Energy principle for magnetohydrodynamic flows and Bogoyavlenskij's transformation

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ABSTRACT

The stability of steady magnetohydrodynamic flows of an inviscid incompressible fluid is studied using the energy method. It is shown that certain symmetry transformations of steady solutions of the equations of ideal magnetohydrodynamics have an important property: if a given steady magnetohydrodynamic flow is stable by the energy method, then certain infinite-dimensional families of steady flows obtained from the given flow by these transformations are also stable. This result is used to obtain new sufficient conditions for linear stability. In particular, it is shown that certain classes of steady magnetohydrodynamic flows in which both the magnetic field and the velocity depend on all three spatial coordinates are stable.

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I Introduction

In this paper, the stability of steady magnetohydrodynamic (MHD) flows of an ideal incompressible perfectly conducting fluid to small three-dimensional perturbations is studied. We employ the well-known energy method, first proposed by Bernstein *et al* [1] for magnetostatic equilibria and later generalized by Frieman and Rotenberg [2] to the case of steady MHD flows. Certain examples of non-trivial MHD flows, which are stable to small three-dimensional perturbations have been given in Refs. 3–6. In this paper we generalize the examples presented in Ref. 5 and give sufficient conditions for stability for the class of steady MHD flows that is identified by the relation $\mathbf{J}\cdot\mathbf{H}=0$ (where $\mathbf{H}$ and $\mathbf{J}=\nabla\times\mathbf{H}$ are the magnetic field and the electric current density in the basic state). All earlier examples of successful application of the energy principle, which we are aware of, represent particular cases of our general criterion. It turns out that the domain of applicability of the general criterion can be extended further with the help of a certain symmetry transformation of steady solutions of equations of ideal magnetohydrodynamics, first introduced by Bogoyavlenskij [7, 8]. This transformation can be used to construct new steady MHD flows from a given flow. In particular, it is possible to construct steady MHD flows which depend on all three spatial coordinates. We show first that Bogoyavlenskij's transformation can be slightly generalized and, second, that it has the following remarkable property: if a steady MHD flow is stable by the energy method, a 'half' of all flows which can be obtained from this flow by Bogoyavlenskij's transformation are also stable. This result gives the extention of our examples of stable flows to wide classes of flows depending on arbitrary functions.

The plan of the paper is as follows. In section 2 we consider steady solutions of the equations of ideal magnetohydrodynamics and introduce Bogoyavlenskij's transformation and its extensions, including an extension to steady MHD flows with current-vortex sheets. In section 3, we discuss the energy principle and show that if a given steady flow is stable by the energy method, then certain infinite dimensional families of flows which can be obtained from the given flow by the symmetry transformations are also stable. In section 4 we discuss the properties of the energy invariant of the linearized equations and formulate sufficient conditions for stability for a general class of flows with $\mathbf{U}=\lambda(\mathbf{x})\mathbf{H}$ and $\mathbf{J}\cdot\mathbf{H}\equiv 0$. Section 5 is devoted to a number of examples of flows with symmetries for which the sufficient conditions for stability take especially simple form. Here we also employ Bogoyavlenskij's transformation to construct certain classes of steady MHD flows which are not symmetric and depend on all three spatial coordinates, but are, nevertheless, stable. Finally, section 6 contains the discussion of the results.

We conclude this introduction with a statement of the governing equations. We suppose that the fluid is inviscid and perfectly conducting and that it is contained in a domain $\mathcal{D}$ with fixed perfectly conducting boundary $\partial\mathcal{D}$ or extends to infinity. We assume also that the flow velocity is much less than the Mach number, so that the effects of compressibility can be ignored, and the fluid can be treated as incompressible. Let $\mathbf{u}(\mathbf{x},t)$ be the velocity field, $\mathbf{h}(\mathbf{x},t)$ the magnetic field (in Alfven velocity units), $p(\mathbf{x},t)$ the pressure (divided by density), and $\mathbf{j}=\nabla\times\mathbf{h}$ the current density. The governing equations are the standard equations of ideal magnetohydrodynamics:

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{h}\cdot\nabla)\mathbf{h} = -\nabla\pi, \tag{1}$$

$$\mathbf{h}_t = (\mathbf{h}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{h} \equiv [\mathbf{u},\mathbf{h}], \tag{2}$$

$$\operatorname{div}\mathbf{u} = 0, \quad \operatorname{div}\mathbf{h} = 0, \tag{3}$$

where $\pi = p + \mathbf{h}^2/2$ is the modified pressure, p being the hydrodynamic pressure. The boundary conditions for $\mathbf{u}$ and $\mathbf{h}$ are

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{h}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{D}, \tag{4}$$

where $\mathbf{n}$ is the unit outward normal on $\partial\mathcal{D}$.

Note that, since we consider an incompressible fluid, the hydrodynamic pressure p (and hence the modified pressure π) is defined up to a constant and can be determined (up to a constant) from the incompressibility condition $\operatorname{div}\mathbf{u} = 0$ and boundary conditions (4). In particular, p (and π) may be negative.

II Symmetry transformations of steady MHD flows

Consider a steady solution to problem (1)–(4) given by

$$\mathbf{u} = \mathbf{U}(\mathbf{x}), \quad \mathbf{h} = \mathbf{H}(\mathbf{x}), \quad \pi = \Pi(\mathbf{x}), \tag{5}$$

where $\Pi = P + \mathbf{H}^2/2$. The steady state (5) represents a solution of the equations

$$(\mathbf{U}\cdot\nabla)\mathbf{U} - (\mathbf{H}\cdot\nabla)\mathbf{H} = -\nabla\Pi, \tag{6}$$

$$[\mathbf{U},\mathbf{H}] = 0, \quad \nabla\cdot\mathbf{U} = \nabla\cdot\mathbf{H} = 0 \quad \text{in} \quad \mathcal{D} \tag{7}$$

with boundary conditions

$$\mathbf{U}\cdot\mathbf{n} = \mathbf{H}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{D}. \tag{8}$$

Recently, Bogoyavlenskij (see Refs. 7 and 8) has discovered a certain symmetry transformation that connects steady solutions of the governing equations of ideal magnetohydrodynamics. He used this transformation to construct new steady MHD flows. For homogeneous (in density) incompressible MHD flows, the transformation is defined as follows.

Let $\mathbf{H}(\mathbf{x})$, $\mathbf{U}(\mathbf{x})$ and $\Pi(\mathbf{x})$ be a steady solution (5) and let functions $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ be constant along both the magnetic field lines and the streamlines, i.e.

$$\mathbf{H}\cdot\nabla\alpha = 0, \quad \mathbf{H}\cdot\nabla\beta = 0, \quad \mathbf{U}\cdot\nabla\alpha = 0, \quad \mathbf{U}\cdot\nabla\beta = 0, \tag{9}$$

and satisfy the equation

$$\alpha^2(\mathbf{x}) - \beta^2(\mathbf{x}) = C \quad \text{in} \quad \mathcal{D} \tag{10}$$

with some constant C.

Then the functions $\tilde{\mathbf{H}}(\mathbf{x})$, $\tilde{\mathbf{U}}(\mathbf{x})$ and $\tilde{\Pi}(\mathbf{x})$, given by

$$\tilde{\mathbf{H}} = \alpha(\mathbf{x})\mathbf{H} + \beta(\mathbf{x})\mathbf{U}, \quad \tilde{\mathbf{U}} = \beta(\mathbf{x})\mathbf{H} + \alpha(\mathbf{x})\mathbf{U}, \quad \tilde{\Pi} = C\Pi, \tag{11}$$

also represent a steady solution of Eqs. (1)–(4). This fact can be easily checked by a straightforward calculation.

Thus, the transformation $\mathbf{H}(\mathbf{x}) \to \tilde{\mathbf{H}}(\mathbf{x})$, $\mathbf{U}(\mathbf{x}) \to \tilde{\mathbf{U}}(\mathbf{x})$, $\Pi(\mathbf{x}) \to \tilde{\Pi}(\mathbf{x})$ is a symmetry transformation for Eqs. (6), (7).

