Stability of incompressible current-vortex sheets

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Abstract

We revisit the study in [15] where an energy a priori estimate for the linearized free boundary value problem for planar current-vortex sheets in ideal incompressible magnetohydrodynamics was proved for a part of the whole stability domain found a long time ago in [14, 1]. In this paper we derive an a priori estimate in the whole stability domain. The crucial point in deriving this estimate is the construction of a symbolic symmetrizer for a nonstandard elliptic problem for the small perturbation of total pressure. This symmetrizer is an analogue of Kreiss' type symmetrizers. As in hyperbolic theory, the failure of the uniform Lopatinski condition, i.e., the fact that current-vortex sheets are only weakly (neutrally) stable yields losses of derivatives in the energy estimate. The result of this paper is a necessary step to prove the local-in-time existence of stable nonplanar incompressible current-vortex sheets by a suitable Nash-Moser type iteration scheme.

1 Introduction

We consider the equations of ideal incompressible magnetohydrodynamics (MHD), i.e., the equations governing the motion of a perfectly conducting inviscid incompressible plasma. In the case of homogeneous plasma (the density $\rho(t,\mathbf{x}) \equiv \text{const} > 0$) these equations in a dimensionless form are

$$\begin{aligned} &\partial_t \mathbf{v} + (\mathbf{v}, \nabla)\mathbf{v} - (\mathbf{H}, \nabla)\mathbf{H} + \nabla q = 0\,, \\ &\partial_t \mathbf{H} + (\mathbf{v}, \nabla)\mathbf{H} - (\mathbf{H}, \nabla)\mathbf{v} = 0\,, \\ &\operatorname{div} \mathbf{v} = 0\,. \end{aligned} \tag{1}$$

where $\mathbf{v} = \mathbf{v}(t,\mathbf{x}) = (v_1, v_2, v_3)$ denotes plasma velocity, $\mathbf{H} = \mathbf{H}(t,\mathbf{x}) = (H_1, H_2, H_3)$ magnetic field (in Alfvén velocity units), $q = p + |\mathbf{H}|^2/2$ total pressure, and $p = p(t,\mathbf{x})$ pressure (divided by ρ). Hereafter we forget about the explicit form for the total pressure and work in terms of the unknowns $\mathbf{U} = (\mathbf{v}, \mathbf{H})$ and q. System (1) is supplemented by the *divergent constraint*

$$\operatorname{div} \mathbf{H} = 0 \tag{2}$$

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on the initial data $\mathbf{U}_{|t=0} = \mathbf{U}_0$ for the Cauchy problem in the whole space $\mathbb{R}^3$.

We are interested in weak solutions of (1) that are smooth on either side of a smooth hypersurface $\Gamma(t) = \{x_1 - f(t, \mathbf{x}') = 0\}$ in $[0, T] \times \mathbb{R}^3$, where $\mathbf{x}' = (x_2, x_3)$ (for technical simplicity we suppose that the density is the same constant on either side of Γ). Such weak solutions should satisfy jump conditions at each point of Γ. If Γ is a tangential discontinuity [7], i.e., the plasma does not flow through the discontinuity and the magnetic field on Γ is tangent to Γ, then the general jump conditions for system (1) take the form

$$\partial_t f = v_{\mathrm{N}}^{\pm}, \quad H_{\mathrm{N}}^{\pm} = 0, \quad [q] = 0 \qquad \text{on} \quad \Gamma(t). \tag{3}$$

Here $v_{\mathrm{N}} = (\mathbf{v}, \mathbf{N})$ and $H_{\mathrm{N}} = (\mathbf{H}, \mathbf{N})$ are normal components of the velocity and the magnetic field, $\mathbf{N} = (1, -\partial_{x_2} f, -\partial_{x_3} f)$ is a normal vector to Γ, and $[g] = g^+_{|\Gamma} - g^-_{|\Gamma}$ denotes the jump of a function g, with $g^{\pm} := g$ in $\Omega^{\pm}(t) = \{x_1 \gtrless f(t, \mathbf{x}')\}$. The tangential components of both the velocity and the magnetic field may undergo any jump. A tangential MHD discontinuity is usually called a *current-vortex sheet* [1, 12]. A current-vortex sheet has vorticity and current ($\operatorname{curl} \mathbf{v}$ and $\operatorname{curl} \mathbf{H}$) concentrated along its surface.

As was shown in [15], the divergent constraint (2) as well as the boundary conditions

$$H_{\mathrm{N}}^{+} = 0\,, \quad H_{\mathrm{N}}^{-} = 0 \qquad \text{on} \quad \Gamma(t) \tag{4}$$

should be regarded as the restrictions only on the initial data

$$\mathbf{U}^{\pm}(0, \mathbf{x}) = \mathbf{U}_0^{\pm}(\mathbf{x}), \quad \mathbf{x} \in \Omega^{\pm}(0), \qquad f(0, \mathbf{x}') = f_0(\mathbf{x}'), \quad \mathbf{x}' \in \mathbb{R}^2, \tag{5}$$

i.e., they are automatically satisfied for all $t > 0$ if they were satisfied at $t = 0$. Our final goal is to find conditions on the initial data (5) providing the existence of current-vortex sheet solutions to the MHD system, i.e., the existence of a solution $(\mathbf{U}^{\pm}, f)$ of the free boundary value problem (1), (3), (5), where $\mathbf{U}^{\pm} := \mathbf{U}$ in $\Omega^{\pm}(t)$. Note that the total pressure q is an "elliptic" unknown defined up to an arbitrary constant.

Recently the local-in-time existence of current-vortex sheet solutions of the equations of ideal *compressible* MHD was proved in [18] provided that a stability condition [16] is satisfied at each point of the initial (nonplanar) discontinuity. This stability condition found in [16] by constructing a dissipative symmetrizer [17] is only sufficient for the weak stability of planar compressible current-vortex sheets and a corresponding necessary and sufficient condition is still unknown (and it cannot be found analytically).

A great advantage of the situation with *incompressible* planar current-vortex sheets is that one can analytically find a necessary and sufficient stability condition for them. This was done a long time ago by Syrovatskij [14] and Axford [1] (and for the 2D case by Michael [12]) by the normal modes analysis. This condition reads [14, 1] (see also [7])

$$|[\mathbf{v}]|^2 < 2\left\{|\mathbf{H}^+|^2 + |\mathbf{H}^-|^2\right\}\,, \tag{6}$$

$$\left|\mathbf{H}^+ \times [\mathbf{v}]\right|^2 + \left|\mathbf{H}^- \times [\mathbf{v}]\right|^2 \leq 2\left|\mathbf{H}^+ \times \mathbf{H}^-\right|^2\,, \tag{7}$$

with $[\mathbf{v}] = \mathbf{v}^+ - \mathbf{v}^-$, $\mathbf{v}^{\pm} = (0, v_2^{\pm}, v_3^{\pm})$, and $\mathbf{H}^{\pm} = (0, H_2^{\pm}, H_3^{\pm})$. All the values in (6), (7) are constants describing a piecewise constant solution of (1), (3), i.e., an unperturbed flow with a planar current-vortex sheet. Without loss of generality the planar discontinuity is given by the equation $x_1 = 0$.

The case when we have equality in (7) corresponds to transition to *violent* instability (ill-posedness of the linearized problem). We exclude this critical case from the consideration and assume that we have strict inequality in (7):

$$\left|\mathbf{H}^{+} \times [\mathbf{v}]\right|^{2} + \left|\mathbf{H}^{-} \times [\mathbf{v}]\right|^{2} < 2\left|\mathbf{H}^{+} \times \mathbf{H}^{-}\right|^{2}. \tag{8}$$

The transition to violent instability includes, in particular, the case $\mathbf{H}^{+} \times \mathbf{H}^{-} = 0$. That is, inequality (8) implies the condition

$$H_2^{+} H_3^{-} - H_3^{+} H_2^{-} \neq 0. \tag{9}$$

Observe that if $\mathbf{H}^{+} = \mathbf{H}^{-} = 0$ we have a planar incompressible vortex sheet, which is always violently unstable. At last, it is worth to note that inequality (6) is *redundant* since it can be shown to follow from (8) (actually, inequality (6) is the 2D stability condition [12], i.e., the stability condition for the case $\mathbf{v}^{\pm} = (0, 0, v_3^{\pm})$ and $\mathbf{H}^{\pm} = (0, 0, H_3^{\pm})$).

The stability condition (8) is always satisfied for current sheets, i.e., for the case $[\mathbf{v}'] = 0$ if (9) holds. If $[\mathbf{v}'] \neq 0$ inequality (8) is rewritten as

$$|[\mathbf{v}']| < \frac{\sqrt{2}\, |\mathbf{H}'^{+}|\, |\mathbf{H}'^{-}|\, |\sin(\varphi^{+} - \varphi^{-})|}{\sqrt{|\mathbf{H}'^{+}|^{2} \sin^{2}\varphi^{+} + |\mathbf{H}'^{-}|^{2} \sin^{2}\varphi^{-}}}\,, \tag{10}$$

where

$$[\mathbf{v}'] = \mathbf{v}'^{+} - \mathbf{v}'^{-}\,, \quad \mathbf{v}'^{\pm} = (v_2^{\pm}, v_3^{\pm}), \quad \mathbf{H}'^{\pm} = (H_2^{\pm}, H_3^{\pm})\,, \quad \cos\varphi^{\pm} = \frac{([\mathbf{v}'], \mathbf{H}'^{\pm})}{|[\mathbf{v}']|\, |\mathbf{H}'^{\pm}|}.$$

If we introduce the dimensionless parameters

$$x = \frac{|[\mathbf{v}']|^{2} \sin^{2}\varphi^{+}}{|\mathbf{H}'^{-}|^{2} \sin^{2}(\varphi^{+} - \varphi^{-})} \quad \text{and} \quad y = \frac{|[\mathbf{v}']|^{2} \sin^{2}\varphi^{-}}{|\mathbf{H}'^{+}|^{2} \sin^{2}(\varphi^{+} - \varphi^{-})}\,,$$

then in the xy-plane inequality (10) determines the domain

$$\mathcal{T} = \{x > 0,\ y > 0,\ x + y < 2\}$$

(interior of a triangle). In [15] an a priori estimate for the linearized problem was proved by the energy method exactly for a *half* of the domain $\mathcal{T}$, namely, for the parameter domain

$$\mathcal{S} = \{x > 0,\ y > 0,\ \max\{x, y\} < 1\}$$

(interior of the square inscribed in the above triangle). Note that the crucial role in the energy method in [15] is played by an "incompressible" analogue of the dissipative symmetrizer [17] proposed in [16] for compressible current-vortex sheets. Moreover, in the incompressibility limit the sufficient stability condition from [16] describes exactly the domain $\mathcal{S}$.

In this paper, our main goal is to derive an energy a priori estimate for planar incompressible current-vortex sheets for the *whole* stability domain $\mathcal{T}$, more precisely, for the range of the parameters $v_2^{\pm}$, $v_3^{\pm}$, $H_2^{\pm}$, $H_3^{\pm}$ satisfying condition (8). For this purpose, from the constant coefficients linearized problem we obtain a nonstandard elliptic problem for the small perturbation of q (denoted again by q). In fact, a reduced problem for an auxiliary unknown $\dot{q}$ is a boundary value problem for the Laplace equation $\Delta \dot{q} = 0$ with nonstandard boundary conditions at $x_1 = 0$ (these boundary conditions differ from those of diffraction problems [6]). Being formally elliptic this problem keeps an information about the evolutional character of the original problem by means of the boundary conditions. Then, we construct a *symbolic symmetrizer* for the Fourier transform of the problem for $\dot{q}$.

Unlike Kreiss' symmetrizers [5, 10] for hyperbolic problems, our symmetrizer gives us only an L^2 estimate of the trace of unknown (in our case $\nabla\dot{q}_{|x_1=0}$) but not an interior L^2 estimate (of $\nabla\dot{q}$). That is, this symmetrizer is, roughly speaking, a kind of "elliptically degenerate Kreiss' symmetrizer." On the other hand, our symmetrizer is also degenerate in the another sense because it is like a *degenerate* Kreiss' symmetrizer constructed by Coulombel and Secchi [3] for the linearized problem for 2D compressible vortex sheets. Since planar incompressible current-vortex sheets are only weakly (neutrally) stable, i.e., the *uniform* Lopatinski condition is violated for the linearized problem, as in [3], we have to consider separately the case of boundary points of the hemisphere of "frequencies." Surprisingly, our construction of an "elliptic" symmetrizer is internally similar to that of Coulombel and Secchi of "hyperbolic" (Kreiss') symmetrizer. It is even much simpler than the construction in [3] because in our case the matrix of the ODE system for the Fourier transform has no poles and is always diagonalizable.

The failure of the uniform Lopatinski condition yields a loss of derivatives with respect to the source terms in the estimate for $\nabla q_{|x_1=0}$. Unlike [15], we consider the case of zero initial data for the linearized problem but introduce source terms to make the interior equations and the boundary conditions inhomogeneous because this is needed to attack the nonlinear problem. The assumption that the initial data are zero is usual and we postpone the case of non-zero initial data to the nonlinear analysis (construction of a so-called approximate solution, etc., see [4, 18]). It should be noted that we have to introduce a source term also in the incompressibility condition because in the future nonlinear analysis we intend to go outside the class of divergence free velocity fields while using the Nash-Moser technique. Motivated however by a special form of accumulated errors of the Nash-Moser type iteration scheme for our problem we consider this source term in a divergence form subordinated to corresponding source terms in the boundary conditions.

