On compressible current-vortex sheets

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Summary. Recent results for ideal compressible current-vortex sheets (tangential MHD discontinuities) are surveyed. A sufficient condition for the weak stability of planar current-vortex sheets is first found for a general case of the unperturbed flow. In astrophysics, this condition can be interpreted as the sufficient condition for the macroscopic stability of the heliopause. The crucial role in finding this stability condition is played by a new symmetric form of the MHD equations. The linear variable coefficients problem for nonplanar current-vortex sheets is studied as well. The fact that the Kreiss-Lopatinski condition is satisfied only in a weak sense yields losses of derivatives in a priori estimates. We prove an a priori tame estimate that is a necessary step to show the local-in-time existence of "stable" current-vortex sheet solutions to the nonlinear equations of ideal compressible MHD by a suitable Nash-Moser-type iteration scheme. Since the tangential discontinuity is characteristic, the functional setting is provided by the anisotropic weighted Sobolev spaces H^m_*.

1 Introduction

We consider the system of ideal compressible magnetohydrodynamics (MHD):

$$\begin{gathered}
\partial_t\rho + \mathrm{div}\,(\rho\mathbf{v}) = 0,\\
\partial_t(\rho\mathbf{v}) + \mathrm{div}\,(\rho\mathbf{v}\otimes\mathbf{v} - \mathbf{H}\otimes\mathbf{H}) + \nabla q = 0,\\
\partial_t\mathbf{H} - \nabla\times(\mathbf{v}{\times}\mathbf{H}) = 0,\\
\partial_t\big(\rho e + (|\mathbf{H}|^2/2)\big) + \mathrm{div}\,\big((\rho e + p)\mathbf{v} + \mathbf{H}{\times}(\mathbf{v}{\times}\mathbf{H})\big) = 0,
\end{gathered}\tag{1}$$

where ρ, $\mathbf{v}$, $\mathbf{H}$, and p are the density, the fluid velocity, the magnetic field, and the pressure respectively, $q = p + (|\mathbf{H}|^2/2)$ is the total pressure, $e = E + (|\mathbf{v}|^2/2)$. With a state equation of medium, $E = E(\rho, S)$, (1) is a closed system, where S is the entropy obeying the Gibbs relation. As the unknown we can fix, for example, the vector $\mathbf{U} = (p, \mathbf{v}, \mathbf{H}, S)$. System (1) is supplemented by the divergent constraint

$$\mathrm{div}\,\mathbf{H} = 0 \tag{2}$$

on the initial data $\mathbf{U}(0,\mathbf{x}) = \mathbf{U}_0(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, x_3)$.

Let $\Gamma(t) = \{x_1 - f(t,\mathbf{y}) = 0\}$ be a smooth hypersurface in $\mathbb{R} \times \mathbb{R}^3$ ($\mathbf{y} = (x_2, x_3)$ are tangential coordinates). We assume that $\Gamma(t)$ is a surface of tangential discontinuity [3, 4] (*current-vortex sheet*) for solutions of the MHD system (1). This is the type of contact discontinuities for which the MHD Rankine-Hugoniot conditions are satisfied in the following way:

$$\partial_t f = v_{\mathrm{N}}^{\pm}, \quad H_{\mathrm{N}}^{\pm} = 0, \quad [q] = 0, \tag{3}$$

where $H_{\mathrm{N}} = (\mathbf{H}, \mathbf{N})$, $v_{\mathrm{N}} = (\mathbf{v}, \mathbf{N})$, $[g] = g^+ - g^-$, $g^{\pm} := g(t, f(t,\mathbf{y}) \pm 0, \mathbf{y})$, and $\mathbf{N} = (1, -\partial_2 f, -\partial_3 f)$ is the space normal vector to $\Gamma(t)$. The tangential components of the velocity and the magnetic field may undergo any jump on $\Gamma(t)$. It is worth noting that (2) as well as the boundary conditions $H_{\mathrm{N}}^{\pm} = 0$ can be regarded as the restrictions only on the initial data

$$f(0,\mathbf{y}) = f_0(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^2; \quad \mathbf{U}(0,\mathbf{x}) = \mathbf{U}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega^{\pm}(0), \tag{4}$$

where $\Omega^{\pm}(t) = \{x_1 \gtrless f(t,\mathbf{y})\}$.

