration scheme. Moreover, in comparison with [8] we have the additional constraints (14) and (15), which actually make most trouble. As vortex sheets, current-vortex sheets are characteristic free boundaries that implies a loss of control on derivatives in the normal direction. But, the peculiarity of our problem consists in the fact that the loss of control on normal derivatives cannot be compensated as it was done in [8] (see also [38]) for gas dynamics by estimating missing normal derivatives through a vorticity-type linearized equation. Therefore, in our case, as for the MHD system in a fixed domain whose boundary is a regular magnetic surface [27,33,36,46], the natural functional setting is provided by the anisotropic weighted Sobolev spaces H^m_* (see the definition below).

In principle, for current-vortex sheets we could straighten the front as was suggested in [8]. For the choice of $\Phi(t,\mathbf{x})$ used in [8] the boundary matrix A_ν would have constant rank in the whole space $\mathbb{R}^3_+$, but not only on the boundary $x_1=0$. Then, we could work in the usual Sobolev spaces H^m with a certain anisotropy in the normal direction [30]. Here, we however prefer to use the function space H^m_* because it provides a little bit more regularity of solutions. Following [27–29,33–37,39,46], this space is defined as follows:

$$H^m_*(\mathbb{R}^3_+) := \left\{ u\in L_2(\mathbb{R}^3_+) \mid \partial^\alpha_*\partial_1^k u\in L_2(\mathbb{R}^3_+) \quad \text{if} \quad |\alpha|+2k\leq m \right\},$$

where $m\in\mathbb{N}$, $\partial^\alpha_* = (\sigma\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$, and $\sigma(x_1)\in C^\infty(\mathbb{R}_+)$ is a monotone increasing function such that $\sigma(x_1)=x_1$ in a neighborhood of the origin and $\sigma(x_1)=1$ for x_1 large enough. The space $H^m_*(\mathbb{R}^3_+)$ is normed by

$$\|u\|^2_{m,*} = \sum_{|\alpha|+2k\leq m} \|\partial^\alpha_*\partial_1^k u\|^2_{L_2(\mathbb{R}^3_+)}.$$

Following [34,37], we also define the space

$$\mathcal{L}^2_T(H^m_*) = \bigcap_{k=0}^m H^k([0,T],H^{m-k}_*)$$

equipped with the norm

$$[u]^2_{m,*,T} = \int_0^T |||u(t)|||^2_{m,*}dt, \quad \text{where} \quad |||u(t)|||^2_{m,*} = \sum_{j=0}^m \|\partial_t^j u(t)\|^2_{m-j,*}$$

For a multi-index $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)$ we shall also use the notation $\partial^\alpha_\star = \partial_t^{\alpha_0}(\sigma\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$.

We are now in a position to state the main result of the present paper.

Theorem 1. *Let $m \in \mathbb{N}$ and $m \geq 12$. Suppose the initial data (13), with*

$$(\mathbf{U}_0^{\pm}, f_0) \in H_*^{2m+19}(\mathbb{R}^3_+) \times H^{2m+19}(\mathbb{R}^2),$$

satisfy the hyperbolicity condition (5) and the divergent constraints (14) for all $\mathbf{x} \in \mathbb{R}^3_+$. Let the initial data at $x_1 = 0$ satisfy the stability condition (9), restriction (10), and constraints (15) for all $\mathbf{x}' \in \mathbb{R}^2$. Assume also that the initial data are compatible up to order $m+9$ in the sense of Definition 1 (see Section 4). Then, there exists a sufficiently short time $T > 0$ such that problem (11)–(13) has a unique solution

$$(\mathbf{U}^{\pm}, f) \in \mathcal{L}^2_T(H_*^m(\mathbb{R}^3_+)) \times H^m([0,T] \times \mathbb{R}^2).$$

The main tool for proving the convergence of the Nash-Moser iteration scheme is a so-called tame estimate [1,8] for the linearized problem. In [43], the basic a priori estimate in H^1_* for the linearized problem was obtained by the energy method thanks to a *new* symmetric form of the MHD equations. The energy method we used in [43] can be formalized as the construction of a so-called dissipative 0-symmetrizer [45]. Here, to get the tame estimate for the linearized equations we use the same idea as in [43]. Since we exploit standard energy arguments and, unlike [7], do not use paradifferential calculus in the linear variable coefficients analysis, we do not need to make the technical assumption from [8] that the initial data are close to a piecewise constant solution associated with a planar discontinuity.

Again, since we use the energy method, there is no problem at all to prove the uniqueness of a solution to problem (11)–(13). Uniqueness follows already from the basic H^1_* estimate deduced in [43] and can be proved by standard argument, exactly in the same manner as was done in [44] for *incompressible* current-vortex sheets.[3] With this short remark, we shall no longer discuss the problem of uniqueness in this paper.

In Theorem 1, we have a big loss of derivatives from the initial data to the solution. Actually, the number of lost derivatives could be two or even three times less if we worked in the function space H^m_{**} (see [35]) instead of H^m_*. In our case, the solution belongs to $\mathcal{L}^2_T(H^m_{**})$ if its "noncharacteristic" part $\mathbf{U}^{\pm}_{\rm nc}$ is such that $\partial_1 \mathbf{U}^{\pm}_{\rm nc} \in \mathcal{L}^2_T(H^{m-1}_*)$, where

$$\mathbf{U}^{\pm}_{\rm nc} = (q^{\pm}, \mathbf{v}^{\pm}_{\rm n}, \mathbf{H}^{\pm}_{\rm n}), \quad v^{\pm}_{\rm n} = v^{\pm}_1 - v^{\pm}_2 \partial_2 \Psi^{\pm} - v^{\pm}_3 \partial_3 \Psi^{\pm}.$$

However, we prefer to not follow this way, because it is connected with extremely big technical difficulties, calculations, etc. In particular, for $\mathbf{U}^{\pm}_{\rm nc}$ we need to estimate the errors of the Nash-Moser iteration separately.

[3] Note that for the energy method the appearance of a zero-order term for the front in the interior equations for the difference of solutions [44] (in terms of a so-called "good unknown" of Alinhac [1], see Section 2) does not make any trouble.

Recall that condition (9) is only sufficient for stability. As was shown in [43, 44], in the incompressibility limit this condition describes exactly the half of the whole parameter domain of stability of planar incompressible current-vortex found by Syrovatskij [40] by the normal modes analysis. For compressible current-vortex sheets, the main difficulty in the normal modes analysis is connected with the fact that the Lopatinski determinant is generically reduced to an algebraic equation of the tenth degree depending on seven dimensionless parameters and one more inner parameter determining the wave vector (see [32, 43]). Moreover, the squaring was applied under the reduction of the Lopatinski determinant to this algebraic equation and, therefore, it can introduce spurious roots. For all these reasons, the problem on finding the necessary and sufficient condition of weak stability of planar compressible current-vortex sheets is still open.

Remark 1. For current-vortex sheets one can also consider associated diffusing (*nonstationary*) viscous profiles for the resistive viscous MHD equations.[4] Since for these equations we can at least expect the existence of global-in-time solutions, it is natural to study Lyupunov's (long-time) stability of "viscous" compressible current-vortex sheets. Some numerical results in this direction can be found in [9].

The plan of the rest of the paper is the following. In Section 2 we formulate the linearized problem and prove its well-posedness under suitable assumptions on the basic state about which we linearize our nonlinear problem (11), (12). The main of these assumptions is the stability condition (9). In Section 3, for the linearized problem we derive an a priori tame estimate in the anisotropic weighted Sobolev spaces. In Section 4, we specify compatibility conditions for the initial data and, by constructing an approximate solution, reduce problem (11)–(13) to that with zero initial data. At last, in section 5 we solve the reduced problem by a suitable Nash-Moser-type iteration scheme.

2. Linearized problem associated to (11), (12)

2.1. The basic state

Let

$$(\widehat{\mathbf{U}}^+(t,\mathbf{x}), \widehat{\mathbf{U}}^-(t,\mathbf{x}), \hat{f}(t,\mathbf{x}')) \tag{16}$$

be a given vector-function, where $\widehat{\mathbf{U}}^\pm = (\hat{p}^\pm, \hat{\mathbf{v}}^\pm, \widehat{\mathbf{H}}^\pm, \widehat{S}^\pm)$ and $\hat{f}$ are supposed to be sufficiently smooth in

$$\Omega_T := (-\infty, T] \times \mathbb{R}^3_+.$$

[4] For a complete analysis of the existence and bifurcation of (stationary) viscous profiles for MHD shock waves we refer to [12].

Later on for the functions defined in Ω_T the time interval $[0,T]$ appearing in the definition of the norm $[\cdot]_{s,*,T}$ given in Section 2 is replaced by $(-\infty,T]$. Moreover, by $H^s_*(\Omega_T)$ we will denote the space that coincides with the space $\mathcal{L}^2_T(H^s_*(\mathbb{R}^3_+))$ defined in Section 2 if we replace the time interval $[0,T]$ by $(-\infty,T]$.

We assume that the basic state (16) about which we shall linearize problem (11), (12) satisfies the hyperbolicity condition (5) in Ω_T,

$$\rho(\hat{p}^{\pm},\widehat{S}^{\pm})>0,\quad \rho_p(\hat{p}^{\pm},\widehat{S}^{\pm})>0, \tag{17}$$

the Rankine-Hugoniot conditions (12),

$$\partial_t\hat{f}-\hat{v}^+_{\rm N}|_{x_1=0}=0,\quad \partial_t\hat{f}-\hat{v}^-_{\rm N}|_{x_1=0}=0,\quad [\hat{q}]=0, \tag{18}$$

and the stability condition (9) together with restriction (10) on $\partial\Omega_T:=(-\infty,T]\times\{x_1=0\}\times\mathbb{R}^2$,

$$G(\widehat{\mathbf{U}}^+|_{x_1=0},\widehat{\mathbf{U}}^-|_{x_1=0})>0,\quad (\widehat{H}^+_2\widehat{H}^-_3-\widehat{H}^+_3\widehat{H}^-_2)|_{x_1=0}\geq\epsilon>0, \tag{19}$$

where $\hat{q}^{\pm}=\hat{p}^{\pm}+\frac{1}{2}|\widehat{\mathbf{H}}^{\pm}|^2$,

$$\hat{v}^{\pm}_{\rm N}=\hat{v}^{\pm}_1-\hat{v}^{\pm}_2\partial_2\hat{f}-\hat{v}^{\pm}_3\partial_3\hat{f}\,,\quad [\hat{q}]=(\hat{q}^+-\hat{q}^-)|_{x_1=0}\,.$$

Let also

$$\begin{gathered}\widehat{\mathbf{U}}^+,\,\widehat{\mathbf{U}}^-\in W^2_\infty(\Omega_T),\quad \hat{f}\in W^3_\infty(\partial\Omega_T),\\ \|\widehat{\mathbf{U}}^+\|_{W^2_\infty(\Omega_T)}+\|\widehat{\mathbf{U}}^-\|_{W^2_\infty(\Omega_T)}+\|\hat{f}\|_{W^3_\infty(\partial\Omega_T)}\leq K,\end{gathered} \tag{20}$$

where $K>0$ is a constant. Moreover, without loss of generality we assume that $\|\hat{f}\|_{L_\infty(\partial\Omega_T)}<1$. This implies

$$\partial_1\widehat{\Phi}^+\geq 1/2,\quad \partial_1\widehat{\Phi}^-\leq -1/2,$$

with

$$\widehat{\Phi}^{\pm}(t,\mathbf{x}):=\pm x_1+\widehat{\Psi}^{\pm}(t,\mathbf{x}),\quad \widehat{\Psi}^{\pm}(t,\mathbf{x}):=\chi(\pm x_1)\hat{f}(t,\mathbf{x}').$$

Observe that (20) yields

$$\|\widehat{\mathbf{W}}\|_{W^2_\infty(\Omega_T)}\leq C(K),$$

where $\widehat{\mathbf{W}}:=(\widehat{\mathbf{U}}^+,\widehat{\mathbf{U}}^-,\nabla_{t,{\rm x}}\widehat{\Psi}^+,\nabla_{t,{\rm x}}\widehat{\Psi}^-)$, $\nabla_{t,{\rm x}}=(\partial_t,\nabla)$, and $C=C(K)>0$ is a constant depending on K.

Later on, for the perturbation of the magnetic field we shall deduce equations associated to the nonlinear constraints (14) and (15). However, to do this it is not enough that these constraints are satisfied by the basic state (16). We need actually that the equation for $\mathbf{H}^{\pm}$ itself contained in (11) is fulfilled for (16) (cf. (197) in Appendix A):

$$\partial_t\widehat{\mathbf{H}}^{\pm}+\frac{1}{\partial_1\widehat{\Phi}^{\pm}}\Big\{(\hat{\mathbf{w}}^{\pm},\nabla)\widehat{\mathbf{H}}^{\pm}-(\hat{\mathbf{h}}^{\pm},\nabla)\hat{\mathbf{v}}^{\pm}+\widehat{\mathbf{H}}^{\pm}{\rm div}\,\hat{\mathbf{u}}^{\pm}\Big\}=0, \tag{21}$$

where

$$\hat{\mathbf{u}}^{\pm} = (\hat{v}_{\rm n}^{\pm}, \hat{v}_2^{\pm}\partial_1\widehat{\Phi}^{\pm}, \hat{v}_3^{\pm}\partial_1\widehat{\Phi}^{\pm}), \quad \hat{v}_{\rm n}^{\pm} = \hat{v}_1^{\pm} - \hat{v}_2^{\pm}\partial_2\widehat{\Psi}^{\pm} - \hat{v}_3^{\pm}\partial_3\widehat{\Psi}^{\pm},$$

$$\hat{\mathbf{h}}^{\pm} = (\widehat{H}_{\rm n}^{\pm}, \widehat{H}_2^{\pm}\partial_1\widehat{\Phi}^{\pm}, \widehat{H}_3^{\pm}\partial_1\widehat{\Phi}^{\pm}), \quad \widehat{H}_{\rm n}^{\pm} = \widehat{H}_1^{\pm} - \widehat{H}_2^{\pm}\partial_2\widehat{\Psi}^{\pm} - \widehat{H}_3^{\pm}\partial_3\widehat{\Psi}^{\pm},$$

and $\hat{\mathbf{w}}^{\pm} = \hat{\mathbf{u}}^{\pm} - (\partial_t\widehat{\Psi}^{\pm}, 0, 0)$. Assume that (16) satisfies (21). Then, it follows from the proof of Proposition 1 (see Appendix A) that constraints (14) and (15) are satisfied for the basic state (16) if they are true for it at $t = 0$:

$$\operatorname{div}\hat{\mathbf{h}}^{\pm}|_{t=0} = 0, \quad \widehat{H}_{\rm N}^{\pm}|_{x_1=t=0} = 0, \tag{22}$$

where $\widehat{H}_{\rm N}^{\pm} = \widehat{H}_1^{\pm} - \widehat{H}_2^{\pm}\partial_2\hat{f} - \widehat{H}_3^{\pm}\partial_3\hat{f}$.

2.2. The linearized equations

The linearized equations for (11), (12) read:

$$\mathbb{L}'(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})(\delta\mathbf{U}^{\pm}, \delta\Psi^{\pm}) := \frac{d}{d\varepsilon}\mathbb{L}(\mathbf{U}_{\varepsilon}^{\pm}, \Psi_{\varepsilon}^{\pm})|_{\varepsilon=0} = \mathbf{f}^{\pm} \quad \text{in } \Omega_T,$$

$$\mathbb{B}'(\widehat{\mathbf{U}}^{+}, \widehat{\mathbf{U}}^{-}, \hat{f})(\delta\mathbf{U}^{+}, \delta\mathbf{U}^{-}, \delta f) := \frac{d}{d\varepsilon}\mathbb{B}(\mathbf{U}_{\varepsilon}^{+}, \mathbf{U}_{\varepsilon}^{-}, f_{\varepsilon})|_{\varepsilon=0} = \mathbf{g} \quad \text{on } \partial\Omega_T$$

where $\mathbf{U}_{\varepsilon}^{\pm} = \widehat{\mathbf{U}}^{\pm} + \varepsilon\,\delta\mathbf{U}^{\pm}$, $f_{\varepsilon} = \hat{f} + \varepsilon\,\delta f$, and

$$\Psi_{\varepsilon}^{\pm}(t, \mathbf{x}) := \chi(\pm x_1) f_{\varepsilon}(t, \mathbf{x}'), \quad \Phi_{\varepsilon}^{\pm}(t, \mathbf{x}) := \pm x_1 + \Psi_{\varepsilon}^{\pm}(t, \mathbf{x}),$$

$$\delta\Psi^{\pm}(t, \mathbf{x}) := \chi(\pm x_1)\delta f(t, \mathbf{x}).$$

Here, as usual, we introduce the source terms $\mathbf{f}^{\pm}(t, \mathbf{x})$ and $\mathbf{g}(t, \mathbf{x}')$ to make the interior equations and the boundary conditions inhomogeneous. To simplify the notations we will below drop δ.

We easily compute the exact form of the linearized operators:

$$\mathbb{L}'(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})(\mathbf{U}^{\pm}, \Psi^{\pm})$$
$$= L(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})\mathbf{U}^{\pm} + \mathcal{C}(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})\mathbf{U}^{\pm} - \left\{L(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})\Psi^{\pm}\right\}\partial_1\widehat{\mathbf{U}}^{\pm},$$

$$\mathbb{B}'(\widehat{\mathbf{U}}^{+}, \widehat{\mathbf{U}}^{-}, \hat{f})(\mathbf{U}^{+}, \mathbf{U}^{-}, f) = \begin{pmatrix} \partial_t f + \hat{v}_2^{+}\partial_2 f + \hat{v}_3^{+}\partial_3 f - v_{\rm N}^{+} \\ \partial_t f + \hat{v}_2^{-}\partial_2 f + \hat{v}_3^{-}\partial_3 f - v_{\rm N}^{-} \\ q^{+} - q^{-} \end{pmatrix},$$

where $q^{\pm} = p^{\pm} + (\widehat{\mathbf{H}}^{\pm}, \mathbf{H}^{\pm})$, $v_{\rm N}^{\pm} = v_1^{\pm} - v_2^{\pm}\partial_2\hat{f} - v_3^{\pm}\partial_3\hat{f}$, and the matrix $\mathcal{C}(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})$ is determined as follows:

$$\mathcal{C}(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})\mathbf{Y} = (\mathbf{Y}, \nabla_y A_0(\widehat{\mathbf{U}}^{\pm}))\partial_t\widehat{\mathbf{U}}^{\pm} + (\mathbf{Y}, \nabla_y \widetilde{A}_1(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm}))\partial_1\widehat{\mathbf{U}}^{\pm}$$
$$+ (\mathbf{Y}, \nabla_y A_2(\widehat{\mathbf{U}}^{\pm}))\partial_2\widehat{\mathbf{U}}^{\pm} + (\mathbf{Y}, \nabla_y A_3(\widehat{\mathbf{U}}^{\pm}))\partial_3\widehat{\mathbf{U}}^{\pm},$$

$$(\mathbf{Y}, \nabla_y A(\widehat{\mathbf{U}}^{\pm})) := \sum_{i=1}^{8} y_i \left(\frac{\partial A(\mathbf{Y})}{\partial y_i} \bigg|_{\mathbf{Y}=\widehat{\mathbf{U}}^{\pm}} \right), \quad \mathbf{Y} = (y_1, \dots, y_8).$$

The differential operator $\mathbb{L}'(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})$ is a first order operator in $\Psi^{\pm}$. This fact can give some trouble in obtaining a priori estimates for the linearized problem. Following [1], we overcome this difficulty by introducing the "good unknown":

$$\dot{\mathbf{U}}^{+} := \mathbf{U}^{+} - \frac{\Psi^{+}}{\partial_1 \widehat{\Phi}^{+}} \partial_1 \widehat{\mathbf{U}}^{+}, \quad \dot{\mathbf{U}}^{-} := \mathbf{U}^{-} - \frac{\Psi^{-}}{\partial_1 \widehat{\Phi}^{-}} \partial_1 \widehat{\mathbf{U}}^{-}. \tag{23}$$

In terms of unknown (23) the linearized interior equations take the form

$$L(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})\dot{\mathbf{U}}^{\pm} + \mathcal{C}(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})\dot{\mathbf{U}}^{\pm} - \frac{\Psi^{\pm}}{\partial_1 \widehat{\Phi}^{\pm}} \partial_1 \big\{ \mathbb{L}(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm}) \big\} = \mathbf{f}^{\pm}. \tag{24}$$

In principle, the zero-order terms in (24) do not give any trouble in the application of the energy method for deducing a priori estimates. But, to prove the existence of solutions to the linearized problem it is better to have standard linear symmetric hyperbolic systems for $\dot{\mathbf{U}}^{+}$ and $\dot{\mathbf{U}}^{-}$. Therefore, as in [1,8,11], we drop the zero-order term in $\Psi^{\pm}$ in (24) and consider the effective linear operators

$$\begin{aligned} \mathbb{L}'_e(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})\dot{\mathbf{U}}^{\pm} &:= L(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})\dot{\mathbf{U}}^{\pm} + \mathcal{C}(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})\dot{\mathbf{U}}^{\pm} \\ &= A_0(\widehat{\mathbf{U}}^{\pm})\partial_t \dot{\mathbf{U}}^{\pm} + \widetilde{A}_1(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})\partial_1 \dot{\mathbf{U}}^{\pm} \\ &+ A_2(\widehat{\mathbf{U}}^{\pm})\partial_2 \dot{\mathbf{U}}^{\pm} + A_3(\widehat{\mathbf{U}}^{\pm})\partial_3 \dot{\mathbf{U}}^{\pm} + \mathcal{C}(\widehat{\mathbf{U}}^{\pm}, \widehat{\Psi}^{\pm})\dot{\mathbf{U}}^{\pm} \end{aligned} \tag{25}$$

In the subsequent nonlinear analysis the dropped term in (23) will be considered as an error term at each Nash-Moser iteration step.

Concerning the boundary differential operator $\mathbb{B}'$, in terms of unknown (23) it reads:

$$\begin{aligned} &\mathbb{B}'_e(\widehat{\mathbf{U}}, \hat{f})(\dot{\mathbf{U}}, f) := \mathbb{B}'(\widehat{\mathbf{U}}, \hat{f})(\mathbf{U}^{+}, \mathbf{U}^{-}, f) \\ &= \begin{pmatrix} \partial_t f + \hat{v}_2^{+} \partial_2 f + \hat{v}_3^{+} \partial_3 f - \dot{v}_{\mathrm{N}}^{+} - f\, \partial_1 \hat{v}_{\mathrm{N}}^{+} \\ \partial_t f + \hat{v}_2^{-} \partial_2 f + \hat{v}_3^{-} \partial_3 f - \dot{v}_{\mathrm{N}}^{-} - f\, \partial_1 \hat{v}_{\mathrm{N}}^{-} \\ \dot{q}^{+} - \dot{q}^{-} + f(\partial_1 \hat{q}^{+} - \partial_1 \hat{q}^{-}) \end{pmatrix}, \end{aligned} \tag{26}$$

where $\widehat{\mathbf{U}} = (\widehat{\mathbf{U}}^{+}, \widehat{\mathbf{U}}^{-})$, $\dot{\mathbf{U}} = (\dot{\mathbf{U}}^{+}, \dot{\mathbf{U}}^{-})$, and $\dot{v}_{\mathrm{N}}^{\pm} = \dot{v}_1^{\pm} - \dot{v}_2^{\pm} \partial_2 \hat{f} - \dot{v}_3^{\pm} \partial_3 \hat{f}$. As in [1,8,11] we keep the zero-order terms in f in (26). Note that the main difficulties in deducing a priori estimates for the linearized problem by the energy method are connected with these terms (see [43] and Section 3). Introducing the notation

$$\mathbb{L}'_e(\widehat{\mathbf{U}}, \widehat{\boldsymbol{\Psi}})\dot{\mathbf{U}} := \begin{pmatrix} \mathbb{L}'_e(\widehat{\mathbf{U}}^{+}, \widehat{\Psi}^{+})\dot{\mathbf{U}}^{+} \\ \mathbb{L}'_e(\widehat{\mathbf{U}}^{-}, \widehat{\Psi}^{-})\dot{\mathbf{U}}^{-} \end{pmatrix}, \tag{27}$$

with $\widehat{\boldsymbol{\Psi}} = (\widehat{\Psi}^+, \widehat{\Psi}^-)$, we write down the linear problem for $(\dot{\mathbf{U}}, f)$:

$$\mathbb{L}_e'(\widehat{\mathbf{U}}, \widehat{\boldsymbol{\Psi}})\dot{\mathbf{U}} = \mathbf{f} \quad \text{in } \Omega_T, \tag{28}$$

$$\mathbb{B}_e'(\widehat{\mathbf{U}}, \hat{f})(\dot{\mathbf{U}}, f) = \mathbf{g} \quad \text{on } \partial\Omega_T, \tag{29}$$

$$(\dot{\mathbf{U}}, f) = 0 \quad \text{for } t < 0, \tag{30}$$

where $\mathbf{f} = (\mathbf{f}^+, \mathbf{f}^-) = (f_1^+, \dots, f_8^+, f_1^-, \dots, f_8^-)$ and $\mathbf{g} = (g_1^+, g_1^-, g_2)$ vanish in the past. Here, unlike [43], we consider the case of zero initial data, that is usual assumption, and postpone the case of nonzero initial data to the nonlinear analysis (construction of a so-called approximate solution, see Section 4).

In Appendix A we prove the following proposition that will play an important role in the proof of well-posedness of problem (28)–(30).

Proposition 2. *Let for the basic state (16) all the assumptions above (in particular, (21), (22)) are fulfilled. Then solutions of problem (28)–(30) satisfy*

$$\operatorname{div} \dot{\mathbf{h}}^+ = r^+, \quad \operatorname{div} \dot{\mathbf{h}}^- = r^- \quad \textit{in } \Omega_T, \tag{31}$$

$$\begin{cases} \widehat{H}_2^+ \partial_2 f + \widehat{H}_3^+ \partial_3 f - \dot{H}_{\mathrm{N}}^+ - f\, \partial_1 \widehat{H}_{\mathrm{N}}^+ = g_3^+, \\ \widehat{H}_2^- \partial_2 f + \widehat{H}_3^- \partial_3 f - \dot{H}_{\mathrm{N}}^- - f\, \partial_1 \widehat{H}_{\mathrm{N}}^- = g_3^-, \quad \textit{on } \partial\Omega_T. \end{cases} \tag{32}$$

Here

$$\dot{\mathbf{h}}^\pm = (\dot{H}_{\mathrm{n}}^\pm, \dot{H}_2^\pm \partial_1 \widehat{\Phi}^\pm, \dot{H}_3^\pm \partial_1 \widehat{\Phi}^\pm), \quad \dot{H}_{\mathrm{n}}^\pm = \dot{H}_1^\pm - \dot{H}_2^\pm \partial_2 \widehat{\Psi}^\pm - \dot{H}_3^\pm \partial_3 \widehat{\Psi}^\pm$$

($\dot{H}_{\mathrm{N}}^\pm|_{x_1=0} = \dot{H}_{\mathrm{n}}^\pm|_{x_1=0}$), and the functions $r^\pm = r^\pm(t, \mathbf{x})$, $g_3^\pm = g_3^\pm(t, \mathbf{x}')$, that vanish in the past, are determined by the source terms and the basic state as solutions to the linear inhomogeneous equations[5]

$$\partial_t a^\pm + \frac{1}{\partial_1 \widehat{\Phi}^\pm} \left\{ (\hat{\mathbf{w}}^\pm, \nabla a^\pm) + a^\pm \operatorname{div} \hat{\mathbf{u}}^\pm \right\} = \mathcal{F}^\pm \quad \textit{in } \Omega_T, \tag{33}$$

$$\partial_t g_3^\pm + \hat{v}_2^\pm \partial_2 g_3^\pm + \hat{v}_3^\pm \partial_3 g_3^\pm + \left(\partial_2 \hat{v}_2^\pm + \partial_3 \hat{v}_3^\pm \right) g_3^\pm = \mathcal{G}^\pm \quad \textit{on } \partial\Omega_T, \tag{34}$$

where $a^\pm = r^\pm / \partial_1 \widehat{\Phi}^\pm$, $\quad \mathcal{F}^\pm = (\operatorname{div} \mathbf{f}_{\mathrm{h}}^\pm) / \partial_1 \widehat{\Phi}^\pm$,

$$\mathbf{f}_{\mathrm{h}}^\pm = (f_{\mathrm{n}}^\pm, f_6^\pm, f_7^\pm), \quad f_{\mathrm{n}}^\pm = f_5^\pm - f_6^\pm \partial_2 \widehat{\Psi}^\pm - f_7^\pm \partial_3 \widehat{\Psi}^\pm,$$

$$\mathcal{G}^\pm = \left\{ \partial_2 \big(\widehat{H}_2^\pm g_1^\pm \big) + \partial_3 \big(\widehat{H}_3^\pm g_1^\pm \big) - f_{\mathrm{n}}^\pm \right\} \big|_{x_1=0}.$$

[5] It follows from (18) that the interior equations (33) do not need boundary conditions because $\hat{w}_1^\pm|_{x_1=0} = 0$.

2.3. Reduction to homogeneous boundary conditions

In [43], the basic a priori estimate for problem (28), (29) was obtained for the case of nonzero initial data but the zero source terms $\mathbf{f}$ and $\mathbf{g}$. This was done thanks to the fact the boundary conditions for an equivalent formulation of problem (28), (29) (it will be presented below) are *dissipative*. There is no problem at all to generalize the estimate from [43] to nonzero source terms. But we will lose derivatives from the source term $\mathbf{g}$ because the boundary conditions are *not strictly* dissipative. Since we anyway lose derivatives from $\mathbf{g}$, we can use the classical argument (see, e.g., [31]) to deal with inhomogeneous boundary conditions. It suggests to subtract from the solution a more regular function satisfying the boundary conditions, and reduce the problem to one with homogeneous boundary conditions. As is known, if we work in usual Sobolev spaces such a way leads to the loss of "1/2 derivative" from $\mathbf{g}$.

However, because of the presence of constraints (14) and (15), that play an important role in the energy method in [43], to prove the well-posedness of the original linear problem (28)–(30) we have to modify a little bit the above mentioned classical argument. Let there exists a solution $(\dot{\mathbf{U}}, f) \in H^s_*(\Omega_T)\times H^s(\partial\Omega_T)$ to problem (28)–(30), with a given $s\in\mathbb{N}$. Consider now a vector-function

$$\widetilde{\mathbf{U}} = (\tilde{p}^+, \tilde{\mathbf{v}}^+, \widetilde{\mathbf{H}}^+, \widetilde{S}^+, \tilde{p}^-, \tilde{\mathbf{v}}^-, \widetilde{\mathbf{H}}^-, \widetilde{S}^-) \in H^{s+2}_*(\Omega_T)$$

that vanishes in the past and on the boundary $\partial\Omega_T$ satisfies not only conditions (29) (with $f=0$) but also (32), (34) (with $f=0$), where $\tilde{q}^\pm$, $\tilde{v}^\pm_{\rm N}$, and $\widetilde{H}^\pm_{\rm N}$ are defined similarly to $\dot{q}^\pm$, $\dot{v}^\pm_{\rm N}$, and $\dot{H}^\pm_{\rm N}$. Then, we define $\tilde{q}^\pm$, $\tilde{v}^\pm_{\rm n}$, and $\widetilde{H}^\pm_{\rm n}$ in the interior domain Ω_T by using a lifting operator

$$\mathcal{R}_T: \quad H^{s+1}(\partial\Omega_T) \longrightarrow H^{s+2}_*(\Omega_T) \tag{35}$$

from the boundary to the interior:

$$\tilde{q}^\pm = \mathcal{R}_T(\tilde{q}^\pm|_{x_1=0}), \quad \tilde{v}^\pm_{\rm n} = \mathcal{R}_T(\tilde{v}^\pm_{\rm N}|_{x_1=0}), \quad \widetilde{H}^\pm_{\rm n} = \mathcal{R}_T(\widetilde{H}^\pm_{\rm N}|_{x_1=0}).$$

Such a lifting operator exists thanks to the trace theorem [28] for the spaces H^m_*.