Having in hand the L^2 estimate of the trace $\nabla q_{|x_1=0}$ and returning to the original linearized problem for the small perturbation of $\mathbf{U}$ (denoted again by $\mathbf{U}$), we get an L^2 estimate of the trace $\mathbf{U}_{|x_1=0}$ and an H^1 estimate of the front perturbation. As for shock waves [2, 9, 11, 17] and compressible vortex and current-vortex sheets [3, 4, 15, 18], for our problem under assumption (9) the symbol associated to the front is elliptic, and this enables us to gain one derivative for the front perturbation. With the estimate for the front perturbation we close the L^2 estimate for ∇q and then easily get an interior L^2 estimate of $\mathbf{U}$.

In this paper, unlike the study in [15], we restrict ourself to the case of constant coefficients of the linearized problem. The variable coefficients and nonlinear analysis is postponed to the future. As in [3], we intend to derive an energy estimate for the variable coefficients linearized problem for nonplanar discontinuities using paradifferential calculus with a parameter (see references in [3]). The local-in-time existence of solutions to problem (1), (3), (5) is planned to be proved by a suitable Nash-Moser type iteration scheme, provided that the initial data (5) satisfy the stability condition (8) at each point of the initial discontinuity (together with all the other necessary conditions, e.g., compatibility conditions). Since the success of the Nash-Moser technique mainly depends on the possibility to derive a tame estimate in the Sobolev space H^m, where m is arbitrarily large (see [3, 18] and references therein), in this paper we prove also higher order estimates for the linearized problem.

In [15], as for compressible current-vortex sheets [16, 18], higher order estimates were derived in the anisotropic weighted Sobolev spaces H^m_* (see [13] and references therein). Indeed, at first sight incompressible current-vortex sheets are like *characteristic* discontinuities for hyperbolic conservation laws that implies a loss of control on derivatives in the normal direction. Nevertheless, we manage to improve the result in [15] and derive higher order estimates in usual Sobolev spaces. This is achieved by using a big advantage of incompressible MHD that enables us to estimate missing normal derivatives through a current-vorticity-type linearized system.

We now describe the content of the rest of the paper. In Section 2 we write down the constant coefficients linearized problem for planar current-vortex sheets and formulate the main result for it that is an L^2 estimate in the whole domain of stability. We also equivalently reformulate the linearized problem as well as the L^2 estimate and higher order estimates in terms of the exponentially weighted unknowns. In Section 3, we prove the basic L^2 estimate assuming that we have already in hand an L^2 estimate of the trace of ∇q. Section 4 is devoted to the construction of a symbolic symmetrizer with the help of which we derive the L^2 estimate of the trace of ∇q. At last, in Section 5, using the current-vorticity-type linearized system, we prove the H^m estimate announced in Section 2.

2 Linearized problem

2.1 Reduction to a fixed domain

To reduce the free boundary value problem (1), (3), (5) to a problem in a fixed domain we straighten, as usual (see, e.g., [9]), the unknown front Γ. That is, the unknowns $(\mathbf{U}^{\pm}, q^{\pm})$ being smooth in $\Omega^{\pm}(t)$ are replaced by the functions

$$(\mathbf{U}^{\pm}_{\sharp}, q^{\pm}_{\sharp})(t,\mathbf{x}) := (\mathbf{U}^{\pm}, q^{\pm})(t, \Phi^{\pm}(t,\mathbf{x}))$$

that are smooth in the fixed domain $\mathbb{R}^3_+ = \{x_1 > 0,\ \mathbf{x}' \in \mathbb{R}^2\}$, where $\Phi^{\pm}(t,\mathbf{x}) := \Phi(t, \pm x_1, \mathbf{x}')$, and Φ is a smooth function satisfying

$$\Phi(t,0,\mathbf{x}') = f(t,\mathbf{x}') \quad \text{and} \quad \partial_{x_1}\Phi \geq \kappa > 0.$$

Dropping the index $\sharp$ in $(\mathbf{U}^{\pm}_{\sharp}, q^{\pm}_{\sharp})$, we get the system

$$\partial_t \mathbf{v}^+ + \frac{1}{\partial_{x_1}\Phi^+}\left\{(\mathbf{w}^+,\nabla)\mathbf{v}^+ - (\widetilde{\mathbf{H}}^+,\nabla)\mathbf{H}^+\right\} + \nabla^+ q^+ = 0, \tag{11}$$

$$\partial_t \mathbf{H}^+ + \frac{1}{\partial_{x_1}\Phi^+}\left\{(\mathbf{w}^+,\nabla)\mathbf{H}^+ - (\widetilde{\mathbf{H}}^+,\nabla)\mathbf{v}^+\right\} = 0, \tag{12}$$

$$\partial_t \mathbf{v}^- + \frac{1}{\partial_{x_1}\Phi^-}\left\{(\mathbf{w}^-,\nabla)\mathbf{v}^- - (\widetilde{\mathbf{H}}^-,\nabla)\mathbf{H}^-\right\} + \nabla^- q^- = 0, \tag{13}$$

$$\partial_t \mathbf{H}^- + \frac{1}{\partial_{x_1}\Phi^-}\left\{(\mathbf{w}^-,\nabla)\mathbf{H}^- - (\widetilde{\mathbf{H}}^-,\nabla)\mathbf{v}^-\right\} = 0, \tag{14}$$

$$\operatorname{div}\widetilde{\mathbf{v}}^+ = 0, \qquad \operatorname{div}\widetilde{\mathbf{v}}^- = 0, \tag{15}$$

in the space domain $\mathbb{R}^3_+$, and the boundary conditions

$$\Phi^+_{|x_1=0} = \Phi^-_{|x_1=0} = f,$$
$$\partial_t f = v^+_{\mathrm{N}|x_1=0} = v^-_{\mathrm{N}|x_1=0}, \quad [q] = 0, \tag{16}$$

where

$$\nabla^\pm = \frac{1}{\partial_{x_1}\Phi^\pm}\mathbf{n}^\pm\partial_{x_1} + \sum_{k=2}^{3}\mathbf{e}_k\partial_{x_k}, \quad \mathbf{n}^\pm = (1, -\partial_{x_2}\Phi^\pm, -\partial_{x_3}\Phi^\pm), \quad \mathbf{e}_k = (0, \delta_{2k}, \delta_{3k}), ,$$

$$\mathbf{w}^\pm = \widetilde{\mathbf{v}}^\pm - (\partial_t\Phi^\pm, 0, 0), \quad \widetilde{\mathbf{v}}^\pm = (v^\pm_{\mathrm{n}}, v^\pm_2\partial_{x_1}\Phi^\pm, v^\pm_3\partial_{x_1}\Phi^\pm), \quad v^\pm_{\mathrm{n}} = (\mathbf{v}^\pm, \mathbf{n}^\pm),$$

$$\widetilde{\mathbf{H}}^\pm = (H^\pm_{\mathrm{n}}, H^\pm_2\partial_{x_1}\Phi^\pm, H^\pm_3\partial_{x_1}\Phi^\pm), \quad H^\pm_{\mathrm{n}} = (\mathbf{H}^\pm, \mathbf{n}^\pm), \qquad [q] = q^+_{|x_1=0} - q^-_{|x_1=0}.$$

In (15) and below we use the notations $\operatorname{div}\mathbf{a}^\pm := \operatorname{div}^\pm\mathbf{a}^\pm$ for vector functions $\mathbf{a}^\pm = (a^\pm_1, a^\pm_2, a^\pm_3)$, where $\operatorname{div}^\pm\mathbf{a} := \pm\partial_{x_1}a_1 + \partial_{x_2}a_2 + \partial_{x_3}a_3$ (usually we will drop the superscripts $\pm$ in the operators $\operatorname{div}^\pm$).

In [15], as in [2, 9], the simple choice $\Phi^\pm(t, \mathbf{x}) := \pm x_1 + f(t, \mathbf{x}')$ was used. Such a choice was suitable for the energy method exploited in [15] in the constant and variable coefficients linear analysis. Since now we are going to derive an a priori estimate for the constant coefficients linearized problem by constructing a symbolic symmetrizer and, in the future, carry this estimate over variable coefficients by using paradifferential calculus, we make a different choice of the functions $\Phi^\pm$. Similarly to the choice of Coulombel and Secchi [3] for 2D compressible vortex sheets, we choose the change of variables $\Phi^\pm$ such that the equations

$$\partial_t\Phi^+ - v^+_{\mathrm{n}} = 0, \quad \partial_t\Phi^- - v^-_{\mathrm{n}} = 0, \tag{17}$$

are satisfied in the whole space domain $\mathbb{R}^3_+$. The main advantage of this choice is that under suitable initial data for $\Phi^\pm$ the so-called *boundary matrix* of the hyperbolic quasilinear operator for $(\mathbf{U}^+, \mathbf{U}^-)$ in (11)–(14) has constant rank not only on the boundary $\{x_1 = 0\}$ but in the whole domain $\mathbb{R}^3_+$. Moreover, this matrix is *identically zero*. Indeed, in view of (17), the first components of the vectors $\mathbf{w}^\pm$ in system (11)–(14) are zeros. But actually the same is true for the first components of the vectors $\widetilde{\mathbf{H}}^\pm$. More precisely, we prove the following proposition.

Proposition 2.1. *Let the initial data for the functions $\Phi^\pm$ satisfy the restrictions*

$$H^+_{\mathrm{n}|t=0} = 0, \quad H^-_{\mathrm{n}|t=0} = 0,$$

then

$$H^+_{\mathrm{n}} = 0, \quad H^-_{\mathrm{n}} = 0 \tag{18}$$

for all $t > 0$ in the whole space domain $\mathbb{R}^3_+$.

Proof. Using (17) and (15), after some algebra from equations (12), (14) we obtain

$$\partial_t H^+_{\mathrm{n}} + v^+_2\partial_{x_2}H^+_{\mathrm{n}} + v^+_3\partial_{x_3}H^+_{\mathrm{n}} + \left(\partial_{x_2}v^+_2 + \partial_{x_3}v^+_3\right)H^+_{\mathrm{n}} = 0,$$

$$\partial_t H^-_{\mathrm{n}} + v^-_2\partial_{x_2}H^-_{\mathrm{n}} + v^-_3\partial_{x_3}H^-_{\mathrm{n}} + \left(\partial_{x_2}v^-_2 + \partial_{x_3}v^-_3\right)H^-_{\mathrm{n}} = 0.$$

Then, by standard method of characteristic curves we get (18) for all $t > 0$. □

Remark 2.1. Since $H^{\pm}_{\mathrm{n}|x_1=0} = H^{\pm}_{\mathrm{N}|x_1=0}$, as a direct corollary of Proposition 2.1 we have the fact that the conditions

$$H^{+}_{\mathrm{N}|x_1=0} = 0, \quad H^{-}_{\mathrm{N}|x_1=0} = 0$$

are just restrictions only on the initial data for problem (11)–(16). Therefore we did not include them into the boundary conditions (16). This fact was proved in [15] for the choice $\Phi^{\pm}(t, \mathbf{x}) := \pm x_1 + f(t, \mathbf{x}')$. Analogously to [15], we can also prove that the divergent constraints

$$\operatorname{div} \widetilde{\mathbf{H}}^{+} = 0, \quad \operatorname{div} \widetilde{\mathbf{H}}^{-} = 0,$$

hold for all $t > 0$ if they were satisfied for $t = 0$.

It follows from (17) and (18) that the nonlinear equations (11)–(14) are rewritten as

$$\partial_t \mathbf{v}^{\pm} + (\mathbf{v}'^{\pm}, \nabla')\mathbf{v}^{\pm} - (\mathbf{H}'^{\pm}, \nabla')\mathbf{H}^{\pm} = -\nabla^{\pm} q^{\pm},$$

$$\partial_t \mathbf{H}^{\pm} + (\mathbf{v}'^{\pm}, \nabla')\mathbf{H}^{\pm} - (\mathbf{H}'^{\pm}, \nabla')\mathbf{v}^{\pm} = 0,$$

with $\mathbf{v}'^{\pm} = (v_2^{\pm}, v_3^{\pm})$, $\mathbf{H}'^{\pm} = (H_2^{\pm}, H_3^{\pm})$, and $\nabla' = (\partial_{x_2}, \partial_{x_3})$, or in a compact form

$$\partial_t \mathbf{U} + A_2(\mathbf{U})\partial_{x_2}\mathbf{U} + A_3(\mathbf{U})\partial_{x_3}\mathbf{U} = -\begin{pmatrix} \mathbf{e} \otimes \nabla^{+} q^{+} \\ \mathbf{e} \otimes \nabla^{-} q^{-} \end{pmatrix}, \tag{19}$$

where $\mathbf{U} := (\mathbf{U}^{+}, \mathbf{U}^{-})$, $\mathbf{e} = (1, 0)$,

$$A_k(\mathbf{U}) = \begin{pmatrix} A_k^{+}(\mathbf{U}^{+}) & 0 \\ 0 & A_k^{-}(\mathbf{U}^{+}) \end{pmatrix}, \quad A_k^{\pm}(\mathbf{U}^{\pm}) = \begin{pmatrix} v_k^{\pm} & -H_k^{\pm} \\ -H_k^{\pm} & v_k^{\pm} \end{pmatrix} \otimes I_3, \quad k = 2, 3. \tag{20}$$

As we can see, the left-hand side of system (19) does not contain normal (x_1-)derivatives, i.e., the boundary matrix for the hyperbolic operator for $\mathbf{U}$ is identically zero.