Our final goal is to find conditions on the initial data (4) providing the existence of current-vortex sheet solutions to the MHD system, i.e., the existence of a solution $(\mathbf{U}, f)$ of the free boundary value problem (1), (3), (4), where $\mathbf{U}$ is smooth in the domains $\Omega^{\pm}(t)$. Because of the general properties of hyperbolic conservation laws it is natural to expect only the local-in-time existence of current-vortex sheet solutions. Therefore, the question on the nonlinear Lyapunov's stability of an ideal current-vortex sheet has no sense. At the same time, the study of the linearized stability of current-vortex sheets is not only a necessary step to prove local-in-time existence but also is of independent interest in connection with various astrophysical applications. In particular, the ideal compressible current-vortex sheet is used for modeling the *heliopause*, which is caused by the interaction of the supersonic solar wind plasma with the local interstellar medium (in some sense, the heliopause is the *boundary of the solar system*). The model of heliopause was suggested in [2], and the heliopause is in fact a current-vortex sheet separating the interstellar plasma compressed at the bow shock from the solar wind plasma compressed at the termination shock wave. Note that in December 2004 the spacecraft *Voyager 1* has crossed the termination shock at the distance of 93 AU from the Sun.

Piecewise constant solutions of (1) satisfying (3) on a planar discontinuity are a simplest case of current-vortex sheet solutions. In astrophysics the linear stability of a planar compressible current-vortex sheet is usually interpreted as the macroscopic stability of the heliopause. One can show that for the constant coefficients linearized problem for planar current-vortex sheets the uniform Kreiss-Lopatinski condition is never satisfied [13]. That is, planar current-vortex sheets can be only neutrally (*weakly*) stable or violently unstable. In the 1960–90's, in a number of works motivated by astrophysical applications (see [10] and references therein) the linear stability of planar compressible current-vortex sheets was examined by the normal modes analysis.

But, because of insuperable technical difficulties neither stability nor instability conditions were found for a general case of the unperturbed flow. We propose an alternative energy method that has first enabled one to find a sufficient condition [13] for the weak stability of compressible current-vortex sheets. For the most general case when for the unperturbed flow $[\mathbf{v}] \neq 0$ and $\mathbf{H}^+ \times \mathbf{H}^- \neq 0$ this condition reads

$$|[\mathbf{v}]| < |\sin(\varphi^+ - \varphi^-)| \min\left\{ \frac{c^+ c_{\mathrm{A}}^+}{|\sin\varphi^-|\sqrt{(c^+)^2 + (c_{\mathrm{A}}^+)^2}}\,, \frac{c^- c_{\mathrm{A}}^-}{|\sin\varphi^+|\sqrt{(c^-)^2 + (c_{\mathrm{A}}^-)^2}} \right\}, \tag{5}$$

where c and $c_{\mathrm{A}} = |\mathbf{H}|/\sqrt{\rho}$ are the sound velocity and the Alfvén velocity respectively (for the unperturbed flow), φ^+ (φ^-) is the angle between the vector $\mathbf{H}^+$ ($\mathbf{H}^-$) and the jump $[\mathbf{v}]$.

Thus, our goal is to prove the local-in-time existence of current-vortex sheet solutions to the MHD system (1), provided that the initial data (4) satisfy the stability condition (5) together with all the other necessary conditions (hyperbolicity condition, compatibility conditions, etc.). The basic a priori estimate for the linearized variable coefficients problem for nonplanar current-vortex sheets was obtained in [13]. Since the Kreiss-Lopatinski condition is satisfied only in a weak sense there appears a loss of derivatives phenomena. In this paper we present an a priori *tame* estimate that can be used to achieve nonlinear local-in-time existence by Nash-Moser iterations.[1] Note that recently the Nash-Moser method was successfully used in [5] for 2D compressible vortex sheets. In our case there is an additional principal difficulty in comparison with compressible vortex sheets. The point is that in MHD the loss of control on normal derivatives cannot be compensated as it was done in [5] for vortex sheets by estimating missing normal derivatives through a vorticity-type linearized equation. Therefore, the natural functional setting is provided by the anisotropic weighted Sobolev spaces H^m_* (see, e.g., [11, 12]).