The constant C entering Eq. (10) may be any real number (positive, negative, or zero). In particular, the value $C = 0$ corresponds to a transformation of any steady MHD flow to the Alfven solutions: $\mathbf{H} \to \tilde{\mathbf{H}} = \alpha(\mathbf{x})(\mathbf{H} \pm \mathbf{U})$, $\mathbf{U} \to \tilde{\mathbf{U}} = \pm\tilde{\mathbf{H}}$, $\Pi \to \tilde{\Pi} = 0$.

For general properties of Bogoyavlenskij's transformation and a discussion of the conditions for existence of functions $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ satisfying (9) we refer to the original papers by Bogoyavlenskij [7, 8]. Here we discuss only some fairly straightforward extensions of Bogoyavlenskij's transformation.

Transformation (11) can be represented by a 2×2 matrix having the form

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \tag{12}$$

where $\alpha = \alpha(\mathbf{x})$ and $\beta = \beta(\mathbf{x})$ are functions satisfying Eqs. (9), (10). The composition of two transformations can be represented as multiplication of 2×2 matrices:

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \beta_1 & \alpha_1 \end{pmatrix}\begin{pmatrix} \alpha_2 & \beta_2 \\ \beta_2 & \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 + \beta_1\beta_2 & \alpha_1\beta_2 + \beta_1\alpha_2 \\ \alpha_1\beta_2 + \beta_1\alpha_2 & \alpha_1\alpha_2 + \beta_1\beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_3 & \beta_3 \\ \beta_3 & \alpha_3 \end{pmatrix}. \tag{13}$$

Note that $C_3 = C_1 C_2$ where $C_i = \alpha_i^2(\mathbf{x}) - \beta_i^2(\mathbf{x})$ $(i = 1, 2, 3)$. It is evident that the operation of composition of transformations (11) is commutative and associative and that a unique inverse transformation exists provided that $C \neq 0$. Therefore, the symmetry transformations (11) with $C \neq 0$ form an Abelian group [8].

In fact, this Abelian group is a subgroup of a larger group of transformations. Indeed, Eqs. (6)–(7) are invariant under the transformation $\mathbf{H}(\mathbf{x}) \to \mathbf{H}(\mathbf{x})$, $\mathbf{U}(\mathbf{x}) \to -\mathbf{U}(\mathbf{x})$, $\Pi(\mathbf{x}) \to \Pi(\mathbf{x})$. Combining this fact with (11), we deduce that Eqs. (6)–(7) are invariant under the transformation

$$\mathbf{H} \to \tilde{\mathbf{H}} = \alpha(\mathbf{x})\mathbf{H} + \beta(\mathbf{x})\mathbf{U}, \quad \mathbf{U} \to \tilde{\mathbf{U}} = -\beta(\mathbf{x})\mathbf{H} - \alpha(\mathbf{x})\mathbf{U}, \quad \Pi \to \tilde{\Pi} = C\Pi, \tag{14}$$

with the same functions $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ and a constant C as before. Transformation (14) can be represented by a matrix

$$\begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}, \tag{15}$$

Transformations (11) and (14) with $C \neq 0$ form an infinite dimensional Lie group which is not Abelian (transformations defined by (11) and (14) do not commute).

In what follows we are especially interested in field-aligned flows in which the velocity is everywhere parallel to the magnetic field, i.e.

$$\mathbf{U}(\mathbf{x}) = \lambda(\mathbf{x})\,\mathbf{H}(\mathbf{x}) \tag{16}$$

with some smooth function $\lambda(\mathbf{x})$. From the incompressibility condition, we have

$$\mathbf{H}\cdot\nabla\lambda = 0. \tag{17}$$

Note also that Eqs. (6), (16) and (17) imply the relation

$$\nabla\Pi = (1-\lambda^2)(\mathbf{H}\cdot\nabla)\mathbf{H}\,. \tag{18}$$

For field-aligned flows, the transformations (11) and (14) reduce to

$$\tilde{\mathbf{H}} = \gamma(\mathbf{x})\mathbf{H}, \quad \tilde{\mathbf{U}} = \pm\tilde{\lambda}(\mathbf{x})\tilde{\mathbf{H}}, \quad \tilde{\Pi} = C\Pi. \tag{19}$$

where

$$\gamma(\mathbf{x}) = \alpha(\mathbf{x}) + \lambda(\mathbf{x})\beta(\mathbf{x}), \quad \tilde{\lambda}(\mathbf{x}) = \frac{\beta(\mathbf{x}) + \lambda(\mathbf{x})\alpha(\mathbf{x})}{\alpha(\mathbf{x}) + \lambda(\mathbf{x})\beta(\mathbf{x})}. \tag{20}$$

It follows from (10) and (20) that

$$\gamma^2(\mathbf{x})\left(1-\tilde{\lambda}^2(\mathbf{x})\right) = C\left(1-\lambda^2(\mathbf{x})\right). \tag{21}$$

Transformation (19) is completely determined by two functions $\gamma(\mathbf{x})$ and $\tilde{\lambda}(\mathbf{x})$ satisfying Eq. (21) with some constant $C \neq 0$ and the conditions

$$\mathbf{H}\cdot\nabla\gamma = 0, \quad \mathbf{H}\cdot\nabla\tilde{\lambda} = 0. \tag{22}$$

It follows from (21) that any steady flow (16) can be transformed to a magnetostatic equilibrium (by choosing $\tilde{\lambda}(\mathbf{x}) = 0$ and $\gamma^2(\mathbf{x}) = C\left(1-\lambda^2(\mathbf{x})\right)$) and, conversely, any magnetostatic equilibrium ($\mathbf{U} = 0$) can be transformed to a steady flow.

As was shown in Refs. 7 and 8, the symmetry transformation (19) can break geometrical symmetries of field-aligned steady MHD flows. Indeed, suppose that a steady MHD flow with the property (16) is invariant with respect to certain geometrical transformations of the flow domain (e.g. rotations or translations). If in this flow the magnetic field lines are closed or extend to infinity, then functions $\gamma(\mathbf{x})$ and $\tilde{\lambda}(\mathbf{x})$ satisfying Eq. (22) depend on two coordinates that are transversal to the magnetic field lines and are, in general, not invariant under the geometrical symmetry transformations of the flow domain. This property has been exploited in Refs. 7 and 8 to construct certain steady MHD flows that depend on all three spatial coordinates. We will give some examples of such flows later.

Discontinuous MHD flows. The symmetry transformations (11) and (14) can be extended to the case of steady MHD flows with current-vortex sheets. (In Refs. 7 and 8, transformation (11) has been used to construct certain examples of steady MHD flows with current-vortex sheets.) We assume that the magnetic field $\mathbf{H}$ and the velocity $\mathbf{U}$ are continuous everywhere in the flow domain except for a smooth surface S at which tangent components of $\mathbf{H}$ and $\mathbf{U}$ may have jump discontinuities. Thus, the surface S divides the flow domain $\mathcal{D}$ in two parts $\mathcal{D}_+$ and $\mathcal{D}_-$. We will use the notation: $\mathbf{H}(\mathbf{x}) = \mathbf{H}^{\pm}(\mathbf{x})$ for $\mathbf{x} \in \mathcal{D}_{\pm}$, $\mathbf{U}(\mathbf{x}) = \mathbf{U}^{\pm}(\mathbf{x})$ for $\mathbf{x} \in \mathcal{D}_{\pm}$ and $\Pi(\mathbf{x}) = \Pi^{\pm}(\mathbf{x})$ for $\mathbf{x} \in \mathcal{D}_{\pm}$. In a steady flow with tangential discontinuity, $\mathbf{U}^{\pm}(\mathbf{x})$ and $\mathbf{H}^{\pm}(\mathbf{x})$ obey Eqs. (6) and (7) in $\mathcal{D}_{\pm}$ and boundary conditions (8) at $\partial\mathcal{D}\cap\partial\mathcal{D}^{\pm}$. In addition, the following conditions at the discontinuity surface S must be satisfied (see, e.g., [9]):

$$\mathbf{U}^{\pm}\cdot\mathbf{n} = 0, \quad \mathbf{H}^{\pm}\cdot\mathbf{n} = 0, \quad \Pi^{+} = \Pi^{-} \quad \text{at} \quad S. \tag{23}$$

Here $\mathbf{n}$ is the unit normal on S directed from $\mathcal{D}_+$ to $\mathcal{D}_-$.