2.2 The constant coefficients linearized problem

For planar current-vortex sheets we know exact solutions of (1), (3). They are piecewise constant solutions of (1), (3), i.e., constant solutions of (19), (16):

$$\mathbf{U}_{\mathrm{c}}^{\pm} = (0, \mathbf{v}'^{\pm}, 0, \mathbf{H}'^{\pm}) = (0, v_2^{\pm}, v_3^{\pm}, 0, H_2^{\pm}, H_3^{\pm}), \quad q_{\mathrm{c}}^{\pm} = \text{const}, \quad \Phi_{\mathrm{c}}^{\pm} = \pm x_1, \tag{21}$$

where $v_2^{\pm}$, $v_3^{\pm}$, $H_2^{\pm}$, and $H_3^{\pm}$ are fixed constants. Linearizing (19), (15), and (16) about the exact solution (21) we get a linear constant coefficients problem for the perturbations $(\delta\mathbf{U}^{\pm}, \delta q^{\pm})$ and δf. If we drop δ and set $\mathbf{U}^{\pm} = (\mathbf{u}^{\pm}, \mathbf{h}^{\pm})$, this problem reads

$$\begin{cases} \partial_t \mathbf{U}^{\pm} + A_2^{\pm}\partial_{x_2}\mathbf{U}^{\pm} + A_3^{\pm}\partial_{x_3}\mathbf{U}^{\pm} + \mathbf{e} \otimes \nabla q^{\pm} = \mathbf{F}^{\pm}, \\ \operatorname{div} \mathbf{u}^{\pm} = \mathcal{F}^{\pm}, \quad \text{in } \{x_1 > 0\}, \end{cases} \tag{22}$$

$$\begin{cases} u_1^{\pm} = \partial_t f + v_2^{\pm}\partial_{x_2} f + v_3^{\pm}\partial_{x_3} f + g^{\pm}, \\ [q] = g, \quad \text{on } \{x_1 = 0\}, \end{cases} \tag{23}$$

where $A_k^\pm := A_k^\pm(\mathbf{U}_c^\pm)$, $k=2,3$ (see (20)),

$$\nabla q^\pm := (\pm\partial_{x_1}q^\pm, \partial_{x_2}q^\pm, \partial_{x_3}q^\pm), \quad \text{and} \quad \operatorname{div}\mathbf{u}^\pm = \pm\partial_{x_1}u_1^\pm + \partial_{x_2}u_2^\pm + \partial_{x_3}u_3^\pm.$$

Here we introduce the source terms

$$\mathbf{F}^\pm = \mathbf{F}^\pm(t,\mathbf{x}) = (\mathbf{F}_1^\pm, \mathbf{F}_2^\pm) = (F_{1,1}^\pm, F_{1,2}^\pm, F_{1,3}^\pm, F_{2,1}^\pm, F_{2,2}^\pm, F_{2,3}^\pm),$$
$$\mathcal{F}^\pm = \mathcal{F}^\pm(t,\mathbf{x}), \quad g^\pm = g^\pm(t,\mathbf{x}'), \quad \text{and} \quad g = g(t,\mathbf{x}').$$

Since the incompressibility conditions (15) are nonlinear, they produce errors of the Nash-Moser iteration scheme for the nonlinear problem (11)–(16) (as was mentioned in Sect. 1 the nonlinear analysis is postponed to the future). Therefore, in the future nonlinear analysis we will have to go outside the class of divergence free velocity fields, and now we must introduce source terms $\mathcal{F}^\pm$ in the linearized incompressibility conditions. At the same time, it follows from the detailed analysis of an exact form of the accumulated errors for the incompressibility conditions (15) and the boundary conditions $f_t = v^\pm_{\mathrm{N}|x_1=0}$ (corresponding arguments are omitted and postponed to the nonlinear analysis) that the source terms $\mathcal{F}^\pm$ and $g^\pm$ have the following special form:

$$\mathcal{F}^\pm = \operatorname{div}\mathbf{b}^\pm, \quad g^\pm = b^\pm_{1|x_1=0}, \tag{24}$$

where $\mathbf{b}^\pm = (b_1^\pm, b_2^\pm, b_3^\pm)$. Performing the change of unknown functions

$$\widetilde{\mathbf{u}}^\pm = \mathbf{u}^\pm - \mathbf{b}^\pm$$

and taking into account (24), for $(\widetilde{\mathbf{u}}^\pm, \mathbf{h}^\pm, q^\pm)$ we get problem (22), (23) with $\mathcal{F}^\pm = 0$, $g^\pm = 0$, and the vector-functions $\mathbf{F}^\pm$ replaced by some $\widetilde{\mathbf{F}}^\pm$. Dropping tildes in $\widetilde{\mathbf{u}}^\pm$ and $\widetilde{\mathbf{F}}^\pm$, we have the problem

$$\begin{cases} L(\mathbf{U},\nabla q) = \mathbf{F}\,, \\ \operatorname{div}\mathbf{u}^\pm = 0\,, \quad \text{in } \{x_1>0\}\,, \\ \mathcal{B}(u_1,q,f) = \mathbf{g}\,, \quad \text{on } \{x_1=0\}, \end{cases} \tag{25}$$

where

$$L(\mathbf{U},\nabla q) := \partial_t\mathbf{U} + A_2\partial_{x_2}\mathbf{U} + A_3\partial_{x_3}\mathbf{U} + \begin{pmatrix} \mathbf{e}\otimes\nabla q^+ \\ \mathbf{e}\otimes\nabla q^- \end{pmatrix}, \qquad A_k = \begin{pmatrix} A_k^+ & 0 \\ 0 & A_k^- \end{pmatrix}, \quad k=2,3\,,$$

$$\mathcal{B}(u_1,q,f) := \underline{M}\begin{pmatrix} u_1^+ \\ q^+ \\ u_1^- \\ q^- \end{pmatrix}_{|\,x_1=0} + \underline{b}\begin{pmatrix} \partial_t f \\ \partial_{x_2} f \\ \partial_{x_3} f \end{pmatrix}, \qquad \underline{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix},$$

$$\underline{b} = \begin{pmatrix} -1 & -v_2^+ & -v_3^+ \\ -1 & -v_2^- & -v_3^- \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{U} := \begin{pmatrix} \mathbf{U}^+ \\ \mathbf{U}^- \end{pmatrix}, \quad \nabla q := \begin{pmatrix} \nabla q^+ \\ \nabla q^- \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{F}^+ \\ \mathbf{F}^- \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}.$$

From now on we concentrate on the boundary value problem (25) in the unbounded space-time domain

$$\Omega := \mathbb{R}\times\mathbb{R}^3_+ = \{t\in\mathbb{R},\ \mathbf{x}\in\mathbb{R}^3_+\}$$

with zero initial data $\mathbf{U}_{|t=0} = 0$ assuming that $\mathbf{U}$, $\mathbf{F}$, and g vanish in the past (for $t\le 0$). The boundary $\partial\Omega$ is identified with $\mathbb{R}^3 = \mathbb{R}\times\mathbb{R}^2 = \{t\in\mathbb{R},\ \mathbf{x}'\in\mathbb{R}^2\}$.

2.3 The main result

Our goal is deriving energy a priori estimates for the constant coefficients linearized problem (25) in the weighted Sobolev spaces $H^m_\gamma(\Omega)$ and $H^m_\gamma(\mathbb{R}^3)$, where $H^0_\gamma := L^2_\gamma$, $L^2_\gamma := e^{\gamma t}L^2$, $H^m_\gamma := e^{\gamma t}H^m$, with $\gamma \geq 1$, and the usual Sobolev spaces $H^m(\Omega)$ and $H^m(\mathbb{R}^3)$ are equipped with the (*weighted*) norms

$$\|v\|^2_{m,\gamma} := \sum_{|\alpha|\leq m} \gamma^{2(m-|\alpha|)}\|\partial^\alpha_{\tan} v\|^2_{L^2(\mathbb{R}^3)} \quad \text{and} \quad |||u|||^2_{m,\gamma} := \sum_{|\beta|\leq m} \gamma^{2(m-|\beta|)}\|\partial^\beta u\|^2_{L^2(\Omega)}$$

respectively ($\partial^\alpha_{\tan} := \partial_t^{\alpha_0}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, with $\alpha = (\alpha_0, \alpha_2, \alpha_3) \in \mathbb{N}^3$). That is, the spaces $H^m_\gamma(\mathbb{R}^3)$ and $H^m_\gamma(\Omega)$ are equipped with the norms

$$\|v\|_{H^m_\gamma(\mathbb{R}^3)} := \|e^{-\gamma t}v\|_{m,\gamma} \quad \text{and} \quad \|u\|^2_{H^m_\gamma(\Omega)} := |||e^{-\gamma t}u|||^2_{m,\gamma}$$

for real numbers m and $\gamma \geq 1$.

Note that the norms $\|\cdot\|_{0,\gamma}$ and $|||\cdot|||_{0,\gamma}$ are usual norms in $L^2(\mathbb{R}^3)$ and $L^2(\Omega)$ respectively. Below we will sometimes use the inequalities

$$\|v\|_{s,\gamma} \leq \frac{1}{\gamma^{r-s}}\|v\|_{r,\gamma}\,, \quad |||u|||_{s,\gamma} \leq \frac{1}{\gamma^{r-s}}|||u|||_{r,\gamma} \quad \text{for } r > s. \tag{26}$$

Observe also that in terms of the weighted norms the trace estimate in H^m reads

$$\|u_{|x_1=0}\|^2_{m,\gamma} \leq \frac{C}{\gamma}|||u|||^2_{m+1,\gamma}.$$

Usually we will use the roughened version of this estimate with C instead of C/γ.

We are now in a position to state the main result of the paper.

Theorem 2.1. *Let* $(\mathbf{v}'^{\pm}, \mathbf{H}'^{\pm})$ *be a given planar current-vortex sheet solution satisfying the stability condition* (8). *Then there exists a positive constant* C *such that for all* $\gamma \geq 1$ *and all sufficiently smooth solutions* $(\mathbf{U}, q, f)$ *of* (25) *the following estimate holds:*

$$\begin{aligned} &\gamma\|\mathbf{U}\|^2_{L^2_\gamma(\Omega)} + \|\nabla q\|^2_{L^2_\gamma(\Omega)} + \|(\mathbf{U}, \nabla q)_{|x_1=0}\|^2_{L^2_\gamma(\mathbb{R}^3)} + \|f\|^2_{H^1_\gamma(\mathbb{R}^3)} \\ &\qquad \leq \frac{C}{\gamma^2}\left(\|L(\mathbf{U}, \nabla q)\|^2_{H^3_\gamma(\Omega)} + \|\mathcal{B}(u_1, q, f)\|^2_{H^2_\gamma(\mathbb{R}^3)}\right). \end{aligned} \tag{27}$$

Moreover, under the same assumptions problem (25) *obeys the a priori estimate*

$$\begin{aligned} &\gamma\|\mathbf{U}\|^2_{H^m_\gamma(\Omega)} + \|\nabla q\|^2_{H^m_\gamma(\Omega)} + \|(\mathbf{U}, \nabla q)_{|x_1=0}\|^2_{H^m_\gamma(\mathbb{R}^3)} + \|f\|^2_{H^{m+1}_\gamma(\mathbb{R}^3)} \\ &\qquad \leq \frac{C}{\gamma^2}\left(\|L(\mathbf{U}, \nabla q)\|^2_{H^{m+3}_\gamma(\Omega)} + \|\mathcal{B}(u_1, q, f)\|^2_{H^{m+2}_\gamma(\mathbb{R}^3)}\right) \end{aligned} \tag{28}$$

for all $m \in \mathbb{N}$.

Remark 2.2. In spite of the fact that (27) and (28) are estimates in weighted Sobolev spaces (in the sense different from that for the H^m_* estimates in [15]), using arguments like those in [11], we can, in principle, derive from them a priori estimates in usual Sobolev spaces. That is, it is not a real mistake when we say in Sect. 1 that in this paper we derive estimates in usual Sobolev spaces. We note that a usual procedure towards the proof of a nonlinear existence theorem by the Nash-Moser method (see, e.g., [3, 4, 18]) provides for the derivation of a basic L^2 estimate for constant coefficients (like estimate (27)), the carrying this estimate over variable coefficients, and then the derivation of a *tame* estimate in Sobolev spaces (for variable coefficients). Of course, the a priori estimate (28) is not a tame estimate. It is only a higher order estimate for constant coefficients, and we present it here just for demonstration that for incompressible current-vortex sheets, unlike compressible ones [16], we have no loss of control of derivatives in the normal direction as well as no loss of control of the trace of the solution at the boundary.