Let also $\tilde{v}^\pm_1$ and $\widetilde{H}^\pm_1$ are such that

$$\tilde{v}^\pm_{\rm n} = \tilde{v}^\pm_1 - \tilde{v}^\pm_2\partial_2\widehat{\Psi}^\pm - \tilde{v}^\pm_3\partial_3\widehat{\Psi}^\pm, \quad \widetilde{H}^\pm_{\rm n} = \widetilde{H}^\pm_1 - \widetilde{H}^\pm_2\partial_2\widehat{\Psi}^\pm - \widetilde{H}^\pm_3\partial_3\widehat{\Psi}^\pm.$$

Since we still have a freedom in the choice of "characteristic unknowns" $\widetilde{H}^\pm_2$ and $\widetilde{H}^\pm_3$, we can define them in such a way that they satisfy equations (33) for $a^\pm = \operatorname{div}\tilde{\mathbf{h}}^\pm/\partial_1\widehat{\Phi}^\pm$, where $\tilde{\mathbf{h}}^\pm = (\widetilde{H}^\pm_{\rm n}, \widetilde{H}^\pm_2\partial_1\widehat{\Phi}^\pm, \widetilde{H}^\pm_3\partial_1\widehat{\Phi}^\pm)$. The rest components of $\widetilde{\mathbf{U}}$, i.e., $\tilde{v}^\pm_{2,3}$ and $\widehat{S}^\pm$ can be taken, for example, as zeros.

If $\dot{\mathbf{U}} = \dot{\mathbf{U}}^\natural + \widetilde{\mathbf{U}}$, then $\dot{\mathbf{U}}^\natural$ satisfies

$$\mathbb{L}'_e(\widehat{\mathbf{U}}, \widehat{\boldsymbol{\Psi}})\dot{\mathbf{U}}^\natural = \mathbf{F} \quad \text{in } \Omega_T, \tag{36}$$

$$\mathbb{B}'_e(\widehat{\mathbf{U}}, \hat{f})(\dot{\mathbf{U}}^\natural, f) = 0 \quad \text{on } \partial\Omega_T, \tag{37}$$

where

$$\mathbf{F} = (F_1^+, \dots F_8^+, F_1^-, \dots F_8^-) = \mathbf{f} - \mathbb{L}_e'(\widehat{\mathbf{U}}, \widehat{\boldsymbol{\Psi}})\widetilde{\mathbf{U}}.$$

Moreover, in view of equations (33) for $a^\pm = \operatorname{div} \tilde{\mathbf{h}}^\pm / \partial_1 \widehat{\Phi}^\pm$, conditions (32) for $\widetilde{\mathbf{U}}$ (with $f = 0$), and (34), it follows from (36), (37) that equations (33) and (34) are satisfied for $a^\pm = \operatorname{div} \dot{\mathbf{h}}^{\natural\pm} / \partial_1 \widehat{\Phi}^\pm$ and $g_3^\pm = (\widehat{H}_2^\pm \partial_2 f + \widehat{H}_3^\pm \partial_3 f - \dot{H}_{\rm N}^{\natural\pm} - f\,\partial_1 \widehat{H}_{\rm N}^\pm)|_{x_1=0}$ with the zero right-hand sides ($\mathcal{F}^\pm = \mathcal{G}^\pm = 0$), where $\dot{\mathbf{h}}^{\natural\pm}$ and $\dot{H}_{\rm N}^{\natural\pm}$ are defined for $\dot{\mathbf{U}}^\natural$ similarly to $\dot{\mathbf{h}}^\pm$ and $\dot{H}_{\rm N}^\pm$. Since $\dot{\mathbf{U}}^\natural$ vanishes in the past, the conditions

$$\operatorname{div} \dot{\mathbf{h}}^{\natural\pm} = 0, \quad (\widehat{H}_2^\pm \partial_2 f + \widehat{H}_3^\pm \partial_3 f - \dot{H}_{\rm N}^{\natural\pm} - f\,\partial_1 \widehat{H}_{\rm N}^\pm)|_{x_1=0} = 0 \tag{38}$$

hold for $t < 0$. Then, by standard method of characteristic curves we get that equations (38) are satisfied for all $t \in (-\infty, T]$.

Since $[\partial_1(\cdot)]_{s,*,T} \leq [\cdot]_{s+2,*,T}$ we get the estimate

$$[\mathbb{L}_e'(\widehat{\mathbf{U}}, \widehat{\boldsymbol{\Psi}})\widetilde{\mathbf{U}}]_{s,*,T} \leq C\,[\widetilde{\mathbf{U}}]_{s+2,*,T}. \tag{39}$$

Here and later on C is a constant that can change from line to line, and sometimes we show the dependence of C from another constants. In particular, in this section we focus on the proof of well-posedness based on the a priori estimate in H_*^1 obtained in [43], and for the case $s = 1$ the constant $C = C(K)$ in (39) depends on the constant K from (20). For the general case $s \geq 1$, if we use rough estimates, the constant C in (39) depends on the W_∞^{s+2} norm of the basic state. However, to get a so-called *tame* a priori estimate in Section 3 we will need more delicate estimates than (39). Such estimates will be deduced by using Moser-type inequalities in H_*^s, and the assumption for the basic state will be much weaker than W_∞^{s+2}. We postpone this analysis to Section 3.

Then, taking into account (39), we have

$$[\mathbf{F}]_{s,*,T} \leq C\left\{[\mathbf{f}]_{s,*,T} + \|\mathbf{g}\|_{H^{s+1}(\partial\Omega_T)} + \|(g_3^+, g_3^-)\|_{H^{s+1}(\partial\Omega_T)}\right\}.$$

Using (34), we easily estimate:

$$\begin{aligned}\|(g_3^+, g_3^-)\|_{H^{s+1}(\partial\Omega_T)} &\leq C\|(\mathcal{G}^+, \mathcal{G}^-)\|_{H^{s+1}(\partial\Omega_T)}\\ &\leq C\left\{[\mathbf{f}]_{s+2,*,T} + \|(g_1^+, g_1^-)\|_{H^{s+2}(\partial\Omega_T)}\right\}.\end{aligned}$$

Then, we finally get the estimate

$$[\mathbf{F}]_{s,*,T} \leq C\left\{[\mathbf{f}]_{s+2,*,T} + \|\mathbf{g}\|_{H^{s+2}(\partial\Omega_T)}\right\}. \tag{40}$$

Dropping for convenience the indices $^\natural$ in (36)–(38), we have the problem

$$\mathbb{L}_e'(\widehat{\mathbf{U}}, \widehat{\boldsymbol{\Psi}})\dot{\mathbf{U}} = \mathbf{F} \quad \text{in } \Omega_T, \tag{41}$$

$$\mathbb{B}_e'(\widehat{\mathbf{U}}, \hat{f})(\dot{\mathbf{U}}, f) = 0 \quad \text{on } \partial\Omega_T, \tag{42}$$

$$(\dot{\mathbf{U}}, f) = 0 \quad \text{for } t < 0, \tag{43}$$

where the source term $\mathbf{F}$ defined above obeys estimate (40) and is such that solutions to (41)–(43) satisfy the constraints

$$\operatorname{div} \dot{\mathbf{h}}^+ = 0, \quad \operatorname{div} \dot{\mathbf{h}}^- = 0 \quad \text{in } \Omega_T, \tag{44}$$

$$\begin{cases} \widehat{H}_2^+ \partial_2 f + \widehat{H}_3^+ \partial_3 f - \dot{H}_{\mathrm{N}}^+ - f\,\partial_1 \widehat{H}_{\mathrm{N}}^+ = 0, \\ \widehat{H}_2^- \partial_2 f + \widehat{H}_3^- \partial_3 f - \dot{H}_{\mathrm{N}}^- - f\,\partial_1 \widehat{H}_{\mathrm{N}}^- = 0, \quad \text{on } \partial\Omega_T. \end{cases} \tag{45}$$

We thus have proved the following result.

Lemma 1. *Let problem (41)–(43) is well-posed and its unique solution* $(\dot{\mathbf{U}}, f)$ *belongs to* $H^s_*(\Omega_T) \times H^s(\partial\Omega_T)$ *for* $\mathbf{F} \in H^s_*(\Omega_T)$*, where* $s \in \mathbb{N}$ *is a given number. Then problem (28)–(30) is well-posed in the same function space* $H^s_*(\Omega_T) \times H^s(\partial\Omega_T)$ *for* $(\mathbf{f}, \mathbf{g}) \in H^{s+2}_*(\Omega_T) \times H^{s+2}(\partial\Omega_T)$*.*

Remark 2. As was already noted above, in this section we concentrate on the case $s = 1$. For this case the assumption (20) is enough for the assertion of Lemma 1. For s large enough for which we prove the tame estimate in Section 3 we will assume that $(\widehat{\mathbf{U}}, \nabla_{t,x'} \hat{f}) \in H^{s+2}_*(\Omega_T) \times H^{s+2}(\partial\Omega_T)$ (see estimate (106) that is a "tame counterpart" of (40)).

2.4. Secondary generalized Friedrichs symmetrizer for the MHD system

From now on we focus on the proof of the well-posedness of problem (41)–(43). Recall that solutions to this problem satisfy (44) and (45). In [43], the basic a priori estimate in H^1_* for problem (41), (42) (with $\mathbf{F} = 0$ and nonzero initial data) was constructed thanks to a new symmetric form of the MHD system. This symmetric form is the result of the application of a *secondary generalized Friedrichs symmetrizer* $\mathbb{S} = (\mathcal{S}, \mathbf{R})$ to system (4):

$$\begin{aligned} \mathcal{S}(\mathbf{U}) A_0(\mathbf{U}) \partial_t \mathbf{U} + \sum_{j=1}^{3} \mathcal{S}(\mathbf{U}) A_j(\mathbf{U}) \partial_j \mathbf{U} + \mathbf{R}(\mathbf{U}) \operatorname{div} \mathbf{H} \\ := B_0(\mathbf{U}) \partial_t \mathbf{U} + \sum_{j=1}^{3} B_j(\mathbf{U}) \partial_j \mathbf{U} = 0. \end{aligned} \tag{46}$$

We call this symmetrizer "secondary" because system (4) was already symmetric and the matrices B_α in (46) are again symmetric, and we call it "generalized" because the symmetrizer $\mathbb{S}$ is not just a matrix, but it also contains the vector $\mathbf{R}$ which appearance is due to taking into account the divergent constraint (2).

The concrete form of $\mathcal{S}$ and $\mathbf{R}$ found in [43] is as follows:

$$\mathcal{S}=\begin{pmatrix} 1 & \dfrac{\lambda H_1}{\rho c^2} & \dfrac{\lambda H_2}{\rho c^2} & \dfrac{\lambda H_3}{\rho c^2} & 0 & 0 & 0 & 0 \\ \lambda H_1\rho & 1 & 0 & 0 & -\rho\lambda & 0 & 0 & 0 \\ \lambda H_2\rho & 0 & 1 & 0 & 0 & -\rho\lambda & 0 & 0 \\ \lambda H_3\rho & 0 & 0 & 1 & 0 & 0 & -\rho\lambda & 0 \\ 0 & -\lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{R}=-\lambda\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ H_1 \\ H_2 \\ H_3 \\ 0 \end{pmatrix},$$

where $\lambda=\lambda(\mathbf{U})$ is an arbitrary function. For the concrete form of the matrices B_j in (46) we refer to [43] and write down here only the matrix B_0:

$$B_0=\mathcal{S}A_0=\begin{pmatrix} \dfrac{1}{\rho c^2} & \dfrac{\lambda H_1}{c^2} & \dfrac{\lambda H_2}{c^2} & \dfrac{\lambda H_3}{c^2} & 0 & 0 & 0 & 0 \\ \dfrac{\lambda H_1}{c^2} & \rho & 0 & 0 & -\rho\lambda & 0 & 0 & 0 \\ \dfrac{\lambda H_2}{c^2} & 0 & \rho & 0 & 0 & -\rho\lambda & 0 & 0 \\ \dfrac{\lambda H_3}{c^2} & 0 & 0 & \rho & 0 & 0 & -\rho\lambda & 0 \\ 0 & -\rho\lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\rho\lambda & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\rho\lambda & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly, systems (4) and (46) coincide for $\lambda=0$.

Remark 3. The new symmetric form (46) for the MHD equations was guessed in [43] by considering the magnetoacoustics system and using for it a "compressible" counterpart of the so-called cross-helicity integral $d/dt\left(\int_{\mathbb{R}^3}(\mathbf{v},\mathbf{H})\,d\mathbf{x}\right)=0$ taking place for incompressible MHD. This gives a new conserved integral for the linearized constant coefficient MHD equations and, respectively, a concrete form of the matrix B_0, etc.

The symmetric system (46) is hyperbolic if $B_0>0$ (this also guarantees that $\det\mathcal{S}\neq 0$). The last condition is satisfied if inequalities (5) hold together with the additional requirement [43]

$$\rho\lambda^2<\frac{1}{1+(c_{\mathrm{A}}^2/c^2)}. \tag{47}$$

As was shown in [43], condition (47) guarantees the equivalence of systems (4) and (46) on smooth solutions provided that $\lambda(\mathbf{U})$ is a smooth function of $\mathbf{U}$. The analogous assertion can also be proved for current-vortex sheet solutions, but we need it only for the linearized equations.

Applying the symmetrizer $\mathbb{S}(\widehat{\mathbf{U}})$ to system (41), i.e., multiplying (41) on the left by the matrix $\mathcal{S}(\widehat{\mathbf{U}})$ and adding to the result the vector

$$\begin{pmatrix} \dfrac{\operatorname{div}\dot{\mathbf{h}}^+}{\partial_1\widehat{\Phi}^+}\,\mathbf{R}(\widehat{\mathbf{U}}^+) \\ \dfrac{\operatorname{div}\dot{\mathbf{h}}^-}{\partial_1\widehat{\Phi}^-}\,\mathbf{R}(\widehat{\mathbf{U}}^-) \end{pmatrix},$$

we get

$$\begin{aligned} B_0(\widehat{\mathbf{U}})\partial_t\dot{\mathbf{U}} + \widetilde{B}_1(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})\partial_1\dot{\mathbf{U}} + B_2(\widehat{\mathbf{U}})\partial_2\dot{\mathbf{U}} \\ + B_3(\widehat{\mathbf{U}})\partial_3\dot{\mathbf{U}} + \widetilde{\mathcal{C}}(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})\dot{\mathbf{U}} = \widetilde{\mathbf{F}}(\widehat{\mathbf{U}}), \end{aligned} \tag{48}$$

where $\widetilde{\mathcal{C}}(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}}) = \mathcal{S}(\widehat{\mathbf{U}})\mathcal{C}(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})$, $\widetilde{\mathbf{F}}(\widehat{\mathbf{U}}) = \mathcal{S}(\widehat{\mathbf{U}})\mathbf{F}$,

$$\mathcal{S}(\widehat{\mathbf{U}}) = \operatorname{diag}(\mathcal{S}(\widehat{\mathbf{U}}^+),\mathcal{S}(\widehat{\mathbf{U}}^-)),\quad B_\alpha(\widehat{\mathbf{U}}) = \operatorname{diag}(B_\alpha(\widehat{\mathbf{U}}^+),B_\alpha(\widehat{\mathbf{U}}^-)),$$

$$\mathcal{C}(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}}) = \operatorname{diag}(\mathcal{C}(\widehat{\mathbf{U}}^+,\widehat{\Psi}^+),\mathcal{C}(\widehat{\mathbf{U}}^-,\widehat{\Psi}^-)),$$

$$\widetilde{B}_1(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}}) = \operatorname{diag}(\widetilde{B}_1(\widehat{\mathbf{U}}^+,\widehat{\Psi}^+),\widetilde{B}_1(\widehat{\mathbf{U}}^-,\widehat{\Psi}^-)),$$

$$\widetilde{B}_1(\mathbf{U}^\pm,\Psi^\pm) = \frac{1}{\partial_1\Phi^\pm}\Big(B_1(\mathbf{U}^\pm) - B_0(\mathbf{U}^\pm)\partial_t\Psi^\pm - \sum_{k=2}^{3}B_k(\mathbf{U}^\pm)\partial_k\Psi^\pm\Big).$$

We now prove the equivalence of problems (41)–(43) and (48), (42), (43) in the following sense.

Lemma 2. *Let assumptions (17), (18), (20)–(22), and condition (47) are fulfilled for the basic state (16). Assume also that problems (41)–(43) and (48), (42), (43) have sufficiently smooth solutions. Then, solutions to these problems coincide.*

Proof. If inequalities (17) and condition (47) written for $\widehat{\mathbf{U}}^+$ and $\widehat{\mathbf{U}}^-$ are satisfied, then system (48) is hyperbolic, i.e., $B_0(\widehat{\mathbf{U}}) > 0$. In particular, it means that the matrix $\mathcal{S}(\widehat{\mathbf{U}})$ is nonsingular. Clearly, we have only to prove that from system (48) we can obtain system (41).

Let us multiply (48) on the left by the matrix $\mathcal{S}^{-1}(\widehat{\mathbf{U}})$. Taking into account that $\mathcal{S}^{-1}(\widehat{\mathbf{U}}^\pm)\mathbf{R}(\widehat{\mathbf{U}}^\pm) = \mathbf{R}(\widehat{\mathbf{U}}^\pm)$, we write down the equation for $\dot{\mathbf{H}}^\pm$ contained in the resulting system. The left-hand side of this equation differs from that of (200) (see the proof of Proposition 2 in Appendix A) only by the appearance of the additional term $-\lambda(\widehat{\mathbf{U}}^\pm)\widehat{\mathbf{H}}^\pm\operatorname{div}\dot{\mathbf{h}}^\pm$ in the expression in braces, whereas its right-hand side is $(F_5^\pm, F_6^\pm, F_7^\pm)$.

Reasoning then as in Appendix A, we can obtain linear *homogeneous* equations for $r^\pm = \operatorname{div}\dot{\mathbf{h}}^\pm$ (the left-hand sides of these equations coincide with those of (33) for $\lambda(\widehat{\mathbf{U}}^\pm) = 0$). By standard method of characteristic curves we get (44). Hence, system (48) multiplied on the left by the matrix $\mathcal{S}^{-1}(\widehat{\mathbf{U}})$ coincides with (41). □

2.5. Well-posedness of the linearized problem

Let us introduce the new unknown $\mathbf{V}=(\mathbf{V}^+,\mathbf{V}^-)$, where

$$\mathbf{V}^\pm=(\dot q^\pm,\dot v_{\rm n}^\pm,\dot v_2^\pm,\dot v_3^\pm,\dot H_{\rm n}^\pm,\dot H_2^\pm,\dot H_3^\pm,\dot S^\pm).$$

We have $\dot{\mathbf{U}}=J\mathbf{V}$, with $J=\mathrm{diag}\,(J^+,J^-)$,

$$J^\pm=\begin{pmatrix} 1 & 0 & 0 & 0 & -\widehat{H}_1^\pm & -\widehat{H}_{\tau_1}^\pm & -\widehat{H}_{\tau_2}^\pm & 0\\ 0 & 1 & \partial_2\widehat{\Psi}^\pm & \partial_3\widehat{\Psi}^\pm & 0 & 0 & 0 & 0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&\partial_2\widehat{\Psi}^\pm&\partial_3\widehat{\Psi}^\pm&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\end{pmatrix},$$

$\widehat{H}_{\tau_k}^\pm=(\widehat{\mathbf{H}}^\pm,\boldsymbol{\tau}_k^\pm)$, $k=1,2$, $\boldsymbol{\tau}_1^\pm=(\partial_2\widehat{\Psi}^\pm,1,0)$, and $\boldsymbol{\tau}_2^\pm=(\partial_2\widehat{\Psi}^\pm,0,1)$. Then, systems (41) and (48) are equivalently rewritten as

$$\mathcal{A}_0(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})\partial_t\mathbf{V}+\sum_{k=1}^3\mathcal{A}_k(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})\partial_k\mathbf{V}+\mathcal{A}_4(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})\mathbf{V}=\boldsymbol{\mathcal{F}}(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}}) \qquad (49)$$

and

$$\mathcal{B}_0(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})\partial_t\mathbf{V}+\sum_{k=1}^3\mathcal{B}_k(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})\partial_k\mathbf{V}+\mathcal{B}_4(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})\mathbf{V}=\widetilde{\boldsymbol{\mathcal{F}}}(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}}) \qquad (50)$$

respectively, where $\mathcal{A}_\alpha=J^{\mathsf T}A_\alpha J$, $\quad\mathcal{B}_\alpha=J^{\mathsf T}B_\alpha J\quad(\alpha=0,2,3)$,

$$\mathcal{A}_1=J^{\mathsf T}\widetilde{A}_1J,\quad \mathcal{B}_1=J^{\mathsf T}\widetilde{B}_1J,\quad \boldsymbol{\mathcal{F}}=J^{\mathsf T}\mathbf{F},\quad \widetilde{\boldsymbol{\mathcal{F}}}=J^{\mathsf T}\widetilde{\mathbf{F}},$$

$$\mathcal{A}_4=J^{\mathsf T}\bigl\{\mathcal{C}+A_0\partial_tJ+\widetilde{A}_1\partial_1J+A_2\partial_2J+A_3\partial_2J\bigr\},$$

$$\mathcal{B}_4=J^{\mathsf T}\bigl\{\widetilde{\mathcal{C}}+B_0\partial_tJ+\widetilde{B}_1\partial_1J+B_2\partial_2J+B_3\partial_2J\bigr\},$$

The matrices $A_\alpha=A_\alpha(\widehat{\mathbf{U}})$ and $\widetilde{A}_1=\widetilde{A}_1(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})$ are defined similarly to the matrices $B_\alpha=B_\alpha(\widehat{\mathbf{U}})$ and $\widetilde{B}_1=\widetilde{B}_1(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})$ in (48).

The boundary matrix $\mathcal{A}_1$ in system (49) has the form

$$\mathcal{A}_1=\mathcal{A}+\mathcal{A}_{(0)},\quad \mathcal{A}=\mathrm{diag}\Bigl(\frac{1}{\partial_1\widehat{\Phi}^+}\,\mathcal{E}_{12},\frac{1}{\partial_1\widehat{\Phi}^-}\,\mathcal{E}_{12}\Bigr),\quad \mathcal{A}_{(0)}|_{x_1=0}=0, \quad (51)$$

where $\mathcal{E}_{ij}$ is the symmetric matrix which (ij)th and (ji)th elements equal to 1 and others are zero. The explicit form of $\mathcal{A}_{(0)}$ is of no interest, and it is only important that, in view of constraints (15) and (18) for the basic state (16), $\mathcal{A}_{(0)}|_{x_1=0}=0$. Therefore, the boundary matrix $\mathcal{A}_1$ on the boundary $x_1=0$, $\mathcal{A}_1|_{x_1=0}=\mathrm{diag}\,(\mathcal{E}_{12},-\mathcal{E}_{12})$, is of constant rank 4 and has two positive and two negative eigenvalues.

That is, (49) is a symmetric hyperbolic system with characteristic boundary of constant multiplicity (in the sense of Rauch [30]). Moreover, we have the correct number of boundary conditions in (42), that is two plus one, because the first or the second boundary condition in (42) is needed to find f. It is also noteworthy that because of (44) not only $\dot{q}^{\pm}$ and $\dot{v}_{\mathrm{n}}^{\pm}$ but also $\dot{H}_{\mathrm{n}}^{+}$ and $\dot{H}_{\mathrm{n}}^{-}$ are "noncharacteristic" unknowns. For the "noncharacteristic" part of the vector $\mathbf{V}$,

$$\mathbf{V}_{\mathrm{n}} := (\mathbf{V}_{\mathrm{n}}^{+}, \mathbf{V}_{\mathrm{n}}^{-}), \quad \mathbf{V}_{\mathrm{n}}^{\pm} = (\dot{q}^{\pm}, \dot{v}_{\mathrm{n}}^{\pm}, \dot{H}_{\mathrm{n}}^{\pm}), \tag{52}$$

we expect to have a better control on the normal (x_1-) derivatives.

Concerning the boundary matrix $\mathcal{B}_1$ in system (50), we will only need its explicit form on the boundary:

$$\mathcal{B}_1|_{x_1=0} = \mathrm{diag}\,\big(\mathcal{B}(\widehat{\mathbf{U}}^{+}|_{x_1=0}), -\mathcal{B}(\widehat{\mathbf{U}}^{-}|_{x_1=0})\big), \quad \mathcal{B}(\mathbf{U}) = \mathcal{E}_{12} - \lambda(\mathbf{U})\mathcal{E}_{15}.$$

Let us now define the function $\lambda = \lambda(\mathbf{U})$. Unlike [43], we shall assume that the stability condition for variable coefficients is satisfied only on the boundary at each point of the nonplanar current-vortex sheet. This is possible thanks to the use of a kind of cut-off function for $\lambda(\mathbf{U})$. Roughly speaking, the hyperbolicity condition (47) can be imposed only on the boundary.

That is, our choice of λ differs a little bit from that in [43]:

$$\lambda(\widehat{\mathbf{U}}^{\pm}) := \eta(x_1)\lambda^{\pm}(t, \mathbf{x}') \tag{53}$$

where $\eta(x_1) \in C^{\infty}(\mathbb{R}_+)$ is such a rapidly decreasing function that $\eta(0) = 1$ and $\eta(x_1) = 0$ for $x_1 > \varepsilon$, with a sufficiently small constant ε; the functions λ^{+} and λ^{-} are chosen exactly as in [43], i.e., at each point $(t, \mathbf{x}') \in \partial\Omega_T$ we set $\lambda^{\pm}(t, \mathbf{x}') = 0$ if the jump $[\hat{\mathbf{v}}'](t, \mathbf{x}') = 0$, and, otherwise, we choose $\lambda^{+}(t, \mathbf{x}')$ and $\lambda^{-}(t, \mathbf{x}')$ such that

$$[\hat{\mathbf{v}}' - \lambda\widehat{\mathbf{H}}'] = 0 \tag{54}$$

(*it is possible* thanks to the second inequality in assumption (19)). Here $\hat{\mathbf{v}}'^{\pm} = (\hat{v}_2^{\pm}, \hat{v}_3^{\pm})$, $\widehat{\mathbf{H}}'^{\pm} = (\widehat{H}_2^{\pm}, \widehat{H}_3^{\pm})$, $[\hat{\mathbf{v}}'] = (\hat{\mathbf{v}}'^{+} - \hat{\mathbf{v}}'^{-})|_{x_1=0}$, $[\lambda] = \lambda^{+} - \lambda^{-}$, etc.

One can show that for the choice taken in (53), (54) condition (47) guaranteeing (together with (17)) the hyperbolicity of the symmetric system (48) (or (50) is satisfied on the boundary $\partial\Omega_T$ for the basic state (16) if and only if the stability condition in (19) holds at each point $(t, \mathbf{x}') \in \partial\Omega_T$. By continuity of state (16) the stability condition in (19) is still satisfied in a small neighborhood $0 \leq x_1 \leq \varepsilon$. Hence, thanks to the properties of $\eta(x_1)$ and the fact that (47) holds for $\lambda = 0$, condition (47) is fulfilled for state (16) in the *whole* domain Ω_T if and only if the first inequality in (19) holds, i.e., the stability condition (9) holds at each point of the straightened unperturbed current-vortex sheet.

From now on we suppose that the function $\lambda(\mathbf{U})$ is given (53), (54).

Lemma 3. *Let assumptions (17)–(22) are fulfilled for the basic state (16). If we set formally that* $(\partial_1\hat{v}_{\rm N}^{\pm})|_{x_1=0} = (\partial_1\widehat{H}_{\rm N}^{\pm})|_{x_1=0} = 0$ *(i.e., we omit the zero-order terms in* f *in (42) and (45)), then the boundary conditions (42) for the symmetric hyperbolic system (50) are maximally dissipative.*

Proof. Recall that assumptions (17) and (19) imply that $B_0(\widehat{\mathbf{U}}) > 0$ and, hence, $\mathcal{B}_0(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}}) > 0$, i.e., the symmetric system (50) is hyperbolic. It follows from the explicit form of the boundary matrix $\mathcal{B}_1(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})$ at $x_1 = 0$ that

$$(\mathcal{B}_1\mathbf{V},\mathbf{V})|_{x_1=0} = 2[\dot{q}(\dot{v}_{\rm N} - \lambda\dot{H}_{\rm N})].$$

By virtue of (42), (45) and the artificial assumption about $\hat{v}_{\rm N}^{\pm}$ and $\widehat{H}_{\rm N}^{\pm}$,

$$[\dot{q}(\dot{v}_{\rm N} - \lambda\dot{H}_{\rm N})] = \dot{q}^{+}([\hat{\mathbf{v}}' - \lambda\widehat{\mathbf{H}}'], \nabla_{\mathbf{x}'}f).$$

Then, thanks to (54)

$$(\mathcal{B}_1\mathbf{V},\mathbf{V})|_{x_1=0} = 0,$$

i.e., the boundary conditions are dissipative and even *conservative*. Moreover, since the number of boundary conditions in (42) is correct, the property of maximality is fulfilled. □

Remark 4. Because of the presence of zero-order terms in f in the boundary conditions, it is *impossible* to get the classical L_2 estimate (with no loss of derivatives from $\mathbf{F}$) for solutions of problem (41)–(43). The basic a priori estimate in H^1_* was proved in [43] by using the result of Lemma 3 and some standard manipulations with lower-order terms appearing in the boundary integral (by passing to the volume integral and integration by parts).

We are now in a position to prove the well-posedness of problem (41)–(43).

Lemma 4. *Let assumptions (17)–(22) are fulfilled for the basic state (16). Let also there exists a constant* $K_1 > 0$ *that* $\|\partial_1\widehat{\mathbf{U}}\|_{W^2_\infty(\Omega_T)} \le K_1$.[6] *Then for all* $\mathbf{F} \in H^1_*(\Omega_T)$ *that vanish in the past problem (41)–(43) has a solution* $(\dot{\mathbf{U}}, f) \in H^1_*(\Omega_T) \times H^1(\partial\Omega_T)$. *Moreover, this solution obeys the a priori estimate*

$$|||\dot{\mathbf{U}}(t)|||_{1,*} + \|f(t)\|_{H^1(\mathbb{R}^2)} \le Ce^{Ct}\,[\mathbf{F}]_{1,*,t} \tag{55}$$

for all $t \in [0,T]$. *Here* $C = C(K, K_1) > 0$ *is a constant independent of the data* $\mathbf{F}$.