Evidently, the symmetry transformations (11) and (14) can be applied separately to $\mathbf{H}^+$, $\mathbf{U}^+$ in $\mathcal{D}^+$ and to $\mathbf{H}^-$, $\mathbf{U}^-$ in $\mathcal{D}^-$ producing $\tilde{\mathbf{H}}^+$, $\tilde{\mathbf{U}}^+$ and $\tilde{\mathbf{H}}^-$, $\tilde{\mathbf{U}}^-$, respectively. All that remains to be done to obtain a discontinuous solution in the whole domain $\mathcal{D}$ is to satisfy the boundary condition

$$\tilde{\Pi}^+ = \tilde{\Pi}^- \quad \text{at} \quad S. \tag{24}$$

Since $\tilde{\Pi}^\pm = C^\pm \Pi^\pm$, Eq. (24) will be satisfied provided that $C^+ = C^- = C$. Thus, for steady MHD flows with current-vortex sheets the analogues of symmetry transformations (11) and (14) are

$$\tilde{\mathbf{H}}^\pm = \alpha^\pm(\mathbf{x})\mathbf{H}^\pm + \beta^\pm(\mathbf{x})\mathbf{U}^\pm, \quad \tilde{\mathbf{U}}^\pm = \beta^\pm(\mathbf{x})\mathbf{H}^\pm + \alpha^\pm(\mathbf{x})\mathbf{U}^\pm, \quad \tilde{\Pi}^\pm = C\Pi^\pm, \tag{25}$$

$$\tilde{\mathbf{H}}^\pm = \alpha^\pm(\mathbf{x})\mathbf{H}^\pm + \beta^\pm(\mathbf{x})\mathbf{U}^\pm, \quad \tilde{\mathbf{U}}^\pm = -\beta^\pm(\mathbf{x})\mathbf{H}^\pm - \alpha^\pm(\mathbf{x})\mathbf{U}^\pm, \quad \tilde{\Pi}^\pm = C\Pi^\pm, \tag{26}$$

where, as before, functions $\alpha^\pm(\mathbf{x})$ and $\beta^\pm(\mathbf{x})$ are constant along the streamlines and the magnetic field lines and satisfy the equations

$$(\alpha^+(\mathbf{x}))^2 - (\beta^+(\mathbf{x}))^2 = C, \quad (\alpha^-(\mathbf{x}))^2 - (\beta^-(\mathbf{x}))^2 = C \tag{27}$$

with some constant $C \neq 0$.

In the next section we employ the energy principle to study the linearized stability problem for steady MHD flows and show that in many situations the steady flows that are connected by the above symmetry transformations have the same stability properties.

III Energy principle

Let $\mathbf{u}'(\mathbf{x},t)$, $\mathbf{h}'(\mathbf{x},t)$, $\pi'(\mathbf{x},t)$ be infinitesimal perturbations of the steady solution (5). Linearization of Eqs. (1)–(3) gives us the following equations

$$\mathbf{u}'_t + (\mathbf{U}\cdot\nabla)\mathbf{u}' + (\mathbf{u}'\cdot\nabla)\mathbf{U} - (\mathbf{H}\cdot\nabla)\mathbf{h}' - (\mathbf{h}'\cdot\nabla)\mathbf{H} = -\nabla\pi', \tag{28}$$

$$\mathbf{h}'_t = [\mathbf{u}',\mathbf{H}] + [\mathbf{U},\mathbf{h}'], \quad \operatorname{div}\mathbf{u}' = 0, \quad \operatorname{div}\mathbf{h}' = 0 \quad \text{in} \quad \mathcal{D}. \tag{29}$$

The boundary conditions for $\mathbf{u}'$ and $\mathbf{h}'$ are

$$\mathbf{u}'\cdot\mathbf{n} = 0, \quad \mathbf{h}'\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{D}. \tag{30}$$

From here on, we omit 'primes' to simplify the notation.

Following [2], we introduce the Lagrangian displacement $\boldsymbol{\xi}(\mathbf{x},t)$ of a fluid particle (i.e. the displacement at the time t of a fluid particle in the perturbed flow relative to its position $\mathbf{x}$ at the same time in the unperturbed flow) satisfying the equations

$$\mathbf{u} = \boldsymbol{\xi}_t + [\boldsymbol{\xi},\mathbf{U}], \quad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{in} \quad \mathcal{D} \tag{31}$$

and the boundary conditions

$$\boldsymbol{\xi}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{D}. \tag{32}$$

It can be shown using Eqs. (29) and (31) that

$$(\mathbf{h} - [\boldsymbol{\xi}, \mathbf{H}])_t = [\mathbf{U}, (\mathbf{h} - [\boldsymbol{\xi}, \mathbf{H}])] \quad \text{in} \quad \mathcal{D}.$$

Therefore, if the relation

$$\mathbf{h} = [\boldsymbol{\xi}, \mathbf{H}] \quad \text{in} \quad \mathcal{D} \tag{33}$$

is satisfied at $t = 0$, then it holds for any $t > 0$. This allows us to introduce a special class of *isomagnetic perturbations* which satisfy (33). Note that for isomagnetic perturbations the boundary condition for the perturbation magnetic field (30) is automatically satisfied. In what follows only isomagnetic perturbations will be considered.

Substitution of (31) and (33) into (28) give us the following equation for the Lagrangian displacement $\boldsymbol{\xi}$ (see also [5]):

$$\boldsymbol{\xi}_{tt} + 2(\mathbf{U} \cdot \nabla)\boldsymbol{\xi}_t + (\mathbf{U} \cdot \nabla)^2\boldsymbol{\xi} - (\mathbf{H} \cdot \nabla)^2\boldsymbol{\xi} + (\boldsymbol{\xi} \cdot \nabla)\nabla\Pi = -\nabla\pi \quad \text{in} \quad \mathcal{D}, \tag{34}$$

Equations (34) and (31) and the boundary condition (32) completely determine the evolution of small isomagnetic perturbations to the basic steady state (5).

Taking a dot-product of Eq. (34) with $\boldsymbol{\xi}_t$, integrating over $\mathcal{D}$ and using the boundary condition (32), we obtain

$$\frac{dE}{dt} = 0, \tag{35}$$

where

$$E = T + W, \quad T = \frac{1}{2}\int\limits_{\mathcal{D}} \boldsymbol{\xi}_t^2 dV, \tag{36}$$

$$W = \frac{1}{2}\int\limits_{\mathcal{D}} \Big(((\mathbf{H} \cdot \nabla)\boldsymbol{\xi})^2 - ((\mathbf{U} \cdot \nabla)\boldsymbol{\xi})^2 + \boldsymbol{\xi} \cdot (\boldsymbol{\xi} \cdot \nabla)\nabla\Pi \Big) dV. \tag{37}$$

Thus, the quadratic integral E is conserved by the linearized equations. We shall refer to it as the energy of the linearized problem.