Theorem 2.1 admits an equivalent formulation in terms of the exponentially weighted unknowns

$$\overline{\mathbf{U}}^{\pm} := e^{-\gamma t}\mathbf{U}^{\pm}, \quad \overline{q}^{\pm} := e^{-\gamma t} q^{\pm}, \quad \overline{f} := e^{-\gamma t} f. \tag{29}$$

To get this formulation, that is much more convenient for the proof, we first restate problem (25) in terms of the new unknowns:

$$\begin{cases} L_\gamma(\overline{\mathbf{U}}, \nabla \overline{q}) = \overline{\mathbf{F}}\,, \\ \operatorname{div} \overline{\mathbf{u}}^{\pm} = 0\,, \quad \text{in } \{x_1 > 0\}\,, \\ \mathcal{B}_\gamma(\overline{u}_1, \overline{q}, \overline{f}) = \overline{\mathbf{g}}\,, \quad \text{on } \{x_1 = 0\}, \end{cases} \tag{30}$$

where

$$L_\gamma(\overline{\mathbf{U}}, \nabla \overline{q}) := L(\overline{\mathbf{U}}, \nabla \overline{q}) + \gamma \overline{\mathbf{U}}\,, \qquad \mathcal{B}_\gamma(\overline{u}_1, \overline{q}, \overline{f}) := \mathcal{B}(\overline{u}_1, \overline{q}, \overline{f}) + \gamma \begin{pmatrix} \overline{f} \\ 0 \\ 0 \end{pmatrix},$$

$\overline{\mathbf{F}} := e^{-\gamma t}\mathbf{F}$, $\overline{\mathbf{g}}^{\pm} = e^{-\gamma t}\mathbf{g}$, and $\overline{\mathbf{u}}^{\pm} := e^{-\gamma t}\mathbf{u}^{\pm}$, etc.

It is clear that if the original unknowns belong to H^m_γ, then the exponentially weighted unknowns belong to the usual Sobolev space H^m endowed with the weighted (m,γ)-norm. We can now equivalently reformulate Theorem 2.1 in terms of the exponentially weighted unknowns $(\overline{\mathbf{U}}, \overline{q}, \overline{f})$.

Theorem 2.2. *Let $m \in \mathbb{N}$, and let $(\mathbf{v}'^{\pm}, \mathbf{H}'^{\pm})$ be a given planar current-vortex sheet solution satisfying the stability condition* (8)*. Then there exists a positive constant C such that for all $\gamma \geq 1$ and all sufficiently smooth solutions $(\overline{\mathbf{U}}, \overline{q}, \overline{f})$ of* (30) *the following estimates hold:*

$$\begin{aligned} \gamma \|\overline{\mathbf{U}}\|^2_{L^2(\Omega)} + \|\nabla \overline{q}\|^2_{L^2(\Omega)} + \|(\overline{\mathbf{U}}, \nabla \overline{q})_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)} + \|\overline{f}\|^2_{1,\gamma} \\ \leq \frac{C}{\gamma^2}\left(|||L_\gamma(\overline{\mathbf{U}}, \nabla \overline{q})|||^2_{3,\gamma} + \|\mathcal{B}_\gamma(\overline{u}_1, \overline{q}, \overline{f})\|^2_{2,\gamma}\right) \end{aligned} \tag{31}$$

and

$$\begin{aligned} \gamma |||\overline{\mathbf{U}}|||^2_{m,\gamma} + |||\nabla \overline{q}|||^2_{m,\gamma} + \|(\overline{\mathbf{U}}, \nabla \overline{q})_{|x_1=0}\|^2_{m,\gamma} + \|\overline{f}\|^2_{m+1,\gamma} \\ \leq \frac{C}{\gamma^2}\left(|||L_\gamma(\overline{\mathbf{U}}, \nabla \overline{q})|||^2_{m+3,\gamma} + \|\mathcal{B}_\gamma(\overline{u}_1, \overline{q}, \overline{f})\|^2_{m+2,\gamma}\right). \end{aligned} \tag{32}$$

In order to simplify the notations, from now on we drop bars in problem (30) and the desired estimates (31) and (32). For the convenience of subsequent references we write down problem (30) in the explicit form

$$\partial_t \mathbf{u}^+ + \gamma \mathbf{u}^+ + (\mathbf{v}'^+, \nabla')\mathbf{u}^+ - (\mathbf{H}'^+, \nabla')\mathbf{h}^+ + \nabla q^+ = \mathbf{F}_1^+ \,, \tag{33}$$

$$\partial_t \mathbf{h}^+ + \gamma \mathbf{h}^+ + (\mathbf{v}'^+, \nabla')\mathbf{h}^+ - (\mathbf{H}'^+, \nabla')\mathbf{u}^+ = \mathbf{F}_2^+ \,, \tag{34}$$

$$\partial_t \mathbf{u}^- + \gamma \mathbf{u}^- + (\mathbf{v}'^-, \nabla')\mathbf{u}^- - (\mathbf{H}'^-, \nabla')\mathbf{h}^- + \nabla q^- = \mathbf{F}_1^- \,, \tag{35}$$

$$\partial_t \mathbf{h}^- + \gamma \mathbf{h}^- + (\mathbf{v}'^-, \nabla')\mathbf{h}^- - (\mathbf{H}'^-, \nabla')\mathbf{u}^- = \mathbf{F}_2^- \,, \tag{36}$$

$$\operatorname{div} \mathbf{u}^+ = 0 \,, \quad \operatorname{div} \mathbf{u}^- = 0 \,, \qquad \text{in } \{x_1 > 0\} \,, \tag{37}$$

$$u_1^+ = \partial_t f + \gamma f + v_2^+ \partial_{x_2} f + v_3^+ \partial_{x_3} f, \tag{38}$$

$$u_1^- = \partial_t f + \gamma f + v_2^- \partial_{x_2} f + v_3^- \partial_{x_3} f, \tag{39}$$

$$[q] = g, \qquad \text{on } \{x_1 = 0\}. \tag{40}$$

In the matrix form equations (33)–(36) read

$$\partial_t \mathbf{U}^\pm + \gamma \mathbf{U}^\pm + A_2^\pm \partial_{x_2} \mathbf{U}^\pm + A_3^\pm \partial_{x_3} \mathbf{U}^\pm = -\mathbf{e} \otimes \nabla q^\pm + \mathbf{F}^\pm. \tag{41}$$

Since for the original nonlinear problem the boundary conditions for the magnetic field and the divergent constraints are just restrictions on the initial data (see Proposition 2.1 and Remark 2.1), we do not include their linearized versions in problem (30) (or (33)–(40)). At the same time, we can easily prove the following important proposition.

Proposition 2.2. *Let* $(\mathbf{U}, f)$ *be a sufficiently smooth solution of problem* (30)*. Then this solution satisfies*

$$h^+_{1|x_1=0} - H_2^+ \partial_{x_2} f - H_3^+ \partial_{x_3} f = G^+, \quad h^-_{1|x_1=0} - H_2^- \partial_{x_2} f - H_3^- \partial_{x_3} f = G^-, \tag{42}$$

and

$$\operatorname{div} \mathbf{h}^+ = \mathcal{F}^+ \,, \quad \operatorname{div} \mathbf{h}^- = \mathcal{F}^-, \tag{43}$$

where $\operatorname{div} \mathbf{h}^\pm = \pm\partial_{x_1} h_1^\pm + \partial_{x_2} h_2^\pm + \partial_{x_3} h_3^\pm$*, and the functions* $G^\pm = G^\pm(t, \mathbf{x}')$ *and* $\mathcal{F}^\pm = \mathcal{F}^\pm(t, \mathbf{x})$ *are determined by the source terms as solutions of the inhomogeneous transport equations*

$$\partial_t G^\pm + \gamma G^\pm + (\mathbf{v}'^\pm, \nabla' G^\pm) = F^\pm_{2,1|x_1=0} \,, \tag{44}$$

$$\partial_t \mathcal{F}^\pm + \gamma \mathcal{F}^\pm + (\mathbf{v}'^\pm, \nabla' \mathcal{F}^\pm) = \operatorname{div} \mathbf{F}_2^\pm \tag{45}$$

(equations (45) *do not need boundary conditions at* $x_1 = 0$*).*

Proof. Restricting to the boundary $\{x_1 = 0\}$ the first scalar equations in (34), (36) and using the boundary conditions (38), (39), we get the transport equations (44) for the functions $G^\pm$ defined in (42). Applying the divergence operators $\operatorname{div}^\pm$ to systems (34) and (36) respectively and using (37), we easily get equations (45) for the functions $\mathcal{F}^\pm$ defined in (43). □

Remark 2.3. By standard arguments from equations (44) and (45) we derive the estimates

$$\gamma \|G^\pm\|^2_{m,\gamma} \leq \frac{C}{\gamma} \big\|F^\pm_{2,1|x_1=0}\big\|^2_{m,\gamma}, \quad \gamma |||\mathcal{F}^\pm|||^2_{m,\gamma} \leq \frac{C}{\gamma} |||\operatorname{div} \mathbf{F}_2^\pm|||^2_{m,\gamma}, \quad \forall\, m \in \mathbb{N}.$$

Here and below $C > 0$ is a constant independent of γ. Then we easily get

$$|||\mathcal{F}^{\pm}|||^2_{m,\gamma} \leq \frac{C}{\gamma^2}|||\mathbf{F}^{\pm}|||^2_{m+1,\gamma}, \quad \forall\, m \in \mathbb{N}, \tag{46}$$

and using the trace theorem, we obtain

$$\|G^{\pm}\|^2_{m,\gamma} \leq \frac{C}{\gamma^2}|||\mathbf{F}^{\pm}|||^2_{m+1,\gamma}, \quad \forall\, m \in \mathbb{N}. \tag{47}$$

In fact, the loss of one derivative in (46) and (47) from $\mathbf{F}^{\pm}$ to $\mathcal{F}^{\pm}$ and $G^{\pm}$ causes the loss of an additional derivative in estimates (31) and (32) with respect to the source terms of the interior equations. This is a natural prize for the non-inclusion of the divergent constraint and the boundary conditions for the magnetic field in the original problem. The same situation takes place for compressible current-vortex sheets [18].

3 Proof of the energy estimate (31)

3.1 Estimating the trace of U and the front through the trace of ∇q

We prove at first the following lemma.

Lemma 3.1. *Let* (9) *holds (recall that it follows from* (8)*). Sufficiently smooth solutions of problem* (30) *obey the estimates*

$$\|\mathbf{U}^{\pm}_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)} \leq \frac{C}{\gamma^2}\left(\|\nabla q^{\pm}_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)} + |||\mathbf{F}^{\pm}|||^2_{1,\gamma}\right) \tag{48}$$

and

$$\|f\|^2_{1,\gamma} \leq \frac{C}{\gamma^2}\left(\|\nabla q_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)} + |||\mathbf{F}|||^2_{1,\gamma}\right) \tag{49}$$

for all $\gamma \geq 1$, with two constants $C > 0$.

Proof. Restricting the interior equations (41) to the boundary $\{x_1 = 0\}$, by standard arguments (we multiply these equations by $\mathbf{U}^{\pm}_{|x_1=0}$, integrate the result over $\mathbb{R}^3$, use Young's inequality, etc.) we get

$$\gamma\|\mathbf{U}^{\pm}_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)} \leq \frac{C}{\gamma}\left(\|\nabla q^{\pm}_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)} + \|\mathbf{F}^{\pm}_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)}\right).$$

Using then the trace theorem in H^1, we obtain estimate (48).

Taking into account (42) and (9), the symbol associated to the front is elliptic. This enables us to get an estimate of the front f in the $(1,\gamma)$-norm through the L^2 norm of $\mathbf{U}_{|x_1=0}$. Here we prefer to not follow usual arguments (see, e.g., [9, 3]), which in our case require the application of the Fourier transform in $(t, \mathbf{x}')$ to conditions (38), (39), and (42). Instead of this we just use simple arguments of the energy method that is more suitable for our goals. Thanks to assumption (9) we can resolve (42) for $\partial_{x_2} f$ and $\partial_{x_3} f$:

$$\begin{aligned} \partial_{x_2} f &= \frac{H_3^-(h^+_{1|x_1=0} - G^+) - H_3^+(h^-_{1|x_1=0} - G^-)}{H_2^+H_3^- - H_3^+H_2^-}, \\ \partial_{x_3} f &= \frac{H_2^-(h^+_{1|x_1=0} - G^+) - H_2^+(h^-_{1|x_1=0} - G^-)}{H_3^+H_2^- - H_2^+H_3^-}. \end{aligned} \tag{50}$$

After the substitution of (50) into the boundary condition (38) we easily get the L^2 estimate

$$\gamma^2\|f\|^2_{L^2(\mathbb{R}^3)} \leq C\Big(\|u^+_{1|x_1=0}\|^2_{L^2(\mathbb{R}^3)} + \sum_{\pm}\big(\|h^\pm_{1|x_1=0}\|^2_{L^2(\mathbb{R}^3)} + \|G^\pm\|^2_{L^2(\mathbb{R}^3)}\big)\Big), \tag{51}$$

where $\sum_\pm a^\pm := a^+ + a^-$.