Proof. The a priori estimate in H^1_* was proved in [43] for the nonzero initial data but for $\mathbf{F} = 0$. Because of the result of Lemma 3 it is clear that we can easily get the same estimate with no loss of derivatives from the nonzero $\mathbf{F}$. That is, we get the a priori estimate (55). To prove the existence of solutions to problem (41)–(43) we use the idea of the works [29,

[6] This additional assumption is automatically fulfilled under the conditions imposed on the basic state in the tame estimate in Section 3.

34] where the existence of solutions of symmetric hyperbolic systems with characteristic boundary of constant multiplicity was established by making use of approximations via noncharacteristic regularization. In fact, problem (41), (42) differs from that studied in [29,34] only by the the presence of the unknown function f in the boundary conditions. Note also that, in view of (43), we do not need to care for compatibility conditions for our problem.

As follows from Lemma 2, instead of (41) we can consider system (48) that is equivalently rewritten as (50). Following [29,34], we now consider the so-called noncharacteristic regularization of system (50):

$$\mathcal{B}_0\partial_t\mathbf{V} + \sum_{k=1}^{3}\mathcal{B}_k\partial_k\mathbf{V} + \mathcal{B}_4\mathbf{V} - \varepsilon\partial_1\mathbf{V} = \widetilde{\boldsymbol{\mathcal{F}}} \quad \text{in } \Omega_T, \tag{56}$$

where ε is a small positive constant, the unknown is again denoted by $\mathbf{V}$, and for short we do not indicate the dependence of $\mathcal{B}_\alpha$ and $\widetilde{\boldsymbol{\mathcal{F}}}$ on the basic state. For problem (56), (42) the boundary $x_1 = 0$ is noncharacteristic, and the number of boundary conditions in (42) is still correct, i.e., the property of maximality takes place. From the problem (56), (42), (43) we can still deduce constraints (44) and (45).

Actually, the boundary conditions (42) with the omitted zero-order terms in f are not just maximally dissipative for system (56). They are even *strictly* dissipative. Using this fact, we can easily treat the zero-order terms in f and get even a L_2 a priori estimate for (56), (42), (43). This estimate will be however *not uniform* in ε. Therefore, we prefer to use the same arguments as in [43] and get an a priori estimate for a prolonged system. Clearly, this estimate is (see also [45])

$$\begin{aligned} |||\dot{\mathbf{U}}(t)|||_{1,*} + \varepsilon|||\dot{\mathbf{U}}(t)|||_{H^1(\mathbb{R}^3_+)} + \varepsilon\|\dot{\mathbf{U}}|_{x_1=0}\|_{H^1(\partial\Omega_t)} \\ + \|f(t)\|_{H^1(\mathbb{R}^2)} + \varepsilon\|f\|_{H^2(\partial\Omega_t)} \leq Ce^{Ct}\,[\mathbf{F}]_{1,*,t}. \end{aligned} \tag{57}$$

In estimate (57) we control the trace of solution in the high norm and, therefore, do not "lose derivatives from the front" f. This estimate is internally the same as was deduced in [25] for uniformly stable shock waves (see also [45]). Taking into account the recent result in [26], we can still apply Kreiss' theory for the MHD system, that is a symmetric hyperbolic system with variable multiplicities. Therefore, since estimate (57) is with no loss of derivatives, it is equivalent to the fact that problem (56), (42) with frozen coefficients satisfies the *uniform* Kreiss-Lopatinski condition.

Moreover, the boundary conditions (42) have almost the same form as the linearized boundary conditions for uniformly stable shock waves. The only difference that the boundary conditions (42) contain zero-order terms in f. It is however not so important because we have only to modify a little bit the definition of a dual problem for (56), (42). In this connection, we refer to [8] where the presence of the mentioned zero-order terms was taken into account under the definition of the dual problem. With the reference to [25], we can now conclude that the uniformly stable problem (56), (42), (43) is well-posed in $H^1(\Omega_T)\times H^2(\partial\Omega_T)$ (actually, even in $H^s(\Omega_T)\times H^{s+1}(\partial\Omega_T)$ with $s\geq 1$). Alternatively, for this problem, having estimate (57), we could also straightforwardly use the classical argument of Lax & Phillips [21].

Now we pass to the limit for $\varepsilon\to 0$. By passing to the limit in (56) we can easily show the existence of $(\mathbf{V},f)\in H^1_*(\Omega_T)\times H^1(\partial\Omega_T)$ that is a solution to problem (50), (42), (43). Then, the corresponding $(\dot{\mathbf{U}},f)\in H^1_*(\Omega_T)\times H^1(\partial\Omega_T)$ is a solution to (41)–(43). □

Taking into account Lemma 1 and estimate (40), from Lemma 4 we conclude the following theorem.

Theorem 2. *Let all the assumptions of Lemma 4 are fulfilled for the basic state (16). Then for all* $(\mathbf{f},\mathbf{g})\in H^3_*(\Omega_T)\times H^3(\partial\Omega_T)$ *that vanish in the past problem (28)–(30) has a unique solution* $(\dot{\mathbf{U}},f)\in H^1_*(\Omega_T)\times H^1(\partial\Omega_T)$. *This solution obeys the a priori estimate*

$$[\dot{\mathbf{U}}]_{1,*,T}+\|f\|_{H^1(\partial\Omega_T)}\leq C\left\{[\mathbf{f}]_{3,*,T}+\|\mathbf{g}\|_{H^3(\partial\Omega_T)}\right\}, \tag{58}$$

where $C=C(K,K_1,T)>0$ *is a constant independent of the data* $(\mathbf{f},\mathbf{g})$.

Remark 5. Strictly speaking, the uniqueness of the solution to problem (28)–(30) follows from estimate (58), provided that our solution belongs to $H^2_*(\Omega_T)\times H^2(\partial\Omega_T)$. We do not present here a formal proof of the existence of solutions having an arbitrary degree of smoothness, and we shall suppose that the existence result of Theorem 2 is also valid for the function spaces $H^s_*(\Omega_T)\times H^s(\partial\Omega_T)$, with $s\geq 2$. In this case exact assumptions about the regularity of the basic state will be made in Section 3, where we prove a tame a priori estimate in $H^s_*(\Omega_T)\times H^s(\partial\Omega_T)$ with s large enough (see also Remark 2 above).

3. Tame estimate for the linearized problem

We now want to derive a tame a priori estimate in H^s_* for problem (41)–(43), with s large enough. This tame estimate being, roughly speaking, linear in how norms (that are multiplied by low norms) will be with no loss of derivatives from the source term $\mathbf{F}$ and with a fixed loss of derivatives with respect to the coefficients, i.e., with respect to the basic state (16). Then, using a “tame counterpart” of estimate (40) (see (106)) we shall get a tame estimate for our main linear problem (28)–(30) with the loss of two derivatives from $(\mathbf{f},\mathbf{g})$ (in the sense of $H^s_*(\Omega_T)\times H^s(\partial\Omega_T)$ norms).

The main idea for deriving the tame estimate for (41)–(43) is the use of the new symmetric form (46) of the MHD system (i.e., the use of a dissipative 0-symmetrizer [45], see Lemma 3), and the main tool is the application of Moser-type inequalities following from a corresponding Gagliardo-Nirenberg inequality for the anisotropic weighted Sobolev space H^s_*. These inequalities and necessary (for our goals) embedding theorems for H^s_* are given in Appendix B. Since the Moser-type inequalities for H^s_* are different for even and odd s (see Appendix B), we first prove the tame estimate for even s and then briefly discuss how to extend it (with minor modifications) to the case when s is odd.

Later on, if even not exactly said, we suppose that all the assumptions (17)–(22) are fulfilled for the basic state (16).

3.1. Estimate of the normal derivative of the "noncharacteristic" unknown

Let s is a positive even number. We first obtain an estimate of $\partial_1\mathbf{V}_{\rm n}$ in H_*^{s-1}, where $\mathbf{V}_{\rm n}$ is the "noncharacteristic" unknown (52). Taking into account equations (44) and decomposition (51) for the boundary matrix $\mathcal{A}_1$ of system (49), we have

$$\partial_1\mathbf{V}_{\rm n} = \begin{pmatrix} \{\boldsymbol{\mathcal{K}}\}_{1,2} \\ -\partial_2(\dot{H}_2^-\partial_1\widehat{\Phi}^+) - \partial_3(\dot{H}_3^+\partial_1\widehat{\Phi}^+) \\ \{\boldsymbol{\mathcal{K}}\}_{9,10} \\ -\partial_2(\dot{H}_2^-\partial_1\widehat{\Phi}^-) - \partial_3(\dot{H}_3^-\partial_1\widehat{\Phi}^-) \end{pmatrix}, \tag{59}$$

where $\{\boldsymbol{\mathcal{K}}\}_{i,j} \in \mathbb{R}^2$ is the vector composed from the ith and the jth components of the vector

$$\boldsymbol{\mathcal{K}} := \widetilde{\mathcal{A}}\Big(\boldsymbol{\mathcal{F}} - \mathcal{A}_0\partial_t\mathbf{V} - \sum_{k=2}^{3}\mathcal{A}_k\partial_k\mathbf{V} - \mathcal{A}_4\mathbf{V} - \mathcal{A}_{(0)}\partial_1\mathbf{V}\Big), \tag{60}$$

with $\widetilde{\mathcal{A}} = \operatorname{diag}\big((\partial_1\widehat{\Phi}^+)\mathcal{E}_{12}, (\partial_1\widehat{\Phi}^-)\mathcal{E}_{12}\big)$ (recall that $\partial_1\widehat{\Phi}^\pm = \pm 1 + \partial_1\widehat{\Psi}^\pm$).

Using inequalities (204) and (205), we estimate the following terms containing in the right-hand side in (60):

$$\begin{aligned}[]
[\widetilde{\mathcal{A}}\boldsymbol{\mathcal{F}}]^2_{s-1,*,t} &\le [\widetilde{\mathcal{A}}J^{\mathsf{T}}\mathbf{F}]^2_{s,*,T} \\
&\le C(K)\left\{[\mathbf{F}]^2_{s,*,T} + \|\mathbf{F}\|^2_{L_\infty(\Omega_T)}\big(1 + [\widehat{\mathbf{W}}]^2_{s,*,T}\big)\right\},
\end{aligned} \tag{61}$$

$$[\widetilde{\mathcal{A}}\mathcal{A}_4\mathbf{V}]^2_{s-1,*,t} \le C(K)\left\{[\mathbf{V}]^2_{s,*,t} + \|\dot{\mathbf{U}}\|^2_{L_\infty(\Omega_T)}\big(1 + [\widehat{\mathbf{W}}]^2_{s+2,*,T}\big)\right\}. \tag{62}$$

Recall that $\widehat{\mathbf{W}} = (\widehat{\mathbf{U}}, \nabla_{t,\rm x}\widehat{\boldsymbol{\Psi}})$ and $C(K)$ is a positive constant depending on the constant K from (20). Likewise, we estimate:

$$\begin{aligned}[]
[\widetilde{\mathcal{A}}\mathcal{A}_j\partial_j\mathbf{V}]^2_{s-1,*,t} &\le C[\widetilde{\mathcal{A}}\mathcal{A}_j\mathbf{V}]^2_{s,*,t} \\
\le C(K)\left\{[\mathbf{V}]^2_{s,*,t} + \|\dot{\mathbf{U}}\|^2_{L_\infty(\Omega_T)}\big(1 + [\widehat{\mathbf{W}}]^2_{s,*,T}\big)\right\}&, \quad j = 0, 2, 3.
\end{aligned} \tag{63}$$

Here and below $\partial_0 := \partial_t$ and we shall use the notations $D^\beta = \partial_\star^\alpha\partial_1^k$, with a multi-index $\beta = (\alpha, k) = (\alpha_0, \dots, \alpha_3, k)$ for which

$$\langle\beta\rangle := |\alpha| + 2k.$$

Using again (204), (205), and the important fact that $\mathcal{A}_{(0)}|_{x_1=0} = 0$, we now estimate the last term in the right-hand side in (60):

$$[\widetilde{\mathcal{A}}\mathcal{A}_{(0)}\partial_1\mathbf{V}]^2_{s-1,*,t} \le C(\varSigma_1 + \varSigma_2), \tag{64}$$

where

$$\varSigma_1 = \sum_{1\le\langle\beta\rangle\le 2} [D^\beta(\widetilde{\mathcal{A}}\mathcal{A}_{(0)})\partial_1\mathbf{V}]^2_{s-1-\langle\beta\rangle,*,t},$$

$$\varSigma_2 = \sum_{\langle\beta\rangle\le s-1} \|\widetilde{\mathcal{A}}\mathcal{A}_{(0)}D^\beta\partial_1\mathbf{V}\|^2_{L_2(\varOmega_t)},$$

$$\varSigma_1 \le C(K)\left\{[\partial_1\mathbf{V}]^2_{s-2,*,t} + \|\dot{\mathbf{U}}\|^2_{W^1_\infty(\varOmega_T)}\big(1 + [\widehat{\mathbf{W}}]^2_{s,*,T}\big)\right\}, \tag{65}$$

$$\varSigma_2 \le \varSigma_2' + \varSigma_2'', \tag{66}$$

$$\begin{aligned}
\varSigma_2' &= \sum_{\langle\beta\rangle\le s-1} \|\eta\widetilde{\mathcal{A}}\mathcal{A}_{(0)}D^\beta\partial_1\mathbf{V}\|^2_{L_2(\varOmega_t)} \\
&= \sum_{\langle\beta\rangle\le s-1} \Big\| \int_0^{x_1} \partial_1\big(\widetilde{\mathcal{A}}\mathcal{A}_{(0)}(t,\xi,\mathbf{x}')\big)d\xi\, D^\beta\partial_1\mathbf{V}\Big\|^2_{L_2(\varOmega_t^\varepsilon)} \\
&\le C\|\partial_1(\widetilde{\mathcal{A}}\mathcal{A}_{(0)})\|^2_{L_\infty(\varOmega_T)} \sum_{\langle\beta\rangle\le s-1} \|x_1 D^\beta\partial_1\mathbf{V}\|^2_{L_2(\varOmega_t^\varepsilon)} \\
&\le C(K)[\mathbf{V}]^2_{s,*,t}.
\end{aligned} \tag{67}$$

$$\varSigma_2' = \sum_{\langle\beta\rangle\le s-1} \|(1-\eta)\widetilde{\mathcal{A}}\mathcal{A}_{(0)}D^\beta\partial_1\mathbf{V}\|^2_{L_2(\varOmega_t)} \le C(K)[\mathbf{V}]^2_{s,*,t}, \tag{68}$$

where $\varOmega_T^\varepsilon := \varOmega_T \cap \{x_1 < \varepsilon\}$ and the function $\eta = \eta(x_1)$ is the same as in (53) (in particular, $(1-\eta)|_{x_1=0} = 0$).

Combining (61)–(68), from (59) and (60) we get the following result.

Proposition 3. *The estimate*

$$[\partial_1\mathbf{V}_{\rm n}]^2_{s-1,*,t} \le C(K)\mathcal{M}(t), \tag{69}$$

with

$$\mathcal{M}(t) = \Big\{[\mathbf{V}]^2_{s,*,t} + \|\dot{\mathbf{U}}\|^2_{W^1_\infty(\varOmega_T)}\big(1 + [\widehat{\mathbf{W}}]^2_{s+2,*,T}\big) + [\mathbf{F}]^2_{s,*,T} + \|\mathbf{F}\|^2_{L_\infty(\varOmega_T)}\big(1 + [\widehat{\mathbf{W}}]^2_{s,*,T}\big)\Big\},$$

holds for problem (41)–(43) for all $t \le T$ if $s \ge 2$ is even.

Later on we will also need the following "layerwise" estimate of $\partial_1\mathbf{V}_{\rm n}$.

Proposition 4. *The estimate*

$$|||\partial_1\mathbf{V}_{\rm n}(t)|||^2_{s-1,*} \le C(K)\left(|||\mathbf{V}(t)|||^2_{s,*} + \mathcal{M}(t)\right) \tag{70}$$

holds for problem (41)–(43) for all $t \le T$, provided that $s \ge 2$ is even.

Proof. We again estimate separately all the terms in the right-hand side of (60). For example, using the elementary inequality

$$|||u(t)|||^2_{s-1,*} \le C[u]^2_{s,*,t}\,, \tag{71}$$

we get

$$|||(\widetilde{\mathcal{A}}\boldsymbol{\mathcal{F}})(t)|||^2_{s-1,*} \le C[\widetilde{\mathcal{A}}J^{\mathsf{T}}\mathbf{F}]^2_{s,*,T}$$

and then refer to (61). We first estimate $\widetilde{\mathcal{A}}\mathcal{A}_{(0)}\partial_1\mathbf{V}$ as follows:

$$|||(\widetilde{\mathcal{A}}\mathcal{A}_{(0)}\partial_1\mathbf{V})(t)|||_{s-1,*} \le C(\Sigma_1+\Sigma_2+\Sigma_3),$$

where

$$\Sigma_1 = \sum_{\langle\beta\rangle\le s-1} \|(\widetilde{\mathcal{A}}\mathcal{A}_{(0)}D^\beta\partial_1\mathbf{V})(t)\|^2_{L_2(\mathbb{R}^3_+)},$$

$$\Sigma_2 = \sum_{\substack{|\beta'|+\langle\beta''\rangle\le s-1\\ |\beta'|=1}} \|(\partial_\star^{\beta'}(\widetilde{\mathcal{A}}\mathcal{A}_{(0)})D^{\beta''}\partial_1\mathbf{V})(t)\|^2_{L_2(\mathbb{R}^3_+)}$$

$$\Sigma_3 = \sum_{\langle\beta\rangle=2} |||(D^\beta(\widetilde{\mathcal{A}}\mathcal{A}_{(0)})\partial_1\mathbf{V})(t)|||^2_{m-3,*}.$$

Then, by applying the same arguments as in (66)–(68), one gets

$$\Sigma_1 \le C(K)|||\mathbf{V}(t)|||^2_{s,*}.$$

The second sum is easily estimated as

$$\Sigma_2 \le C(K)|||\partial_1\mathbf{V}(t)|||^2_{s-2,*} \le C(K)|||\mathbf{V}(t)|||^2_{s,*}$$

Using first (71) and then inequalities (204), (205), we estimate Σ_3 from above by $C(K)\mathcal{M}(t)$. The estimation of the rest terms in the right-hand side of (60) is even simpler, and we get (70). $\square$

3.2. Estimate of weighted derivatives

We now get an estimate of weighted derivatives, i.e., the terms in the form $D^\beta\mathbf{V} = \partial_\star^\alpha\partial_1^k\mathbf{V}$, with $\alpha_1 \ne 0$ and $\langle\beta\rangle = |\alpha| + 2k \le s$. To estimate such terms we do not need boundary conditions. Observe that in the differential operator $\partial_\star^\alpha$ we can replace $(\sigma\partial_1)^{\alpha_1}$ by $\sigma^{\alpha_1}\partial_1^{\alpha_1}$ because the corresponding norms are equivalent [29]. That is, we can suppose that $\partial_\star^\alpha = \sigma^{\alpha_1}\partial^\alpha_{t,x}$, where $\partial^\alpha_{t,x} := \partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$.

Proposition 5. *The following estimate holds for problem (41)–(43) for all $t \le T$:*

$$\sum_{\langle\beta\rangle\le s,\ \alpha_1\ne 0} \|D^\beta\mathbf{V}(t)\|^2_{L_2(\mathbb{R}^3_+)} \le C(K)\mathcal{M}(t), \tag{72}$$

where $s \ge 2$ is even and $\mathcal{M}(t)$ is given in Proposition 3.

Proof. By applying to system (49) the operator D^β, using standard arguments of the energy method, and taking into account the identity

$$\sigma^n\partial_1^{m+1}=\partial_1(\sigma^n\partial_1^m)-n\sigma'\sigma^{n-1}\partial_1^m$$

and the fact that $D^\beta\mathbf{V}|_{x_1=0}=0$ (with $\alpha_1\neq 0$), one gets

$$\int_{\mathbb{R}^3_+}(\mathcal{A}_0D^\beta\mathbf{V},D^\beta\mathbf{V})d\mathbf{x}=\int_{\Omega_t}\big((\mathrm{div}\mathbb{A}\,D^\beta\mathbf{V}+2\boldsymbol{\mathcal{R}}),D^\beta\mathbf{V}\big)\,d\mathbf{x}d\tau,$$

where

$$\mathrm{div}\mathbb{A}=\sum_{j=0}^{3}\partial_j\mathcal{A}_j\quad(\partial_0:=\partial_t),\qquad \boldsymbol{\mathcal{R}}=D^\beta\boldsymbol{\mathcal{F}}+\boldsymbol{\mathcal{R}}_0+\boldsymbol{\mathcal{R}}_1,$$

$$\boldsymbol{\mathcal{R}}_0=\alpha_1\sigma'\mathcal{A}_1\sigma^{\alpha_1-1}\partial^\alpha_{t,x}\partial_1^k\mathbf{V},\quad \boldsymbol{\mathcal{R}}_1=-\sum_{j=0}^{3}[D^\beta,\mathcal{A}_j]\partial_j\mathbf{V}-D^\beta(\mathcal{A}_4\mathbf{V}),$$

and we use the notation of commutator: $[a,b]c:=a(bc)-b(ac)$. Since $\mathcal{A}_0>0$, we come to the inequality

$$\|D^\beta\mathbf{V}(t)\|^2_{L_2(\mathbb{R}^3_+)}\leq C(K)\left([\mathbf{V}]^2_{s,*,t}+\|\boldsymbol{\mathcal{R}}\|^2_{L_2(\Omega_t)}\right). \tag{73}$$

Using (204), (205), we easily estimate:

$$\begin{aligned}&\|D^\beta(\mathcal{A}_4\mathbf{V})\|^2_{L_2(\Omega_t)}\\&\qquad\leq C(K)\left\{[\mathbf{V}]^2_{s,*,t}+\|\dot{\mathbf{U}}\|^2_{L_\infty(\Omega_T)}\big(1+[\widehat{\mathbf{W}}]^2_{s+2,*,T}\big)\right\},\end{aligned}\tag{74}$$

$$\|D^\beta\boldsymbol{\mathcal{F}}\|^2_{L_2(\Omega_t)}\leq C(K)\left\{[\mathbf{F}]^2_{s,*,T}+\|\mathbf{F}\|^2_{L_\infty(\Omega_T)}\big(1+[\widehat{\mathbf{W}}]^2_{s,*,T}\big)\right\}. \tag{75}$$

In view of decomposition (51), reasoning as in (66)–(68) and utilizing estimate (69), one has

$$\begin{aligned}\|\boldsymbol{\mathcal{R}}_0\|^2_{L_2(\Omega_t)}\leq C(K)\Big\{&[\partial_1\mathbf{V}_{\rm n}]^2_{s-1,*,t}+\|x_1\sigma^{\alpha_1-1}\partial^\alpha_{t,x}\partial_1^k\mathbf{V}\|_{L_2(\Omega_t^\varepsilon)}\\&+[\mathbf{V}]^2_{s,*,t}+\|\dot{\mathbf{U}}\|^2_{L_\infty(\Omega_T)}\big(1+[\widehat{\mathbf{W}}]^2_{s,*,T}\big)\Big\}\leq C(K)\mathcal{M}(t).\end{aligned}\tag{76}$$

The commutators can be preliminary estimated as follows:

$$\begin{aligned}\big\|[D^\beta,\mathcal{A}_j]\partial_j\mathbf{V}\big\|^2_{L_2(\Omega_t)}\leq&\sum_{\substack{|\alpha'|+\langle\beta'\rangle\leq s\\|\alpha'|=1}}\|\partial_\star^{\alpha'}(\mathcal{A}_j)D^{\beta'}\partial_j\mathbf{V}\|^2_{L_2(\Omega_t)}\\&+C\sum_{\langle\beta'\rangle=2}[D^{\beta'}(\mathcal{A}_j)\partial_j\mathbf{V}]^2_{s-2,*,t}\,.\end{aligned}\tag{77}$$

Applying then (204), (205), for tangential derivatives, i.e., for the cases $j = 0, 2, 3$ we have

$$\big\|[D^\beta, \mathcal{A}_j]\partial_j \mathbf{V}\big\|^2_{L_2(\Omega_t)} \le C(K)\left\{[\mathbf{V}]^2_{s,*,t} + \|\dot{\mathbf{U}}\|^2_{W^1_\infty(\Omega_T)}\big(1 + [\widehat{\mathbf{W}}]^2_{s,*,T}\big)\right\}.$$

Concerning the rest commutator (the case $j = 1$), to estimate the first sum in (77) we use decomposition (51), reason as in (66)–(68) and (76), and apply estimate (69). The second sum in (77) is easily estimated, as for the cases $j = 0, 2, 3$, by making use inequalities (204), (205) (recall also that $[\partial_1(\cdot)]_{s-2,*,t} \le [\cdot]_{s,*,t}$). That is, in view of (74), we finally obtain the estimate

$$\|\boldsymbol{\mathcal{R}}_1\|^2_{L_2(\Omega_t)} \le C(K)\mathcal{M}(t). \tag{78}$$

At last, by summing up estimates (75), (76), (78) and recalling inequality (73), one gets estimate (72). □

3.3. Estimate of nonweighted normal derivatives

Let us estimate nonweighted normal derivatives of $\mathbf{V}$, i.e., the terms in the form $D^\beta \mathbf{V}$, with $\alpha_1 = 0$ and $k \ge 1$. Since for this case $D^\beta = \partial^\alpha_{\tan}\partial_1^k$ ($\partial^\alpha_{\tan} := \partial_t^{\alpha_0}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$) and $|\beta| = |\alpha| + k \le s - k < s$, we again do not need boundary conditions for estimating such terms.

Proposition 6. *The estimate*

$$\sum_{\substack{\langle\beta\rangle\le s\\ \alpha_1=0,\ k\ge1}} \|D^\beta \mathbf{V}(t)\|^2_{L_2(\mathbb{R}^3_+)} \le C(K)\mathcal{M}(t) \tag{79}$$

holds for problem (41)–(43) for all $t \le T$, with even $s \ge 2$ and $\mathcal{M}(t)$ given in Proposition 3.

Proof. Using the same arguments as in the proof of Proposition 5, we obtain the inequality

$$\|D^\beta \mathbf{V}(t)\|^2_{L_2(\mathbb{R}^3_+)} \le C(K)\left([\mathbf{V}]^2_{s,*,t} + \|\boldsymbol{\mathcal{R}}\|^2_{L_2(\Omega_t)} + \mathcal{J}(t)\right),$$

where

$$\mathcal{J}(t) = \left|\int_{\partial\Omega_t} (\mathcal{A}_1 D^\beta \mathbf{V}, D^\beta \mathbf{V})|_{x_1=0}\, d\mathbf{x}' d\tau\right| = \left|\int_{\partial\Omega_t} \left[D^\beta \dot{q}\, D^\beta \dot{v}_{\rm n}\right]\, d\mathbf{x}' d\tau\right|$$

and the vector $\boldsymbol{\mathcal{R}}$ is the same as in (73), with $\boldsymbol{\mathcal{R}}_0 = 0$ ($\alpha_1 = 0$). Likewise, as above we can get the estimate

$$\|\boldsymbol{\mathcal{R}}\|^2_{L_2(\Omega_t)} \le C(K)\mathcal{M}(t)$$

that yields

$$\|D^\beta \mathbf{V}(t)\|^2_{L_2(\mathbb{R}^3_+)} \le C(K)(\mathcal{M}(t) + \mathcal{J}(t)). \tag{80}$$

To estimate the boundary integral we use (59), (60):

$$\mathcal{J}(t) \le \|\partial^{\alpha}_{\tan}\partial_1^{k-1}\boldsymbol{\mathcal{K}}\|^2_{L_2(\partial\Omega_t)}.$$

Since $\partial_j^i\widetilde{\mathcal{A}}|_{x_1=0} = 0$ ($j=\overline{0,3}$), $\|\widetilde{\mathcal{A}}|_{x_1=0}\| = 1$, and $\mathcal{A}_{(0)}|_{x_1=0} = 0$ we further estimate as follows:

$$\mathcal{J}(t) \le C\Big\{\varSigma_1(t) + \varSigma_2(t) + \varSigma_3(t) + \|\partial^{\alpha}_{\tan}\partial_1^{k-1}(\mathcal{A}_4\mathbf{V})\|^2_{L_2(\partial\Omega_t)} + \|\partial^{\alpha}_{\tan}\partial_1^{k-1}\boldsymbol{\mathcal{F}}\|^2_{L_2(\partial\Omega_t)}\Big\}, \tag{81}$$

where

$$\varSigma_1(t) = \sum_{j=0,2,3} \|\mathcal{A}_j\partial^{\alpha}_{\tan}\partial_1^{k-1}\partial_j\mathbf{V}\|^2_{L_2(\partial\Omega_t)},$$

$$\varSigma_2(t) = \sum_{j=0,2,3}\ \sum_{\substack{\langle\beta'\rangle\ge 1\\ \langle\beta'\rangle+\langle\beta''\rangle\le s-2}} \|D^{\beta'}\mathcal{A}_j\, D^{\beta''}\partial_j\mathbf{V}\|^2_{L_2(\partial\Omega_t)},$$

$$\varSigma_3(t) = \sum_{\substack{k'+k''\le k\\ k',k''\ge 1}} \|\partial^{\alpha}_{\tan}(\partial_1^{k'}\mathcal{A}_{(0)}\,\partial_1^{k''}\mathbf{V})\|^2_{L_2(\partial\Omega_t)}.$$

The sums $\varSigma_2$ and $\varSigma_3$ are estimated by using the trace property and the calculus inequalities (204), (205):

$$\begin{aligned}\varSigma_2(t) &\le C \sum_{\substack{j=0,2,3\\ \langle\beta'\rangle=1}} \Big([D^{\beta'}\mathcal{A}_j\,\partial_j\mathbf{V}]^2_{s-3,*,t} + [\partial_1(D^{\beta'}\mathcal{A}_j\,\partial_j\mathbf{V})]^2_{s-3,*,t}\Big)\\ &\le C(K)\Big\{[\mathbf{V}]^2_{s,*,t} + \|\dot{\mathbf{U}}\|^2_{W^1_\infty(\Omega_T)}\big(1+[\widehat{\mathbf{W}}]^2_{s,*,T}\big)\Big\},\end{aligned} \tag{82}$$

$$\begin{aligned}\varSigma_3(t) &\le C\left([\partial_1\mathcal{A}_{(0)}\partial_1\mathbf{V}]^2_{s-4,*,t} + [\partial_1(\partial_1\mathcal{A}_{(0)}\partial_1\mathbf{V})]^2_{s-4,*,t}\right)\\ &\le C(K)\Big\{[\mathbf{V}]^2_{s,*,t} + \|\dot{\mathbf{U}}\|^2_{W^1_\infty(\Omega_T)}\big(1+[\widehat{\mathbf{W}}]^2_{s,*,T}\big)\Big\}.\end{aligned} \tag{83}$$

To estimate the sum $\varSigma_1$ we pass to the volume integral and then integrate by parts:

$$\begin{aligned}\varSigma_1(t) &= -\sum_{j=0,2,3}\int_{\Omega_t}\partial_1|\mathcal{A}_j\partial^{\alpha}_{\tan}\partial_1^{k-1}\partial_j\mathbf{V}|^2 d\mathbf{x}d\tau\\ &= -2\sum_{j=0,2,3}\int_{\Omega_t}\big\{\big(\mathcal{A}_j^2\partial^{\alpha}_{\tan}\partial_1^{k-1}\partial_j\mathbf{V}, D^{\beta}\partial_j\mathbf{V}\big)\\ &\quad + \big(\mathcal{A}_j\partial^{\alpha}_{\tan}\partial_1^{k-1}\partial_j\mathbf{V}, \partial_1(\mathcal{A}_j)\partial^{\alpha}_{\tan}\partial_1^{k-1}\partial_j\mathbf{V}\big)\big\}d\mathbf{x}d\tau = \widetilde{\varSigma}_1(t) + \mathcal{J}_0(t),\end{aligned}$$

where

$$\widetilde{\varSigma}_1(t) = 2\sum_{j=0,2,3}\int_{\Omega_t}\big\{\big(\mathcal{A}_j^2\partial^{\alpha}_{\tan}\partial_1^{k-1}\partial_j^2\mathbf{V}, D^{\beta}\mathbf{V}\big)$$

$$+ \big(\{\mathcal{A}_j\partial_j(\mathcal{A}_j) + \partial_j(\mathcal{A}_j)\mathcal{A}_j\}\partial^{\alpha}_{\rm tan}\partial_1^{k-1}\partial_j\mathbf{V}, D^{\beta}\mathbf{V}\big)$$

$$- \big(\mathcal{A}_j\partial^{\alpha}_{\rm tan}\partial_1^{k-1}\partial_j\mathbf{V}, \partial_1(\mathcal{A}_j)\partial^{\alpha}_{\rm tan}\partial_1^{k-1}\partial_j\mathbf{V}\big)\big\}d\mathbf{x}d\tau,$$

$$\mathcal{J}_0(t) = -2\int_{\mathbb{R}^3_+}\big(\mathcal{A}_0^2\partial^{\alpha}_{\rm tan}\partial_1^{k-1}\partial_t\mathbf{V}, D^{\beta}\mathbf{V}\big)\,d\mathbf{x}.$$

Since $|\alpha| + 2 + 2(k-1) \leq s$ we easily obtain the estimate

$$\widetilde{\Sigma}_1(t) \leq C(K)[V]^2_{s,*,t}, \tag{84}$$

and by using the Young inequality, one gets

$$\mathcal{J}_0(t) \leq C(K)\Big(\varepsilon\|D^{\beta}\mathbf{V}(t)\|^2_{L_2(\mathbb{R}^3_+)} + \frac{1}{\varepsilon}\,|||\mathbf{V}(t)|||^2_{s-1,*}\Big), \tag{85}$$

where ε is a positive constant. In view of (71), it follows from (84), (85) that

$$\Sigma_1(t) \leq C(K)\left(\varepsilon\|D^{\beta}\mathbf{V}(t)\|^2_{L_2(\mathbb{R}^3_+)} + \frac{1}{\varepsilon}\,[\mathbf{V}]^2_{s,*,t}\right). \tag{86}$$

The last two terms in the expression in braces in (81) are easily estimated from above by $C(K)\mathcal{M}(t)$ by using the trace property and (204), (205). Combining then (80)–(83) and (86) and choosing the constant ε small enough, we get estimate (79). □

Remark 6. In fact, as in [29,34], our preparatory estimates (69), (72), and (79) are with no loss of derivatives from the coefficients of the matrices A_j and $\mathcal{C}$. In these estimates the loss of two derivatives from the basic state (in the sense of $H^s_*(\Omega_T) \times H^s(\partial\Omega_T)$ norms) is caused only by the fact that the matrix $\mathcal{C}$ (or $\mathcal{A}_4$) depends on the first-order derivatives of $\widehat{\mathbf{W}}$. Recall that the loss of one derivative in sense of usual Sobolev norms gives the loss of two derivatives in H^s_* norms because $[\partial_1(\cdot)]_{s,*,T} \leq [\cdot]_{s+2,*,T}$.