Note that W can be written in the following form

$$W = \frac{1}{2}\int\limits_{\mathcal{D}} \Big([\boldsymbol{\xi}, \mathbf{H}]^2 + [\boldsymbol{\xi}, \mathbf{H}] \cdot (\mathbf{J} \times \boldsymbol{\xi}) - [\boldsymbol{\xi}, \mathbf{U}]^2 - [\boldsymbol{\xi}, \mathbf{U}] \cdot (\boldsymbol{\Omega} \times \boldsymbol{\xi}) \Big) dV, \tag{38}$$

where $\mathbf{J} = \nabla \times \mathbf{H}$ and $\boldsymbol{\Omega} = \nabla \times \mathbf{U}$ are the electric current and the vorticity in the basic flow. If, in the basic state, there is no flow ($\mathbf{U} = 0$), then W reduces to the classic expression of Bernstein et al [1].

E is a quadratic functional of $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_t$. If it is positive definite, then the fact that it is an invariant of the linearized problem (34), (31) and (32) implies the stability of the basic state (5) with respect to small isomagnetic perturbations. Evidently, E is positive definite if and only if the 'potential energy' W is positive definite. The stability problem is thus reduced to the analysis of W.

Energy principle and the symmetry transformations. Now we show that in certain situations the stability properties of the steady states that are connected by the symmetry transformations (11) and (14) are the same.

To do this, we consider the potential energy $\tilde{W}$ (given by Eq. (37)) corresponding to a basic state (with the magnetic field $\tilde{\mathbf{H}}(\mathbf{x})$, the velocity $\tilde{\mathbf{U}}(\mathbf{x})$ and the modified pressure $\tilde{\Pi}(\mathbf{x})$) that has been obtained by transformations (11) or (14) from a given basic state with $\mathbf{H}(\mathbf{x})$, $\mathbf{U}(\mathbf{x})$ and $\Pi(\mathbf{x})$. It is easy to see that, according to (9)–(11) (or (9)–(10) and (14)),

$$\left(\tilde{\mathbf{H}} \cdot \boldsymbol{\xi}\right)^2 - \left(\tilde{\mathbf{U}} \cdot \boldsymbol{\xi}\right)^2 = C\left((\mathbf{H} \cdot \boldsymbol{\xi})^2 - (\mathbf{U} \cdot \boldsymbol{\xi})^2\right), \qquad \tilde{\Pi} = C\Pi.$$

Substituting these into Eq. (37), we obtain

$$\tilde{W} = CW, \tag{39}$$

where $\tilde{W}$ and W are considered as functionals of $\boldsymbol{\xi}(\mathbf{x})$. It follows that if W is positive definite (semidefinite) and $C > 0$, then $\tilde{W}$ is also positive definite (semidefinite). Another consequence of Eq. (39) is that if W is negative definite (semidefinite) and $C < 0$, then $\tilde{W}$ is positive definite (semidefinite).

Thus, we have proved the following proposition.

Proposition 1 *(i) If a steady MHD flow is stable according to the energy principle, then all steady flows which are obtained form this flow by the symmetry transformations (11) and (14) with $C > 0$ are also stable; (ii) if for a given steady flow W is negative definite (semidefinite), then all steady flows which are obtained from this flow by the symmetry transformations (11) and (14) with $C < 0$ are stable.*

Discontinuous MHD flows. The energy principle is also applicable to steady MHD flows with current-vortex sheets. In this case, the potential energy W includes an additional term W_S that appears due to the presence of discontinuity and is given by [6]

$$W_{\mathrm{S}} = -\frac{1}{2}\int\limits_S (\boldsymbol{\xi} \cdot \mathbf{n}) \{\boldsymbol{\xi} \cdot \nabla\Pi\} \, dS. \tag{40}$$

Here the braces denote the jump of the corresponding quantity at the discontinuity surface S (e.g., $\{\Pi\} = \Pi^+ - \Pi^-$). Since under the discontinuous version of the symmetry transformations (11) and (14) $\Pi^\pm \to \tilde{\Pi}^\pm = C\,\Pi^\pm$, we obtain

$$\tilde{W}_{\mathrm{S}} = C\,W_{\mathrm{S}}.$$

It follows that Proposition 1 is also valid for steady flows with current-vortex sheets.

IV Sufficient conditions for stability

In this section, we identify a particular class of steady MHD flows for which we can formulate sufficient conditions for stability and give explicit examples of stable flows.

It is known (see, e.g., [4, 5]) that W is indefinite in sign if there is a region in the flow domain where $\mathbf{U}$ is non-zero and non-parallel to $\mathbf{H}$. Certain non-trivial examples of steady MHD flows with $\mathbf{U}$ parallel to $\mathbf{H}$ everywhere in $\mathcal{D}$ (field-aligned flows) for which W is non-negative have been given in [5]. Here we extend the examples of [5] to more general classes of steady MHD flows. As in [5], we suppose that *the non-negativeness of W is sufficient for linear stability*, i.e. our sufficient conditions will guarantee, at least, that there is no exponential instability.

So, we assume that in the steady state the velocity and the magnetic field are parallel, i.e. satisfy Eq. (16). It has been shown in Ref. 5 that for field-aligned flows W can be transformed to

$$W = \frac{1}{2}\int_{\mathcal{D}} \left\{(1-\lambda^2)\left(\mathbf{h}^2 + \mathbf{h}\cdot(\mathbf{J}\times\boldsymbol{\xi})\right) - 2\lambda(\boldsymbol{\xi}\cdot\nabla\lambda)\,(\boldsymbol{\xi}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H})\right\} dV. \tag{41}$$

Now we assume that $\mathbf{J}\times\mathbf{H}\neq 0$ in $\mathcal{D}$ and define the vector field $\mathbf{N}(\mathbf{x})$ by

$$\mathbf{N} = \mathbf{J}\times\mathbf{H}. \tag{42}$$

Note that, with this notation, equation (18) can be rewritten as

$$\mathbf{N} = \frac{1}{1-\lambda^2}\nabla\Pi - \nabla\left(\frac{\mathbf{H}^2}{2}\right). \tag{43}$$

Now let

$$\boldsymbol{\xi} = a(\mathbf{x},t)\mathbf{N} + b(\mathbf{x},t)\mathbf{J} + c(\mathbf{x},t)\mathbf{H}. \tag{44}$$

It can be shown (see Appendix) that the 'potential energy' W can be presented in the form

$$W = W_1 + W_2, \tag{45}$$

$$W_1 = \frac{1}{2}\int_{\mathcal{D}} (1-\lambda^2)\,(\mathbf{h} + a\mathbf{J}\times\mathbf{N})^2\, dV, \tag{46}$$

$$W_2 = \frac{1}{2}\int_{\mathcal{D}} \left(Aa^2 + Bb^2 + 2Fab\right) dV \tag{47}$$

where

$$A = -(1-\lambda^2)\left[2(\mathbf{J}\times\mathbf{N})\cdot(\mathbf{H}\cdot\nabla)\mathbf{N} + (\mathbf{M}\cdot\mathbf{N})(\mathbf{J}\cdot\mathbf{H})\right], \tag{48}$$

$$B = -(\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{J}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}), \tag{49}$$

$$F = -(\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{N}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}). \tag{50}$$

In Eq. (48), $\mathbf{M} = \nabla\times\mathbf{N}$.