Relations (50) give us estimates of the derivatives $\partial_{x_2} f$ and $\partial_{x_3} f$ through the traces of $h_1^\pm$. Using (51), from (38) we easily derive an estimate of $\partial_t f$. Collecting these estimates, we obtain

$$\sum_{|\alpha|=1}\|\partial^\alpha_{\tan} f\|^2_{L^2(\mathbb{R}^3)} \leq C\Big(\|u^+_{1|x_1=0}\|^2_{L^2(\mathbb{R}^3)} + \sum_{\pm}\big(\|h^\pm_{1|x_1=0}\|^2_{L^2(\mathbb{R}^3)} + \|G^\pm\|^2_{L^2(\mathbb{R}^3)}\big)\Big). \tag{52}$$

With estimates (47) and (48) inequalities (51) and (52) give us the desired estimate (49). □

3.2 Elliptic problem for the total pressure

From problem (30) we now obtain an elliptic problem for the total pressure $q^\pm$. Introducing the differential operators

$$\mathcal{L}_\pm = \partial_t + v_2^\pm\partial_{x_2} + v_3^\pm\partial_{x_3} + \gamma I \quad \text{and} \quad \mathcal{B}_\pm = H_2^\pm\partial_{x_2} + H_3^\pm\partial_{x_3},$$

we first rewrite equations (33) and (35) in the form

$$\mathcal{L}_+\mathbf{u}^+ - \mathcal{B}_+\mathbf{h}^+ + \nabla q^+ = \mathbf{F}_1^+, \qquad \mathcal{L}_-\mathbf{u}^- - \mathcal{B}_-\mathbf{h}^- + \nabla q^- = \mathbf{F}_1^-. \tag{53}$$

Applying the divergence operators $\mathrm{div}^\pm$ to (53) and using (37), (43), we get the Poisson equations $\Delta q^\pm = \mathfrak{F}^\pm$, with

$$\mathfrak{F}^\pm := \mathrm{div}\,\mathbf{F}_1^\pm + \mathcal{B}_\pm\mathcal{F}^\pm. \tag{54}$$

Substituting (38), (39), and (42),

$$u^\pm_{1|x_1=0} = \mathcal{L}_\pm f, \quad h^\pm_{1|x_1=0} = \mathcal{B}_\pm f + G^\pm,$$

into the first scalar equations in systems (53) restricted to the boundary $\{x_1 = 0\}$, we get boundary conditions for $\partial_{x_1} q^\pm$. Taking into account the boundary condition (40), we finally obtain the following problem for $q^\pm$:

$$\Delta q^+ = \mathfrak{F}^+, \quad \Delta q^- = \mathfrak{F}^-, \quad \text{in } \{x_1 > 0\}, \tag{55}$$

$$q^+ - q^- = g, \tag{56}$$

$$\partial_{x_1} q^+ + (\mathcal{L}_+^2 - \mathcal{B}_+^2)f = g_+, \tag{57}$$

$$-\partial_{x_1} q^- + (\mathcal{L}_-^2 - \mathcal{B}_-^2)f = g_-, \quad \text{on } \{x_1 = 0\}, \tag{58}$$

where $\Delta = \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3}$ is the Laplace operator and

$$g_\pm := F^\pm_{1,1|x_1=0} + \mathcal{B}_\pm G^\pm. \tag{59}$$

This problem contains the unknown front f. In principle, we can exclude it from the boundary conditions by applying the operator $(\mathcal{L}_-^2 - \mathcal{B}_-^2)$ to (57) and the operator $(\mathcal{L}_+^2 - \mathcal{B}_+^2)$ to (58) and subtracting the results.

3.3 Estimating ∇q through its trace

Using the concrete form of the source terms $\mathfrak{F}^{\pm}$ and $g_{\pm}$, cf. (54), (59), we can estimate ∇q in $L^2(\Omega)$ through its trace $\nabla q_{|x_1=0}$ and the front f (and the data $(\mathbf{F}, g)$). Then, with estimate (49) we get the following result.

Lemma 3.2. *There exists a constant $C > 0$ such that the estimate*

$$\|\nabla q\|^2_{L^2(\Omega)} \le C\left(\|\nabla q_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{\gamma}|||\mathbf{F}|||^2_{1,\gamma} + \frac{1}{\gamma^2}\|g\|^2_{1,\gamma}\right) \tag{60}$$

holds for sufficiently smooth solutions of problem (30) *for all* $\gamma \ge 1$.

Proof. Let us integrate the equation $q^+\Delta q^+ + q^-\Delta q^- = q^+\mathfrak{F}^+ + q^-\mathfrak{F}^-$ following from (55) over the domain Ω. Integration by parts leads us to the energy identity

$$\|\nabla q\|^2_{L^2(\Omega)} = \sum_{\pm}\left(\int_{\mathbb{R}^3}\left(\pm q^{\pm}F^{\pm}_{1,1} - q^{\pm}\partial_{x_1}q^{\pm}\right)_{|x_1=0}dtd\mathbf{x}' + \int_{\Omega}\left((\mathbf{F}^{\pm}_1, \nabla q^{\pm}) + \mathcal{F}^{\pm}\mathcal{B}_{\pm}q^{\pm}\right)dtd\mathbf{x}\right), \tag{61}$$

where $\sum_{\pm}(\pm a^{\pm}) := a^+ - a^-$, etc. It follows from the boundary conditions (56)–(58) that

$$\begin{aligned}\sum_{\pm}\left(\pm q^{\pm}F^{\pm}_{1,1} - q^{\pm}\partial_{x_1}q^{\pm}\right)_{|x_1=0} &= q^+_{|x_1=0}(F^+_{1,1} - \partial_{x_1}q^+)_{|x_1=0} - q^-_{|x_1=0}(F^-_{1,1} + \partial_{x_1}q^-)_{|x_1=0}\\ &= q^+_{|x_1=0}\left(F^+_{1,1} - \partial_{x_1}q^+ - (F^-_{1,1} + \partial_{x_1}q^-)\right)_{|x_1=0} - g(F^-_{1,1} + \partial_{x_1}q^-)_{|x_1=0}\\ &= -g(F^-_{1,1} + \partial_{x_1}q^-)_{|x_1=0} + q^+_{|x_1=0}\sum_{\pm}\left\{\pm\mathcal{A}_{\pm}(\mathcal{L}_{\pm}f + \partial_t f + \gamma f) \mp \mathcal{B}_{\pm}(\mathcal{B}_{\pm}f + G^{\pm})\right\},\end{aligned} \tag{62}$$

where $\mathcal{A}_{\pm} = v_2^{\pm}\partial_{x_2} + v_3^{\pm}\partial_{x_3}$.

Substituting (62) into (61) and again integrating by parts, we obtain

$$\begin{aligned}\|\nabla q\|^2_{L^2(\Omega)} = \sum_{\pm}\Bigg(&\int_{\mathbb{R}^3_+}\left(\pm(\mathcal{L}_{\pm}f + \partial_t f + \gamma f)\mathcal{A}_{\pm}q^+_{|x_1=0} \mp (\mathcal{B}_{\pm}f + G^{\pm})\mathcal{B}_{\pm}q^+_{|x_1=0}\right)dtd\mathbf{x}'\\ &+ \int_{\Omega}\left((\mathbf{F}^{\pm}_1, \nabla q^{\pm}) + \mathcal{F}^{\pm}\mathcal{B}_{\pm}q^{\pm}\right)dtd\mathbf{x}\Bigg) - \int_{\mathbb{R}^3_+} g(F^-_{1,1} + \partial_{x_1}q^-)_{|x_1=0}dtd\mathbf{x}'.\end{aligned} \tag{63}$$

Using Cauchy-Schwarz's and Young's inequalities, inequality (26) (for the function g), estimates (46) and (47), and the trace theorem in H^1 (for the function $F^-_{1,1}$), from (63) we derive the estimate

$$\|\nabla q\|^2_{L^2(\Omega)} \le C\left(\|\nabla q_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)} + \|f\|^2_{1,\gamma} + \frac{1}{\gamma}|||\mathbf{F}|||^2_{1,\gamma} + \frac{1}{\gamma^2}\|g\|^2_{1,\gamma}\right).$$

Taking into account (49), this estimate implies the desired estimate (60). □

Note that for further arguments we will need the roughened version of estimate (60),

$$\|\nabla q\|^2_{L^2(\Omega)} \le C\left(\|\nabla q_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{\gamma^2}|||\mathbf{F}|||^2_{2,\gamma} + \frac{1}{\gamma^2}\|g\|^2_{1,\gamma}\right), \tag{64}$$

that is obtained by using inequality (26). From systems (41) we easily derive the estimates

$$\gamma\|\mathbf{U}^{\pm}\|^2_{L^2(\Omega)} \le \frac{C}{\gamma}\left(\|\nabla q^{\pm}\|^2_{L^2(\Omega)} + \frac{1}{\gamma^2}|||\mathbf{F}^{\pm}|||^2_{1,\gamma}\right). \tag{65}$$

Combining then estimates (48), (49), (64), and (65), one gets

$$\begin{aligned}\gamma\|\mathbf{U}\|^2_{L^2(\Omega)}+\|\nabla q\|^2_{L^2(\Omega)}+\|\mathbf{U}_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)}+\|f\|^2_{1,\gamma}\\ \leq C\left(\|\nabla q_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{\gamma^2}|||\mathbf{F}|||^2_{2,\gamma}+\frac{1}{\gamma^2}\|g\|^2_{1,\gamma}\right).\end{aligned}\tag{66}$$

Thus, to prove the desired a priori estimate (31) in Theorem 2.2 it remains to derive the estimate

$$\|\nabla q_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)}\leq C\left(\frac{1}{\gamma^2}|||\mathbf{F}|||^2_{3,\gamma}+\frac{1}{\gamma^2}\|g\|^2_{2,\gamma}\right)\tag{67}$$

for the trace of ∇q.

3.4 Reduced problem for the Laplace equation

Our present goal is deriving estimate (67) for the nonstandard elliptic problem (55)–(58) with the source terms $\mathfrak{F}^\pm$ and $g_\pm$ given by (54) and (59) provided that the stability condition (8) is fulfilled. We first reduce this problem to that for the Laplace equation.

Consider the auxiliary problem

$$\begin{cases}\Delta\widetilde{q}^{\,+}=\mathfrak{F}^+,\quad \Delta\widetilde{q}^{\,-}=\mathfrak{F}^-,\quad \text{in } \{x_1>0\},\\ \widetilde{q}^{\,+}=\partial_{x_1}\widetilde{q}^{\,+}-F^+_{1,1|x_1=0},\\ \widetilde{q}^{\,-}=\partial_{x_1}\widetilde{q}^{\,-}+F^-_{1,1|x_1=0},\quad \text{on } \{x_1=0\}.\end{cases}\tag{68}$$

It implies the equation $\widetilde{q}^{\,+}\Delta\widetilde{q}^{\,+}+\widetilde{q}^{\,-}\Delta\widetilde{q}^{\,-}=\widetilde{q}^{\,+}\mathfrak{F}^++\widetilde{q}^{\,-}\mathfrak{F}^-$ which we integrate over the domain Ω. Then, integrating by parts, using the boundary conditions in (68), and accounting for (54), we obtain

$$\sum_{\pm}\Big\{\frac{1}{2}\Big(\|\widetilde{q}^{\,\pm}_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)}+\|\mathfrak{g}_\pm\|^2_{L^2(\mathbb{R}^3)}\Big)+\|\nabla\widetilde{q}^{\,\pm}\|^2_{L^2(\Omega)}-\int_\Omega\big((\mathbf{F}_1^\pm,\nabla\widetilde{q}^{\,\pm})+\mathcal{F}^\pm\mathcal{B}_\pm\widetilde{q}^{\,\pm}\big)\,dtd\mathbf{x}\Big\}=0,\tag{69}$$

where

$$\mathfrak{g}_\pm:=F^\pm_{1,1|x_1=0}\mp\partial_{x_1}\widetilde{q}^{\,\pm}_{|x_1=0}\,.\tag{70}$$

Taking into account estimate (46), from the energy identity (69) we get by standard arguments the estimate

$$\|\widetilde{q}^{\,+}_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)}+\|\widetilde{q}^{\,-}_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)}+\|\mathfrak{g}_+\|^2_{L^2(\mathbb{R}^3)}+\|\mathfrak{g}_-\|^2_{L^2(\mathbb{R}^3)}\leq\frac{C}{\gamma^2}|||\mathbf{F}|||^2_{1,\gamma},\tag{71}$$

Clearly, the tangential differentiation of problem (68) gives us also the estimate

$$\|\widetilde{q}^{\,+}_{|x_1=0}\|^2_{m,\gamma}+\|\widetilde{q}^{\,-}_{|x_1=0}\|^2_{m,\gamma}+\|\mathfrak{g}_+\|^2_{m,\gamma}+\|\mathfrak{g}_-\|^2_{m,\gamma}\leq\frac{C}{\gamma^2}|||\mathbf{F}|||^2_{m+1,\gamma},\quad\forall\, m\in\mathbb{N}.\tag{72}$$

Moreover, by using the boundary conditions in (68), estimates (71), (72), the trace theorem in H^1, and the inequality (26), one gets

$$\|\nabla\widetilde{q}_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)}\leq\frac{C}{\gamma^2}|||\mathbf{F}|||^2_{2,\gamma}\,,\tag{73}$$

where $\nabla\widetilde{q}=(\nabla\widetilde{q}^{\,+},\nabla\widetilde{q}^{\,-})$.