3.4. Estimate of nonweighted tangential derivatives

We now proceed to the case of nonweighted tangential derivatives ($\alpha_1 = 0$ and $k = 0$), i.e., we estimate the terms $\partial^{\alpha}_{\rm tan}\mathbf{V}$, with $|\alpha| \leq s$. This is the most important case because we shall use the boundary conditions. This gives the loss of two additional derivatives in comparison with estimates (69), (72), and (79). That is, in the final tame estimate we will have the "$s+4,*,T$" loss of derivatives from the coefficients. This loss is caused by the presence of zero-order terms in f in the boundary conditions and the fact that when these terms are omitted the boundary conditions (43) are dissipative but *not strictly* dissipative for system (50) (see Lemma 3).

Proposition 7. *The following estimate holds for problem (41)–(43) for all $t\leq T$ and even $s\geq 2$:*

$$
\begin{aligned}
\sum_{|\alpha|\leq s}\|\partial^\alpha_{\rm tan}\mathbf{V}(t)\|^2_{L_2(\mathbb{R}^3_+)} \leq C(K)\Big\{[\mathbf{V}]^2_{s,*,t}+\|f\|^2_{H^{s-1}(\partial\Omega_t)}\\
+\|\dot{\mathbf{U}}\|^2_{W^1_\infty(\Omega_T)}\big(1+[\widehat{\boldsymbol{\mathcal{W}}}]^2_{s+3,*,T}\big)+\|f\|^2_{W^1_\infty(\partial\Omega_T)}\big(1+[\widehat{\boldsymbol{\mathcal{W}}}]^2_{s+4,*,T}\big)\\
+[\mathbf{F}]^2_{s,*,T}+\|\mathbf{F}\|^2_{L_\infty(\Omega_T)}\big(1+[\widehat{\mathbf{W}}]^2_{s,*,T}\big)\Big\}\\
+\varepsilon\,\widetilde{C}(K)\,\Big(|||\mathbf{V}(t)|||^2_{s,*}+|||f(t)|||^2_{H^{s-1}(\mathbb{R}^2)}\Big),
\end{aligned}
\tag{87}
$$

where ε is an arbitrary positive constant (it will be chosen later on), $\widetilde{C}(K)$ is a positive constant depending on K and independent on ε,[7] *and $\widehat{\boldsymbol{\mathcal{W}}}:=(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})$.*

Proof. First of all, in view of the elementary inequality

$$
\sum_{|\alpha|\leq s-1}\|\partial^\alpha_{\rm tan}\mathbf{V}(t)\|^2_{L_2(\mathbb{R}^3_+)}\leq C[\mathbf{V}]^2_{s,*,t}, \tag{88}
$$

we only need to estimate the higher-order tangential derivatives (with $|\alpha|=s$). Taking into account Lemma 2 we can use system (48) or equivalently (50) for obtaining estimate (87). Applying to system (50) the same "energy" arguments as in the proofs of Propositions 5 and 6, we come to the inequality

$$
\|\partial^\alpha_{\rm tan}\mathbf{V}(t)\|^2_{L_2(\mathbb{R}^3_+)}\leq C(K)(\mathcal{M}(t)+\mathcal{J}(t)) \tag{89}
$$

for a multi-index $\alpha=(\alpha_0,\alpha_2,\alpha_3)$ with $|\alpha|=s$, where

$$
\mathcal{J}(t)=\left|\int_{\partial\Omega_t}(\mathcal{B}_1\partial^\alpha_{\rm tan}\mathbf{V},\partial^\alpha_{\rm tan}\mathbf{V})|_{x_1=0}\,d\mathbf{x}'d\tau\right|.
$$

By virtue of (42), (45) and the choice of λ in (54), the quadratic form in the boundary integral is explicitly written as follows:

$$
\begin{aligned}
(\mathcal{B}_1\partial^\alpha_{\rm tan}\mathbf{V},\partial^\alpha_{\rm tan}\mathbf{V})|_{x_1=0}=\Big[(\partial^\alpha_{\rm tan}\dot{v}_{\rm N}-\lambda\partial^\alpha_{\rm tan}\dot{H}_{\rm N})\partial^\alpha_{\rm tan}\dot{q}\Big]=[a\partial^\alpha_{\rm tan}\dot{q}]\\
+[\partial^\alpha_{\rm tan}w\,\partial^\alpha_{\rm tan}\dot{q}]=[a\partial^\alpha_{\rm tan}\dot{q}]+\partial^\alpha_{\rm tan}w^-_{|x_1=0}\,[\partial^\alpha_{\rm tan}\dot{q}]+\partial^\alpha_{\rm tan}\dot{q}^+_{|x_1=0}\,[\partial^\alpha_{\rm tan}w]\\
=[a\partial^\alpha_{\rm tan}\dot{q}]-\partial^\alpha_{\rm tan}w^-_{|x_1=0}\partial^\alpha_{\rm tan}\big([\partial_1\hat{q}]f\big)-\partial^\alpha_{\rm tan}\dot{q}^+_{|x_1=0}\partial^\alpha_{\rm tan}\big(\big[\partial_1\hat{v}_{\rm N}-\lambda\partial_1\widehat{H}_{\rm N}\big]f\big),
\end{aligned}
$$

where

$$
a^\pm=\sum_{\substack{|\alpha'|+|\alpha''|=s\\|\alpha'|\geq 1}}\partial^{\alpha'}_{\rm tan}\lambda(\widehat{\mathbf{U}}^\pm)\partial^{\alpha''}_{\rm tan}H^\pm_{\rm n},\quad w^\pm=v^\pm_{\rm n}-\lambda(\widehat{\mathbf{U}}^\pm)H^\pm_{\rm n},
$$

[7] The constant $C(K)$ in (87) depends linearly on $1/\varepsilon$, but this is of now importance for the subsequent application of (87).

$$w^{\pm}_{|x_1=0} = (v^{\pm}_{\rm N} - \lambda^{\pm} H^{\pm}_{\rm N})|_{x_1=0}, \quad [\partial_1 \hat{q}] = (\partial_1 \hat{q}^+)|_{x_1=0} - (\partial_1 \hat{q}^-)|_{x_1=0}, \quad \text{etc.}$$

That is, the boundary integral is estimated by lower-order terms[8]:

$$\mathcal{J}(t) \le \sum_{\pm} \sum_{i=1}^{4} \mathcal{J}^{\pm}_i(t), \tag{90}$$

with

$$\mathcal{J}^{\pm}_1(t) = \left| \int_{\partial\Omega_t} \big(a^{\pm}\partial^{\alpha}_{\rm tan}\dot{q}^{\pm}\big)\big|_{x_1=0}\, d\mathbf{x}'d\tau \right|,$$

$$\mathcal{J}^{\pm}_2(t) = \left| \int_{\partial\Omega_t} \big(\partial^{\alpha}_{\rm tan}w^{-}\, \partial^{\alpha}_{\rm tan}\big(f\partial_1\hat{q}^{\pm}\big)\big)\big|_{x_1=0}\, d\mathbf{x}'d\tau \right|,$$

$$\mathcal{J}^{\pm}_3(t) = \left| \int_{\partial\Omega_t} \big(\partial^{\alpha}_{\rm tan}\dot{q}^{+}\partial^{\alpha}_{\rm tan}\big(f\partial_1\hat{v}^{\pm}_{\rm n}\big)\big)\big|_{x_1=0}\, d\mathbf{x}'d\tau \right|,$$

$$\mathcal{J}^{\pm}_4(t) = \left| \int_{\partial\Omega_t} \Big(\partial^{\alpha}_{\rm tan}\dot{q}^{+}\partial^{\alpha}_{\rm tan}\big(f\lambda(\widehat{\mathbf{U}}^{\pm})\partial_1\widehat{H}^{\pm}_{\rm n}\big)\Big)\Big|_{x_1=0}\, d\mathbf{x}'d\tau \right|.$$

Since $|\alpha| = s \ge 2$, we have $\partial^{\alpha}_{\rm tan} = \partial_{\ell} l \partial^{\gamma}_{\rm tan}$, where $\ell = 2$ or $\ell = 3$ if $\alpha_0 \neq s$ and $\ell = 0$ otherwise, and $\gamma = (\gamma_0, \gamma_2, \gamma_3)$ is a multi-index with $|\gamma| = s-1 \ge 1$. Consider the case $\alpha_0 = s$, i.e., $\partial^{\alpha}_{\rm tan} = \partial_t\partial^{\gamma}_{\rm tan}$. Passing to the volume integral, integrating by parts, and using inequalities (69), (70) and (71), we estimate $\mathcal{J}^{+}_1(t)$ as follows (we omit detailed calculations):

$$\mathcal{J}^{+}_1(t) = \left| \int_{\Omega_t} \big(\partial_t a^{+}\partial_1\partial^{\gamma}_{\rm tan}q^{+} - \partial_1 a^{+}\partial^{\alpha}_{\rm tan}q^{+}\big)\, d\mathbf{x}d\tau - \int_{\mathbb{R}^3_+} a^{+}\partial_1\partial^{\gamma}_{\rm tan}q^{+} d\mathbf{x} \right|$$

$$\le C\Big([q^+]^2_{s,*,t} + [\partial_1 q^+]^2_{s-1,*,t} + \|\partial_t a^+\|^2_{L_2(\Omega_t)} + \|\partial_1 a^+\|^2_{L_2(\Omega_t)}$$

$$+\frac{1}{\varepsilon}\|a^+(t)\|^2_{L_2(\mathbb{R}^3_+)} + \varepsilon|||\partial_1 q^+(t)|||^2_{s-1,*}\Big) \le C(K)\mathcal{M}(t) + \varepsilon\widetilde{C}(K)|||\mathbf{V}(t)|||^2_{s,*}\,.$$

Here and below ε is an arbitrary positive constant, $\widetilde{C}(K)$ is independent of ε, and $C(K)$ may depend linearly on $1/\varepsilon$. Similarly, we estimate $\mathcal{J}^{-}_1(t)$, i.e., one has

$$\mathcal{J}^{+}_1(t) + \mathcal{J}^{-}_1(t) \le C(K)\mathcal{M}(t) + \varepsilon\widetilde{C}(K)|||\mathbf{V}(t)|||^2_{s,*}, \tag{91}$$

where clearly $\widetilde{C}(K) = 0$ for the case $\alpha_0 \neq s$.

The boundary integrals $\mathcal{J}^{\pm}_2(t)$, $\mathcal{J}^{\pm}_3(t)$, and $\mathcal{J}^{\pm}_4(t)$ are estimated in the same way. Consider, for example, $\mathcal{J}^{+}_3(t)$:

$$\mathcal{J}^{+}_3(t) \le \mathcal{J}_0(t) + \Sigma(t),$$

where

$$\mathcal{J}_0(t) = \left| \int_{\partial\Omega_t} \big(f\,\partial^{\alpha}_{\rm tan}\dot{q}^{+}\,\partial^{\alpha}_{\rm tan}\partial_1\hat{v}^{+}_{\rm n}\big)\big|_{x_1=0}\, d\mathbf{x}'d\tau \right|,$$

[8] The higher-order boundary terms vanish thanks to the fact that the boundary conditions are dissipative (in the sense of Lemma 3).

$$\Sigma(t) = \sum_{\substack{|\alpha'|+|\alpha''|=s\\ |\alpha'|\geq 1}} \left| \int_{\partial\Omega_t} \left(\partial^{\alpha}_{\tan}\dot{q}^+\, \partial^{\alpha'}_{\tan} f\, \partial^{\alpha''}_{\tan}\partial_1\hat{v}^+_{\rm n}\right)\Big|_{x_1=0} d\mathbf{x}' d\tau \right|.$$

Then, we again pass to the volume integral and integrate by parts:

$$\begin{aligned}\mathcal{J}_0(t) = \Bigg| \int_{\Omega_t} \Big(&(\partial_1\partial_\ell\partial^{\alpha}_{\tan}\hat{v}^+_{\rm n})\, f\partial_1\partial^{\gamma}_{\tan}\dot{q}^+ - (\partial_1^2\partial^{\alpha}_{\tan}\hat{v}^+_{\rm n})\, f\partial^{\alpha}_{\tan}\dot{q}^+ \\ &-(\partial_1\partial^{\alpha}_{\tan}\hat{v}^+_{\rm n})\,\partial_\ell f\partial_1\partial^{\gamma}_{\tan}\dot{q}^+\Big) d\mathbf{x}d\tau \\ &-(\ell-2)(\ell-3)\int_{\mathbb{R}^3_+} (\partial_1\partial^{\alpha}_{\tan}\hat{v}^+_{\rm n})\, f\partial_1\partial^{\gamma}_{\tan}\dot{q}^+ d\mathbf{x}\Bigg|,\end{aligned}$$

where ℓ and γ are the same as above. Here we do not pass from f to Ψ^+ while passing to the volume integral because we can then estimate f in the W^1_∞ norm. We see that the biggest loss of derivatives from the coefficients in the estimate of $\mathcal{J}_0(t)$ will be caused by the term $\partial_1^2\partial^{\alpha}_{\tan}\hat{v}^+_{\rm n}$ and is "$s+4,*,T$". Omitting detailed calculations, we get the estimate

$$\mathcal{J}_0(t) \leq C(K)\left(\mathcal{M}(t) + \|f\|^2_{W^1_\infty(\partial\Omega_T)}\big(1+[\widehat{\boldsymbol{\mathcal{W}}}]^2_{s+4,*,T}\big)\right) + \varepsilon\widetilde{C}(K)|||\mathbf{V}(t)|||^2_{s,*}.$$

To estimate the sum $\Sigma(t)$ we use the fact that $|\alpha'|\geq 1$. It allows to write down $\partial^{\alpha'}_{\tan} f = \partial^{\gamma'}_{\tan}(\partial_j f)$, with $|\gamma'|\leq s-1$ and $j=0$ or $j=2$ or $j=3$. Then we use the important fact that the boundary conditions (42), (45) can be resolved for $\nabla_{t,x'}f$.[9] Indeed, thanks to the second inequality in (19) conditions (45) are resolved for $\partial_2 f$ and $\partial_3 f$, and then from the first or the second boundary condition in (42) we find $\partial_t f$. We write the result in the compact form

$$\nabla_{t,{\rm x}'} f = \frac{1}{\kappa}\, G(\widehat{\mathbf{U}},\hat{f})\mathbf{V}_{\rm n}\big|_{x_1=0}, \tag{92}$$

where $\kappa = \kappa(\widehat{\mathbf{U}}|_{x_1=0}) = (\widehat{H}^+_2\widehat{H}^-_3 - \widehat{H}^+_3\widehat{H}^-_2)|_{x_1=0} \geq \epsilon > 0$ for all $(t,\mathbf{x}')\in\partial\Omega_T$ and the matrix G depends smoothly on $\widehat{\mathbf{U}}|_{x_1=0}$, $\partial_1\widehat{\mathbf{U}}|_{x_1=0}$, and $\nabla_{t,{\rm x}'}\hat{f}$.

That is, from (92) we express $\partial_j f$ through the trace of the "noncharacteristic" unknown $\mathbf{V}_{\rm n}$ and then substitute the result $\partial^{\alpha'}_{\tan} f = \partial^{\gamma'}_{\tan}(\ldots)$ into the sum $\Sigma(t)$. Then, $\Sigma(t)$ is the sum of the terms like

$$(\partial^{\alpha}_{\tan}\dot{q}^+\,\partial^{\gamma'}_{\tan}(\hat{b}\,\dot{H}_{\rm n})\partial^{\alpha''}_{\tan}\partial_1\hat{v}^+_{\rm n})|_{x_1=0},\quad (\partial^{\alpha}_{\tan}\dot{q}^+\,\partial^{\gamma'}_{\tan}(\hat{b}\,\Psi^+)\partial^{\alpha''}_{\tan}\partial_1\hat{v}^+_{\rm n})|_{x_1=0},$$

etc., with $|\gamma'|+|\alpha''| = s-1$ (recall that $|\alpha|=s$), where $\hat{b}$ is a coefficient of the matrix G. We use then the same arguments as in the estimation of $\mathcal{J}_1^{\pm}(t)$ and $\mathcal{J}_0(t)$. Omitting the details, we just calculate the biggest loss of derivatives from the coefficients in the estimate of $\Sigma(t)$. While passing to the volume integral this loss will be caused by the term $\partial^{\alpha''}_{\tan}\partial_1^2\hat{v}^+_{\rm n}$. For

[9] Using the terminology of paradifferential calculus, we can say that the symbol associated to the front is *elliptic* [7,25].

$|\alpha''| = s-1$ it gives the "$s+3,*,T$" loss of derivatives from the coefficients. As a result, we estimate $\Sigma(t)$ and, consequently, $\mathcal{J}_3^+(t)$ from above by the right-hand side in (87). Similarly, we handle $\mathcal{J}_3^-(t)$, $\mathcal{J}_2^\pm$, and $\mathcal{J}_4^\pm$. At last, with the reference to (88)–(91), we obtain the desired inequality (87). □

3.5. Estimate of the front

To close the estimate obtained by combining inequalities (72), (79), and (87) we need also to estimate the front perturbation f in $H^{s-1}(\mathbb{R}^2)$. But, in view of (92), by using the trace theorem [28] (or, alternatively, by passing to the volume integral and applying (69)) we are able to estimate $\nabla_{t,x'}f$ in $H^{s-1}(\partial\Omega_t)$ as well.

Proposition 8. *The following estimates hold for problem (41)–(43) for all $t \le T$ and even $s \ge 2$:*

$$\begin{aligned}&|||f(t)|||^2_{H^{s-1}(\mathbb{R}^2)} \le C(K)\Big\{[\mathbf{V}]^2_{s,*,t} + \|f\|^2_{H^{s-1}(\partial\Omega_t)}\\ &+\|\dot{\mathbf{U}}\|^2_{L_\infty(\Omega_T)}\big(1+[\widehat{\mathbf{W}}]^2_{s,*,t}\big) + \|f\|^2_{W^1_\infty(\partial\Omega_T)}\big(1+[\widehat{\mathbf{W}}]^2_{s+2,*,t}\big)\Big\},\end{aligned} \tag{93}$$

$$\begin{aligned}\|\nabla_{t,x'}f\|^2_{H^{s-1}(\partial\Omega_T)} \le C(K)\Big\{&[\mathbf{V}]^2_{s,*,T}\\ &+\|\dot{\mathbf{U}}\|^2_{L_\infty(\Omega_T)}\big(1+[\widehat{\mathbf{W}}]^2_{s+2,*,T}\big)\Big\}.\end{aligned} \tag{94}$$

Proof. By using standard "energy" arguments for the first boundary condition in (42), one gets

$$|||f(t)|||^2_{H^{s-1}(\mathbb{R}^2)} \le \|\dot{v}_{\rm n}^+|_{x_1=0}\|^2_{H^{s-1}(\partial\Omega_t)} + \|f\|^2_{H^{s-1}(\partial\Omega_t)} + \sum_{|\alpha|\le s-1}\|G_\alpha\|^2_{L_2(\partial\Omega_t)},$$

where

$$G_\alpha = \partial^\alpha_{\tan}(f\partial_1\hat{v}^+_{\rm N}) - \sum_{j=2}^3\big([\partial^\alpha_{\tan},\hat{v}^+_j]\partial_j f + (1/2)\partial_j\hat{v}^+_j\partial^\alpha_{\tan}f\big).$$

Applying the Moser-type calculus inequalities in the usual Sobolev space $H^{s-1}(\partial\Omega_t)$ (counterparts of (204) and (205)) for G_α and using the trace theorem in H^s_* (cf. (35)), we obtain estimate (93). From (92) we derive estimate (94) by applying the trace theorem in H^s_* and the calculus inequalities (204) and (205). □

3.6. Tame estimates for problems (41)–(43) and (28)–(30)

Propositions 5–8 yield the following result.

Proposition 9. *Problem (41)–(43) obeys the a priori estimate*

$$[\dot{\mathbf{U}}]^2_{s,*,T} + \|f\|^2_{H^s(\partial\Omega_T)} \leq C(K)Te^{C(K)T}\mathcal{N}(T) \tag{95}$$

for even $s \geq 2$, with

$$\mathcal{N}(T) = [\mathbf{F}]^2_{s,*,T} + \big(\|\dot{\mathbf{U}}\|^2_{W^1_\infty(\Omega_T)} + \|f\|^2_{W^1_\infty(\partial\Omega_T)}$$
$$+\|\mathbf{F}\|^2_{L_\infty(\Omega_T)}\big)\big(1 + [\widehat{\mathbf{U}}]^2_{s+4,*,T} + \|\hat{f}\|^2_{H^{s+4}(\partial\Omega_T)}\big)$$

Proof. Summing up inequalities (72), (79), (87), and (93) and choosing the constant ε in (87) small enough, we obtain

$$\mathcal{I}(t) \leq C(K)\Big(\mathcal{N}(T) + \int_0^t \mathcal{I}(\tau)d\tau\Big), \tag{96}$$

where

$$\mathcal{I}(t) = |||\mathbf{V}(t)|||^2_{s,*} + |||f(t)|||^2_{H^{s-1}(\mathbb{R}^2)} \qquad (\mathcal{I}(0) = 0,\ \text{cf. (43)}).$$

Since only the biggest loss of derivatives from the coefficients will play the role for obtaining the final tame estimate, we have roughened inequality (96) by choosing the biggest loss. Applying Gronwall's lemma to (96), one gets

$$\mathcal{I}(t) \leq C(K)\, e^{C(K)T}\mathcal{N}(T).$$

Taking into account (43) and integrating the last inequality over the interval $[0,T]$, we come to the estimate

$$[\mathbf{V}]^2_{s,*,T} + \|f\|^2_{H^{s-1}(\partial\Omega_T)} \leq C(K)Te^{C(K)T}\mathcal{N}(T). \tag{97}$$

Recall that $\dot{\mathbf{U}} = J\mathbf{V}$. Consider the decomposition $J(\widehat{\mathbf{W}}) = I + J_0(\widehat{\mathbf{W}})$. Since $J_0(0) = 0$ we can use the calculus inequality (206). Applying (204) and (206), we obtain

$$\begin{aligned}[\dot{\mathbf{U}}]^2_{s,*,T} = [\mathbf{V} + J_0\mathbf{V}]^2_{s,*,T} &\leq C(K)\big([\mathbf{V}]^2_{s,*,T} + \|\dot{\mathbf{U}}\|^2_{L_\infty(\Omega_T)}[\widehat{\mathbf{W}}]^2_{s,*,T}\big)\\ &\leq C(K)[\mathbf{V}]^2_{s,*,T} + TC(K)\|\dot{\mathbf{U}}\|^2_{L_\infty(\Omega_T)}[\widehat{\mathbf{W}}]^2_{s+1,*,T}\,.\end{aligned} \tag{98}$$

Here we also used Sobolev's embedding in one space dimension:

$$[\widehat{\mathbf{W}}]^2_{s,*,T} \leq T\|\widehat{\mathbf{W}}\|^2_{L_\infty([0,T],H^s_*(\mathbb{R}^3_+))} \leq TC[\widehat{\mathbf{W}}]^2_{s+1,*,T}$$

(since $\mathbf{V}|_{t<0} = 0$ we can suppose that the norm $[\widehat{\mathbf{W}}]^2_{s,*,T}$ appearing in (98) is the norm in the space $\mathcal{L}^2_T(H^s_*)$ but not in the space $H^s_*(\Omega_T)$). Inequalities (97) and (98) imply

$$[\dot{\mathbf{U}}]^2_{s,*,T} + \|f\|^2_{H^{s-1}(\partial\Omega_T)} \leq C(K)Te^{C(K)T}\mathcal{N}(T). \tag{99}$$

Similarly to (98) we also get the inequality

$$[\mathbf{V}]^2_{s,*,T} \leq C(K)[\dot{\mathbf{U}}]^2_{s,*,T} + TC(K)\|\dot{\mathbf{U}}\|^2_{L_\infty(\Omega_T)}[\widehat{\mathbf{W}}]^2_{s+1,*,T} \tag{100}$$

Clearly, (94), (100), and (99) yield the desired estimate (95). □

With estimate (95) we are now in a position to prove the tame estimate for problem (41)–(43).