Evidently, W is positive semidefinite if $|\lambda|\leq 1$ and W_2 is positive semidefinite. The latter is true if functions $A(\mathbf{x})$, $B(\mathbf{x})$ and $F(\mathbf{x})$ defined by Eqs. (48)–(50) satisfy the conditions:

$$A\geq 0, \quad AB - F^2 \geq 0 \quad \text{in} \quad \mathcal{D}. \tag{51}$$

Thus, *a general steady flow satisfying Eq. (16) is stable to small three-dimensional perturbations provided that (i) the flow is sub-alfvenic and (ii) inequalities (51) are satisfied.*

In what follows, we consider a particular class of steady MHD flows satisfying Eq. (16) for which the electric current is orthogonal to the magnetic field, i.e.

$$\mathbf{J}\cdot\mathbf{H} = 0 \quad \text{in} \quad \mathcal{D}. \tag{52}$$

In this case,

$$\mathbf{J}\times\mathbf{N} = \mathbf{J}\times(\mathbf{J}\times\mathbf{H}) = \mathbf{J}(\mathbf{J}\cdot\mathbf{H}) - \mathbf{H}(\mathbf{J}^2) = -\mathbf{H}(\mathbf{J}^2),$$

and Eq. (48) simplifies to

$$A = -\left(2((1-\lambda^2)\mathbf{J}^2 + \mathbf{N}\cdot\nabla\lambda^2\right)(\mathbf{N}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}). \tag{53}$$

Another equivalent form of A is obtained by using the fact that

$$\mathbf{N}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H} = \mathbf{N}\cdot\left(\mathbf{J}\times\mathbf{H} + \nabla(\mathbf{H}^2/2)\right) = \mathbf{N}^2 + \mathbf{N}\cdot\nabla(\mathbf{H}^2/2) = \mathbf{J}^2\mathbf{H}^2 + \mathbf{N}\cdot\nabla(\mathbf{H}^2/2).$$

Then,

$$A = -\left(2(1-\lambda^2)\mathbf{J}^2 + \mathbf{N}\cdot\nabla\lambda^2\right)\left(\mathbf{J}^2\mathbf{H}^2 + \mathbf{N}\cdot\nabla(\mathbf{H}^2/2)\right). \tag{54}$$

Consider now the integral W_2 given by (47). As was mentioned above, W_2 is positive semi-definite if inequalities (51) are satisfied. It turns out that the second of these inequalities always holds, namely,

$$AB - F^2 \equiv 0$$

for flows satisfying Eq. (52). Therefore, the sufficient condition for stability of such flows reduces to the only inequality $A \geq 0$. Below we prove this fact.

We have

$$AB - F^2 = (\mathbf{J}\cdot\nabla\lambda^2)\left(\mathbf{J}^2\mathbf{H}^2 + \mathbf{N}\cdot\nabla(\mathbf{H}^2/2)\right)\cdot X \tag{55}$$

where

$$X = \left(2(1-\lambda^2)\mathbf{J}^2 + \mathbf{N}\cdot\nabla\lambda^2\right)\left(\mathbf{J}\cdot\nabla\left(\mathbf{H}^2/2\right)\right) - (\mathbf{J}\cdot\nabla\lambda^2)\left(\mathbf{J}^2\mathbf{H}^2 + \mathbf{N}\cdot\nabla(\mathbf{H}^2/2)\right). \tag{56}$$

Simple manipulations yield

$$\begin{aligned} X &= \mathbf{J}^2(\mathbf{J}\cdot\nabla)\left[(1-\lambda^2)\mathbf{H}^2\right] + (\mathbf{N}\cdot\nabla\lambda^2)\left(\mathbf{J}\cdot\nabla\left(\mathbf{H}^2/2\right)\right) - \left(\mathbf{N}\cdot\nabla\left(\mathbf{H}^2/2\right)\right)(\mathbf{J}\cdot\nabla\lambda^2) \\ &= \mathbf{J}^2(\mathbf{J}\cdot\nabla)\left[(1-\lambda^2)\mathbf{H}^2\right] + \nabla\left(\mathbf{H}^2/2\right)\cdot\left(\nabla\lambda^2\times(\mathbf{J}\times\mathbf{N})\right) \\ &= \mathbf{J}^2(\mathbf{J}\cdot\nabla)\left[(1-\lambda^2)\mathbf{H}^2\right] - \mathbf{J}^2\nabla\left(\mathbf{H}^2/2\right)\cdot\left(\nabla\lambda^2\times\mathbf{H}\right). \end{aligned} \tag{57}$$

It follows from Eq. (18) that

$$(1-\lambda^2)\mathbf{J}\times\mathbf{H} = \nabla P + \lambda^2\nabla(\mathbf{H}^2/2).$$

Taking inner product of the curl of this equation with $\mathbf{H}$ results in

$$(1-\lambda^2)\left(\mathbf{H}\cdot(\mathbf{H}\cdot\nabla)\mathbf{J} - \mathbf{H}\cdot(\mathbf{J}\cdot\nabla)\mathbf{H}\right) + \mathbf{H}^2(\mathbf{J}\cdot\nabla\lambda^2) = \mathbf{H}\cdot\left(\nabla\lambda^2\times\nabla(\mathbf{H}^2/2)\right).$$

Since $\mathbf{H}\cdot(\mathbf{H}\cdot\nabla)\mathbf{J} = -\mathbf{J}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H} = \mathbf{J}\cdot\nabla(\mathbf{H}^2/2)$, we obtain

$$\mathbf{H}\cdot\left(\nabla\lambda^2\times\nabla(\mathbf{H}^2/2)\right) = -(1-\lambda^2)(\mathbf{J}\cdot\nabla\mathbf{H}^2) + \mathbf{H}^2(\mathbf{J}\cdot\nabla\lambda^2) = -(\mathbf{J}\cdot\nabla)\left[(1-\lambda^2)\mathbf{H}^2\right]. \tag{58}$$

Finally, it follows from (57) and (58) that $X = 0$ and, hence, $AB - F^2 = 0$.

Thus, we can formulate the following proposition.

Proposition 2 *A steady MHD flow (5) with* $\mathbf{J}\cdot\mathbf{H} = 0$ *and* $\mathbf{U} = \lambda\mathbf{H}$ *is linearly stable provided that the flow is subalfvenic (*$|\lambda| < 1$*) and*

$$\left(2(1-\lambda^2)\mathbf{J}^2 + \mathbf{N}\cdot\nabla\lambda^2\right)\left(\mathbf{J}^2\mathbf{H}^2 + \mathbf{N}\cdot\nabla(\mathbf{H}^2/2)\right) \leq 0 \quad in \quad \mathcal{D}. \tag{59}$$

Combining Proposition 2 with the result formulated in Proposition 1, we arrive at the following conclusions.

Proposition 3 *If there is a steady MHD flow satisfying the conditions of Proposition 2, then any steady MHD flow obtained from this flow by the symmetry transformation (19) with $C > 0$ is stable.*

Proposition 4 *If there is a steady MHD flow with $\mathbf{J}\cdot\mathbf{H} = 0$ and $\mathbf{U} = \lambda\mathbf{H}$ which is superalfvenic ($|\lambda| \geq 1$) and satisfies the inequality*

$$\left(2(1-\lambda^2)\mathbf{J}^2 + \mathbf{N}\cdot\nabla\lambda^2\right)\left(\mathbf{J}^2\mathbf{H}^2 + \mathbf{N}\cdot\nabla(\mathbf{H}^2/2)\right) \geq 0 \quad in \quad \mathcal{D}, \tag{60}$$

then any steady MHD flow obtained from this flow by the symmetry transformation (19) with $C < 0$ is stable.

Note also that class of steady field-aligned flows defined by Eq. (52) is invariant under the symmetry transformation (19). Indeed, in view of (22), $\tilde{\mathbf{J}}\cdot\tilde{\mathbf{H}} = \gamma^2\mathbf{J}\cdot\mathbf{H}$ and, therefore, the equality $\mathbf{J}\cdot\mathbf{H} = 0$ implies that $\tilde{\mathbf{J}}\cdot\tilde{\mathbf{H}} = 0$. This fact explains why the sufficient conditions for stability of an infinite-dimensional family of steady flows (obtained by transformation (19) from a given flow) involve only inequality (59) (or (60)) rather than two inequalities (51).

In the next section we present particular classes of steady MHD flows for which sufficient conditions for stability can be formulated explicitly.