Let us define the new unknowns $\dot{q}^{\pm} := q^{\pm} - \widetilde{q}^{\,\pm}$. It follows from (55)–(59), (68), and (70) that they satisfy the following problem for the Laplace equations:

$$\begin{cases} \quad \Delta\dot{q}^{+} = 0, \quad \Delta\dot{q}^{-} = 0, \quad \text{in } \{x_1 > 0\}, \\ \dot{q}^{+} - \dot{q}^{-} = \mathfrak{g} := g + \widetilde{g}, \\ \partial_{x_1}\dot{q}^{+} + (\mathcal{L}_{+}^2 - \mathcal{B}_{+}^2)f = \mathcal{B}_{+}G^{+} + \mathfrak{g}_{+}, \\ -\partial_{x_1}\dot{q}^{-} + (\mathcal{L}_{-}^2 - \mathcal{B}_{-}^2)f = \mathcal{B}_{-}G^{-} + \mathfrak{g}_{-}, \quad \text{on } \{x_1 = 0\}, \end{cases} \tag{74}$$

where $\widetilde{g} := \widetilde{q}^{\,-}_{|x_1=0} - \widetilde{q}^{\,+}_{|x_1=0}$ and, in view of (72) with $m = 2$,

$$\|\widetilde{g}\|^2_{2,\gamma} + \|\mathfrak{g}_{+}\|^2_{2,\gamma} + \|\mathfrak{g}_{-}\|^2_{2,\gamma} \leq \frac{C}{\gamma^2}|||\mathbf{F}|||^2_{3,\gamma}. \tag{75}$$

Now it is clear that if for problem (74) we manage to prove the estimate

$$\|\nabla\dot{q}_{|x_1=0}\|^2_{L^2(\mathbb{R}^3)} \leq \frac{C}{\gamma^2}\Big(\|\mathfrak{g}\|^2_{2,\gamma} + \sum_{\pm}\big(\|G^{\pm}\|^2_{2,\gamma} + \|\mathfrak{g}_{\pm}\|^2_{2,\gamma}\big)\Big), \tag{76}$$

with $\nabla\dot{q} = (\nabla\dot{q}^{+}, \nabla\dot{q}^{-})$, then, by virtue of (47) for $m = 2$, (73), and (75), we obtain the desired estimate (67). Recall that (67) and (66) imply estimate (31). That is, if we assume that (76) holds, then the proof of estimate (31) in Theorem 2.2 is complete. In the next section we derive the a priori estimate (76) by constructing a symbolic symmetrizer for problem (74).

4 Construction of a symbolic symmetrizer

4.1 Statement of a boundary problem for Fourier transforms

Applying a Fourier transform in $(t, \mathbf{x}')$ to problem (74) we obtain

$$\begin{cases} \quad \dfrac{d^2\widehat{\dot{q}^{\pm}}}{dx_1^2} - \omega^2\widehat{\dot{q}^{\pm}} = 0, \quad x_1 > 0, \\ \widehat{\dot{q}^{+}}(0) - \widehat{\dot{q}^{-}}(0) = \widehat{\mathfrak{g}}, \\ \dfrac{d\widehat{\dot{q}^{+}}(0)}{dx_1} + ((l^{+})^2 + (b^{+})^2)\widehat{f} = ib^{+}\widehat{G^{+}} + \widehat{\mathfrak{g}_{+}}, \\ -\dfrac{d\widehat{\dot{q}^{-}}(0)}{dx_1} + ((l^{-})^2 + (b^{-})^2)\widehat{f} = ib^{-}\widehat{G^{-}} + \widehat{\mathfrak{g}_{-}}, \end{cases} \tag{77}$$

where $\widehat{v} = \widehat{v}(x_1) = \widehat{v}(\xi, x_1, \boldsymbol{\omega})$ is the Fourier transform of a function $v = v(t, \mathbf{x})$, with the dual variables ξ and $\boldsymbol{\omega} = (\omega_2, \omega_3)$ for t and $\mathbf{x}'$;

$$\omega = |\boldsymbol{\omega}|, \quad l^{\pm} = \tau + ia^{\pm}, \quad \tau = \gamma + i\xi, \quad a^{\pm} = \omega_2 v_2^{\pm} + \omega_3 v_3^{\pm}, \quad b^{\pm} = \omega_2 H_2^{\pm} + \omega_3 H_3^{\pm}. \tag{78}$$

Observe that τ is, in fact, the Laplace dual variable because the change of unknowns in (29) together with performing a Fourier transform in t amounts to performing a Laplace transform with respect to t.

Let us define the new unknowns

$$\mathbf{Y}^+ = \begin{pmatrix} y_1^+ \\ y_2^+ \end{pmatrix} = \begin{pmatrix} \dfrac{d\widehat{\dot q^+}}{dx_1} \\ \omega \widehat{\dot q^+} \end{pmatrix}, \qquad \mathbf{Y}^- = \begin{pmatrix} y_1^- \\ y_2^- \end{pmatrix} = \begin{pmatrix} \dfrac{d\widehat{\dot q^-}}{dx_1} \\ \omega \widehat{\dot q^-} \end{pmatrix}.$$

In terms of $\mathbf{Y}^\pm$ the interior equations in (77) are rewritten as the first order systems

$$\frac{d\mathbf{Y}^\pm}{dx_1} = A^\pm \mathbf{Y}^\pm \quad x_1 > 0, \tag{79}$$

with

$$A^\pm = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}.$$

Eliminating the front in the last two boundary conditions in (77), we get

$$\begin{cases} \sigma^- y_1^+(0) + \sigma^+ y_1^-(0) = \mathcal{G}_1, \\ y_2^+(0) - y_2^-(0) = \mathcal{G}_2, \end{cases} \tag{80}$$

where

$$\sigma^\pm = \frac{(l^\pm)^2 + (b^\pm)^2}{|\tau|^2 + \omega^2}, \qquad \theta^\pm = ib^\pm \sigma^\mp,$$

$$\mathcal{G}_1 = \theta^+(\tau,\boldsymbol{\omega})\widehat{G^+} - \theta^-(\tau,\boldsymbol{\omega})\widehat{G^-} + \sigma^-(\tau,\boldsymbol{\omega})\widehat{\mathfrak{g}_+} - \sigma^+(\tau,\boldsymbol{\omega})\widehat{\mathfrak{g}_-}\,, \qquad \mathcal{G}_2 = \omega\widehat{\mathfrak{g}}.$$

The functions $\omega(\tau,\boldsymbol{\omega}) = |\boldsymbol{\omega}|$, $\sigma^\pm(\tau,\boldsymbol{\omega})$, and $\theta^\pm(\tau,\boldsymbol{\omega})$ are homogeneous (of order 1, 0, and 1 respectively). As usual (see, e.g., [3, 11]), in order to take such homogeneity properties into account, we define the hemisphere

$$\Sigma := \{(\tau,\boldsymbol{\omega}) \in \mathbb{C}\times\mathbb{R}^2 : \, |\tau|^2 + \omega^2 = 1, \, \Re\tau \geq 0\},$$

and denote by Ξ the set of "frequencies"

$$\Xi := \{(\gamma,\xi,\boldsymbol{\omega}) \in [0,+\infty[\times\mathbb{R}^3 : \, (\gamma,\xi,\boldsymbol{\omega}) \neq (0,0,0,0)\} =]0,+\infty[\cdot\Sigma.$$

We always identify $(\gamma,\xi) \in \mathbb{R}^2$ with $\tau = \gamma + i\xi \in \mathbb{C}$. Using the uniform boundedness of ω, $\sigma^\pm$, and $\theta^\pm$ on Σ, we can estimate the source term $\boldsymbol{\mathcal{G}} = (\mathcal{G}_1,\mathcal{G}_2) \in \mathbb{C}^2$ by $\widehat{\mathfrak{g}}$, $\widehat{G^\pm}$, and $\widehat{\mathfrak{g}_\pm}$:

$$\forall\,(\tau,\boldsymbol{\omega}) \in \Xi, \quad |\boldsymbol{\mathcal{G}}|^2 \leq C\Big(\omega^2|\widehat{\mathfrak{g}}|^2 + \sum_\pm \Big(|\widehat{\mathfrak{g}_\pm}|^2 + (|\tau|^2+\omega^2)|\widehat{G^\pm}|^2\Big)\Big). \tag{81}$$

It is convenient to rewrite the boundary problem (79), (80) in the compact form

$$\begin{cases} \dfrac{d\mathbf{Y}}{dx_1} = \mathcal{A}(\boldsymbol{\omega})\mathbf{Y}, \quad x_1 > 0, \\ \beta(\tau,\boldsymbol{\omega})\mathbf{Y}(0) = \boldsymbol{\mathcal{G}}, \end{cases} \tag{82}$$

where

$$\mathbf{Y} := \begin{pmatrix} \mathbf{Y}^+ \\ \mathbf{Y}^- \end{pmatrix}, \qquad \mathcal{A}(\boldsymbol{\omega}) = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix},$$

and

$$\beta(\tau,\boldsymbol{\omega}) = \begin{pmatrix} \sigma^- & 0 & \sigma^+ & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \tag{83}$$

which is a symbol homogeneous of order 0. Note that the matrix $\mathcal{A}(\boldsymbol{\omega})$ is diagonalizable for all $\boldsymbol{\omega}$. More precisely,

$$T\mathcal{A}(\boldsymbol{\omega})T^{-1} = \omega \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

4.2 Normal mode analysis for problem (82)

The ODE system in (82) does not contain the Laplace variable τ. However, the evolutional (time-dependent) character of our problem is hidden in the boundary conditions in (82) which do contain τ. Therefore, as for hyperbolic problems [5, 10], we should distinguish between the Lopatinski condition and the *uniform* Lopatinski condition. Recall that, the uniform Lopatinski condition is satisfied if the Lopatinski condition holds up to the boundary of the hemisphere Σ, i.e., not only for $\Re\tau > 0$ but also for $\Re\tau = 0$. In this paragraph, we show that planar current-vortex sheets can be only *weakly* stable. More precisely, we prove that under the stability condition (8) the boundary problem (82) satisfies the Lopatinski condition but violates the uniform Lopatinski condition. Of course, our calculations are similar to those of Syrovatskij [14] and Axford [1] (see also [7]), and our additional goal is to show that the Lopatinski determinant has only simple roots on Σ.

As in hyperbolic theory [5, 10], for problem (82) we define the Lopatinski determinant

$$\Delta(\tau,\boldsymbol{\omega}) = \det[\beta(\tau,\boldsymbol{\omega})(E^+,E^-)] \tag{84}$$

associated with $\beta(\tau,\boldsymbol{\omega})$ given by (83), where

$$E^+ = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad E^- = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \tag{85}$$

are the eigenvectors associated to the (stable) eigenvalue $\lambda = -\omega$ of negative real part of A^+ and A^- respectively. We get

$$\beta(\tau,\boldsymbol{\omega})(E^+,E^-) = \begin{pmatrix} \sigma^- & \sigma^+ \\ -1 & 1 \end{pmatrix} \tag{86}$$

from which we derive

$$\Delta(\tau,\boldsymbol{\omega}) = \sigma^+ + \sigma^- \quad \forall\,(\tau,\boldsymbol{\omega}) \in \Xi,$$

which reduces to

$$\Delta(\tau,\boldsymbol{\omega}) = (l^+)^2 + (b^+)^2 + (l^-)^2 + (b^-)^2, \quad \forall\,(\tau,\boldsymbol{\omega}) \in \Sigma.$$

The Lopatinski determinant Δ is defined in the whole hemisphere Σ and is continuous with respect to $\tau,\boldsymbol{\omega}$. If the Lopatinski determinant vanishes for $\Re\tau > 0$, then the constant coefficients linearized problem

(22), (23) is *ill-posed*, i.e., the corresponding planar current-vortex sheet is *unstable*. As follows from [14, 1] (see also just below), this never happens if and only if the stability condition (6), (7) is satisfied.

Proposition 4.1. *Assume that* (8) *holds. Then the equation* $\Delta(\tau, \boldsymbol{\omega}) = 0$ *has only simple roots* $(\tau, \boldsymbol{\omega}) \in \Sigma$ *(and for these roots* $\Re\tau = 0$*).*

Proof. Let us rewrite the equation $\Delta(\tau, \boldsymbol{\omega}) = 0$ in terms of $s = i\tau \in \mathbb{C}$:

$$(s - a^+)^2 + (s - a^-)^2 = (b^+)^2 + (b^-)^2 \tag{87}$$

($a^{\pm}, b^{\pm}$ are defined in (78)). Equation (87) is a quadratic equation for s and has two roots

$$s_{1,2} = \frac{1}{2}\left((a^+ + a^-)^2 \pm \sqrt{D(\boldsymbol{\omega})}\right),$$

where $D(\boldsymbol{\omega}) = 2((b^+)^2 + (b^-)^2) - (a^+ - a^-)^2$. Clearly, the equation $\Delta(\tau, \boldsymbol{\omega}) = 0$ has no unstable roots τ (of positive real part) if and only if both the roots $s_{1,2}$ are real, i.e., the quadratic form $D(\boldsymbol{\omega})$ (for ω_2 and ω_3) is nonnegative. This is so if and only if the stability condition (6), (7) is satisfied.