Theorem 3. *Let $T > 0$ and s is an even number, $s \geq 6$. Assume that the basic state $(\widehat{\mathbf{U}}, \hat{f}) \in H^{s+4}_*(\Omega_T) \times H^{s+4}(\partial\Omega_T)$ satisfies assumptions (17)–(22) and*

$$[\widehat{\mathbf{U}}]_{10,*,T} + \|\hat{f}\|_{H^{10}(\partial\Omega_T)} \leq \widehat{K}, \tag{101}$$

where $\widehat{K} > 0$ is a constant. Assume also that $\mathbf{F} \in H^s_*(\Omega_T)$ vanishes in the past. Then there exists a positive constant K_0, that does not depend on s and T, and there exists a constant $C(K_0) > 0$ such that, if $\widehat{K} \leq K_0$, then there exists a unique solution $(\dot{\mathbf{U}}, f) \in H^s_*(\Omega_T) \times H^s(\partial\Omega_T)$ to problem (41)–(43) that obeys the following a priori tame estimate for T small enough:

$$\begin{aligned}[\dot{\mathbf{U}}]_{s,*,T} + \|f\|_{H^s(\partial\Omega_T)} \leq C(K_0)\Big\{&[\mathbf{F}]_{s,*,T} + \\ &+[\mathbf{F}]_{6,*,T}\big([\widehat{\mathbf{U}}]_{s+4,*,T} + \|\hat{f}\|_{H^{s+4}(\partial\Omega_T)}\big)\Big\}.\end{aligned} \tag{102}$$

Proof. Taking into account Lemma 4 and Remark 5, we have the well-posedness of problem (41)–(43) in $H^s_*(\Omega_T) \times H^s(\partial\Omega_T)$. Applying Sobolev's embedding (207) (the second inequality in (207)), from (95) with $s \geq 6$ we get

$$\begin{aligned}[\dot{\mathbf{U}}]_{s,*,T} + \|f\|_{H^s(\partial\Omega_T)} \leq C(K)T^{1/2}e^{C(K)T}\Big\{[\mathbf{F}]_{s,*,T} + \\ \big([\dot{\mathbf{U}}]_{6,*,T} + \|f\|_{H^6(\partial\Omega_T)} + [\mathbf{F}]_{6,*,T}\big)\big([\widehat{\mathbf{U}}]_{s+4,*,T} + \|\hat{f}\|_{H^{s+4}(\partial\Omega_T)}\big)\Big\},\end{aligned} \tag{103}$$

where we have absorbed some norms $[\dot{\mathbf{U}}]_{6,*,T}$ and $\|f\|_{H^6(\partial\Omega_T)}$ in the left-hand side by choosing T small enough. Considering (103) for $s = 6$ and using (101), we obtain for T small enough that

$$[\dot{\mathbf{U}}]_{6,*,T} + \|f\|_{H^6(\partial\Omega_T)} \leq C(K_0)[\mathbf{F}]_{6,*,T}\,. \tag{104}$$

It is natural to assume that $T < 1$ and, hence, we can suppose that the constant $C(K_0)$ does not depend on T. Inequalities (103) and (104) imply (102). □

Corollary 1. *Let all the assumptions of Theorem 3 are satisfied. Let also the data* $(\mathbf{f},\mathbf{g})\in H^{s+2}_*(\Omega_T)\times H^{s+2}(\partial\Omega_T)$ *vanish in the past. Then there exists a unique solution* $(\dot{\mathbf{U}},f)\in H^s_*(\Omega_T)\times H^s(\partial\Omega_T)$ *to problem (28)–(30) that obeys the a priori tame estimate*

$$\begin{aligned}[\dot{\mathbf{U}}]_{s,*,T}+\|f\|_{H^s(\partial\Omega_T)}\leq C(K_0)\Big\{&[\mathbf{f}]_{s+2,*,T}+\|\mathbf{g}\|_{H^{s+2}(\partial\Omega_T)}\\ &+\big([\mathbf{f}]_{8,*,T}+\|\mathbf{g}\|_{H^8(\partial\Omega_T)}\big)\big([\widehat{\mathbf{U}}]_{s+4,*,T}+\|\hat{f}\|_{H^{s+4}(\partial\Omega_T)}\big)\Big\}\end{aligned}\tag{105}$$

for T *small enough.*

Proof. Utilizing the Moser-type inequalities (204) and (205), we easily derive the refined variant of the estimate (39)

$$[\mathbb{L}_e'(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})\widetilde{\mathbf{U}}]_{s,*,T}\leq C(K)\left\{[\widetilde{\mathbf{U}}]_{s+2,*,T}+\|\widetilde{\mathbf{U}}\|_{W^1_\infty(\Omega_T)}\big(1+[\widehat{\mathbf{W}}]_{s+2,*,T}\big)\right\}$$

from which, by applying the second inequality in (207), we get the estimate

$$[\mathbb{L}_e'(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})\widetilde{\mathbf{U}}]_{s,*,T}\leq C(K)\left\{[\widetilde{\mathbf{U}}]_{s+2,*,T}+[\widetilde{\mathbf{U}}]_{6,*,T}[\widehat{\mathbf{W}}]_{s+2,*,T}\right\}.$$

By using this estimate, one gets (cf. (40))

$$\begin{aligned}[\mathbf{F}]_{s,*,T}\leq C(K)\Big\{&[\mathbf{f}]_{s+2,*,T}+\|\mathbf{g}\|_{H^{s+2}(\partial\Omega_T)}\\ &+\big([\mathbf{f}]_{8,*,T}+\|\mathbf{g}\|_{H^8(\partial\Omega_T)}\big)[\widehat{\mathbf{W}}]_{s+2,*,T}\Big\}.\end{aligned}\tag{106}$$

Assumption (101) and inequality (106) yield

$$[\mathbf{F}]_{6,*,T}\leq C(K_0)\big([\mathbf{f}]_{8,*,T}+\|\mathbf{g}\|_{H^8(\partial\Omega_T)}\big).\tag{107}$$

At last, using (106) and (107), from (102) we get estimate (105). □

Consider now the case when s is an odd number. We can repeat all the previous arguments of this section using inequalities (208) and (209) instead of inequalities (204)–(207). Then, for odd s in the counterpart of the a priori estimate (95) we will have $W^{1,\tan}_\infty$ and $W^{2,\tan}_\infty$ norms (see Appendix B) instead of L_∞ and W^1_∞ norms respectively. We finally obtain the following tame estimate for odd s.

Corollary 2. *Let* s *is an odd number,* $s\geq 7$*. Assume also that*

$$[\widehat{\mathbf{U}}]_{11,*,T}+\|\hat{f}\|_{H^{11}(\partial\Omega_T)}\leq\widehat{K}$$

and all the remaining assumptions of Theorem 3 and Corollary 1 are fulfilled. Then the a priori tame estimate

$$\begin{aligned}[\dot{\mathbf{U}}]_{s,*,T}+\|f\|_{H^s(\partial\Omega_T)}\leq C(K_0)\Big\{&[\mathbf{f}]_{s+2,*,T}+\|\mathbf{g}\|_{H^{s+2}(\partial\Omega_T)}\\ &+\big([\mathbf{f}]_{9,*,T}+\|\mathbf{g}\|_{H^9(\partial\Omega_T)}\big)\big([\widehat{\mathbf{U}}]_{s+4,*,T}+\|\hat{f}\|_{H^{s+4}(\partial\Omega_T)}\big)\Big\}\end{aligned}\tag{108}$$

hold for problem (28)–(30) for T *small enough.*

Let us just collect the results of Corollaries 1 and 2 and formulate the main theorem for problem (28)–(30).

Theorem 4. *Let $T>0$ and $s\in\mathbb{N}$, with $s\geq 6$. Assume that the basic state $(\widehat{\mathbf{U}},\hat{f})\in H^{s+4}_*(\Omega_T)\times H^{s+4}(\partial\Omega_T)$ satisfies assumptions (17)–(22) and*

$$[\widehat{\mathbf{U}}]_{11,*,T}+\|\hat{f}\|_{H^{11}(\partial\Omega_T)}\leq\widehat{K}, \tag{109}$$

where $\widehat{K}>0$ is a constant. Let also the data $(\mathbf{f},\mathbf{g})\in H^{s+2}_*(\Omega_T)\times H^{s+2}(\partial\Omega_T)$ vanish in the past. Then there exists a positive constant K_0, that does not depend on s and T, and there exists a constant $C(K_0)>0$ such that, if $\widehat{K}\leq K_0$, then there exists a unique solution $(\dot{\mathbf{U}},f)\in H^s_*(\Omega_T)\times H^s(\partial\Omega_T)$ to problem (28)–(30) that obeys the a priori tame estimate

$$\begin{aligned}&[\dot{\mathbf{U}}]_{s,*,T}+\|f\|_{H^s(\partial\Omega_T)}\leq C(K_0)\Big\{[\mathbf{f}]_{s+2,*,T}+\|\mathbf{g}\|_{H^{s+2}(\partial\Omega_T)}\\&\quad+\big([\mathbf{f}]_{s_0,*,T}+\|\mathbf{g}\|_{H^{s_0}(\partial\Omega_T)}\big)\big([\widehat{\mathbf{U}}]_{s+4,*,T}+\|\hat{f}\|_{H^{s+4}(\partial\Omega_T)}\big)\Big\}\end{aligned} \tag{110}$$

for a sufficiently short time T, where $s_0=8$ if s is even and $s_0=9$ if s is odd.

4. Compatibility conditions and approximate solution

4.1. The compatibility conditions for the initial data (13)

Suppose the initial data (13),

$$(\mathbf{U}^{\pm}_0,f_0)=(p^{\pm}_0,v^{\pm}_{1,0},v^{\pm}_{2,0},v^{\pm}_{3,0},H^{\pm}_{1,0},H^{\pm}_{2,0},H^{\pm}_{3,0},S^{\pm}_0,f_0),$$

satisfy constraints (15) and condition (10) at $x_1=0$ for all $\mathbf{x}'\in\mathbb{R}^2$. Thanks to assumption (10) we can resolve (15) for $\partial_2 f$ and $\partial_3 f$:

$$\partial_2 f=\mu_2(\mathbf{U})|_{x_1=0},\quad \partial_3 f=\mu_3(\mathbf{U})|_{x_1=0}, \tag{111}$$

where $\mathbf{U}:=(\mathbf{U}^+,\mathbf{U}^-)$,

$$\mu_2(\mathbf{U})=\frac{H_1^+H_3^--H_1^-H_3^+}{H_2^+H_3^--H_2^-H_3^+},\quad \mu_3(\mathbf{U})=-\frac{H_1^+H_2^--H_1^-H_2^+}{H_2^+H_3^--H_2^-H_3^+}.$$

Then, from the first boundary condition in (12) we find

$$\partial_t f=\eta(\mathbf{U})|_{x_1=0}, \tag{112}$$

with

$$\eta(\mathbf{U})=v_1^+-v_2^+\mu_2(\mathbf{U})-v_3^+\mu_3(\mathbf{U}).$$

Thanks to the hyperbolicity condition (5) system (11) reads

$$\partial_t\mathbf{U}=-(A_0(\mathbf{U}))^{-1}\Big(\widetilde{A}_1(\mathbf{U},\boldsymbol{\Psi})\partial_1\mathbf{U}+\sum_{i=2}^{3}A_i(\mathbf{U})\partial_i\mathbf{U}\Big), \tag{113}$$

where $\boldsymbol{\Psi} := (\Psi^+, \Psi^-)$, and the block-diagonal matrices A_0, A_i, and $\widetilde{A}_1$ were defined in Section 2. The traces

$$\mathbf{U}_j = (p_j^+, v_{1,j}^+, \dots, H_{3,j}^-, S_j^-) = \partial_t^j \mathbf{U}|_{t=0} \quad \text{and} \quad f_j = \partial_t^j f|_{t=0},$$

with $j \geq 1$, are recursively defined by the formal application of the differential operator ∂_t^{j-1} to (112) and (113) and evaluating $\partial_t^j \mathbf{U}$ and $\partial_t^j f$ at $t = 0$. Moreover, $\Psi_j^\pm = \partial_t^j \Psi^\pm|_{t=0} = \chi(\pm x_1) f_j$.

In particular, $f_1 = \eta(\mathbf{U}_0)|_{x_1=0}$, where $\mathbf{U}_0 := (\mathbf{U}_0^+, \mathbf{U}_0^-)$. Then we define $\mathbf{U}_1$ by evaluating (113) at $t = 0$ and taking into account that $\partial_t \Psi^\pm|_{t=0} = \chi(\pm x_1) f_1$ and $\partial_i \Psi^\pm|_{t=0} = \chi(\pm x_1)\partial_i f_0$. We define the zero-order compatibility conditions as follows

$$[v_{2,0}]\partial_2 f_0 + [v_{3,0}]\partial_3 f_0 = 0, \quad [p_0 + (|\mathbf{H}_0|^2/2)] = 0 \tag{114}$$

(recall that by brackets we denote the jumps, e.g., $[v_{2,0}] = (v_{2,0}^+ - v_{2,0}^-)|_{x_1=0}$). From (111) and (112) evaluated at $t = 0$ and (114) we get

$$f_1 = (v_{1,0}^\pm - v_{2,0}^\pm \partial_2 f_0 - v_{3,0}^\pm \partial_3 f_0)|_{x_1=0}. \tag{115}$$

Using (115), from (113) we obtain

$$(H_{\rm N}^\pm)_1 = -\sum_{i=2}^{3} \left(v_{i,0}^\pm \partial_i (H_{\rm N}^\pm)_0 + \partial_i v_{i,0}^\pm (H_{\rm N}^\pm)_0 \right)\big|_{x_1=0}$$

(see the proof of Proposition 1 in Appendix A), where $(H_{\rm N}^\pm)_j = \partial_t^j (H_{\rm N}^\pm)|_{t=0}$, in particular,

$$(H_{\rm N}^\pm)_1 = H_{1,1}^\pm - \sum_{j=2}^{3} \left(H_{i,1}^\pm \partial_i f_0 + H_{i,0}^\pm \partial_i f_1 \right)\big|_{x_1=0},$$

In view of (15) evaluated at $t = 0$, $(H_{\rm N}^\pm)_0|_{x_1=0} = 0$ that yields $(H_{\rm N}^\pm)_1|_{x_1=0} = 0$. Knowing $\mathbf{U}_1$ and f_1 we can then find $\mathbf{U}_2$, f_2, etc. Moreover, at each jth step we can prove that

$$(H_{\rm N}^\pm)_j|_{x_1=0} = 0, \tag{116}$$

provided that $\mathbf{U}_j$ and f_j satisfy the compatibility conditions (they will be defined below).

The following lemma is the analogue of Lemma 4.2.1 in [25] and Lemma 2 in [8].

Lemma 5. *Let $\mu \in \mathbb{N}$, $\mu \geq 3$, $\mathbf{U}_0 \in H_*^{2\mu+1}(\mathbb{R}_+^3)$, and $f_0 \in H^{2\mu+1}(\mathbb{R}^2)$. Then, the procedure described above determines $\mathbf{U}_j \in H_*^{2(\mu-j)+1}(\mathbb{R}_+^3)$ and $f_j \in H^{2(\mu-j+1)}(\mathbb{R}^2)$ for $j = 1, \dots, \mu$. Moreover,*

$$\sum_{j=1}^{\mu} \left(\|\mathbf{U}_j\|_{2(\mu-j)+1,*} + \|f_j\|_{H^{2(\mu-j+1)}(\mathbb{R}^2)} \right) \leq C M_0, \tag{117}$$

where

$$M_0 = \|\mathbf{U}_0\|_{2\mu+1,*} + \|f_0\|_{H^{2\mu+1}(\mathbb{R}^2)}, \tag{118}$$

the constant $C > 0$ *depends only on* μ *and the norms* $\|\mathbf{U}_0\|_{W^{2,\tan}_{\infty}(\mathbb{R}^3_+)}$ *and* $\|f_0\|_{W^2_{\infty}(\mathbb{R}^2_+)}$.

The proof is absolutely analogous to that in [25,8]. We only need to use the calculus inequality (see, e.g., [29,34])

$$\|uv\|_{s,*} \leq C\|u\|_{s,*}\|v\|_{r,*}$$

for $u \in H^s_*(\mathbb{R}^3_+)$ and $v \in H^r_*(\mathbb{R}^3_+)$, with $r = \max\{s, 5\}$, instead of the analogous calculus inequality in the usual Sobolev space H^s. Moreover, it needs to remember that each differentiation with respect to x_1 "takes two derivatives" in the anisotropic weighted Sobolev space H^s_*.

Definition 1. Let $\mu \in \mathbb{N}$, $\mu \geq 3$. The initial data $(\mathbf{U}_0, f_0) \in H^{2\mu+1}_*(\mathbb{R}^3_+) \times H^{2\mu+1}(\mathbb{R}^2)$ are said to be compatible up to order μ when $(\mathbf{U}_j, f_j)$ satisfy (114) for $j = 0$ and

$$\sum_{i=2}^{3}\sum_{l=0}^{j}[v_{i,j-l}]\partial_i f_l = 0, \quad [p_j] + \sum_{l=0}^{j-1} C^l_{j-1}[(\mathbf{H}_l, \mathbf{H}_{j-l})] = 0 \tag{119}$$

for $j = 1, \dots, \mu$.

Remark 7. As for the case of a perfectly conducting wall boundary condition [33,46], in our case the compatibility conditions associated with the condition that the magnetic field is parallel to the boundary follow from the remaining compatibility conditions. That is, the compatibility conditions (116) associated with (15) are automatically satisfied by the condition $(H^{\pm}_{\mathrm{N}})_0|_{x_1=0} = 0$ and (119).

4.2. Construction of the approximate solution to problem (11)–(13)

To use the tame estimate (110) for the proof of convergence of the Nash-Moser iteration we need to reduce our nonlinear problem (11)–(13) on $[0, T] \times \mathbb{R}^3_+$ to that on Ω_T which solutions vanish in the past. This is achieved by the classical argument suggesting to absorb the initial data into the interior equations by constructing a so-called approximate solution. In our case the only nonstandard point is that this approximate solution should satisfy not only the system resulting from taking jth time derivatives of (11) and evaluating at $t = 0$ but also the equations for $\mathbf{H}$ contained in (11) (see (197)) for all times t. Moreover, as in [8], the boundary conditions should also be satisfied by the approximate solution for all times. Below we will use the notation

$$\mathbb{L}(\mathbf{U}, \boldsymbol{\Psi}) := \begin{pmatrix} \mathbb{L}(\mathbf{U}^+, \Psi^+) \\ \mathbb{L}(\mathbf{U}^-, \Psi^-) \end{pmatrix}.$$

Lemma 6. *Suppose the initial data (13) are compatible up to order μ and satisfy the assumptions of Theorem 1 (i.e., (5), (9), (10), (14), and (15)). Then there exists a vector-function $(\mathbf{U}^a, f^a) \in H^{\mu+1}_*(\Omega_T) \times H^{\mu+1}(\partial\Omega_T)$, that is further called the approximate solution to problem (11)–(13), such that*

$$\partial_t^j \mathbb{L}(\mathbf{U}^a, \boldsymbol{\Psi}^a)|_{t=0} = 0 \quad \text{for } j = 0, \dots, \mu - 1, \tag{120}$$

and it satisfies equations (197) and the boundary conditions (12) and (15), where $\boldsymbol{\Psi}^a$ is associated to f^a like $\boldsymbol{\Psi}$ is associated to f. Moreover, the approximate solution obeys the estimate

$$[\mathbf{U}^a]_{\mu+1,*,T} + \|f^a\|_{H^{\mu+1}(\partial\Omega_T)} \le C_1(M_0) \tag{121}$$

and satisfies the hyperbolicity condition (5) and the divergent constraints (14) on Ω_T as well as the stability condition (9) and restriction (10) on $\partial\Omega_T$, where $C_1 = C_1(M_0) > 0$ is a constant depending on M_0 (see (118)).

Proof. We use the notations $\mathbf{U}^a = (\mathbf{U}^{a+}, \mathbf{U}^{a-})$, $\mathbf{v}^a = (\mathbf{v}^{a+}, \mathbf{v}^{a-})$, etc. Consider functions $\mathbf{v}^a, \mathbf{H}^a, S^a \in H^{\mu+1}_*(\mathbb{R} \times \mathbb{R}^3_+)$ and $f^a \in H^{\mu+1}(\mathbb{R}^3)$ such that

$$\partial_t^j (\mathbf{v}^a, \mathbf{H}^a, S^a)|_{t=0} = (\mathbf{v}_j, \mathbf{H}_j, S_j) \in H^{2(\mu-j)+1}_*(\mathbb{R}^3_+) \quad \text{for } j = 0, \dots, \mu,$$

$$f^a|_{t=0} = f_0 \in H^{2\mu+1}(\mathbb{R}^2), \quad \partial_t^j f_j|_{t=0} \in H^{2(\mu-j+1)}(\mathbb{R}^2) \quad \text{for } j = 1, \dots, \mu,$$

where $\mathbf{U}_j$ and f_j are given by Lemma 5 ($\mathbf{v}_j = (v^+_{1,j}, \dots, v^-_{3,j})$, $S_j = (S^+_j, S^-_j)$, etc.). Thanks to the compatibility conditions (114), that imply (115), and the definition $\mathbf{U}_j$ and f_j, we can choose $\mathbf{v}^a$, $\mathbf{H}^a$, and f^a that satisfy

$$\partial_t f^a = (v_1^{a\pm} - v_2^{a\pm}\partial_2 f^a - v_3^{a\pm}\partial_3 f^a)|_{x_1=0}\,, \tag{122}$$

$$\mathbb{L}_{\mathrm{H}}(\mathbf{v}^a, \mathbf{H}^a, \boldsymbol{\Psi}^a) = 0, \tag{123}$$

where (123) is the compact form of equations (197) written for $(\mathbf{v}^a, \mathbf{H}^a, f^a)$. From (122) and (123) we get that $(\mathbf{v}^a, \mathbf{H}^a, f^a)$ satisfies (198) and (199) (see the proof of Proposition 1 in Appendix A). Since $(\mathbf{v}^a, \mathbf{H}^a, f^a)|_{t=0} = (\mathbf{v}_0, \mathbf{H}_0, f_0)$, it follows from the assumptions made for the initial data that $(\mathbf{v}^a, \mathbf{H}^a, f^a)$ satisfies the divergent constraints (14) and the boundary conditions (15).

Then we define $p^a \in H^{\mu+1}_*(\mathbb{R} \times \mathbb{R}^3_+)$ such that

$$\partial_t^j p^a|_{t=0} = p_j \in H^{2(\mu-j)+1}_*(\mathbb{R}^3_+) \quad \text{for } j = 0, \dots, \mu.$$

Thanks to the compatibility conditions we can choose p^a that

$$[p^a + (|\mathbf{H}^a|/2)] = 0$$

($[p^a] = p^{a+}_{|x_1=0} - p^{a-}_{|x_1=0}$, etc.). By using a cut-off C_0^∞ function we can suppose that $(\mathbf{U}^a, f^a)$ vanishes outside of the interval $[-T, T]$, i.e., $(\mathbf{U}^a, f^a) \in$

$H^{\mu+1}_*(\Omega_T) \times H^{\mu+1}(\partial\Omega_T)$. Applying Sobolev's embeddings (in particular, inequalities (209)), we rewrite estimate (117) as

$$\sum_{j=1}^{\mu} \left(\|\mathbf{U}_j\|_{2(\mu-j)+1,*} + \|f_j\|_{H^{2(\mu-j+1)}(\mathbb{R}^2)} \right) \le C(M_0), \tag{124}$$

where $C = C(M_0) > 0$ is a constant depending on M_0. The estimate (121) follows from (124) and the continuity of the lifting operators from the hyperplane $t = 0$ to $\mathbb{R} \times \mathbb{R}^3_+$.

Conditions (120) hold thanks to the properties of $(\mathbf{U}_j, f_j)$ given by Lemma 5. At last, since $(\mathbf{U}^a, f^a)$ satisfies the hyperbolicity condition (5) and the stability conditions (9), (10) at $t = 0$, in the above procedure we can choose $(\mathbf{U}^a, f^a)$ that it satisfies (9), (10) (at $x_1 = 0$), and (5) for all times $t \in [-T, T]$. □

Without loss of generality we can suppose that

$$\|\mathbf{U}_0\|_{2\mu+1,*} + \|f_0\|_{H^{2\mu+1}(\mathbb{R}^2)} \le 1, \tag{125}$$

Also, without loss of generality, we assume that

$$\|f_0\|_{H^{2\mu+1}(\mathbb{R}^2)} \le 1/2.$$

Then for a sufficiently short time interval $[0, T]$ the smooth solution which existence we are going to prove satisfies $\|f\|_{L_\infty([0,T]\times\mathbb{R}^2)} \le 1$ that implies

$$\partial_1 \Phi^+ \ge 1/2, \quad \partial_1 \Phi^- \le -1/2 \tag{126}$$

(recall that $\|\chi'\|_{L_\infty(\mathbb{R})} < 1/2$, see Section 1). Let μ is an integer number that will appear in the regularity assumption for the initial data in the existence theorem for problem (11)–(13). Running ahead, we take $\mu = m + 9$, with $m \ge 12$ (see Theorem 1). In the end of Section 5 we will see that this choice is suitable. Taking into account (125), we rewrite (121) as

$$[\mathbf{U}^a]_{m+10,*,T} + \|f^a\|_{H^{m+10}(\partial\Omega_T)} \le C_*, \tag{127}$$

where $C_* = C_1(1)$.

Let us introduce

$$\mathbf{F}^a := \begin{cases} -\mathbb{L}(\mathbf{U}^a, \boldsymbol{\Psi}^a) & \text{for } t > 0, \\ 0 & \text{for } t > 0, \end{cases} \tag{128}$$

Since $(\mathbf{U}^a, f^a) \in H^{m+10}_*(\Omega_T) \times H^{m+10}(\partial\Omega_T)$, exploiting (120), one gets $\mathbf{F}^a \in H^{m+8}_*(\Omega_T)$ and

$$[\mathbf{F}^a]_{m+8,*} \le \delta_0(T), \tag{129}$$

where the constant $\delta_0(T) \to 0$ as $T \to 0$. To prove estimate (129) we use the Moser-type and embedding inequalities from Appendix B, estimate (127), and the fact that $\mathbf{F}^a$ vanishes in the past.

Then, given the approximate solution defined in Lemma 6, $(\mathbf{U},f) = (\mathbf{U}^a,f^a)+(\widetilde{\mathbf{U}},\tilde{f})$ is a solution of the original problem (11)–(13) on $[0,T]\times\mathbb{R}^3_+$ if $(\widetilde{\mathbf{U}},\tilde{f})$ satisfies the following problem on Ω_T (tildes are dropped):

$$\mathcal{L}(\mathbf{U},\boldsymbol{\Psi}) = \mathbf{F}^a \quad \text{in } \Omega_T, \tag{130}$$

$$\mathcal{B}(\mathbf{U},f) = 0 \quad \text{on } \partial\Omega_T, \tag{131}$$

$$(\mathbf{U},f) = 0 \quad \text{for } t<0, \tag{132}$$

where

$$\begin{aligned} \mathcal{L}(\mathbf{U},\boldsymbol{\Psi}) &:= \mathbb{L}(\mathbf{U}^a+\mathbf{U},\boldsymbol{\Psi}^a+\boldsymbol{\Psi}) - \mathbb{L}(\mathbf{U}^a,\boldsymbol{\Psi}^a),\\ \mathcal{B}(\mathbf{U},f) &= \mathcal{B}(\mathbf{U}^a+\mathbf{U},f^a+f) - \mathcal{B}(\mathbf{U}^a,f^a). \end{aligned} \tag{133}$$

From now on we concentrate on the proof of the existence of solutions to problem (130)–(132).

5. Nash-Moser iteration

5.1. Iteration scheme for solving problem (130)–(132)

We solve problem (130)–(132) by a suitable Nash-Moser-type iteration scheme. The general description of the Nash-Moser method can be find, for example, in [14] (see also references therein). The main idea is to solve the equation $\mathcal{F}(\mathbf{u}) = 0$ by the iteration scheme

$$\mathcal{F}'(S_{\theta_n}\mathbf{u}_n)(\mathbf{u}_{n+1}-\mathbf{u}_n) = -\mathcal{F}(\mathbf{u}_n),$$

where $\mathcal{F}'$ is the linearization of $\mathcal{F}$ and S_{θ_n} is a sequence of smoothing operators, with $S_{\theta_n}\to I$ as $n\to\infty$. This scheme is the classical Newton's scheme if $S_{\theta_n} = I$.

Errors of a classical Nash-Moser iteration are the "quadratic" error of Newton's scheme and the "substitution" error caused by the application of smoothing operators. As in [8], in our case the Nash-Moser procedure is not completely standard and we have the additional error caused by the introduction of an intermediate state $\mathbf{u}_{n+1/2}$ satisfying some nonlinear constraints and the error caused by dropping the zero-order term in $\Psi^{\pm}$ in the linearized interior equations written in terms of the "good unknown" (see (24) and (25)). In our case the intermediate (or modified) state should satisfy the same constraints/assumptions that were required to be fulfilled for the basic state (16), i.e., the mentioned nonlinear constraints are (17)–(19), (21), and (22).

Here we closely follow the plan and notations of [8]. The main differences from [8] are that we work in the anisotropic weighted Sobolev spaces H^s_* and instead of the assumption that the initial data are close to the piecewise constant solution, that allows to construct small approximate solutions, we make the assumption on the smallness of the time interval. Moreover, we have additional constraints associated to (14) and (15). We first list the important properties of smoothing operators. The following proposition is the analogue of the corresponding ones from [1,8,14].

Proposition 10. *There exists such a family $\{S_\theta\}_{\theta\geq 1}$ of smoothing operators in $H^s_*(\Omega_T)$ acting on the class of functions vanishing in the past that*

$$[S_\theta u]_{\beta,*,T} \leq C\theta^{(\beta-\alpha)_+}[u]_{\alpha,*,T}, \qquad \alpha,\beta\geq 0, \tag{134}$$

$$[S_\theta u - u]_{\beta,*,T} \leq C\theta^{\beta-\alpha}[u]_{\alpha,*,T}, \qquad 0\leq\beta\leq\alpha, \tag{135}$$

$$\Big[\frac{d}{d\theta}S_\theta u\Big]_{\beta,*,T} \leq C\theta^{\beta-\alpha-1}[u]_{\alpha,*,T}, \qquad \alpha,\beta\geq 0, \tag{136}$$

where $C>0$ is a constant, and $(\beta-\alpha)_+ := \max(0,\beta-\alpha)$. Moreover, there is another family of smoothing operators (still denoted S_θ) acting on functions defined on the boundary $\partial\Omega_T$ and meeting properties (134)–(136), with the norms $\|\cdot\|_{H^\alpha(\partial\Omega_T)}$.

Now, following [8] (with necessary modifications), we describe the iteration scheme for problem (130)–(132). We choose

$$\mathbf{U}_0 = 0, \quad f_0 = 0$$

and assume that $(\mathbf{U}_k, f_k)$ are already given for $k = 0,\dots,n$. Moreover, let $(\mathbf{U}_k, f_k)$ vanish in the past, i.e., they satisfy (132). Below we again use the notations like $\mathbf{U}_k = (\mathbf{U}_k^+, \mathbf{U}_k^-)$, $\boldsymbol{\Psi}_k = (\Psi_k^+, \Psi_k^-)$, etc., and in spite of the fact that these notations are somewhere inapplicable, for short we usually drop the $\pm$ superscripts. We define

$$\mathbf{U}_{n+1} = \mathbf{U}_n + \delta\mathbf{U}_n, \quad f_{n+1} = f_n + \delta f_n,$$

where the differences $\delta\mathbf{U}_n$ and δf_n solve the linear problem

$$\begin{aligned}
&\mathbb{L}'_e(\mathbf{U}^a + \mathbf{U}_{n+1/2}, \boldsymbol{\Psi}^a + \boldsymbol{\Psi}_{n+1/2})\delta\dot{\mathbf{U}}_n = \mathbf{f}_n && \text{in } \Omega_T,\\
&\mathbb{B}'_{n+1/2}(\delta\dot{\mathbf{U}}_n, \delta f_n) = \mathbf{g}_n && \text{on } \partial\Omega_T,\\
&(\delta\dot{\mathbf{U}}_n, \delta f_n) = 0 && \text{for } t<0.
\end{aligned} \tag{137}$$

Here

$$\delta\dot{\mathbf{U}}_n := \delta\mathbf{U}_n - \frac{\delta\Psi_n}{\partial_1(\Phi^a + \Psi_{n+1/2})}\,\partial_1(\mathbf{U}^a + \mathbf{U}_{n+1/2}) \tag{138}$$

is the "good unknown" (cf. (23)),

$$\mathbb{B}'_{n+1/2} := \mathbb{B}'_e((\mathbf{U}^a + \mathbf{U}_{n+1/2})|_{x_1=0}, f^a + f_{n+1/2}),$$

the operators $\mathbb{L}'_e$ and $\mathbb{B}'_e$ are defined in (25)–(27), and $(\mathbf{U}_{n+1/2}, f_{n+1/2})$ is a smooth modified state such that $(\mathbf{U}^a + \mathbf{U}_{n+1/2}, f^a + f_{n+1/2})$ satisfies constraints (17)–(19), (21), and (22) ($\boldsymbol{\Psi}_{n+1/2}$ is associated to $f_{n+1/2}$ like $\boldsymbol{\Psi}$ is associated to f). The right-hand sides $\mathbf{f}_n$ and $\mathbf{g}_n$ are defined through the accumulated errors at the step n.