V Examples

A. Translationally invariant flow and field. Let the flow domain $\mathcal{D}$ be an infinite cylinder (of arbitrary cross-section) parallel to the z-axis. We consider a steady discontinuous MHD flow (5) in which both the velocity and the magnetic field are along the z-axis and depend only upon the transverse coordinates x and y:

$$\mathbf{H} = H_0(x,y)\mathbf{e}_z, \quad \mathbf{U} = \lambda(x,y)\mathbf{H} = U_0(x,y)\mathbf{e}_z. \tag{61}$$

In this case, $\mathbf{J}\cdot\mathbf{H} = 0$ and inequality (59) is always satisfied. Therefore, the following criterion is valid [5]:

Proposition 5 *The steady MHD flow (61) is linearly stable provided that the flow is subalfvenic, i.e.*

$$|\mathbf{U}(x,y)| \leq |\mathbf{H}(x,y)| \quad in \quad \mathcal{D}. \tag{62}$$

Note that, in view of the invariance of the basic state under translations along the z axis, it suffices that inequality (62) is satisfied in some reference frame moving with constant velocity along z axis.

Transformation (19) applied to the flow (61) produces a flow of the form (61). Therefore in this case Propositions 3 and 4 do not lead to new stable MHD flows.

B. Steady two-dimensional flows. Consider now a general two-dimensional steady MHD flow. Let the flow domain $\mathcal{D}$ be the same as in the previous example. We suppose that both the velocity and the magnetic field are independent of z and parallel to the (x, y)-plane. Then

$$\mathbf{H} = \nabla\Phi \times \mathbf{e}_z, \quad \mathbf{U} = \lambda(\Phi)\mathbf{H}, \tag{63}$$

where $\Phi(x, y)$ is the flux function for the magnetic field H. We assume that $\mathbf{H}$, λ and $\mathbf{U}$ are sufficiently smooth everywhere in $\mathcal{D}$. The flux function Φ satisfies the Grad-Shafranov equation

$$\triangle\Phi - \frac{\lambda\lambda'}{1-\lambda^2}\mathbf{H}^2 = G(\Phi) \quad \text{in} \quad \mathcal{D} \tag{64}$$

with some function $G(\Phi)$. Here we used the notation $\lambda' = d\lambda/d\Phi$.

Inequality (59) takes the form

$$G(Q - G) \leq 0 \quad \text{in} \quad \mathcal{D}. \tag{65}$$

where

$$Q = \frac{\nabla\Phi \cdot \nabla((1-\lambda^2)\mathbf{H}^2)}{2(1-\lambda^2)\mathbf{H}^2}. \tag{66}$$

So, we can formulate the following sufficient conditions for stability [5].

Proposition 6 *The steady state (61) is stable to small three-dimensional perturbations provided that the flow is sub-alfvenic ($|\lambda| \leq 1$) and condition (65) is valid.*

Now we apply transformation (19) to the steady flow (63). This yields

$$\tilde{\mathbf{H}} = \gamma(\Phi, z)\nabla\Phi \times \mathbf{e}_z, \quad \tilde{\mathbf{U}} = \tilde{\lambda}(\Phi, z)\tilde{\mathbf{H}}, \tag{67}$$

where functions $\gamma(\Phi, z)$, $\tilde{\lambda}(\Phi, z)$ and $\lambda(\Phi)$ obey the condition (21).

It immediately follows from Propositions 3 and 4 that (i) *if the steady flow (63) is stable, then any flow of the form (67) obtained from (63) by transformation (19) with $C > 0$ is also stable*, and (ii) *if the steady flow (63) is superalfvenic ($|\lambda| \geq 1$) and $G(Q - G) \geq 0$ everywhere in $\mathcal{D}$, then any flow of the form (67) obtained from (63) by transformation (19) with $C < 0$ is stable.*

Note that the steady flow (67) depends on three spatial coordinates x, y and z, while the original flow (61) is invariant under translations along the z axis. Here we have an example of symmetry breaking by transformation (19).

C. Axisymmetric, purely poloidal flow. Let $\mathcal{D}$ be an axisymmetric domain. In the cylindrical coordinates (r, θ, z), the velocity and the magnetic field are given by

$$\mathbf{U} = \left(-\frac{1}{r}\frac{\partial\Psi}{\partial z}\right)\mathbf{e}_r + \frac{1}{r}\frac{\partial\Psi}{\partial r}\mathbf{e}_z, \quad \mathbf{H} = \left(-\frac{1}{r}\frac{\partial\Phi}{\partial z}\right)\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial r}\mathbf{e}_z, \tag{68}$$

where $\Phi(r, z)$ is the flux function for the magnetic field and $\Psi(r, z)$ is the streamfunction. We assume that $\Psi = \Psi(\Phi)$ and, therefore,

$$\mathbf{U} = \lambda(\Phi)\mathbf{H} = \Psi'(\Phi)\mathbf{H}. \tag{69}$$

We have

$$\mathbf{\Omega} = (-r\hat{K}\Psi)\,\mathbf{e}_\theta, \quad \mathbf{J} = (-r\hat{K}\Phi)\,\mathbf{e}_\theta, \quad \hat{K} = \frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial z^2}. \tag{70}$$

As in the two-dimensional case, the flux function A satisfies the Grad-Shafranov equation

$$\hat{K}\Phi - \frac{\lambda\lambda'}{1-\lambda^2}\mathbf{H}^2 = G(\Phi), \quad \lambda' = \frac{d\lambda}{d\Phi}. \tag{71}$$

with some function $G(\Phi)$. Since $\mathbf{J}\cdot\mathbf{H} = 0$ in the basic state, we can use Proposition 1 and obtain the following criterion.

Proposition 7 *The steady state (68) is stable to small three-dimensional perturbations provided that the flow is sub-alfvenic ($|\lambda| \le 1$) and*

$$G(Q-G) \le 0 \quad in \quad \mathcal{D} \tag{72}$$

where

$$Q = \frac{\nabla\Phi\cdot\nabla((1-\lambda^2)\mathbf{H}^2)}{2r^2(1-\lambda^2)\mathbf{H}^2}, \quad \nabla = \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right). \tag{73}$$

Now we apply transformation (19) to the steady flow (68). This yields

$$\tilde{\mathbf{H}} = \frac{1}{r}\gamma(\Phi,\theta)\nabla\Phi\times\mathbf{e}_z, \quad \tilde{\mathbf{U}} = \tilde{\lambda}(\Phi,\theta)\tilde{\mathbf{H}}, \tag{74}$$

where functions $\gamma(\Phi,\theta)$, $\tilde{\lambda}(\Phi,\theta)$ and $\lambda(\Phi)$ obey the condition (21).

It follows from Propositions 3 and 4 that (i) *if the steady flow (68) is stable, then any flow of the form (74) obtained from (68) by transformation (19) with $C > 0$ is also stable*, and (ii) *if the steady flow (68) is superalfvenic ($|\lambda| \ge 1$) and $G(Q-G) \ge 0$ everywhere in $\mathcal{D}$, then any flow of the form (74) obtained from (68) by transformation (19) with $C < 0$ is stable.*

Note that (74) is no longer an axisymmetric flow. It depends on all three spatial coordinates r, z and θ.

D. Purely toroidal flow. Let, as in the previous example, $\mathcal{D}$ be an axisymmetric domain. In the basic state, the magnetic field and the velocity are given by

$$\mathbf{H} = H_0(r,z)\mathbf{e}_\theta, \quad \mathbf{U} = \lambda(r,z)\mathbf{H} = U_0(r,z)\mathbf{e}_\theta \tag{75}$$

where functions $H_0(r,z)$ and $U_0(r,z)$ satisfy the relation

$$\frac{\partial}{\partial z}\left(\frac{U_0^2}{r} - \frac{H_0^2}{r}\right) = 0. \tag{76}$$

Proposition 1 leads to the following criterion.

Proposition 8 *The steady state (75) is stable to small three-dimensional perturbations provided that the flow is sub-alfvenic ($|\lambda| \le 1$) and*

$$\frac{d}{dr}\left(r^2(H_0^2 - U_0^2)\right) \le 0 \quad in \quad \mathcal{D}. \tag{77}$$

If we apply transformation (19) to the steady flow (75), this will produce again a flow of the form (75). Therefore here, as in the first example, Propositions 3 and 4 do not lead to new stable MHD flows.