Under the sharpened stability condition (8) the quadratic form $D(\boldsymbol{\omega})$ is positive definite (recall that (8) implies (6)) and the roots $s_{1,2}$ are distinct. These roots correspond to simple roots $\tau_{1,2} = -is_{1,2}$ of the equation $\Delta(\tau, \boldsymbol{\omega}) = 0$ with $\Re\tau = 0$. That is, the uniform Lopatinski condition is violated. □

4.3 Construction of a degenerate symmetrizer

This subsection will be entirely devoted to the construction of a symbolic symmetrizer of (82). A general idea of symmetrizer for our (nonstandard) elliptic problem is inspired by the idea of Kreiss' symmetrizers [5] for hyperbolic problems and is, breafly speaking, the following. We first reduce the ODE system in (82) to a diagonal form with the matrix $T\mathcal{A}T^{-1}$ (see Subsection 4.1). Then, multiplying the resulting system by a Herminian matrix $r(\tau, \boldsymbol{\omega})$ (*symmetrizer*) and using the boundary conditions and special properties of r, we derive the estimate

$$|\mathbf{Y}(\xi, 0, \boldsymbol{\omega})|^2 \leq \frac{C}{\gamma^2}|\boldsymbol{\mathcal{G}}|^2(|\tau|^2 + |\omega|^2) \tag{88}$$

by standard "energy" arguments. Taking into account (81), integrating estimate (88) with respect to $(\xi, \boldsymbol{\omega}) \in \mathbb{R}^3$, recalling the definition of $\mathbf{Y}$, and using Plancherel's theorem, we obtain the desired estimate (76).

While constructing the symmetrizer we closely follow the plan and notations of Coulombel and Secchi in [3]. The symbolic symmetrizer $r(\tau, \boldsymbol{\omega})$ of (82) is sought to be a homogeneous function of degree zero with respect to $(\tau, \boldsymbol{\omega}) \in \Xi$. Thus, it is enough to construct $r(\tau, \boldsymbol{\omega})$ in the unit hemisphere Σ. Since the latter is a compact set, by the use of a smooth partition of unity we still reduce the construction of $r(\tau, \boldsymbol{\omega})$ to that in a neighborhood of each point of Σ. The analysis performed in Subsection 4.2 shows that we have to distinguish between three different subclasses of frequencies $(\tau, \boldsymbol{\omega}) \in \Sigma$ in the construction of $r(\tau, \boldsymbol{\omega})$.

i. The interior points $(\tau_0, \boldsymbol{\omega}_0)$ of Σ such that $\Re\tau_0 > 0$.

ii. The boundary points $(\tau_0, \boldsymbol{\omega}_0)$ of Σ where the Lopatinski condition is satisfied (i.e., $\Delta(\tau_0, \boldsymbol{\omega}_0) \neq 0$).

iii. The boundary points $(\tau_0, \boldsymbol{\omega}_0)$ where the Lopatinski condition breaks down (i.e., $\Delta(\tau_0, \boldsymbol{\omega}_0) = 0$).

The symmetrizer we are going to construct is degenerate in the sense that the uniform Lopatinski condition is violated and we have to treat case iii.

4.3.1 Construction of the symmetrizer: the interior points (case i)

Let us consider a point $(\tau_0, \boldsymbol{\omega}_0) \in \Sigma$ with $\Re\tau_0 > 0$. Recall the matrix $\mathcal{A}(\boldsymbol{\omega})$ is diagonalizable for all $\boldsymbol{\omega}$. In a neighborhood $\mathcal{V}$ of $(\tau_0, \boldsymbol{\omega}_0)$ the symmetrizer is defined by

$$r(\tau, \boldsymbol{\omega}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & K \end{pmatrix} \qquad \forall\, (\tau, \boldsymbol{\omega}) \in \mathcal{V},$$

where $K \geq 1$ is a positive real number, to be fixed large enough. Let us set $\Re M := \frac{M+M^*}{2}$ for every complex square matrix M. The matrix $r(\tau, \boldsymbol{\omega})$ is Hermitian and satisfies

$$\forall (\tau, \boldsymbol{\omega}) \in \mathcal{V}, \quad \Re(r(\tau, \boldsymbol{\omega}) T \mathcal{A}(\boldsymbol{\omega}) T^{-1}) \geq \omega I, \tag{89}$$

where I denotes the identity matrix of order 4. The principal distinction from the construction in [3] is that we have ω in the right-hand side of inequality (89). This is a kind of "elliptical degeneracy."

Furthermore, as in [3, Section 4.4], for $K \geq 1$ sufficiently large the following inequality holds true

$$\forall (\tau, \boldsymbol{\omega}) \in \mathcal{V}, \quad r(\tau, \boldsymbol{\omega}) + C(\widetilde{\beta}(\tau, \boldsymbol{\omega}))^* \widetilde{\beta}(\tau, \boldsymbol{\omega}) \geq I, \tag{90}$$

with a suitable positive constant C and $\widetilde{\beta}(\tau, \boldsymbol{\omega}) := \beta(\tau, \boldsymbol{\omega}) T^{-1}$ (we shrink the neighborhood $\mathcal{V}$ if necessary). We note that the first and the third columns of the matrix T^{-1} are E^+ and E^- in (85), and the crucial point in obtaining inequality (90) is that the matrix $\beta(\tau, \boldsymbol{\omega})(E^+, E^-)$ is invertible because the Lopatinski determinant does not vanish at $(\tau_0, \boldsymbol{\omega}_0)$ (see [3]).

4.3.2 The boundary points (case ii)

Let $(\tau_0, \boldsymbol{\omega}_0)$ belong to the subclass ii of Σ, namely, $\Re\tau_0 = 0$, and $\Delta(\tau_0, \boldsymbol{\omega}_0) \neq 0$. The symmetrizer $r(\tau, \boldsymbol{\omega})$ is defined in a neighborhood of $(\tau_0, \boldsymbol{\omega}_0)$ in a completely similar manner as in case i.

4.3.3 The boundary points (case iii)

Let $(\tau_0, \boldsymbol{\omega}_0) \in \Sigma$ be a point of type iii and denote by $\mathcal{V}$ a neighborhood of $(\tau_0, \boldsymbol{\omega}_0)$ in Σ. We define the symmetrizer in $\mathcal{V}$ by

$$r(\tau, \boldsymbol{\omega}) = \begin{pmatrix} -\gamma^2 & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & -\gamma^2 & 0 \\ 0 & 0 & 0 & K \end{pmatrix} \qquad \forall\, (\tau, \boldsymbol{\omega}) \in \mathcal{V},$$

where $K \geq 1$ is a positive real number, to be fixed large enough. The matrix $r(\tau, \boldsymbol{\omega})$ above is Hermitian and we have

$$\Re(r(\tau, \boldsymbol{\omega}) T \mathcal{A}(\boldsymbol{\omega}) T^{-1}) \geq \omega \begin{pmatrix} \gamma^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{91}$$

Following [3], we also get that there exists a constant $C > 0$ such that

$$r(\tau, \boldsymbol{\omega}) + C \widetilde{\beta}^*(\tau, \boldsymbol{\omega}) \widetilde{\beta}(\tau, \boldsymbol{\omega}) \geq \gamma^2 I \qquad \forall\, (\tau, \boldsymbol{\omega}) \in \mathcal{V}. \tag{92}$$

For the detailed proof of (92) we refer to [3]. For our problem the proof is entirely the same and based on the following result concerning the vanishing of the Lopatinski determinant.

Lemma 4.1. *Let $(\tau_0, \boldsymbol{\omega}_0) \in \Sigma$ be a root of $\Delta(\tau, \boldsymbol{\omega}) = 0$. Then there exist a neighborhood $\mathcal{V}$ of $(\tau_0, \boldsymbol{\omega}_0)$ in Σ and a constant $k_0 > 0$ such that for all $(\tau, \boldsymbol{\omega}) \in \mathcal{V}$ we have*

$$\left|\beta(\tau, \boldsymbol{\omega})(E^+, E^-)\mathbf{Z}\right|^2 \geq k_0 \gamma^2 |\mathbf{Z}|^2 \qquad \forall\, \mathbf{Z} \in \mathbb{C}^2.$$

We omit the proof that is similar to (or even technically simpler than) the proof of [3, Lemma 4.5]. It relies on the facts that the roots of the Lopatinski determinant Δ are simple (see Proposition 4.1) and the lower right corner coefficient of $\beta(E^+, E^-)$ is nonzero (see (86)).

4.4 Derivation of estimate (88)

We are now ready to derive estimate (88). Following [3], we introduce a smooth partition of unity $\{\chi_j\}_{j=1}^J$ related to a given finite open covering $\{\mathcal{V}_j\}_{j=1}^J$ of Σ. Namely, we have

$$\chi_j \in C^\infty, \quad \operatorname{supp}(\chi_j) \subseteq \mathcal{V}_j,\ j = \overline{1, J}, \quad \text{and} \quad \sum_{j=1}^J \chi_j^2 \equiv 1.$$

Fix an arbitrary point $(\tau_0, \boldsymbol{\omega}_0) \in \Sigma$ belonging to one of the classes (i, ii or iii) analyzed before and let $\mathcal{V}_j$ be an open neighborhood of this point. We derive a local energy estimate in $\mathcal{V}_j$ and then, by adding the resulting estimates over all $j = \overline{1, J}$, we obtain the desired global estimate.

1^st^ case. Let $(\tau_0, \boldsymbol{\omega}_0)$ belongs to class i or class ii. We know from paragraphs 4.3.1 and 4.3.2 (see (89) and (90)) that there exist a C^∞ mapping $r_j(\tau, \boldsymbol{\omega})$ defined on $\mathcal{V}_j$ such that

- $r_j(\tau, \boldsymbol{\omega})$ is Hermitian,
- the estimates
$$\begin{aligned} &\Re\left(r_j(\tau, \boldsymbol{\omega}) T \mathcal{A}(\boldsymbol{\omega}) T^{-1}\right) \geq \omega I, \\ &r_j(\tau, \boldsymbol{\omega}) + C_j \left(\beta(\tau, \boldsymbol{\omega}) T^{-1}\right)^* \beta(\tau, \boldsymbol{\omega}) T^{-1} \geq I \end{aligned} \tag{93}$$
hold for all $(\tau, \boldsymbol{\omega}) \in \mathcal{V}_j$, with some positive constant C_j.

We set $\mathbf{U}_j(\tau, x_1, \boldsymbol{\omega}) := \chi_j(\tau, \boldsymbol{\omega}) T \mathbf{Y}(\xi, x_1, \boldsymbol{\omega})$. Since χ_j is supported on $\mathcal{V}_j$, we may think about r_j extended by zero to the whole of Σ. Then we extend χ_j and r_j to the whole set of frequencies Ξ as homogeneous mappings of degree zero with respect to $(\tau, \boldsymbol{\omega})$. Thus, from equations (82) we obtain that $\mathbf{U}_j$ satisfies

$$\begin{cases} \dfrac{d\mathbf{U}_j}{dx_1} = T\mathcal{A}(\boldsymbol{\omega})T^{-1}\mathbf{U}_j, \quad x_1 > 0, \\ \beta(\tau, \boldsymbol{\omega})T^{-1}\mathbf{U}_j(0) = \chi_j\, \boldsymbol{\mathcal{G}}. \end{cases} \tag{94}$$

Taking the scalar product of the ODE system in (94) with $r_j\mathbf{U}_j$, integrating over $\mathbb{R}^+$ with respect to x_1, and considering the real part of the resulting equality, we are led to

$$-\frac{1}{2}(r_j(\tau, \boldsymbol{\omega})\mathbf{U}_j(\tau, 0, \boldsymbol{\omega}), \mathbf{U}_j(\tau, 0, \boldsymbol{\omega})) = \int_0^{+\infty} \Re\left(r_j(\tau, \boldsymbol{\omega})T\mathcal{A}(\boldsymbol{\omega})T^{-1}\mathbf{U}_j(\tau, x_1, \boldsymbol{\omega}), \mathbf{U}_j(\tau, x_1, \boldsymbol{\omega})\right) dx_1.$$

Then, by using estimates (93) and the boundary condition in (94), one gets

$$\omega \int_0^{+\infty} |\mathbf{U}_j(\tau, x_1, \boldsymbol{\omega})|^2 dx_1 + \frac{1}{2}|\mathbf{U}_j(\tau, 0, \boldsymbol{\omega})|^2 \leq \frac{C_j}{2}\chi_j^2(\tau, \boldsymbol{\omega})|\boldsymbol{\mathcal{G}}|^2.$$

Recalling the definition of $\mathbf{U}_j$, we obtain

$$\chi_j^2(\tau, \boldsymbol{\omega})\omega \int_0^{+\infty} |\mathbf{Y}(\xi, x_1, \boldsymbol{\omega})|^2 dx_1 + \chi_j^2(\tau, \boldsymbol{\omega})|\mathbf{Y}(\xi, 0, \boldsymbol{\omega})|^2 \leq C_j\chi_j^2(\tau, \boldsymbol{\omega})|\boldsymbol{\mathcal{G}}|^2. \tag{95}$$

2nd case. It remains to prove a counterpart of estimate (95) for a neighborhood of $(\tau_0, \boldsymbol{\omega}_0) \in \Sigma$ such that $\Re\,\tau_0 = 0$ and $\Delta(\tau_0, \boldsymbol{\omega}_0) = 0$. Let $\mathcal{V}_j$ be an open neighborhood of this $(\tau_0, \boldsymbol{\omega}_0)$ and χ_j the associated cut-off function. As was shown in paragraph 4.3.3 (see (91) and (92)), there exists a C^∞ mapping $r_j(\tau, \boldsymbol{\omega})$ defined in $\mathcal{V}_j$, such that the following holds true

- $r_j(\tau, \boldsymbol{\omega})$ is Hermitian,
- the estimates

$$\begin{aligned} &\Re\left(r_j(\tau, \boldsymbol{\omega})T\mathcal{A}(\boldsymbol{\omega})T^{-1}\right) \geq \gamma^2\omega I, \\ &r_j(\tau, \boldsymbol{\omega}) + C_j\left(\beta(\tau, \boldsymbol{\omega})T^{-1}\right)^* \beta(\tau, \boldsymbol{\omega})T^{-1} \geq \gamma^2 I \end{aligned} \tag{96}$$

  hold for all $(\tau, \boldsymbol{\omega}) \in \mathcal{V}_j$, with some positive constant C_j.