Let us now specify the errors of the iteration scheme. They are defined from the following chains of decompositions:

$$\begin{aligned}
&\mathcal{L}(\mathbf{U}_{n+1},\boldsymbol{\Psi}_{n+1})-\mathcal{L}(\mathbf{U}_n,\boldsymbol{\Psi}_n)\\
&=\mathbb{L}'(\mathbf{U}^a+\mathbf{U}_n,\boldsymbol{\Psi}^a+\boldsymbol{\Psi}_n)(\delta\mathbf{U}_n,\delta\boldsymbol{\Psi}_n)+\mathbf{e}'_n\\
&=\mathbb{L}'(\mathbf{U}^a+S_{\theta_n}\mathbf{U}_n,\boldsymbol{\Psi}^a+S_{\theta_n}\boldsymbol{\Psi}_n)(\delta\mathbf{U}_n,\delta\boldsymbol{\Psi}_n)+\mathbf{e}'_n+\mathbf{e}''_n\\
&=\mathbb{L}'(\mathbf{U}^a+\mathbf{U}_n,\boldsymbol{\Psi}^a+\boldsymbol{\Psi}_{n+1/2})(\delta\mathbf{U}_n,\delta\boldsymbol{\Psi}_n)+\mathbf{e}'_n+\mathbf{e}''_n+\mathbf{e}'''_n\\
&=\mathbb{L}'_e(\mathbf{U}^a+\mathbf{U}_{n+1/2},\boldsymbol{\Psi}^a+\boldsymbol{\Psi}_{n+1/2})\delta\dot{\mathbf{U}}_n+\mathbf{e}'_n+\mathbf{e}''_n+\mathbf{e}'''_n+\mathbb{D}_{n+1/2}\delta\boldsymbol{\Psi}_n
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{B}(\mathbf{U}_{n+1}|_{x_1=0},f_{n+1})-\mathcal{B}(\mathbf{U}_n|_{x_1=0},f_n)\\
&=\mathbb{B}'((\mathbf{U}^a+\mathbf{U}_n)|_{x_1=0},f^a+f_n)(\delta\mathbf{U}_n|_{x_1=0},\delta f_n)+\tilde{\mathbf{e}}'_n\\
&=\mathbb{B}'((\mathbf{U}^a+S_{\theta_n}\mathbf{U}_n)|_{x_1=0},f^a+S_{\theta_n}f_n)(\delta\mathbf{U}_n|_{x_1=0},\delta f_n)+\tilde{\mathbf{e}}'_n+\tilde{\mathbf{e}}''_n\\
&=\mathbb{B}'_{n+1/2}(\delta\dot{\mathbf{U}}_n,\delta f_n)+\tilde{\mathbf{e}}'_n+\tilde{\mathbf{e}}''_n+\tilde{\mathbf{e}}'''_n,
\end{aligned}$$

where S_{θ_n} are smoothing operators enjoying the properties of Proposition 10, with the sequence (θ_n) defined by

$$\theta_0\geq 1,\quad \theta_n=\sqrt{\theta_0+n},$$

and we use the notations

$$S_{\theta_n}\boldsymbol{\Psi}_n:=\begin{pmatrix}\chi(x_1)S_{\theta_n}f_n\\ \chi(-x_1)S_{\theta_n}f_n\end{pmatrix},\quad \mathbb{D}_{n+1/2}\delta\boldsymbol{\Psi}_n:=\begin{pmatrix}\mathbf{D}^+_{n+1/2}\delta\Psi^+_n\\ \mathbf{D}^-_{n+1/2}\delta\Psi^-_n\end{pmatrix},$$

with

$$\mathbf{D}_{n+1/2}:=\frac{1}{\partial_1(\Phi^a+\Psi_{n+1/2})}\,\partial_1\left\{\mathbb{L}(\mathbf{U}^a+\mathbf{U}_{n+1/2},\Psi^a+\Psi_{n+1/2})\right\}.$$

The errors $\mathbf{e}'_n$ and $\tilde{\mathbf{e}}'_n$ are the usual quadratic errors of Newton's method, and $\mathbf{e}''_n$, $\tilde{\mathbf{e}}''_n$ and $\mathbf{e}'''_n$, $\tilde{\mathbf{e}}'''_n$ are the first and the second substitution errors respectively.

Let

$$\mathbf{e}_n:=\mathbf{e}'_n+\mathbf{e}''_n+\mathbf{e}'''_n+\mathbb{D}_{n+1/2}\delta\boldsymbol{\Psi}_n,\quad \tilde{\mathbf{e}}_n:=\tilde{\mathbf{e}}'_n+\tilde{\mathbf{e}}''_n+\tilde{\mathbf{e}}'''_n,\tag{139}$$

then the accumulated errors at the step $n\geq 1$ are

$$\mathbf{E}_n=\sum_{k=0}^{n-1}\mathbf{e}_k,\quad \widetilde{\mathbf{E}}_n=\sum_{k=0}^{n-1}\tilde{\mathbf{e}}_k,\tag{140}$$

with $\mathbf{E}_0:=0$ and $\widetilde{\mathbf{E}}_0:=0$. The right-hand sides $\mathbf{f}_n$ and $\mathbf{g}_n$ are recursively computed from the equations

$$\sum_{k=0}^{n}\mathbf{f}_k+S_{\theta_n}\mathbf{E}_n=S_{\theta_n}\mathbf{F}^a,\quad \sum_{k=0}^{n}\mathbf{g}_k+S_{\theta_n}\widetilde{\mathbf{E}}_n=0,\tag{141}$$

where $\mathbf{f}_0:=S_{\theta_0}\mathbf{F}^a$ and $\mathbf{g}_0:=0$.

Since $S_{\theta_N}\to I$ as $N\to\infty$, one can show (see [8]) that we formally obtain the solution to problem (130)–(132) from $\mathcal{L}(\mathbf{U}_N,\boldsymbol{\Psi}_N)\to\mathbf{F}^a$ and $\mathcal{B}(\mathbf{U}_N|_{x_1=0},f_N)\to 0$, provided that $(\mathbf{e}_N,\tilde{\mathbf{e}}_N)\to 0$.

5.2. Inductive hypothesis

Given a small number $\delta > 0$, the integer $\alpha := m+1$, and an integer $\tilde{\alpha}$, our inductive hypothesis reads:

$$(H_{n-1}) \quad \begin{cases} a) \ \forall\, k=0,\dots,n-1, \quad \forall s \in [6,\tilde{\alpha}] \cap \mathbb{N}, \\ \quad [\delta \mathbf{U}_k]_{s,*,T} + \|\delta f_k\|_{H^s(\partial\Omega_T)} \le \delta \theta_k^{s-\alpha-1} \Delta_k, \\ b) \ \forall\, k=0,\dots,n-1, \quad \forall s \in [6,\tilde{\alpha}-2] \cap \mathbb{N}, \\ \quad [\mathcal{L}(\mathbf{U}_k, \boldsymbol{\Psi}_k) - \mathbf{F}^a]_{s,*,T} \le 2\delta \theta_k^{s-\alpha-1}, \\ c) \ \forall\, k=0,\dots,n-1, \quad \forall s \in [7,\alpha] \cap \mathbb{N}, \\ \quad \|\mathcal{B}(\mathbf{U}_k|_{x_1=0}, f_k)\|_{H^s(\partial\Omega_T)} \le \delta \theta_k^{s-\alpha-1}, \end{cases}$$

where $\Delta_k = \theta_{k+1} - \theta_k$. Note that the sequence (Δ_n) is decreasing and tends to zero, and

$$\forall\, n \in \mathbb{N}, \quad \frac{1}{3\theta_n} \le \Delta_n = \sqrt{\theta_n^2+1} - \theta_n \le \frac{1}{2\theta_n}.$$

Recall that $(\mathbf{U}_k, f_k)$ for $k = 0,\dots,n$ are also assumed to satisfy (132). Running a few steps forward, we observe that we will need to use inequalities (127) and (129) with $m = \tilde{\alpha} - 6$. That is, we now choose $\tilde{\alpha} = m + 6$.

Our goal is to prove that (H_{n-1}) implies (H_n) for a suitable choice of parameters $\theta_0 \ge 1$ and $\delta > 0$, and for a sufficiently short time $T > 0$. After that we shall prove (H_0). From now on we assume that (H_{n-1}) holds. As in [8], we have the following consequences.

Lemma 7. *If θ_0 is big enough, then for every $k = 0,\dots,n$ and for every integer $s \in [6,\tilde{\alpha}]$ we have*

$$[\mathbf{U}_k]_{s,*,T} + \|f_k\|_{H^s(\partial\Omega_T)} \le \delta \theta_k^{(s-\alpha)_+}, \quad \alpha \ne s, \tag{142}$$

$$[\mathbf{U}_k]_{\alpha,*,T} + \|f_k\|_{H^\alpha(\partial\Omega_T)} \le \delta \log \theta_k, \tag{143}$$

$$[(I - S_{\theta_k})\mathbf{U}_k]_{s,*,T} + \|(1 - S_{\theta_k}) f_k\|_{H^s(\partial\Omega_T)} \le C\delta \theta_k^{s-\alpha}. \tag{144}$$

For every $k = 0,\dots,n$ and for every integer $s \in [6,\tilde{\alpha}+6]$ we have

$$[S_{\theta_k}\mathbf{U}_k]_{s,*,T} + \|S_{\theta_k} f_k\|_{H^s(\partial\Omega_T)} \le C\delta \theta_k^{(s-\alpha)_+}, \quad \alpha \ne s, \tag{145}$$

$$[S_{\theta_k}\mathbf{U}_k]_{\alpha,*,T} + \|S_{\theta_k} f_k\|_{H^\alpha(\partial\Omega_T)} \le C\delta \log \theta_k. \tag{146}$$

Observe that (144)–(146) follow from (142), (143), and Proposition 10. Moreover, (144) and (145) hold actually for every integer $s \ge 6$ but below we will need them only for $s \in [6,\tilde{\alpha}]$ and $s \in [6,\tilde{\alpha}+6]$ respectively.

5.3. Estimate of the quadratic errors

Recall that the quadratic errors read

$$\mathbf{e}'_k = \mathcal{L}(\mathbf{U}_{k+1},\boldsymbol{\Psi}_{k+1}) - \mathcal{L}(\mathbf{U}_k,\boldsymbol{\Psi}_k) - \mathcal{L}'(\mathbf{U}_k,\boldsymbol{\Psi}_k)(\delta\mathbf{U}_k,\delta\boldsymbol{\Psi}_k),$$

$$\tilde{\mathbf{e}}'_k = \big(\mathcal{B}(\mathbf{U}_{k+1},f_{k+1}) - \mathcal{B}(\mathbf{U}_k,f_k) - \mathcal{B}'(\mathbf{U}_k,f_k)(\delta\mathbf{U}_k,\delta f_k)\big)|_{x_1=0},$$

where $\mathcal{L}$ and $\mathcal{B}$ are given in (133). We define the second derivatives of the operators $\mathbb{L}$ and $\mathbb{B}$:

$$\mathbb{L}''(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})((\mathbf{U}',\boldsymbol{\Psi}'),(\mathbf{U}'',\boldsymbol{\Psi}'')) := \frac{d}{d\varepsilon}\mathbb{L}'(\mathbf{U}_\varepsilon,\boldsymbol{\Psi}_\varepsilon)(\mathbf{U}',\boldsymbol{\Psi}')|_{\varepsilon=0},$$

$$\mathbb{B}''((\mathbf{W}',f'),(\mathbf{W}'',f'')) := \frac{d}{d\varepsilon}\mathbb{B}'(\mathbf{W}_\varepsilon,f_\varepsilon)(\mathbf{W}',f')|_{\varepsilon=0},$$

where $\mathbf{U}_\varepsilon = \widehat{\mathbf{U}} + \varepsilon\mathbf{U}''$, $\mathbf{W}_\varepsilon = \widehat{\mathbf{U}}|_{x_1=0} + \varepsilon\mathbf{W}''$, $f_\varepsilon = \hat{f} + \varepsilon f''$, and $\boldsymbol{\Psi}'$ and $\boldsymbol{\Psi}''$ are associated to f' and f'' respectively like $\boldsymbol{\Psi}$ is associated to f. Moreover, we easily find the explicit form of $\mathbb{B}''$, that do not depend on the state $(\widehat{\mathbf{U}},\hat{f})$:

$$\mathbb{B}''((\mathbf{W}',f'),(\mathbf{W}'',f'')) = \begin{pmatrix} v_2'^{+}\partial_2 f'' + v_3'^{+}\partial_3 f'' + v_2''^{+}\partial_2 f' + v_3''^{+}\partial_3 f' \\ v_2'^{-}\partial_2 f'' + v_3'^{-}\partial_3 f'' + v_2''^{-}\partial_2 f' + v_3''^{-}\partial_3 f' \\ [(\mathbf{H}',\mathbf{H}'')] \end{pmatrix}.$$

Then, the quadratic errors can be rewritten as

$$\mathbf{e}'_k = \int_0^1 (1-\tau)\mathbb{L}''(\mathbf{U}^a + \mathbf{U}_k + \tau\delta\mathbf{U}_k, \boldsymbol{\Psi}^a + \boldsymbol{\Psi}_k + \tau\delta\boldsymbol{\Psi}_k)\big((\delta\mathbf{U}_k,\delta\boldsymbol{\Psi}_k),(\delta\mathbf{U}_k,\delta\boldsymbol{\Psi}_k)\big)d\tau, \tag{147}$$

$$\tilde{\mathbf{e}}'_k = \frac{1}{2}\,\mathbb{B}''\big((\delta\mathbf{U}_k|_{x_1=0},\delta f_k),(\delta\mathbf{U}_k|_{x_1=0},\delta f_k)\big). \tag{148}$$

To estimate the quadratic errors by utilizing representations (147) and (148) we need estimates for $\mathbb{L}''$ and $\mathbb{B}''$. They can easily be obtained from the explicit forms of $\mathbb{L}''$ and $\mathbb{B}''$ by applying the Moser-type and embedding inequalities from Appendix B (as well as such inequalities for classical Sobolev spaces) . Omitting detailed calculations, we get the following result.

Proposition 11. *Let $T>0$ and $s\in\mathbb{N}$, with $s\geq 6$. Assume that $(\widehat{\mathbf{U}},\hat{f}) \in H^{s+2}_*(\Omega_T)\times H^{s+2}(\partial\Omega_T)$ and*

$$[\widehat{\mathbf{U}}]_{7,*,T} + \|\hat{f}\|_{H^7(\partial\Omega_T)} \leq \widetilde{K}.$$

Then there exists a positive constant $\widetilde{K}_0$, that does not depend on s and T, and there exists a constant $C(\widetilde{K}_0)>0$ such that, if $\widetilde{K}\leq\widetilde{K}_0$ and $(\mathbf{U}',f')$, $(\mathbf{U}'',f'')\in H^{s+2}_*(\Omega_T)\times H^{s+2}(\partial\Omega_T)$, then

$$\big[\mathbb{L}''(\widehat{\mathbf{U}},\widehat{\boldsymbol{\Psi}})((\mathbf{U}',\boldsymbol{\Psi}'),(\mathbf{U}'',\boldsymbol{\Psi}''))\big]_{s,*,T} \leq C(\widetilde{K}_0)\Big\{ \langle\!\langle(\widehat{\mathbf{U}},\hat{f})\rangle\!\rangle_{s+2}\langle\!\langle(\mathbf{U}',f')\rangle\!\rangle_7\langle\!\langle(\mathbf{U}'',f'')\rangle\!\rangle_7 + \langle\!\langle(\mathbf{U}',f')\rangle\!\rangle_{s+2}\langle\!\langle(\mathbf{U}'',f'')\rangle\!\rangle_7 + \langle\!\langle(\mathbf{U}'',f'')\rangle\!\rangle_{s+2}\langle\!\langle(\mathbf{U}',f')\rangle\!\rangle_7\Big\},$$

where $\langle\!\langle(\mathbf{U},f)\rangle\!\rangle_\ell := [\mathbf{U}]_{\ell,*,T} + \|f\|_{H^\ell(\partial\Omega_T)}$.

If $(\mathbf{W}',f')$, $(\mathbf{W}'',f'')\in H^s_*(\partial\Omega_T)\times H^{s+1}(\partial\Omega_T)$, *then*

$$\begin{aligned}\|\mathbb{B}''((\mathbf{W}',f'),(\mathbf{W}'',f''))\|_{H^s(\partial\Omega_T)} &\le C(\widetilde{K}_0)\Big\{\|\mathbf{W}'\|_{H^s(\partial\Omega_T)}\|f''\|_{H^3(\partial\Omega_T)}\\ &+\|\mathbf{W}'\|_{H^2(\partial\Omega_T)}\|f''\|_{H^{s+1}(\partial\Omega_T)} + \|\mathbf{W}''\|_{H^s(\partial\Omega_T)}\|f'\|_{H^3(\partial\Omega_T)}\\ &+\|\mathbf{W}''\|_{H^2(\partial\Omega_T)}\|f'\|_{H^{s+1}(\partial\Omega_T)} + \|\mathbf{W}'\|_{H^s(\partial\Omega_T)}\|\mathbf{W}''\|_{H^2(\partial\Omega_T)}\\ &+\|\mathbf{W}'\|_{H^2(\partial\Omega_T)}\|\mathbf{W}''\|_{H^s(\partial\Omega_T)}\Big\}.\end{aligned}$$

Without loss of generality we assume that the constant $\widetilde{K}_0 = 2C_*$, where C_* is the constant from (127). Using (147), (148), and Proposition 11, we have the following result.

Lemma 8. *Let* $\alpha \ge 8$. *There exist* $\delta > 0$ *sufficiently small, and* $\theta_0 \ge 1$ *sufficiently large, such that for all* $k = 0,\dots n-1$, *and for all integer* $s \in [6,\widetilde{\alpha}-2]$, *we have the estimates*

$$[\mathbf{e}'_k]_{s,*,T} \le C\delta^2\theta_k^{L_1(s)-1}\Delta_k, \tag{149}$$

$$\|\tilde{\mathbf{e}}'_k\|_{H^s(\partial\Omega_T)} \le C\delta^2\theta_k^{L_1(s)-1}\Delta_k, \tag{150}$$

where $L_1(s) = \max\{(s+2-\alpha)_+ + 10 - 2\alpha, s+6-2\alpha\}$.

Proof. Taking into account (127) (recall that $m = \tilde{\alpha}-6$), (H_{n-1}), and (142), we estimate the "coefficient" of $\mathbb{L}''$ in (147) as follows:

$$\begin{aligned}\sup_{\tau\in[0,1]} &\langle\!\langle(\mathbf{U}^a+\mathbf{U}_k+\tau\delta\mathbf{U}_k, f^a+f_k+\tau\delta f_k)\rangle\!\rangle_7\\ &\le C_* + \delta\theta_k^{(7-\alpha)_+} + \delta\theta_k^{6-\alpha}\Delta_k \le C_* + C\delta \le 2C_*\end{aligned}$$

for δ sufficiently small. Hence, we may apply Proposition 11:

$$\begin{aligned}[\mathbf{e}'_k]_{s,*,T} \le C\Big(&\delta^2\theta_k^{12-2\alpha}\Delta_k^2\big(C_* + \langle\!\langle(\mathbf{U}_k,f_k)\rangle\!\rangle_{s+2} + \langle\!\langle(\delta\mathbf{U}_k,\delta f_k)\rangle\!\rangle_{s+2}\big)\\ &+\delta^2\theta_k^{s+7-2\alpha}\Delta_k^2\Big)\end{aligned}$$

for $s\in[6,\widetilde{\alpha}-2]$. If $s+2\neq\alpha$, it follows from (142) that

$$[\mathbf{e}'_k]_{s,*,T} \le C\delta^2\Delta_k^2\left\{\theta_k^{(s+2-\alpha)_+ + 12-2\alpha} + \theta_k^{s+7-2\alpha}\right\} \le C\delta^2\theta_k^{L_1(s)-1}\Delta_k$$

(here we have utilized the inequality $\theta_k\Delta_k \le 1/2$). If $s+2=\alpha$ and $\alpha\ge 8$,

$$\begin{aligned}[\mathbf{e}'_k]_{s,*,T} &\le C\delta^2\Delta_k^2\left\{(C_* + \delta\log\theta_k + \delta\theta_k^{-1}\Delta_k)\theta_k^{12-\alpha} + \theta_k^{5-\alpha}\right\}\\ &\le C\delta^2\Delta_k^2\theta_k^{5-\alpha} \le C\delta^2\theta_k^{L_1(\alpha-2)-1}\Delta_k.\end{aligned}$$

By using (148), the trace theorem [28] (cf. (35)), and Proposition 11, we estimate the quadratic error on the boundary as

$$\begin{aligned}[\tilde{\mathbf{e}}'_k]_{s,*,T} &\le C\big\{[\delta\mathbf{U}_k]_{s+1,*,T}\|\delta f_k\|_{H^6(\partial\Omega_T)} + [\delta\mathbf{U}_k]_{6,*,T}\|\delta f_k\|_{H^{s+1}(\partial\Omega_T)}\\ &+[\delta\mathbf{U}_k]_{6,*,T}[\delta\mathbf{U}_k]_{s+1,*,T}\big\} \le C\delta\theta_k^{s-\alpha}\Delta_k\delta\theta^{5-\alpha}\Delta_k \le C\delta^2\theta_k^{L_1(s)-1}\Delta_k,\end{aligned}$$

that completes the proof. □

5.4. Estimate of the first substitution errors

The first substitution errors can be rewritten as follows:

$$\begin{aligned}\mathbf{e}''_k &= \mathcal{L}'(\mathbf{U}_k,\boldsymbol{\Psi}_k)(\delta\mathbf{U}_k,\delta\boldsymbol{\Psi}_k) - \mathcal{L}'(S_{\theta_k}\mathbf{U}_k,S_{\theta_k}\boldsymbol{\Psi}_k)(\delta\mathbf{U}_k,\delta\boldsymbol{\Psi}_k)\\ &= \int_0^1 \mathbb{L}''\big(\mathbf{U}^a + S_{\theta_k}\mathbf{U}_k+\tau(I-S_{\theta_k})\mathbf{U}_k,\boldsymbol{\Psi}^a+S_{\theta_k}\boldsymbol{\Psi}_k \\ &+\tau(I-S_{\theta_k})\boldsymbol{\Psi}_k\big)\big((\delta\mathbf{U}_k,\delta\boldsymbol{\Psi}_k),((I-S_{\theta_k})\mathbf{U}_k,(I-S_{\theta_k})\boldsymbol{\Psi}_k)\big)d\tau,\end{aligned} \tag{151}$$

$$\begin{aligned}\tilde{\mathbf{e}}''_k &= \big(\mathcal{B}'(\mathbf{U}_k,f_k)(\delta\mathbf{U}_k,\delta f_k)-\mathcal{B}'(S_{\theta_k}\mathbf{U}_k,S_{\theta_k}f_k)(\delta\mathbf{U}_k,\delta f_k)\big)|_{x_1=0}\\ &= \mathbb{B}''\big((\delta\mathbf{U}_k|_{x_1=0},\delta f_k),((\mathbf{U}_k-S_{\theta_k}\mathbf{U}_k)|_{x_1=0},f_k-S_{\theta_k}f_k)\big).\end{aligned} \tag{152}$$

Lemma 9. *Let $\alpha\geq 8$. There exist $\delta>0$ sufficiently small, and $\theta_0\geq 1$ sufficiently large, such that for all $k=0,\dots n-1$, and for all integer $s\in[6,\widetilde{\alpha}-2]$, one has*

$$[\mathbf{e}''_k]_{s,*,T}\leq C\delta^2\theta_k^{L_2(s)-1}\Delta_k, \tag{153}$$

$$\|\tilde{\mathbf{e}}''_k\|_{H^s(\partial\Omega_T)}\leq C\delta^2\theta_k^{L_2(s)-1}\Delta_k, \tag{154}$$

where $L_2(s)=\max\{(s+2-\alpha)_+ + 14-2\alpha,\ s+9-2\alpha\}$.

Proof. It follows from (127), (H_{n-1}), (144), and (145) that

$$\sup_{\tau\in[0,1]}\langle\!\langle(\mathbf{U}^a+S_{\theta_k}\mathbf{U}_k+\tau(I-S_{\theta_k})\mathbf{U}_k,f^a+S_{\theta_k}f_k+\tau(I-S_{\theta_k})f_k)\rangle\!\rangle_7 \leq 2C_*$$

for δ sufficiently small, i.e., we may apply Proposition 11 for estimating $\mathbb{L}''$ in (151). Using again (127), (H_{n-1}), (144), and (145), we obtain for $s+2\neq\alpha$ and $s+2\leq\tilde{\alpha}$ that

$$\begin{aligned}[\mathbf{e}''_k]_{s,*,T}&\leq C\Big\{\delta^2\theta_k^{13-2\alpha}\Delta_k\big(C_*+\delta\theta_k^{(s+2-\alpha)_+}+\delta\theta_k^{s+2-\alpha}\big)+\delta^2\theta_k^{s+8-2\alpha}\Delta_k\Big\}\\ &\leq C\delta^2\Delta_k\Big\{\theta_k^{13-2\alpha+(s+2-\alpha)_+}+\theta_k^{s+8-2\alpha}\Big\}\leq C\delta^2\theta_k^{L_2(s)-1}\Delta_k.\end{aligned}$$

Similarly, but exploiting (146) instead of (145), for the case $s+2=\alpha$ we get

$$\begin{aligned}[\mathbf{e}''_k]_{s,*,T}&\leq C\Big\{\delta^2\theta_k^{13-2\alpha}\Delta_k(C_*+\delta\log\theta_k+\delta)+\delta^2\theta_k^{6-\alpha}\Delta_k\Big\}\\ &\leq C\delta^2\Delta_k\left\{\theta_k^{13-2\alpha}+\theta_k^{6-\alpha}\right\}\leq C\delta^2\theta_k^{L_2(\alpha-2)-1}\Delta_k\end{aligned}$$

for $\alpha\geq 8$.

In view of (152), the trace theorem, and Proposition 11, one has

$$\begin{aligned}[\tilde{\mathbf{e}}''_k]_{s,*,T}\leq C\Big\{&[\delta\mathbf{U}_k]_{s+1,*,T}\|(1-S_{\theta_k})f_k\|_{H^6(\partial\Omega_T)}\\ &+[\delta\mathbf{U}_k]_{6,*,T}\|(1-S_{\theta_k})f_k\|_{H^{s+1}(\partial\Omega_T)}+[(I-S_{\theta_k})\mathbf{U}_k]_{s+1,*,T}\|\delta f_k\|_{H^6(\partial\Omega_T)}\\ &+[(I-S_{\theta_k})\mathbf{U}_k]_{6,*,T}\|\delta f_k\|_{H^{s+1}(\partial\Omega_T)}+[\delta\mathbf{U}_k]_{s+1,*,T}[(I-S_{\theta_k})\mathbf{U}_k]_{6,*,T}\\ &+[\delta\mathbf{U}_k]_{6,*,T}[(I-S_{\theta_k})\mathbf{U}_k]_{s+1,*,T}\Big\}.\end{aligned}$$

Using then (H_{n-1}) and (144), we obtain (154). □

5.5. Construction and estimate of the modified state

As in [8], at this stage we need to construct a smooth modified state $(\mathbf{U}_{n+1/2}, f_{n+1/2})$ satisfying certain nonlinear constraints. In our case these constraints are (17)–(19), (21), and (22). More precisely, the state $(\mathbf{U}^a + \mathbf{U}_{n+1/2}, f^a + f_{n+1/2})$ should meet the same requirements as the basic state $(\widehat{\mathbf{U}}, \hat{f})$ in Section 2. Since the approximate solution satisfies constraints (14) and (15) as well as the strict inequalities (17) and (19) (see Lemma 6) and since we shall require that the smooth modified state vanishes in the past, the state $(\mathbf{U}^a+\mathbf{U}_{n+1/2}, f^a+f_{n+1/2})$ will satisfy (14) and (15) for $t = 0$ (that is (22)) and (17) and (19) for a sufficiently short time $T > 0$. Moreover, it follows from (21) and (22) that this state obeys constraints (14) and (15) for all $t \in [0, T]$. Therefore, while constructing the modified state we may focus only on constraints (18) and (21), i.e., (7) and (197).

Proposition 12. *Let $\alpha \geq 10$. The exist some functions $\mathbf{U}_{n+1/2}$ and $f_{n+1/2}$, that vanish in the past, and such that $(\mathbf{U}^a + \mathbf{U}_{n+1/2}, f^a + f_{n+1/2})$ satisfies (5), (12), (197), (14), (15), and (at $x_1 = 0$) inequalities (9) and (10) for a sufficiently short time T. Moreover, these functions satisfy*

$$f_{n+1/2} = S_{\theta_n} f_n, \quad \Psi^{\pm}_{n+1/2} := \chi(\pm x_1) f_{n+1/2}, \tag{155}$$

$$v^{\pm}_{j,n+1/2} = S_{\theta_n} v^{\pm}_{j,n}, \quad j = 2, 3, \tag{156}$$

$$S^{\pm}_{n+1/2} = S_{\theta_n} S^{\pm}_n, \tag{157}$$

and

$$[\mathbf{U}_{n+1/2} - S_{\theta_n} \mathbf{U}_n]_{s,*,T} \leq C\delta\theta_n^{s+2-\alpha} \quad \text{for } s \in [6, \tilde{\alpha} + 4]. \tag{158}$$

for sufficiently small $\delta > 0$ and $T > 0$, and a sufficiently large $\theta_0 \geq 1$.

Proof. Let $f_{n+1/2}$, the entropies $S^{\pm}_{n+1/2}$, and the tangential components of the velocities $\mathbf{v}^{\pm}_{n+1/2}$ are defined by (155)–(157). Taking into account the discussion above, it is enough to construct such functions $p^{\pm}_{n+1/2}$, $v^{\pm}_{1,n+1/2}$, and $\mathbf{H}^{\pm}_{n+1/2}$ that

$$\mathbb{B}((\mathbf{U}^a + \mathbf{U}_{n+1/2})|_{x_1=0}, f^a + f_{n+1/2}) = 0, \tag{159}$$

$$\mathbb{L}_{\mathrm{H}}(\mathbf{v}^a + \mathbf{v}_{n+1/2}, \mathbf{H}^a + \mathbf{H}_{n+1/2}, \boldsymbol{\Psi}^a + \boldsymbol{\Psi}_{n+1/2}) = 0, \tag{160}$$

where (160) is the compact form of (197) (see also (123)),

$$\boldsymbol{\Psi}_{n+1/2} = (\Psi^+_{n+1/2}, \Psi^-_{n+1/2}), \quad \mathbf{v}_{n+1/2} = (\mathbf{v}^+_{n+1/2}, \mathbf{v}^-_{n+1/2}), \quad \text{etc.}$$

Since $(\mathbf{U}^a, f^a)$ satisfies (123),

$$\mathbb{L}_{\mathrm{H}}(\mathbf{v}^{a\pm} + \mathbf{v}^{\pm}_{n+1/2}, \mathbf{H}^{a\pm} + \mathbf{H}^{\pm}_{n+1/2}, \boldsymbol{\Psi}^{a\pm} + \boldsymbol{\Psi}^{\pm}_{n+1/2})$$

$$= \partial_t \mathbf{H}^{\pm}_{n+1/2} + \frac{1}{1 + \partial_1 \Psi^{\pm}_{n+1/2}} (\mathbf{w}^{a\pm} + \mathbf{w}^{\pm}_{n+1/2}, \nabla) \mathbf{H}^{\pm}_{n+1/2} + \ldots = 0,$$

where $\mathbf{w}^{a\pm}$ and $\mathbf{w}^{\pm}_{n+1/2}$ are defined similarly to $\mathbf{w}^{\pm}$ in (197). Analogously, we can explicitly write down (159) taking into account that $(\mathbf{U}^a, f^a)$ satisfies (12).