VI Conclusions

We employed the energy method to study the linear stability of steady MHD flows of an inviscid incompressible fluid. We have generalized all previous examples of non-trivial steady flows for which the energy of the linearized problem is positive semi-definite and which are, therefore, stable to small three-dimensional perturbations. We have obtained sufficient conditions for stability for the class of steady MHD flows in which the velocity $\mathbf{U}(\mathbf{x})$ is parallel to the magnetic field $\mathbf{H}(\mathbf{x})$ and which satisfy the relation $\mathbf{J}\cdot\mathbf{H}=0$. Also, we have shown that if a given steady MHD flow is stable by energy principle, then certain infinite dimensional families of steady flows obtained by Bogoyavlenskij's transformation from this flow are stable. In particular, we have constructed two new families of stable MHD flows which depend on all three spatial coordinates (so that they are not symmetric in any sense). For each family, we have identified the conditions under which all members of the family are stable.

We have demonstrated that the similarity in the stability properties of steady MHD flows connected by Bogoyavlenskij's transformation holds even for flows with current-vortex sheets, so that the examples of Ref. 6 can be generalized in the same manner as we have done it for continuous MHD flows in this paper.

Much remains to be done in this area. At present, there are no nontrivial examples of *stable* field-aligned flows with $\mathbf{J}\cdot\mathbf{H}\neq 0$. This, however, does not mean that all such flows are unstable. It would be interesting therefore to find an example of a *stable* steady flow in which $\mathbf{J}\cdot\mathbf{H}\neq 0$. This is the subject of a continuing investigation.

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VII Appendix. Derivation of Eqs. (45)–(50)

Here we show that W can be written in the form of Eqs. (45)–(50). First we note that the incompressibility condition $\mathrm{div}\boldsymbol{\xi}=0$ and Eq. (44) imply the relationship

$$\nabla\cdot(a\mathbf{N})+\mathbf{J}\cdot\nabla b+\mathbf{H}\cdot\nabla c=0. \tag{78}$$

From (44), we have

$$\boldsymbol{\xi}\times\mathbf{H}=a\mathbf{N}\times\mathbf{H}+b\mathbf{N},\quad \mathbf{J}\times\boldsymbol{\xi}=a\mathbf{J}\times\mathbf{N}+c\mathbf{N}.$$

Therefore, in view of (33),

$$\mathbf{h}=\nabla\times(a\mathbf{N}\times\mathbf{H}+b\mathbf{N})$$

and

$$\begin{aligned}
I &\equiv \frac{1}{2}\int_{\mathcal{D}}(1-\lambda^2)\left(\mathbf{h}^2+\mathbf{h}\cdot(\mathbf{J}\times\boldsymbol{\xi})\right)dV \\
&= \frac{1}{2}\int_{\mathcal{D}}(1-\lambda^2)\left\{\mathbf{h}^2+\mathbf{h}\cdot(a\mathbf{J}\times\mathbf{N}+c\mathbf{N})\right\}dV \\
&= \frac{1}{2}\int_{\mathcal{D}}(1-\lambda^2)\Big\{\left(\mathbf{h}+a\mathbf{J}\times\mathbf{N}\right)^2-a^2(\mathbf{J}\times\mathbf{N})^2+\mathbf{h}\cdot(c\mathbf{N}-a\mathbf{J}\times\mathbf{N})\Big\}dV
\end{aligned} \tag{79}$$

Since $\mathbf{A}\cdot\operatorname{curl}\mathbf{B}-\mathbf{B}\cdot\operatorname{curl}\mathbf{A}=\operatorname{div}(\mathbf{B}\times\mathbf{A})$ for any vector fields $\mathbf{A}$ and $\mathbf{B}$, we have

$$\mathbf{h}\cdot(c\mathbf{N})=c\mathbf{N}\cdot\nabla\times(a\mathbf{N}\times\mathbf{H}+b\mathbf{N})=(a\mathbf{N}\times\mathbf{H}+b\mathbf{N})\cdot\nabla\times(c\mathbf{N})+\nabla\cdot((a\mathbf{N}\times\mathbf{H}+b\mathbf{N})\times(c\mathbf{N}))\,.$$

Substituting this in (79) and integrating by parts, we obtain

$$I=\frac{1}{2}\int_{\mathcal{D}}(1-\lambda^2)\Big\{\left(\mathbf{h}+a\mathbf{J}\times\mathbf{N}\right)^2-a^2(\mathbf{J}\times\mathbf{N})^2+Y_1+Y_2\Big\}dV \tag{80}$$

where

$$Y_1=\nabla\times(c\mathbf{N})\cdot(a\mathbf{N}\times\mathbf{H}+b\mathbf{N})\,,\quad Y_2=-a(\mathbf{J}\times\mathbf{N})\cdot(a\mathbf{N}\times\mathbf{H}+b\mathbf{N})\,.$$

Let $\mathbf{M}\equiv\nabla\times\mathbf{N}$. Then we have

$$\begin{aligned}
Y_1 &= a\mathbf{N}\cdot(\mathbf{H}\times(c\mathbf{M}+\nabla c\times\mathbf{N}))+b\mathbf{N}\cdot(c\mathbf{M}+\nabla c\times\mathbf{N}) \\
&= ac\mathbf{M}\cdot(\mathbf{N}\times\mathbf{H})-a\mathbf{N}^2(\mathbf{H}\cdot\nabla c)+bc\mathbf{M}\cdot\mathbf{N}, \\
Y_2 &= -a^2(\mathbf{J}\times\mathbf{N})\cdot\nabla\times(\mathbf{N}\times\mathbf{H})-a(\mathbf{J}\times\mathbf{N})\cdot(\nabla a\times(\mathbf{N}\times\mathbf{H}))-ab\mathbf{M}\cdot(\mathbf{J}\times\mathbf{N})-a(\mathbf{J}\times\mathbf{N})\cdot(\nabla b\times\mathbf{N}) \\
&= -a^2(\mathbf{J}\times\mathbf{N})\cdot\nabla\times(\mathbf{N}\times\mathbf{H})-a\mathbf{N}^2\left(\mathbf{N}\nabla\cdot a+\mathbf{J}\nabla\cdot b\right)-ab\mathbf{M}\cdot(\mathbf{J}\times\mathbf{N}).
\end{aligned}$$

It follows that

$$Y_1+Y_2=-a^2\left((\mathbf{J}\times\mathbf{N})\cdot\nabla\times(\mathbf{N}\times\mathbf{H})-\mathbf{N}^2\nabla\cdot\mathbf{N}\right)-ab\mathbf{J}\cdot(\mathbf{N}\times\mathbf{M})-ac\mathbf{H}\cdot(\mathbf{N}\times\mathbf{M})+bc\mathbf{M}\cdot\mathbf{N}. \tag{81}$$

Since $\mathbf{H}\cdot\mathbf{N}=0$, we have $\nabla(\mathbf{H}\cdot\mathbf{N})=0$. The latter identity has a consequence that

$$(\mathbf{N}\cdot\nabla)\mathbf{H}=-(\mathbf{H}\cdot\nabla)\mathbf{N}+\mathbf{M}\times\mathbf{H}+\mathbf{J}\times\mathbf{N}.$$

With help of this identity, we obtain

$$\begin{aligned}
(\mathbf{J}\times\mathbf{N})\cdot\nabla\times(\mathbf{N}\times\mathbf{H}) &= (\mathbf{J}\times\mathbf{N})\cdot((\mathbf{H}\cdot\nabla)\mathbf{N}-\mathbf{H}(\nabla\cdot\mathbf{N})-(\mathbf{N}\cdot\nabla)\mathbf{H}) \\
&= (\mathbf{J}\times\mathbf{N})\cdot(2(\mathbf{H}\cdot\nabla)\mathbf{N}-\mathbf{M}\times\mathbf{H}-\mathbf{J}\times\mathbf{N})+\mathbf{N}^2\nabla\cdot\mathbf{N} \\
&= 2(\mathbf{J}\times\mathbf{N})\cdot(\mathbf{H}\cdot\nabla)\mathbf{N}+(\mathbf{M}\cdot\mathbf{N})(\mathbf{J}\cdot\mathbf{H})-(\mathbf{J}\times\mathbf{N})^2+\mathbf{N}^2\nabla\cdot\mathbf{N}.
\end{aligned}$$