Recall that $r_j(\tau, \boldsymbol{\omega})$, $\mathcal{A}(\boldsymbol{\omega})$, and $\beta(\tau, \boldsymbol{\omega})$ are assumed to be zeros outside $\mathcal{V}_j$. Then, they are extended to the whole of Ξ as homogeneous mappings of degree 2, 1, and 0 respectively. Hence, inequalities (96) become

$$\begin{aligned} &\Re\left(r_j(\tau, \boldsymbol{\omega})T\mathcal{A}(\boldsymbol{\omega})T^{-1}\right) \geq \gamma^2\omega I, \\ &r_j(\tau, \boldsymbol{\omega}) + C_j(|\tau|^2 + \omega^2)(\beta(\tau, \boldsymbol{\omega})T^{-1})^*\beta(\tau, \boldsymbol{\omega})T^{-1} \geq \gamma^2 I \end{aligned} \tag{97}$$

for all $(\tau, \boldsymbol{\omega}) \in \Xi$.

We again define $\mathbf{U}_j(\tau, x_1, \boldsymbol{\omega}) := \chi_j(\tau, \boldsymbol{\omega})T\mathbf{Y}(\xi, x_1, \boldsymbol{\omega})$. Reasoning as above, we derive the estimate

$$\chi_j^2(\tau, \boldsymbol{\omega})\omega \int_0^{+\infty} |\mathbf{Y}(\xi, x_1, \boldsymbol{\omega})|^2 dx_1 + \chi_j^2(\tau, \boldsymbol{\omega})|\mathbf{Y}(\xi, 0, \boldsymbol{\omega})|^2 \leq \frac{C_j}{\gamma^2}\,\chi_j^2(\tau, \boldsymbol{\omega})(|\tau|^2 + \omega^2)|\boldsymbol{\mathcal{G}}|^2, \tag{98}$$

with a suitable positive constant C_j.

We now add up estimates (95) and (98) and use the fact that $\{\chi_j\}$ is a partition of unity. This leads us to the global estimate

$$\omega \int_0^{+\infty} |\mathbf{Y}(\xi, x_1, \boldsymbol{\omega})|^2 dx_1 + |\mathbf{Y}(\xi, 0, \boldsymbol{\omega})|^2 \leq C|\boldsymbol{\mathcal{G}}|^2 + \frac{C}{\gamma^2}\,|\boldsymbol{\mathcal{G}}|^2(|\tau|^2 + \omega^2).$$

Because of the inequality $|\tau|^2+\omega^2\geq\gamma^2$ we finally get

$$\omega\int_0^{+\infty}|\mathbf{Y}(\xi,x_1,\boldsymbol{\omega})|^2dx_1+|\mathbf{Y}(\xi,0,\boldsymbol{\omega})|^2\leq\frac{C}{\gamma^2}\,|\boldsymbol{\mathcal{G}}|^2(|\tau|^2+\omega^2).$$

The last estimate yields the desired estimate (88) which in turn (see Subsection 4.3) gives us (76). This completes the proof of estimate (31) in Theorem 2.2.

5 Higher order estimates

We now prove the higher order estimate (32). Clearly, the problem for $(\partial^\beta_{\rm tan}\mathbf{U},\partial^\beta_{\rm tan}\nabla q,\partial^\beta_{\rm tan}f)$ obtained by the tangential differentiation (with respect to $(t,\mathbf{x}')$) of problem (30) has the same properties as the original problem (30) and, therefore, obeys an estimate like (31). Collecting such estimates for all the multiindices β with $|\beta|\leq m$, we easily obtain

$$\gamma|||\mathbf{U}|||^2_{L^2(H^m)}+|||\nabla q|||^2_{L^2(H^m)}+\|(\mathbf{U},\nabla q)_{|x_1=0}\|^2_{m,\gamma}+\|f\|^2_{m+1,\gamma}\leq\frac{C}{\gamma^2}\left(|||\mathbf{F}|||^2_{m+3,\gamma}+\|g\|^2_{m+2,\gamma}\right). \quad (99)$$

where $|||u|||^2_{L^2(H^m)}:=\sum_{|\alpha|\leq m}\gamma^{2(m-|\alpha|)}\|\partial^\alpha_{\rm tan}u\|^2_{L^2(\Omega)}$.

To derive (32) we still need to estimate normal derivatives of $(\mathbf{U},\nabla q)$. Consider first the case $m=1$. From the divergence-free conditions (37) and equations (43) we express the normal derivatives of the first components of $\mathbf{u}$ and $\mathbf{h}$:

$$\partial_{x_1}u_1^\pm=\mp\partial_{x_2}u_2^\pm\mp\partial_{x_3}u_3^\pm,\qquad\partial_{x_1}h_1^\pm=\mp\partial_{x_2}h_2^\pm\mp\partial_{x_3}h_3^\pm\pm\mathcal{F}^\pm. \quad (100)$$

Hence, it follows from (99) for $m=1$ and (46) that

$$\gamma\|(\partial_{x_1}u_1^\pm,\partial_{x_1}h_1^\pm)\|^2_{L^2(\Omega)}\leq\frac{C}{\gamma^2}\left(|||\mathbf{F}|||^2_{4,\gamma}+\|g\|^2_{3,\gamma}\right). \quad (101)$$

An L^2 estimate for the normal derivatives of the remaining components of $\mathbf{U}$ is derived from the equations for the vorticity $\boldsymbol{\zeta}^\pm:=\operatorname{curl}\mathbf{u}^\pm$ and the current $\mathbf{z}^\pm:=\operatorname{curl}\mathbf{h}^\pm$, where

$$\begin{aligned}\boldsymbol{\zeta}^\pm&=(\zeta_1^\pm,\zeta_2^\pm,\zeta_3^\pm)=(\partial_{x_2}u_3^\pm-\partial_{x_3}u_2^\pm,\partial_{x_3}u_1^\pm\mp\partial_{x_1}u_3^\pm,\pm\partial_{x_1}u_2^\pm-\partial_{x_2}u_1^\pm),\\ \mathbf{z}^\pm&=(z_1^\pm,z_2^\pm,z_3^\pm)=(\partial_{x_2}h_3^\pm-\partial_{x_3}h_2^\pm,\partial_{x_3}h_1^\pm\mp\partial_{x_1}h_3^\pm,\pm\partial_{x_1}h_2^\pm-\partial_{x_2}h_1^\pm).\end{aligned} \quad (102)$$

Applying the curl operator to (33)–(36), we obtain that $\boldsymbol{\zeta}^\pm,\mathbf{z}^\pm$ satisfy

$$\begin{cases}\partial_t\boldsymbol{\zeta}^\pm+\gamma\boldsymbol{\zeta}^\pm+(\mathbf{v}'^\pm,\nabla')\boldsymbol{\zeta}^\pm-(\mathbf{H}'^\pm,\nabla')\mathbf{z}^\pm=\operatorname{curl}\mathbf{F}_1^\pm\,,\\ \partial_t\mathbf{z}^\pm+\gamma\mathbf{z}^\pm+(\mathbf{v}'^\pm,\nabla')\mathbf{z}^\pm-(\mathbf{H}'^\pm,\nabla')\boldsymbol{\zeta}^\pm=\operatorname{curl}\mathbf{F}_2^\pm.\end{cases} \quad (103)$$

Equations (103) do not need boundary conditions at $x_1=0$ and we easily get the L^2 estimate

$$\gamma\|(\boldsymbol{\zeta}^\pm,\mathbf{z}^\pm)\|^2_{L^2(\Omega)}\leq\frac{C}{\gamma}|||\mathbf{F}^\pm|||^2_{1,\gamma}. \quad (104)$$

Expressing from (102) missing normal derivatives,

$$\partial_{x_1}u_2^\pm=\pm\zeta_3^\pm\pm\partial_{x_2}u_1^\pm,\quad\partial_{x_1}u_3^\pm=\mp\zeta_2^\pm\pm\partial_{x_3}u_1^\pm,\quad\partial_{x_1}h_2^\pm=\pm z_3^\pm\pm\partial_{x_2}h_1^\pm,\quad\partial_{x_1}h_3^\pm=\mp z_2^\pm\pm\partial_{x_3}h_1^\pm,$$

and using (99) for $m = 1$ and (104) we obtain

$$\gamma \|(\partial_{x_1} u^{\pm}_{2,3}, \partial_{x_1} h^{\pm}_{2,3})\|^2_{L^2(\Omega)} \leq \frac{C}{\gamma^2} \left(|||\mathbf{F}|||^2_{4,\gamma} + \|g\|^2_{3,\gamma} \right). \tag{105}$$

By adding up (101), (105), and (99) for $m = 1$, one gets

$$\gamma |||\mathbf{U}|||^2_{1,\gamma} + |||\nabla q|||^2_{L^2(H^1)} + \|(\mathbf{U}, \nabla q)_{|x_1=0}\|^2_{1,\gamma} + \|f\|^2_{2,\gamma} \leq \frac{C}{\gamma^2} \left(|||\mathbf{F}|||^2_{4,\gamma} + \|g\|^2_{3,\gamma} \right). \tag{106}$$

Thus, to have in hand estimate (32) for $m = 1$ we miss only $\|\partial_{x_1} \nabla q\|_{L^2(\Omega)}$ in the left-hand side of estimate (106). In fact, we need only to estimate $\partial^2_{x_1} q^{\pm}$ in $L^2(\Omega)$ because other derivatives are already presented in the $L^2(H^1)$ norm.

We express $\partial^2_{x_1} q^{\pm}$ from the Poisson equations (55):

$$\partial^2_{x_1} q^{\pm} = -\partial^2_{x_2} q^{\pm} - \partial^2_{x_3} q^{\pm} + \operatorname{div} \mathbf{F}^{\pm}_1 + \mathcal{B}^{\pm} \mathcal{F}^{\pm}.$$

Using then (106) and (46) (for $m = 1$), we get the estimate

$$\|\partial^2_{x_1} q^{\pm}\|^2_{L^2(\Omega)} \leq \frac{C}{\gamma^2} \left(|||\mathbf{F}|||^2_{4,\gamma} + \|g\|^2_{3,\gamma} \right).$$

That is, the proof of estimate (32) is complete for $m = 1$.

Now we can proceed by finite induction. Estimate (32) has already been proved for $m = k = 1$. Assume that it holds for $m = 1, \ldots, k-1$. Then we easily obtain an estimate in the form of (32) for $(\partial^{\beta}_{\tan} \mathbf{U}, \partial^{\beta}_{\tan} \nabla q, \partial^{\beta}_{\tan} f)$, with $m = k - 1$ and $|\beta| = 1$. That is, to derive (32) for $m = k$ we need only to estimate $\partial^k_{x_1} \mathbf{U}$ and $\partial^k_{x_1} \nabla q$ in $L^2(\Omega)$. Since the current-vorticity system (103) does not need boundary conditions at $x_1 = 0$, by differentiating it (and (100)) $k - 1$ times with respect to x_1 and using (46) for $m = k - 1$ we estimate $\partial^k_{x_1} \mathbf{U}$ by the right-hand side of inequality (32) for $m = k$. Using then the Poisson equations (55), we also get an L^2 estimate of $\partial^k_{x_1} \nabla q$. Thus, we get (32) for $m = k$ and this completes the proof of estimate (32) for $m \in \mathbb{N}$.

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