Let us now define $v^{\pm}_{1,n+1/2}$ as follows:

$$v^{\pm}_{1,n+1/2} := S_{\theta_n} v^{\pm}_{1,n} + \mathcal{R}_T \mathcal{G}^{\pm}, \tag{161}$$

where

$$\mathcal{G}^{\pm} = \partial_t f_{n+1/2} - (S_{\theta_n} v^{\pm}_{1,n})|_{x_1=0} + \sum_{j=2}^{3} \big((v^{a\pm}_j + v^{\pm}_{j,n+1/2}) \partial_j f_{n+1/2} + v^{\pm}_{j,n+1/2} \partial_j f^a \big)\big|_{x_1=0},$$

and $\mathcal{R}_T$ is the lifting operator, cf. (35). With (155) and (156) we have thus defined $\mathbf{v}_{n+1/2}$ and $\boldsymbol{\Psi}_{n+1/2}$. Then, we can define $\mathbf{H}_{n+1/2}$ as a function that vanishes in the past and solves equation (160). Note that, in view of (161), the fist two boundary conditions in (159) are fulfilled. Hence, $(w^{a\pm}_1 + w^{\pm}_{1,n+1/2})|_{x_1=0} = 0$, i.e., (160) considered as the equation for $\mathbf{H}_{n+1/2}$ does not need boundary conditions at $x_1 = 0$. At last, we define $p^{\pm}_{n+1/2}$ by

$$\begin{aligned} p^{\pm}_{n+1/2} := S_{\theta_n} p^{\pm}_n \mp \frac{1}{2}\, \mathcal{R}_T \epsilon_n - \frac{1}{2}\, |\mathbf{H}^{\pm}_{n+1/2}|^2 - (\mathbf{H}^{\pm}_{n+1/2}, \mathbf{H}^{a\pm}) \\ + \frac{1}{2}\, S_{\theta_n} |\mathbf{H}^{\pm}_n|^2 + S_{\theta_n} (\mathbf{H}^{\pm}_n, \mathbf{H}^{a\pm}), \end{aligned} \tag{162}$$

where

$$\begin{aligned} \epsilon_n = \Big(S_{\theta_n} p^+_n - S_{\theta_n} p^-_n + \frac{1}{2}\, S_{\theta_n} |\mathbf{H}^+_n|^2 + S_{\theta_n} (\mathbf{H}^+_n, \mathbf{H}^{a+}) \\ - \frac{1}{2}\, S_{\theta_n} |\mathbf{H}^-_n|^2 - S_{\theta_n} (\mathbf{H}^-_n, \mathbf{H}^{a-}) \Big)\Big|_{x_1=0}. \end{aligned}$$

It should be noted that (161) and (162) considered at $x_1 = 0$ yield (159).

We first get the estimate of $v_{1,n+1/2} - S_{\theta_n} v_{1,n}$. To that end we use the following decompositions (we drop the $\pm$ superscripts, cf. (161)):

$$\begin{aligned} \mathcal{G} = S_{\theta_n} \mathbb{B}_v (\mathbf{U}_n|_{x_1=0}, f_n) - \partial_t (1 - S_{\theta_n}) f_n + (1 - S_{\theta_n}) \partial_t f_n \\ \sum_{j=2}^{3} \big((v^a_j + S_{\theta_n} v_{j,n}) \partial_j S_{\theta_n} f_n - S_{\theta_n} ((v^a_j + v_{j,n}) \partial_j f_n) \\ + (S_{\theta_n} v_{j,n}) \partial_j f^a - S_{\theta_n} (v_{j,n} \partial_j f_n) \big)\big|_{x_1=0} \end{aligned}$$

and

$$\begin{aligned} \mathbb{B}_v (\mathbf{U}_n|_{x_1=0}, f_n) = \mathbb{B}_v (\mathbf{U}_{n-1}|_{x_1=0}, f_{n-1}) + \partial_t (\delta f_{n-1}) \\ + \sum_{j=2}^{3} \big((v^a_j + v_{j,n-1}) \partial_j (\delta f_{n-1}) + \delta v_{j,n-1} \partial_j (f^a + f_n) - \delta v_{1,n-1} \big)\big|_{x_1=0}\, , \end{aligned}$$

where $\mathbb{B}_v$ denotes the first (for $\mathcal{G}^+$) or the second (for $\mathcal{G}^-$) row of the boundary operator $\mathbb{B}$ in (12).

Exploiting point c) of (H_{n-1}), one has

$$[\mathcal{R}_T(S_{\theta_n}\mathbb{B}_v(\mathbf{U}_{n-1}|_{x_1=0},f_{n-1}))]_{s,*,T} \le C\|S_{\theta_n}\mathbb{B}_v(\mathbf{U}_{n-1}|_{x_1=0},f_{n-1})\|_{H^s(\partial\Omega_T)}$$

$$\le \begin{cases} C\theta_n^{s-\alpha+1}\|\mathbb{B}_v(\mathbf{U}_{n-1}|_{x_1=0},f_{n-1})\|_{H^\alpha(\partial\Omega_T)} & \text{for } s\in[\alpha,\tilde{\alpha}+4],\\ C\|\mathbb{B}_v(\mathbf{U}_{n-1}|_{x_1=0},f_{n-1})\|_{H^{s+1}(\partial\Omega_T)} & \text{for } s\in[6,\alpha-1]\end{cases}$$

$$\le C\delta\theta_n^{s-\alpha} \quad \text{for } s\in[6,\tilde{\alpha}+4].$$

Using (134) and point a) of (H_{n-1}), we get

$$[\mathcal{R}_T(S_{\theta_n}\partial_t(\delta f_{n-1}))]_{s,*,T} \le C\|S_{\theta_n}\partial_t(\delta f_{n-1})\|_{H^s(\partial\Omega_T)}$$
$$\le C\theta_n^{s-6}\|\delta f_{n-1}\|_{H^7(\partial\Omega_T)} \le C\theta_n^{s-6}\delta\theta_{n-1}^{6-\alpha}\theta_{n-1}^{-1} \le C\delta\theta_n^{s-\alpha-1}$$

for $s\in[6,\tilde{\alpha}+4]$. We also obtain

$$\big[\mathcal{R}_T\big(S_{\theta_n}((v_j^a+v_{j,n-1})|_{x_1=0}\,\partial_j(\delta f_{n-1}))\big)\big]_{s,*,T}$$
$$\le C\theta_n^{s-6}\|(v_j^a+v_{j,n-1})|_{x_1=0}\,\partial_j(\delta f_{n-1})\|_{H^6(\partial\Omega_T)}$$
$$\le C\theta_n^{s-6}\Big\{\|\delta f_{n-1}\|_{H^7(\partial\Omega_T)}[\mathbf{U}^a+\mathbf{U}_{n-1}]_{6,*,T}$$
$$+\|\delta f_{n-1}\|_{H^6(\partial\Omega_T)}[\mathbf{U}^a+\mathbf{U}_{n-1}]_{7,*,T}\Big\} \le C\theta_n^{s-6}\delta\theta_n^{5-\alpha}C_* \le C\delta\theta_n^{s-\alpha-1}$$

for $j=2,3$ and $s\in[6,\tilde{\alpha}+4]$. Estimating similarly the remaining terms containing in $\mathcal{R}_T(S_{\theta_n}\mathbb{B}_v(\mathbf{U}_n|_{x_1=0},f_n))$, we finally derive the estimate

$$[\mathcal{R}_T(S_{\theta_n}\mathbb{B}_v(\mathbf{U}_n|_{x_1=0},f_n))]_{s,*,T} \le C\delta\theta_n^{s-\alpha}, \quad s\in[6,\tilde{\alpha}+4].$$

We now need to get estimates for the remaining terms containing in $\mathcal{R}_T\mathcal{G}$. For $s\in[\alpha,\tilde{\alpha}+4]$ one has

$$\big[\mathcal{R}_T(-\partial_t(1-S_{\theta_n})f_n+(1-S_{\theta_n})\partial_t f_n)\big]_{s,*,T}$$
$$\le C\big\{\|\partial_t(S_{\theta_n}f_n)\|_{H^s(\partial\Omega_T)}+\|S_{\theta_n}(\partial_t f_n)\|_{H^s(\partial\Omega_T)}\big\}$$
$$\le C\big\{\|S_{\theta_n}f_n\|_{H^{s+1}(\partial\Omega_T)}+\theta_n^{s-\alpha}\|f_n\|_{H^{\alpha+1}(\partial\Omega_T)}\big\} \le C\delta\theta_n^{s+1-\alpha},$$

while for $s\in[6,\tilde{\alpha}-1]$ we obtain (recall that $\tilde{\alpha}=\alpha+5$)

$$\big[\mathcal{R}_T(\partial_t(1-S_{\theta_n})f_n)\big]_{s,*,T} \le C\delta\theta_n^{s+1-\alpha},$$

$$\big[\mathcal{R}_T((1-S_{\theta_n})\partial_t f_n)\big]_{s,*,T} \le C\theta_n^{s-\alpha}\|f_n\|_{H^{\alpha+1}(\partial\Omega_T)} \le C\delta\theta_n^{s+1-\alpha}.$$

Here we have, in particular, used Lemma 7. We do not get estimates for all the remaining terms containing in $\mathcal{R}_T\mathcal{G}$ and leave corresponding calculations to the reader. Collecting these estimates and the estimates above, we finally have

$$[v_{1,n+1/2}-S_{\theta_n}v_{1,n}]_{s,*,T} \le C\delta\theta_n^{s+1-\alpha}, \quad s\in[6,\tilde{\alpha}+4]. \tag{163}$$

Observe that in the calculations above it was enough to assume that $\alpha\ge 8$.

Now we turn to the estimate of $\mathbf{H}_{n+1/2}-S_{\theta_n}\mathbf{H}_n$ that, in view of (160), solves the equation

$$\mathcal{L}_{\rm H}(\mathbf{v}_{n+1/2},\mathbf{H}_{n+1/2}-S_{\theta_n}\mathbf{H}_n,\boldsymbol{\Psi}_{n+1/2})=\mathbf{F}_{\rm H}^{n+1/2},\tag{164}$$

where $\mathcal{L}_{\rm H}(\mathbf{v},\mathbf{H},\boldsymbol{\Psi})=\mathbb{L}_{\rm H}(\mathbf{v}^a+\mathbf{v},\mathbf{H}^a+\mathbf{H},\boldsymbol{\Psi}^a+\boldsymbol{\Psi})$ and the right-hand side is decomposed as follows:

$$\mathbf{F}_{\rm H}^{n+1/2}=\widetilde{\mathbf{F}}_{\rm H}^{n+1/2}-S_{\theta_n}\mathcal{L}_{\rm H}(\mathbf{W}_n,\boldsymbol{\Psi}_n),\tag{165}$$

with $\mathbf{W}_n:=(\mathbf{v}_n,\mathbf{H}_n)$ and

$$\begin{aligned}\widetilde{\mathbf{F}}_{\rm H}^{n+1/2}&=-\mathcal{L}_{\rm H}(\mathbf{v}_{n+1/2}-S_{\theta_n}\mathbf{v}_n,S_{\theta_n}\mathbf{H}_n,S_{\theta_n}\boldsymbol{\Psi}_n)\\&\qquad+S_{\theta_n}\mathcal{L}_{\rm H}(\mathbf{W}_n,\boldsymbol{\Psi}_n)-\mathcal{L}_{\rm H}(S_{\theta_n}\mathbf{W}_n,S_{\theta_n}\boldsymbol{\Psi}_n).\end{aligned}$$

We first consider the second term in the right-hand side in (165). We decompose it as

$$S_{\theta_n}\mathcal{L}_{\rm H}(\mathbf{W}_n,\boldsymbol{\Psi}_n)=S_{\theta_n}\mathcal{L}_{\rm H}(\mathbf{W}_{n-1},\boldsymbol{\Psi}_{n-1})+S_{\theta_n}\mathbf{Z}_{n-1},$$

where

$$\mathbf{Z}_{n-1}=\mathcal{L}_{\rm H}(\mathbf{W}_{n-1}+\delta\mathbf{W}_{n-1},\boldsymbol{\Psi}_{n-1}+\delta\boldsymbol{\Psi}_{n-1})-\mathcal{L}_{\rm H}(\mathbf{W}_{n-1},\boldsymbol{\Psi}_{n-1}).$$

Thanks to the definition of the approximate solution given in Lemma 6 we have $\mathbf{F}_{\rm H}^a=0$, where $\mathbf{F}_{\rm H}^a$ is the right-hand side in (130) corresponding to the equations for $\mathbf{H}$. Using this fact and point *b*) of (H_{n-1}), one gets

$$[S_{\theta_n}\mathcal{L}_{\rm H}(\mathbf{W}_{n-1},\boldsymbol{\Psi}_{n-1})]_{s,*,T}\le C\theta_n^{s-6}[\mathcal{L}_{\rm H}(\mathbf{W}_{n-1},\boldsymbol{\Psi}_{n-1})]_{6,*,T}\le C\delta\theta_n^{s-\alpha-1}$$

for $s\in[6,\tilde{\alpha}+4]$. The typical term containing in $S_{\theta_n}\mathbf{Z}_{n-1}$ is, for example,

$$S_{\theta_n}\left(\frac{\partial_1(v_1^a+v_{1,n})}{1+\partial_1(\Psi^a+\Psi_n)}\delta\mathbf{H}_{n-1}\right)\tag{166}$$

that is estimated by using the properties of smoothing operators, the tame estimate for a product (see Appendix B), etc.:

$$\begin{aligned}&[S_{\theta_n}(\ldots)]_{s,*,T}\le C\theta_n^{s-6}\Big[\frac{\partial_1(v_1^a+v_{1,n})}{1+\partial_1(\Psi^a+\Psi_n)}\delta\mathbf{H}_{n-1}\Big]_{6,*,T}\\&\quad\le C\theta_n^{s-6}\Bigg\{\Big\|\frac{\delta\mathbf{H}_{n-1}}{1+\partial_1(\Psi^a+\Psi_n)}\Big\|_{L_\infty(\Omega_T)}[v_1^a+v_{1,n}]_{8,*,T}\\&\qquad\Big[\frac{\delta\mathbf{H}_{n-1}}{1+\partial_1(\Psi^a+\Psi_n)}\Big]_{6,*,T}\|v_1^a+v_{1,n}\|_{W^1_\infty(\Omega_T)}\Bigg\}\le\ldots\le C\delta\theta_n^{s-\alpha-1}\end{aligned}$$

for $s\in[6,\tilde{\alpha}+4]$. Treating analogously the remaining terms containing in $S_{\theta_n}\mathbf{Z}_{n-1}$, we come to the estimate

$$[S_{\theta_n}\mathcal{L}_{\rm H}(\mathbf{W}_n,\boldsymbol{\Psi}_n)]_{s,*,T}\le C\delta\theta_n^{s-\alpha-1},\quad s\in[6,\tilde{\alpha}+4].\tag{167}$$

To obtain the estimate of $\widetilde{\mathbf{F}}_{\mathrm{H}}^{n+1/2}$, that is the remaining term in the right-hand side in (165) to be estimated, we treat the case $s \geq \alpha + 1$ separately. For $s \in [\alpha + 1, \tilde{\alpha} + 4]$ we proceed as follows:

$$[\widetilde{\mathbf{F}}_{\mathrm{H}}^{n+1/2}]_{s,*,T} \leq [\mathcal{L}_{\mathrm{H}}(\mathbf{v}_{n+1/2} - S_{\theta_n}\mathbf{v}_n, S_{\theta_n}\mathbf{H}_n, S_{\theta_n}\boldsymbol{\Psi}_n)]_{s,*,T}$$
$$+[\mathcal{L}_{\mathrm{H}}(S_{\theta_n}\mathbf{W}_n, S_{\theta_n}\boldsymbol{\Psi}_n)]_{s,*,T} + [S_{\theta_n}\mathcal{L}_{\mathrm{H}}(\mathbf{W}_n, \boldsymbol{\Psi}_n)]_{s,*,T}.$$

The last term in the right-hand side of the above inequality has been already estimated in (167) for the general case $s \in [6, \tilde{\alpha}+4]$. It is clear that we have only to estimate the second term there because for the first one we can get a better estimate by using (156) and (163). To obtain the estimate of $\mathcal{L}_{\mathrm{H}}(S_{\theta_n}\mathbf{W}_n, S_{\theta_n}\boldsymbol{\Psi}_n)$ we exploit the arguments similar to those used for the term in (166). Omitting detailed calculations, for $s \in [\alpha + 1, \tilde{\alpha} + 4]$ we thus obtain

$$[\widetilde{\mathbf{F}}_{\mathrm{H}}^{n+1/2}]_{s,*,T} \leq C\delta\theta_n^{s+2-\alpha}. \tag{168}$$

For the case $s \in [6, \alpha]$ we use the decomposition

$$\widetilde{\mathbf{F}}_{\mathrm{H}}^{n+1/2} = -\mathcal{L}_{\mathrm{H}}(\mathbf{v}_{n+1/2} - S_{\theta_n}\mathbf{v}_n - (I - S_{\theta_n})\mathbf{v}_n, S_{\theta_n}\mathbf{H}_n, S_{\theta_n}\boldsymbol{\Psi}_n)$$
$$-\mathcal{L}_{\mathrm{H}}(\mathbf{v}_n, (I - S_{\theta_n})\mathbf{H}_n, S_{\theta_n}\boldsymbol{\Psi}_n) + (\mathcal{L}_{\mathrm{H}}(\mathbf{W}_n, \boldsymbol{\Psi}_n) - \mathcal{L}_{\mathrm{H}}(\mathbf{W}_n, S_{\theta_n}\boldsymbol{\Psi}_n).$$

Omitting details, we get estimate (168) for the case $s \in [6, \alpha]$ as well. Thus, we have derived the estimate

$$[\mathbf{F}_{\mathrm{H}}^{n+1/2}]_{s,*,T} \leq C\delta\theta_n^{s+2-\alpha}, \quad s \in [6, \tilde{\alpha} + 4]. \tag{169}$$

Equation (164), which right-hand side we have just estimated, is actually written as

$$\partial_t\mathbf{Y} + \sum_{j=1}^{3}\mathcal{D}_j(\mathbf{b})\partial_j\mathbf{Y} + Q(\mathbf{b})\mathbf{Y} = \mathbf{F}_{\mathrm{H}}^{n+1/2}, \tag{170}$$

where $\mathbf{Y} = \mathbf{H}_{n+1/2} - S_{\theta_n}\mathbf{H}_n, \quad \mathbf{b} = (\mathbf{v}^a + \mathbf{v}_{n+1/2}, \boldsymbol{\Psi}^a + S_{\theta_n}\boldsymbol{\Psi}_n)$, and $\mathcal{D}_j$ and Q are matrices. Moreover, the matrices $\mathcal{D}_j$ are diagonal and, what is important, the matrix $\mathcal{D}_1|_{x_1=0} = 0$. That is, system (170) does not need boundary conditions at $x_1 = 0$. Then, with the help of the arguments like those used in Section 3 for deriving the tame estimates (102) and (110), we obtain the tame a priori estimate

$$[\mathbf{Y}]_{s,*,T} \leq C(\widetilde{K}_0)\left\{[\mathbf{F}_{\mathrm{H}}^{n+1/2}]_{s,*,T} + [\mathbf{F}_{\mathrm{H}}^{n+1/2}]_{s_1,*,T}[\mathbf{b}]_{s+2,*,T}\right\} \tag{171}$$

for the solution $\mathbf{Y}$ of (170) that vanishes in the past, where $s_1 = 6$ if s is even and $s_1 = 7$ if s is odd. Estimate (171) holds for the coefficients satisfying the assumption

$$[\mathbf{v}^a + \mathbf{v}_{n+1/2}]_{9,*,T} + \|f^a + S_{\theta_n}f_n\|_{H^9(\partial\Omega_T)} \leq K, \tag{172}$$

provided that $K \leq \widetilde{K}_0$. The constant $\widetilde{K}_0 = 2C_*$, that does not depend on s and T, can be taken the same as in Proposition 11 for δ sufficiently small.

Taking into account (156) and (163) and using, in particular, (145), we check that assumption (172) holds for $s \in [6, \tilde{\alpha}+4]$, $\alpha \geq 10$, and δ sufficiently small:

$$[\mathbf{v}^a + \mathbf{v}_{n+1/2}]_{9,*,T} + \|f^a + S_{\theta_n} f_n\|_{H^9(\partial\Omega_T)} \leq [\mathbf{v}_{n+1/2} - S_{\theta_n}\mathbf{v}_n]_{9,*,T}$$
$$+[S_{\theta_n}(\mathbf{v}_{n-1} + \delta\mathbf{v}_{n-1})]_{9,*,T} + \|S_{\theta_n}(f_{n-1} + \delta f_{n-1})\|_{H^9(\partial\Omega_T)} + [\mathbf{v}^a]_{9,*,T}$$
$$+\|f^a\|_{H^9(\partial\Omega_T)} \leq C(\delta\theta_n^{10-\alpha} + \delta\theta_n^{(9-\alpha)_+}) + C_* \leq C_* + C\delta \leq 2C_*.$$

Using (169), one has

$$[\mathbf{F}_{\mathrm{H}}^{n+1/2}]_{s_1,*,T}[\mathbf{b}]_{s+2,*,T} \leq C\delta\theta_n^{s_1+2-\alpha}(C_* + C\delta\theta_n^{s+3-\alpha} + \delta\theta_n^{\kappa}),$$

where $\kappa = (s+2-\alpha)_+$ for $s \neq \alpha - 2$ and $\kappa = 1$ for $s = \alpha - 2$. One can check that

$$s_1 + 2 - \alpha + \max\{s+3-\alpha, \kappa\} \leq s + 2 - \alpha \quad \text{for } s \in [6, \tilde{\alpha}+4] \quad \text{and} \quad \alpha \geq 10.$$

That is,

$$[\mathbf{F}_{\mathrm{H}}^{n+1/2}]_{s_1,*,T}[\mathbf{b}]_{s+2,*,T} \leq C\delta\theta_n^{s+2-\alpha}. \tag{173}$$

Estimates (169), (171), and (173) yield

$$[\mathbf{H}_{s+1/2} - S_{\theta_n}\mathbf{H}_n]_{s,*,T} \leq C\delta\theta_n^{s+2-\alpha}, \quad s \in [6, \tilde{\alpha}+4]. \tag{174}$$

At last, using (174), from (162) we can derive the estimate

$$[p_{s+1/2} - S_{\theta_n}p_n]_{s,*,T} \leq C\delta\theta_n^{s+2-\alpha}, \quad s \in [6, \tilde{\alpha}+4]. \tag{175}$$

To avoid overloading the paper we omit the calculations and only note that the process of deriving estimate (175) is much simpler than that of obtaining estimates (163) and, especially, (174). In view of (156) and (157), estimates (163), (174), and (175) imply (158). This completes the proof. □

5.6. Estimate of the second substitution errors

The second substitution errors

$$\mathbf{e}_k''' = \mathcal{L}'(S_{\theta_k}\mathbf{U}_k, S_{\theta_k}\boldsymbol{\Psi}_k)(\delta\mathbf{U}_k, \delta\boldsymbol{\Psi}_k) - \mathcal{L}'(\mathbf{U}_{k+1/2}, \boldsymbol{\Psi}_{k+1/2})(\delta\mathbf{U}_k, \delta\boldsymbol{\Psi}_k)$$

and

$$\tilde{\mathbf{e}}_k''' = \big(\mathcal{B}'(S_{\theta_k}\mathbf{U}_k, S_{\theta_k}f_k)(\delta\mathbf{U}_k, \delta f_k) - \mathcal{B}'(\mathbf{U}_{k+1/2}, f_{k+1/2})(\delta\mathbf{U}_k, \delta f_k)\big)|_{x_1=0}$$

can be written as

$$\mathbf{e}_k''' = \int_0^1 \mathbb{L}''\big(\mathbf{U}^a + \mathbf{U}_{k+1/2} + \tau(S_{\theta_k}\mathbf{U}_k - \mathbf{U}_{k+1/2}),\; \boldsymbol{\Psi}^a + S_{\theta_k}\boldsymbol{\Psi}_k\big)\big((\delta\mathbf{U}_k, \delta\boldsymbol{\Psi}_k), (S_{\theta_k}\mathbf{U}_k - \mathbf{U}_{k+1/2}, 0)\big)d\tau, \tag{176}$$

$$\tilde{\mathbf{e}}_k''' = \mathbb{B}''\big((\delta\mathbf{U}_k|_{x_1=0}, \delta f_k), ((S_{\theta_k}\mathbf{U}_k - \mathbf{U}_{k+1/2})|_{x_1=0}, 0)\big). \tag{177}$$

Exploiting expressions (176) and (177), we get the following result.

Lemma 10. *Let $\alpha \geq 10$. There exist $\delta > 0$, $T > 0$ sufficiently small, and $\theta_0 \geq 1$ sufficiently large, such that for all $k = 0, \dots n-1$, and for all integer $s \in [6, \widetilde{\alpha} - 2]$, one has*

$$[\mathbf{e}_k'''\,]_{s,*,T} \leq C\delta^2 \theta_k^{L(s)-1} \Delta_k, \tag{178}$$

$$\|\tilde{\mathbf{e}}_k'''\|_{H^s(\partial\Omega_T)} \leq C\delta^2 \theta_k^{L(s)-1} \Delta_k, \tag{179}$$

where $L(s) = \max\{(s+2-\alpha)_+ + 18 - 2\alpha, s + 11 - 2\alpha\}$.

Proof. Using Lemma 7 and Proposition 12, we obtain the estimate

$$\sup_{\tau\in[0,1]} \langle\!\langle(\mathbf{U}^a + \mathbf{U}_{k+1/2} + \tau(S_{\theta_k}\mathbf{U}_k - \mathbf{U}_{k+1/2}), f^a + S_{\theta_k} f_k)\rangle\!\rangle_{7,*,T} \leq 2C_*$$

for δ sufficiently small, i.e., we can apply Proposition 11. Similarly, one gets

$$\langle\!\langle(\mathbf{U}^a + \mathbf{U}_{k+1/2} + \tau(S_{\theta_k}\mathbf{U}_k - \mathbf{U}_{k+1/2}), f^a + S_{\theta_k} f_k)\rangle\!\rangle_{s+2,*,T} \\ \leq C\big\{C_* + \delta\theta_k^{s+4-\alpha} + \delta\theta_k^{(s+2-\alpha)_+ +1}\big\} \leq C\delta\theta_n^{(s+2-\alpha)_+ +2}.$$

Applying Proposition 11, we get (178):

$$[\mathbf{e}_k'''\,]_{s,*,T} \leq C\Big\{\delta\theta_k^{(s+2-\alpha)_+ +2}\delta\theta_k^{6-\alpha}\Delta_k\delta\theta_k^{9-\alpha} + \delta\theta_k^{s+1-\alpha}\Delta_k\delta\theta_k^{9-\alpha} \\ +\delta\theta_k^{6-\alpha}\Delta_k\delta\theta_k^{s+4-\alpha}\Big\} \leq C\delta^2\theta_k^{L(s)-1}\Delta_k.$$

Using (155), (156), and the explicit form of $\mathbb{B}''$ (see Proposition 11), one has

$$\tilde{\mathbf{e}}_k''' = \begin{pmatrix} 0 \\ 0 \\ [(\delta\mathbf{H}_k, S_{\theta_k}\mathbf{H}_k - \mathbf{H}_{k+1/2})] \end{pmatrix}.$$

Applying then (H_{n-1}) and (158), we easily get (179). □

5.7. Estimate of the last error term

It remains to estimate the last error term $\mathbb{D}_{k+1/2}\boldsymbol{\Psi}_k$, i.e., we should get the estimate of

$$\mathbf{D}_{k+1/2}\delta\Psi_k = \frac{\delta\Psi_k}{\partial_1(\Phi^a + \Psi_{n+1/2})}\,\mathbf{R}_k,$$

where $\mathbf{R}_k := \partial_1\left\{\mathbb{L}(\mathbf{U}^a + \mathbf{U}_{k+1/2}, \Psi^a + \Psi_{k+1/2})\right\}$, and the $\pm$ superscripts are dropped. Observe that

$$|\partial_1(\Phi^a + \Psi_{n+1/2})| = |1 + \partial_1(\Psi^a + \Psi_{n+1/2})| \geq 1/2,$$

provided that T and δ are small enough, cf. (126).

Lemma 11. *Let $\alpha \geq 10$. There exist $\delta > 0$, $T > 0$ sufficiently small, and $\theta_0 \geq 1$ sufficiently large, such that for all $k = 0, \dots n-1$, and for all integer $s \in [6, \widetilde{\alpha} - 2]$, one has*

$$[\mathbb{D}_{k+1/2}\boldsymbol{\Psi}_k]_{s,*,T} \leq C\delta^2\theta_k^{L(s)-1}\Delta_k, \tag{180}$$

where the function $L(s)$ is defined in Lemma 10.

Proof. The proof follows from the arguments as in [1,8]. Using the Moser-type and embedding inequalities from Appendix B, we obtain

$$\begin{aligned}[\mathbf{D}_{k+1/2}\delta\Psi_k]_{s,*,T} \leq C\Big\{ &\|\delta f_k\|_{H^s(\partial\Omega_T)}[\mathbf{R}]_{6,*,T} \\ &\|\delta f_k\|_{H^6(\partial\Omega_T)}\big([\mathbf{R}]_{s,*,T} + [\mathbf{R}]_{6,*,T}\|f^a + f_{k+1/2}\|_{H^s(\partial\Omega_T)}\big)\Big\}\end{aligned} \tag{181}$$

(note that $[\partial_1(\Psi^a + \Psi_{n+1/2})]_{s,*,T} \leq C\|f^a + f_{k+1/2}\|_{H^s(\partial\Omega_T)}$). To estimate $\mathbf{R}_k$ we utilize the decomposition

$$\begin{aligned}&\mathbb{L}(\mathbf{U}^a + \mathbf{U}_{k+1/2}, \Psi^a + \Psi_{k+1/2}) = \mathcal{L}(\mathbf{U}_k, \Psi_k) - \mathbf{F}^a \\ &\quad + \mathbb{L}(\mathbf{U}^a + \mathbf{U}_{k+1/2}, \Psi^a + \Psi_{k+1/2}) - \mathbb{L}(\mathbf{U}^a + \mathbf{U}_k, \Psi^a + \Psi_k) \\ &\quad = \mathcal{L}(\mathbf{U}_k, \Psi_k) - \mathbf{F}^a + \int_0^1 \mathbb{L}'\big(\mathbf{U}^a + \mathbf{U}_k + \tau(\mathbf{U}_{k+1/2} - \mathbf{U}_k), \\ &\qquad\quad \boldsymbol{\Psi}^a + \boldsymbol{\Psi}_k + \tau(\boldsymbol{\Psi}_{k+1/2} - \boldsymbol{\Psi}_k)\big)(\mathbf{U}_{k+1/2} - \mathbf{U}_k, \boldsymbol{\Psi}_{k+1/2} - \boldsymbol{\Psi}_k)d\tau.\end{aligned}$$

Clearly,

$$[\mathbf{R}]_{s,*,T} \leq [\mathcal{L}(\mathbf{U}_k, \Psi_k) - \mathbf{F}^a]_{s+2,*,T} + \sup_{\tau\in[0,1]}[\mathbb{L}'(\dots)(\dots)]_{s+2,*,T} \tag{182}$$

(for short we drop the arguments of $\mathbb{L}'$). Point $b)$ of (H_{n-1}) implies

$$[\mathcal{L}(\mathbf{U}_k, \Psi_k) - \mathbf{F}^a]^{s+2,*,T} \leq 2\delta\theta_k^{s+1-\alpha} \tag{183}$$

for $s \in [6, \tilde{\alpha} - 4]$. We estimate $\mathbb{L}'$ similarly to $\mathbb{L}''$ (see Proposition 11). We have

$$\sup_{\tau\in[0,1]} \langle\!\langle(\mathbf{U}^a + \mathbf{U}_k + \tau(\mathbf{U}_{k+1/2} - \mathbf{U}_k), f^a + f_k + \tau(f_{k+1/2} - f_k))\rangle\!\rangle_{7,*,T} \leq 2C_*$$

for δ small enough. Then, omitting detailed calculations, we get the estimate

$$[\mathbb{L}'(\dots)(\dots)]_{s+2,*,T} \leq C\delta(\theta_k^{s+6-\alpha} + \theta_k^{(s+2-\alpha)_+ + 13-\alpha})$$

for $s \in [6, \tilde{\alpha} - 4]$. This estimate, (182), and (183) yield

$$[\mathbf{R}]_{s,*,T} \leq C\delta(\theta_k^{s+6-\alpha} + \theta_k^{(s+2-\alpha)_+ + 13-\alpha}) \tag{184}$$

for $s \in [6, \tilde{\alpha} - 4]$.