Substitution of this into Eq. (81) yields

$$\begin{aligned}
Y_1+Y_2 &= -a^2\left(2(\mathbf{J}\times\mathbf{N})\cdot(\mathbf{H}\cdot\nabla)\mathbf{N}+(\mathbf{M}\cdot\mathbf{N})(\mathbf{J}\cdot\mathbf{H})-(\mathbf{J}\times\mathbf{N})^2\right) \\
&\quad -ab\mathbf{J}\cdot(\mathbf{N}\times\mathbf{M})-ac\mathbf{H}\cdot(\mathbf{N}\times\mathbf{M})+bc\mathbf{M}\cdot\mathbf{N}.
\end{aligned} \tag{82}$$

In view of (82), Eq. (80) can be written as

$$I = \frac{1}{2}\int\limits_{\mathcal{D}} (1-\lambda^2)\Big\{(\mathbf{h} + a\mathbf{J}\times\mathbf{N})^2 \quad + \quad (-2(\mathbf{J}\times\mathbf{N})\cdot(\mathbf{H}\cdot\nabla)\mathbf{N} - (\mathbf{M}\cdot\mathbf{N})(\mathbf{J}\cdot\mathbf{H}))\, a^2$$
$$\quad - \quad ab\mathbf{J}\cdot(\mathbf{N}\times\mathbf{M}) - ac\mathbf{H}\cdot(\mathbf{N}\times\mathbf{M}) + bc\mathbf{M}\cdot\mathbf{N}\Big\} dV. \qquad (83)$$

Consider now the integral

$$Q = -\int\limits_{\mathcal{D}} \lambda(\boldsymbol{\xi}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}) dV. \qquad (84)$$

Substitution of (44) into (84) yields

$$\begin{aligned} Q \quad = \quad & -\frac{1}{2}\int\limits_{\mathcal{D}} \Big\{a^2(\mathbf{N}\cdot\nabla\lambda^2)(\mathbf{N}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}) + b^2(\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{J}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}) \\ & +ab\left[(\mathbf{N}\cdot\nabla\lambda^2)(\mathbf{J}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}) + (\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{N}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H})\right] \\ & +ac(\mathbf{N}\cdot\nabla\lambda^2)(\mathbf{J}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}) + bc(\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{H}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H})\Big\} dV. \end{aligned} \qquad (85)$$

It follows from (83) and (85) that the expression (41) for W can be written as

$$W = W_1 + W_2 + R, \qquad (86)$$

$$W_1 = \frac{1}{2}\int\limits_{\mathcal{D}} (1-\lambda^2)\,(\mathbf{h} + a\mathbf{J}\times\mathbf{N})^2\, dV, \quad W_2 = \frac{1}{2}\int\limits_{\mathcal{D}} \left(Aa^2 + Bb^2 + 2Fab\right) dV, \qquad (87)$$

$$\begin{aligned} R = -\frac{1}{2}\int\limits_{\mathcal{D}} \Big\{ & ac\left[(1-\lambda^2)\mathbf{H}\cdot(\mathbf{N}\times\mathbf{M}) + (\mathbf{N}\cdot\nabla\lambda^2)(\mathbf{H}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H})\right] \\ & +bc\left[-(1-\lambda^2)\mathbf{M}\cdot\mathbf{H} + (\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{H}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H})\right]\Big\} dV, \end{aligned} \qquad (88)$$

where

$$A = -(1-\lambda^2)\left[2(\mathbf{J}\times\mathbf{N})\cdot(\mathbf{H}\cdot\nabla)\mathbf{N} + (\mathbf{M}\cdot\mathbf{N})(\mathbf{J}\cdot\mathbf{H})\right], \qquad (89)$$

$$B = -(\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{J}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}), \qquad (90)$$

$$F = -\frac{1}{2}\left[\mathbf{J}(\mathbf{N}\times\mathbf{M}) + (\mathbf{N}\cdot\nabla\lambda^2)(\mathbf{J}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}) + (\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{N}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H})\right]. \qquad (91)$$

Now we show that $R = 0$. From (43), we have

$$\mathbf{M} = \nabla\times\mathbf{N} = \nabla\times\left(\frac{\nabla\Pi}{1-\lambda^2} - \nabla\left(\frac{\mathbf{H}^2}{2}\right)\right) = \frac{\nabla\lambda^2\times\nabla\Pi}{(1-\lambda^2)^2}. \qquad (92)$$

It follows from (43) and (92) that

$$\mathbf{M}\cdot\mathbf{N} = \frac{\nabla\Pi\cdot(\nabla\lambda^2\times\nabla(\mathbf{H}^2/2))}{(1-\lambda^2)^2}. \qquad (93)$$

Also, we have

$$(\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{H}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}) = \frac{(\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{H}\cdot\nabla\Pi)}{1-\lambda^2} = \frac{\nabla\lambda^2\cdot(\nabla\Pi\times(\mathbf{J}\times\mathbf{H}))}{1-\lambda^2} = -\frac{\nabla\lambda^2\cdot(\nabla\Pi\times\nabla(\mathbf{H}^2/2))}{1-\lambda^2}. \qquad (94)$$

Equations (93) and (94) give us the relation

$$-(1-\lambda^2)\mathbf{M}\cdot\mathbf{N} + (\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{H}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}) = 0. \qquad (95)$$

Similarly,

$$\mathbf{H}\cdot(\mathbf{N}\times\mathbf{M}) = -\mathbf{N}\cdot(\mathbf{H}\times\mathbf{M}) = -\frac{(\mathbf{N}\cdot\nabla\lambda^2)(\mathbf{H}\cdot\nabla\Pi)}{(1-\lambda^2)^2},$$
$$(\mathbf{N}\cdot\nabla\lambda^2)(\mathbf{H}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}) = \frac{(\mathbf{N}\cdot\nabla\lambda^2)(\mathbf{H}\cdot\nabla\Pi)}{1-\lambda^2}.$$

It follows that

$$(1-\lambda^2)\mathbf{H}\cdot(\mathbf{N}\times\mathbf{M}) + (\mathbf{N}\cdot\nabla\lambda^2)(\mathbf{H}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}) = 0. \tag{96}$$

In view of Eqs. (95) and (96), the integral R is identically zero.

Thus, we have shown that the 'potential energy' W can be written as

$$W = W_1 + W_2, \tag{97}$$
$$W_1 = \frac{1}{2}\int_{\mathcal{D}} (1-\lambda^2)\,(\mathbf{h} + a\mathbf{J}\times\mathbf{N})^2\, dV, \tag{98}$$
$$W_2 = \frac{1}{2}\int_{\mathcal{D}} \left(Aa^2 + Bb^2 + 2Fab\right) dV \tag{99}$$

with functions A, B and F defined by (89)–(91).

Note that, with help of Eqs. (18) and (92), we can simplify the expression for F as follows. From (92), we obtain

$$\mathbf{J}\cdot(\mathbf{N}\times\mathbf{M}) = \frac{\mathbf{J}\cdot(\mathbf{N}\times(\nabla\lambda^2\times\nabla\Pi))}{(1-\lambda^2)^2} = \frac{(\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{N}\cdot\nabla\Pi)) - (\mathbf{N}\cdot\nabla\lambda^2)(\mathbf{J}\cdot\nabla\Pi))}{(1-\lambda^2)^2}.$$

Substituting this into (91) and using (18), we find that

$$F = -(\mathbf{J}\cdot\nabla\lambda^2)(\mathbf{N}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}). \tag{100}$$

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