For $s=\tilde{\alpha}-3$ and $s=\tilde{\alpha}-2$ we estimate as follows:

$$
\begin{aligned}
[\mathbf{R}]_{s,*,T} &\le [\mathbb{L}(\mathbf{U}^a+\mathbf{U}_{k+1/2},\Psi^a+\Psi_{k+1/2})]_{s+2,*,T}\\
&\le C\langle\!\langle(\mathbf{U}^a+(\mathbf{U}_{k+1/2}-S_{\theta_n}\mathbf{U}_k)+-S_{\theta_n}\mathbf{U}_k,f^a+S_{\theta_n}f_k)\rangle\!\rangle_{s+4,*,T}\\
&\le C(C_*+\delta\theta_k^{s+6-\alpha}+\delta\theta_k^{s+4-\alpha})\le C\delta\theta_k^{s+6-\alpha}.
\end{aligned}
$$

Here we assumed that $s\ge\alpha-3$. This is true for $s=\tilde{\alpha}-3$ and $s=\tilde{\alpha}-2$ (recall that $\tilde{\alpha}=\alpha+5$). That is, we get estimate (184) for $s\in[6,\tilde{\alpha}-2]$. Using then (181), we derive (180). □

Observe that estimate (180) is rather rough and we could, in principle, specify the function $L(s)$ more accurately, but we take $L(s)$ as in Lemma 10 because finally we will need only the estimate of the sum of errors given in (139).

5.8. Convergence of the iteration scheme

Lemmas 8–11 yield the estimate of $\mathbf{e}_n$ and $\tilde{\mathbf{e}}_n$ defined in (139) as the sum of all the errors of the kth step.

Lemma 12. *Let $\alpha\ge 10$. There exist $\delta>0$, $T>0$ sufficiently small, and $\theta_0\ge 1$ sufficiently large, such that for all $k=0,\dots n-1$, and for all integer $s\in[6,\widetilde{\alpha}-2]$, one has*

$$
[\mathbf{e}_k]_{s,*,T}+\|\tilde{\mathbf{e}}_k\|_{H^s(\partial\Omega_T)}\le C\delta^2\theta_k^{L(s)-1}\Delta_k, \tag{185}
$$

where $L(s)$ is defined in Lemma 10.

In one's turn, Lemma 12 gives the estimate of the accumulated errors $\mathbf{E}_n$ and $\widetilde{\mathbf{E}}_n$.

Lemma 13. *Let $\alpha\ge 13$. There exist $\delta>0$, $T>0$ sufficiently small, and $\theta_0\ge 1$ sufficiently large, such that*

$$
[\mathbf{E}_n]_{\alpha+3,*,T}+\|\widetilde{\mathbf{E}}_n\|_{H^{\alpha+3}(\partial\Omega_T)}\le C\delta^2\theta_n, \tag{186}
$$

where $L(s)$ is defined in Lemma 10.

Proof. One can check that $L(\alpha+3)\le 1$ if $\alpha\ge 13$. It follows from (185) that

$$
\langle\!\langle(\mathbf{E}_n,\widetilde{\mathbf{E}}_n)\rangle\!\rangle_{\alpha+3}\le\sum_{k=0}^{n-1}\langle\!\langle(\mathbf{e}_k,\widetilde{\mathbf{e}}_k)\rangle\!\rangle_{\alpha+3}\le\sum_{k=0}^{n-1}C\delta^2\Delta_k\le C\delta^2\theta_n
$$

for $\alpha\ge 13$ and $\alpha+3\in[6,\tilde{\alpha}-2]$, i.e., $\tilde{\alpha}\ge\alpha+5$. The minimal possible $\tilde{\alpha}$ is $\alpha+5$, i.e., our choice $\tilde{\alpha}=\alpha+5$ is suitable. □

We now get the estimates of the source terms $\mathbf{f}_n$ and $\mathbf{g}_n$ defined in (141).

Lemma 14. *Let $\alpha \geq 13$. There exist $\delta > 0$, $T > 0$ sufficiently small, and $\theta_0 \geq 1$ sufficiently large, such that for all integer $s \in [6, \widetilde{\alpha} + 2]$, one has*

$$[\mathbf{f}_n]_{s,*,T} \leq C\Delta_n\big\{\theta_n^{s-\alpha-3}\big([\mathbf{F}^a]_{\alpha+2,*,T} + \delta^2\big) + \delta^2\theta_n^{L(s)-1}\big\}, \tag{187}$$

$$\|\mathbf{g}_n\|_{H^s(\partial\Omega_T)} \leq C\delta^2\Delta_n\big(\theta_n^{L(s)-1} + \theta_n^{s-\alpha-3}\big). \tag{188}$$

Proof. It follows from (141) that

$$\mathbf{F}_n = (S_{\theta_n} - S_{\theta_{n-1}})\mathbf{F}^a - (S_{\theta_n} - S_{\theta_{n-1}})\mathbf{E}_{n-1} - S_{\theta_n}\mathbf{e}_{n-1}.$$

Using (134), (136), (185), and (186), we obtain the estimates

$$[(S_{\theta_n} - S_{\theta_{n-1}})\mathbf{F}^a]_{s,*,T} \leq C\theta_{n-1}^{s-\alpha-3}[\mathbf{F}^a]_{\alpha+2,*,T}\Delta_{n-1},$$

$$[S_{\theta_n} - S_{\theta_{n-1}})\mathbf{E}_{n-1}]_{s,*,T} \leq C\theta_{n-1}^{s-\alpha-4}[\mathbf{E}_{n-1}]_{\alpha+3,*,T}\Delta_{n-1} \leq C\delta^2\theta_{n-1}^{s-\alpha-3}\Delta_{n-1},$$

$$[S_{\theta_n}\mathbf{e}_{n-1}]_{s,*,T} \leq C\delta^2\theta_n^{L(s)-1}\Delta_{n-1}.$$

Taking into account the inequalities $\theta_{n-1} \leq \theta_n \leq \sqrt{2}\theta_{n-1}$ and $\theta_{n-1} \leq 3\theta_n$, the above estimates yield (187). Similarly, we get (188). □

We are now in a position to obtain the estimate of the solution to problem (137) by exploiting the tame estimate (110). Then the estimate of $(\mathbf{U}_n, f_n)$ follows from formula (138).

Lemma 15. *Let $\alpha \geq 13$. There exist $\delta > 0$, $T > 0$ sufficiently small, and $\theta_0 \geq 1$ sufficiently large, such that for all integer $s \in [6, \widetilde{\alpha}]$, one has*

$$[\delta\mathbf{U}_n]_{s,*,T} + \|\delta f_n\|_{H^s(\partial\Omega_T)} \leq \delta\theta_n^{s-\alpha-1}\Delta_n. \tag{189}$$

Proof. Without loss of generality we can take the constant K_0 appearing in estimate (110) that $K_0 = 2C_*$, where C_* is the constant from (127). In order to apply Theorem 4, by using (145) and (158), one checks that

$$[\mathbf{U}^a + \mathbf{U}_{n+1/2}]_{11,*,T} + \|f^a + S_{\theta_n}f_n\|_{H^{11}(\partial\Omega_T)} \leq 2C_*$$

for $\alpha \geq 13$ and δ small enough. That is, assumption (109) is satisfied for the coefficients of problem (137). By applying the tame estimate (110), for T small enough one has

$$\begin{aligned}[\delta\dot{\mathbf{U}}_n]_{s,*,T} + \|\delta f_n\|_{H^s(\partial\Omega_T)} \leq C\Big\{&[\mathbf{f}_n]_{s+2,*,T} + \|\mathbf{g}_n\|_{H^{s+2}(\partial\Omega_T)} \\ &+\big([\mathbf{f}_n]_{s_0,*,T} + \|\mathbf{g}_n\|_{H^{s_0}(\partial\Omega_T)}\big)\big([\mathbf{U}^a + \mathbf{U}_{n+1/2}]_{s+4,*,T} \\ &\quad+\|f^a + S_{\theta_n}f_n\|_{H^{s+4}(\partial\Omega_T)}\big)\Big\}.\end{aligned} \tag{190}$$

Using (204) and (208), from formula (138) we have

$$[\delta \mathbf{U}_n]_{s,*,T} \le [\delta \dot{\mathbf{U}}_n]_{s,*,T} + C\big\{\|\delta f_n\|_{H^s(\partial\Omega_T)} + \|\delta f_n\|_{H^6(\partial\Omega_T)}\|f^a + S_{\theta_n} f_n\|_{H^s(\partial\Omega_T)}\big\}.$$

Then (190) implies

$$[\delta \mathbf{U}_n]_{s,*,T} + \|\delta f_n\|_{H^s(\partial\Omega_T)} \le C\Big\{\theta_n [\mathbf{f}_n]_{s+2,*,T} + \|\mathbf{g}_n\|_{H^{s+2}(\partial\Omega_T)} + \big([\mathbf{f}_n]_{s_0,*,T} + \|\mathbf{g}_n\|_{H^{s_0}(\partial\Omega_T)}\big)\big([\mathbf{U}^a + \mathbf{U}_{n+1/2}]_{s+4,*,T} + \|f^a + S_{\theta_n} f_n\|_{H^{s+4}(\partial\Omega_T)}\big)\Big\} \tag{191}$$

for all integer $s \in [6, \widetilde{\alpha}]$. Applying Lemma 14, (145), and Proposition 12, from (191) we get the estimate

$$[\delta \mathbf{U}_n]_{s,*,T} + \|\delta f_n\|_{H^s(\partial\Omega_T)} \le C\big\{\theta_n^{s-\alpha-1}\left([\mathbf{F}^a]_{\alpha+2,*,T} + \delta^2\right) + \delta^2\theta_n^{L(s+2)-1}\big\}\Delta_n + C\delta\Delta_n\big\{\theta_n^{s_0-3-\alpha}\left([\mathbf{F}^a]_{\alpha+2,*,T} + \delta^2\right) + \delta^2\theta_n^{L(s_0)-1}\big\}\big\{C_* + \theta_n^{(s+4-\alpha)_+} + \theta_n^{s+6-\alpha}\big\}. \tag{192}$$

Let first s is even, then $s_0 = 8$, $L(s_0) - 1 = 18 - 2\alpha$, and $s_0 - 3 - \alpha = 5 - \alpha$. We can check that the inequalities

$$\begin{aligned} &L(s+2) \le s - \alpha, \quad (s+4-\alpha)_+ + 5 + \ell - \alpha \le s - \alpha - 1, \\ &(s+4-\alpha)_+ + 18 + \ell - 2\alpha \le s - \alpha - 1, \\ &s + 11 + \ell - 2\alpha \le s - \alpha - 1, \quad s + 24 + \ell - 3\alpha \le s - \alpha - 1. \end{aligned} \tag{193}$$

hold with $\ell = 0$ for $\alpha \ge 13$ and $s \in [6, \tilde{\alpha}]$. If s is odd, then $s_0 = 9$, $L(s_0) - 1 = 19 - 2\alpha$, and $s_0 - 3 - \alpha = 6 - \alpha$. Then, it is easily verified that inequalities (193) with $\ell = 1$ are satisfied for $\alpha \ge 13$ and $s \in [7, \tilde{\alpha}]$. Thus, (192) and (129) yield

$$[\delta \mathbf{U}_n]_{s,*,T} + \|\delta f_n\|_{H^s(\partial\Omega_T)} \le C\left(\delta_0(T) + \delta^2\right)\theta_n^{s-\alpha-1}\Delta_n \le \delta\theta_n^{s-\alpha-1}\Delta_n$$

for δ and T small enough. □

Inequality (189) is point *a*) of (H_n). It remains to prove points *b*) and *c*) of (H_n).

Lemma 16. *Let $\alpha \ge 13$. There exist $\delta > 0$, $T > 0$ sufficiently small, and $\theta_0 \ge 1$ sufficiently large, such that for all integer $s \in [6, \widetilde{\alpha} - 2]$*

$$[\mathcal{L}(\mathbf{U}_n, \boldsymbol{\Psi}_n) - \mathbf{F}^a]_{s,*,T} \le 2\delta\theta_n^{s-\alpha-1}. \tag{194}$$

Moreover, for all integer $s \in [7, \alpha]$ one has

$$\|\mathcal{B}(\mathbf{U}_n|_{x_1=0}, f_n)\|_{H^s(\partial\Omega_T)} \le \delta\theta_n^{s-\alpha-1}. \tag{195}$$

Proof. One can show that

$$\mathcal{L}(\mathbf{U}_n, \boldsymbol{\Psi}_n) - \mathbf{F}^a = (S_{\theta_{n-1}} - I)\mathbf{F}^a + (I - S_{\theta_{n-1}})\mathbf{E}_{n-1} + \mathbf{e}_{n-1}. \qquad (196)$$

For $s \in [\alpha+1, \tilde{\alpha}-2]$, by using (134), we get

$$(I - S_{\theta_{n-1}})\mathbf{F}^a]_{s,*,T} \leq \theta_n^{s-\alpha-1}(C[\mathbf{F}^a]_{\alpha+1,*,T} + [\mathbf{F}^a]_{s,*,T}) \leq C\delta_0(T)\theta_n^{s-\alpha-1},$$

while for $s \in [6, \alpha+1]$, applying (135), we have

$$(I - S_{\theta_{n-1}})\mathbf{F}^a]_{s,*,T} \leq C\theta_{n-1}^{s-\alpha-1}[\mathbf{F}^a]_{\alpha+1,*,T} \leq C\delta_0(T)\theta_n^{s-\alpha-1}.$$

As follows from Lemma 13 and (135), for $6 \leq s \leq \alpha+3 = \tilde{\alpha}-2$

$$[(I - S_{\theta_{n-1}})\mathbf{E}_{n-1}]_{s,*,T} \leq C\theta_{n-1}^{s-\alpha-3}[\mathbf{E}_{n-1}]_{\alpha+3,*,T} \leq C\delta^2\theta_n^{s-\alpha-1}.$$

Applying (185), we obtain

$$[\mathbf{e}_{n-1}]_{s,*,T} \leq C\delta^2\theta_{n-1}^{L(s)-1}\Delta_{n-1} \leq C\delta^2\theta_{n-1}^{s-\alpha-3}\Delta_{n-1} \leq C\delta^2\theta_n^{s-\alpha-1}.$$

From the above estimates and decomposition (196), by choosing $T > 0$ and $\delta > 0$ sufficiently small, we derive (194). Similarly, by using the decomposition

$$\mathcal{B}(\mathbf{U}_n|_{x_1=0}, f_n) = (I - S_{\theta_{n-1}})\widetilde{E}_{n-1} + \tilde{\mathbf{e}}_{n-1},$$

we can prove estimate (195). □

As follows from Lemmas 15 and 16, we have proved that (H_{n-1}) implies (H_{n-1}), provided that $\alpha \geq 13$, $\tilde{\alpha} = \alpha + 5$, the constant $\theta_0 \geq 1$ is large enough, and $T > 0$, $\delta > 0$ are small enough. Fixing now the constants α, δ, and θ_0, we prove (H_0).

Lemma 17. *If the time $T > 0$ is sufficiently small, then (H_0) is true.*

Proof. We recall that $(\mathbf{U}_0, f_0) = 0$. Then, by the definition of the approximate solution in Lemma 6 the state $(\mathbf{U}^a + \mathbf{U}_0, f^a + f_0) = 0$ satisfies already (5), (12), (197), (14), (15), and (at $x_1 = 0$) inequalities (9) and (10). That is, it follows from the construction of Proposition 12 that $(\mathbf{U}_{n+1/2}, f_{n+1/2}) = 0$. Consequently, $(\delta\dot{\mathbf{U}}_0, \delta f_0)$ solves the linear problem (28)–(30) with the coefficients $(\widehat{\mathbf{U}}, \hat{f}) = (\mathbf{U}^a, f^a)$ and the source terms $\mathbf{f} = S_{\theta_0}\mathbf{F}^a$ and $\mathbf{g} = 0$. Thanks to (127) the assumption (109) is satisfied (recall that $K_0 = 2C_*$). Applying (110), we obtain the estimate

$$[\delta\dot{\mathbf{U}}_0]_{s,*,T} + \|\delta f_0\|_{H^s(\partial\Omega_T)} \leq C[S_{\theta_0}\mathbf{F}^a]_{s+2,*,T}.$$

With (130) and formula (138) this estimate yields

$$\begin{aligned}[\delta\mathbf{U}_0]_{s,*,T} + \|\delta f_0\|_{H^s(\partial\Omega_T)} &\leq C[S_{\theta_0}\mathbf{F}^a]_{s+2,*,T} \\ &\leq C\theta_0^{(s-\alpha)_+}\delta_0(T) \leq \delta\theta_0^{s-\alpha-1}\Delta_0\end{aligned}$$

for all integer $s \in [6, \tilde{\alpha}]$, provided that T is sufficiently small. Likewise, points b) and c) of (H_0) can be shown to be satisfied for a sufficiently short time $T > 0$. □

The proof of Theorem 1. Let us consider initial data $(\mathbf{U}_0^{\pm}, f_0) \in H_*^{2m+19}(\mathbb{R}^3_+) \times H^{2m+19}(\mathbb{R}^2)$ satisfying all the assumptions of Theorem 1. In particular, they satisfy the compatibility conditions up to order $\mu = m+9$ (see Definition 1). Then, thanks to Lemmas 5 and 6 we can construct an approximate solution $(\mathbf{U}^a, f^a) \in H_*^{m+10}(\Omega_T) \times H^{m+10}(\partial\Omega_T)$ that satisfies (127). As follows from Lemmas 15–17, (H_n) holds for all integer $n \geq 0$, provided that $\alpha \geq 13$, $\tilde{\alpha} = \alpha + 5$, the constant $\theta_0 \geq 1$ is large enough, and the time $T > 0$ and the constant $\delta > 0$ are small enough. In particular, it follows from (H_n) that

$$\sum_{n=0}^{\infty} \left\{ [\delta \mathbf{U}_n]_{m,*,T} + \|\delta f_n\|_{H^m(\partial\Omega_T)} \right\} \leq \infty.$$

Hence, the sequence $(\mathbf{U}_n, f_n)$ converges in $H_*^m(\Omega_T) \times H^m(\partial\Omega_T)$ to some limit $(\mathbf{U}, f)$. Recall that $m = \alpha - 1 \geq 12$. Passing to the limit in (194) and (195) with $s = m$, we get (130)–(132). Consequently, $\mathbf{U} := \mathbf{U} + \mathbf{U}^a$, $f := f + f^a$ is a solution of problem (11)–(13). This completes the proof of Theorem 1.

Appendix A

The proof of Proposition 1. The equation for $\mathbf{H}^{\pm}$ contained in (11) reads

$$\partial_t \mathbf{H}^{\pm} + \frac{1}{\partial_1 \Phi^{\pm}} \left\{ (\mathbf{w}^{\pm}, \nabla)\mathbf{H}^{\pm} - (\mathbf{h}^{\pm}, \nabla)\mathbf{v}^{\pm} + \mathbf{H}^{\pm} \operatorname{div} \mathbf{u}^{\pm} \right\} = 0, \tag{197}$$

where

$$\mathbf{u}^{\pm} = (v_{\mathrm{n}}^{\pm}, v_2^{\pm}\partial_1\Phi^{\pm}, v_3^{\pm}\partial_1\Phi^{\pm}), \quad v_{\mathrm{n}}^{\pm} = v_1^{\pm} - v_2^{\pm}\partial_2\Psi^{\pm} - v_3^{\pm}\partial_3\Psi^{\pm},$$
$$v_{\mathrm{n}}^{\pm}|_{x_1=0} = v_{\mathrm{N}}^{\pm}|_{x_1=0}, \quad \mathbf{w}^{\pm} = \mathbf{u}^{\pm} - (\partial_t\Psi^{\pm}, 0, 0).$$

Analogous equation contained in system (1) written in the straightened variables differs from (197) by the additional term $\mathbf{v}^{\pm}\operatorname{div}\mathbf{h}^{\pm}$ in the expression in braces. After long, but straightforward calculations (applying, in particular, div to a consequence of (197)) we get

$$\partial_t a^{\pm} + \frac{1}{\partial_1 \Phi^{\pm}} \left\{ (\mathbf{w}^{\pm}, \nabla a^{\pm}) + a^{\pm} \operatorname{div} \mathbf{u}^{\pm} \right\} = 0 \tag{198}$$

for $a^{\pm} = \operatorname{div}\mathbf{h}^{\pm}/\partial_1\Phi^{\pm}$. Analogous equation following from system (1) does not contain the last term in the expression in braces.

In view of the boundary conditions (12),

$$w_1^{\pm}|_{x_1=0} = (v_{\mathrm{N}}^{\pm} - \partial_t f)|_{x_1=0} = 0.$$

Therefore, equation (198) does not need a boundary condition for $a^{\pm}$. Then by standard method of characteristic curves, we get (14) for all $t \in [0, T]$.

Considering (197) on the boundary $x_1 = 0$, using the boundary conditions (12), and omitting detailed calculations, we obtain

$$\partial_t H_{\mathrm{N}}^{\pm} + v_2^{\pm}\partial_2 H_{\mathrm{N}}^{\pm} + v_3^{\pm}\partial_3 H_{\mathrm{N}}^{\pm} + \left(\partial_2 v_2^{\pm} + \partial_3 v_3^{\pm}\right) H_{\mathrm{N}}^{\pm} = 0 \quad \text{on } x_1 = 0. \tag{199}$$

Using again the standard method of characteristic curves, we conclude that (15) is fulfilled for all $t \in [0, T]$ if it is satisfied for $t = 0$. This completes the proof.

The proof of Proposition 2. We write down the equation for $\dot{\mathbf{H}}^{\pm}$ contained in (28):

$$\begin{aligned}\partial_t \dot{\mathbf{H}}^{\pm} + \frac{1}{\partial_1 \widehat{\Phi}^{\pm}} \Big\{ & (\hat{\mathbf{w}}^{\pm}, \nabla)\dot{\mathbf{H}}^{\pm} - (\hat{\mathbf{h}}^{\pm}, \nabla)\dot{\mathbf{v}}^{\pm} + \widehat{\mathbf{H}}^{\pm}\mathrm{div}\,\dot{\mathbf{u}}^{\pm} \\ & + (\dot{\mathbf{u}}^{\pm}, \nabla)\widehat{\mathbf{H}}^{\pm} - (\dot{\mathbf{h}}^{\pm}, \nabla)\hat{\mathbf{v}}^{\pm} + \dot{\mathbf{H}}^{\pm}\mathrm{div}\,\hat{\mathbf{u}}^{\pm} \Big\} = \mathbf{f}_{\mathrm{H}}^{\pm},\end{aligned} \tag{200}$$

where

$$\dot{\mathbf{u}}^{\pm} = (\dot{v}_{\mathrm{n}}^{\pm}, \dot{v}_2^{\pm}\partial_1\widehat{\Phi}^{\pm}, \dot{v}_3^{\pm}\partial_1\widehat{\Phi}^{\pm}), \quad \dot{v}_{\mathrm{n}}^{\pm} = \dot{v}_1^{\pm} - \dot{v}_2^{\pm}\partial_2\widehat{\Psi}^{\pm} - \dot{v}_3^{\pm}\partial_3\widehat{\Psi}^{\pm},$$

$$\dot{v}_{\mathrm{n}}^{\pm}|_{x_1=0} = \dot{v}_{\mathrm{N}}^{\pm}|_{x_1=0}, \quad \mathbf{f}_{\mathrm{H}}^{\pm} = (f_5^{\pm}, f_6^{\pm}, f_7^{\pm}).$$

Recall that (21) and (22) imply

$$\mathrm{div}\,\hat{\mathbf{h}}^{+} = 0, \quad \mathrm{div}\,\hat{\mathbf{h}}^{-} = 0, \tag{201}$$

$$\widehat{H}_{\mathrm{N}}^{+}|_{x_1=0} = \widehat{H}_{\mathrm{N}}^{-}|_{x_1=0} = 0. \tag{202}$$

Using (21) and (201), after long calculations, which are omitted, from (200) we obtain that $r^{\pm} = \mathrm{div}\,\dot{\mathbf{h}}^{\pm}$ satisfy equations (33) (where $a^{\pm} = r^{\pm}/\partial_1\widehat{\Phi}^{\pm}$). It is worth noting that to get (33) we need not only the divergent constraints (201) but also the equations for $\widehat{\mathbf{H}}^{\pm}$ themselves, i.e., equations (21). Similarly, using the boundary conditions (29), system (21) at $x_1 = 0$, and the constraints (202), from (200) being considered at $x_1 = 0$ we get equations (34). That is, the proof of Proposition 2 is complete.

Appendix B

Gagliardo-Nirenberg inequality for H_*^s. First of all we note that if s is *even*, the anisotropic weighted Sobolev spaces H_*^s coincide with the spaces $\widetilde{E}_s$ introduced by Alinhac [1]. Then, referring to [1], we have the following variant of the Gagliardo-Nirenberg inequality for H_*^s:

$$\|\partial_\star^\alpha \partial_1^k u\|_{L_{2p}(\Omega_T)} \leq C\|u\|_{L_\infty(\Omega_T)}^{1-1/p}[u]_{s,*,T}^{1/p}, \quad \frac{1}{p} = \frac{|\alpha| + 2k}{s}, \tag{203}$$

where $s \geq 0$ is even, $|\alpha| + 2k \leq s$, and u is supposed to belong to $H_*^s(\Omega_T) \cap L_\infty(\Omega_T)$.

Moser-type inequalities for H^s_*. The Gagliardo-Nirenberg inequality (203) implies the following Moser-type calculus inequalities for H^s_* with even $s \geq 0$ (see [1]):

$$[uv]_{s,*,T} \leq C\left([u]_{s,*,T}\|v\|_{L_\infty(\Omega_T)} + \|u\|_{L_\infty(\Omega_T)}[v]_{s,*,T}\right), \tag{204}$$

$$[F(u)]_{s,*,T} \leq C(M)\left(1 + [u]_{s,*,T}\right), \tag{205}$$

where the functions u and v are supposed to belong to $H^s_*(\Omega_T) \cap L_\infty(\Omega_T)$, the function F is a C^∞ function of u, and M is such a positive constant that

$$\|u\|_{L_\infty(\Omega_T)} \leq M.$$

As for the usual Sobolev spaces inequality (205) can be refined if we assume that $F(0) = 0$:

$$[F(u)]_{s,*,T} \leq C(M)[u]_{s,*,T}. \tag{206}$$

Embedding theorems for H^s_*. With the reference to [1], for the domain $\Omega_T \subset \mathbb{R}^4$ we have the embeddings

$$\widetilde{E}_s \subset W^r_\infty \qquad \text{if} \quad s > \frac{5}{2} + 2r.$$

In particular, $\widetilde{E}_3 \subset L_\infty$ and $\widetilde{E}_5 \subset W^1_\infty$. For the exact definition of $\widetilde{E}_s$ we refer to [1], and for us it is only important that $\widetilde{E}_s = H^s_*$ if $s \geq 0$ is even. Then, we have the embeddings

$$H^4_*(\Omega_T) \subset L_\infty(\Omega_T) \quad \text{and} \quad H^6_*(\Omega_T) \subset W^1_\infty(\Omega_T).$$

That is, the following inequalities hold:

$$\begin{aligned} \|u\|_{L_\infty(\Omega_T)} &\leq C[u]_{4,*,T} \qquad \forall u \in H^4_*(\Omega_T), \\ \|u\|_{W^1_\infty(\Omega_T)} &\leq C[u]_{6,*,T} \qquad \forall u \in H^6_*(\Omega_T). \end{aligned} \tag{207}$$

The case of odd s. It follows from the definition of the space H^s_* that for odd positive s

$$[u]_{s,*,T} \leq C\Big([u]_{s-1,*,T} + \sum_{|\alpha|=1} [\partial^\alpha_\star u]_{s-1,*,T}\Big).$$

Therefore, from inequalities (204), (205) we deduce their counterparts for H^s_* with odd positive s:

$$\begin{aligned} [uv]_{s,*,T} &\leq C\left([u]_{s,*,T}\|v\|_{W^{1,\tan}_\infty(\Omega_T)} + \|u\|_{W^{1,\tan}_\infty(\Omega_T)}[v]_{s,*,T}\right), \\ [F(u)]_{s,*,T} &\leq C(M_1)\left(1 + [u]_{s,*,T}\right), \end{aligned} \tag{208}$$

where M_1 is such a positive constant that $\|u\|_{W^{1,\tan}_\infty(\Omega_T)} \leq M_1$, and

$$\|u\|_{W^{1,\tan}_\infty(\Omega_T)} := \|u\|_{L_\infty(\Omega_T)} + \sum_{|\alpha|=1} \|\partial^\alpha_\star u\|_{L_\infty(\Omega_T)}.$$

Likewise, it follows from (207) that

$$\begin{aligned} \|u\|_{W^{1,\tan}_\infty(\Omega_T)} &\le C[u]_{5,*,T} \qquad \forall u \in H^5_*(\Omega_T), \\ \|u\|_{W^{2,\tan}_\infty(\Omega_T)} &\le C[u]_{7,*,T} \qquad \forall u \in H^7_*(\Omega_T), \end{aligned} \tag{209}$$

with

$$\|u\|_{W^{2,\tan}_\infty(\Omega_T)} := \|u\|_{W^1_\infty(\Omega_T)} + \sum_{|\alpha|=1} \|\partial^\alpha_\star u\|_{W^1_\infty(\Omega_T)}.$$

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Sobolev Institute of Mathematics,
Koptyug avenue 4, 630090 Novosibirsk, Russia
e-mail: trakhin@math.nsc.ru