2 A secondary generalized Friedrichs symmetrizer for the compressible MHD equations

The crucial role in obtaining the stability condition (5) and proving the a priori estimate for the linearized problem associated to (1), (3), (4) (see Sect. 3, 4) is played by a new symmetric form [13] of the MHD equations. Using the linear analog of this symmetrization makes the linearized constant coefficients

[1] The work towards the proof of nonlinear local-in-time existence by Nash-Moser iterations has been recently finished [15].

boundary conditions conservative. The hyperbolicity condition for the corresponding linearized interior equations gives us the sufficient stability condition (5).

Let us first consider a general situation. Consider a system of N conservation laws ($t>0$, $\mathbf{x}\in\mathbb{R}^n$)

$$\partial_t\boldsymbol{\mathcal{P}}^0(\mathbf{U})+\sum_{j=1}^{n}\partial_j\boldsymbol{\mathcal{P}}^j(\mathbf{U})=0,$$

which is rewritten as the quasilinear system

$$B_0(\mathbf{U})\partial_t\mathbf{U}+\sum_{j=1}^{n}B_j(\mathbf{U})\partial_j\mathbf{U}=0, \tag{6}$$

where $\boldsymbol{\mathcal{P}}^\alpha=(\mathcal{P}_1^\alpha,\dots,\mathcal{P}_N^\alpha)$, $\mathbf{U}(t,\mathbf{x})=(u_1,\dots,u_N)$, $B_\alpha=(\partial\boldsymbol{\mathcal{P}}^\alpha/\partial\mathbf{U})$. Using an *additional* (and usually a priori known) conservation law

$$\partial_t\Phi^0(\mathbf{U})+\operatorname{div}\boldsymbol{\Phi}(\mathbf{U})=0\qquad(\boldsymbol{\Phi}=(\Phi^1,\dots,\Phi^n)),$$

we can perform Godunov's symmetrization [6]: $\mathbf{U}\to\mathbf{Q}=\partial\Phi^0/\partial\boldsymbol{\mathcal{P}}^0$. At the same time, it gives us the Friedrichs symmetrizer $S=(\partial\mathbf{Q}/\partial\mathbf{U})^{\mathsf{T}}$. That is, multiplying (6) from the left by the matrix S we get the symmetric system

$$A_0(\mathbf{U})\partial_t\mathbf{U}+\sum_{j=1}^{n}A_j(\mathbf{U})\partial_j\mathbf{U}=0, \tag{7}$$

with $A_\alpha=SB_\alpha=A_\alpha^{\mathsf{T}}$.

The situation is a little bit different if system (6) is supplemented by a set of K *divergent constrains*

$$\operatorname{div}\boldsymbol{\Psi}_j(\mathbf{U})=0,\quad j=\overline{1,K}\qquad(\boldsymbol{\Psi}_j=(\Psi_j^1,\dots,\Psi_j^n))$$

(e.g., for the MHD system (1) we have the sole divergent constraint (2)). In this case we have to introduce a *generalized Friedrichs symmetrizer* that can be found from the modified Godunov's symmetrization [7]. Such a symmetrizer is now the set $\mathbb{S}=(S,\mathbf{R}_1,\dots,\mathbf{R}_K)$ containing the same matrix S as above and the vectors $\mathbf{R}_j$ determined from the relations

$$\mathbf{R}_j=S\,\frac{\partial r_j}{\partial\mathbf{Q}}\,,\qquad d\Phi^k=(\mathbf{Q},d\boldsymbol{\mathcal{P}}^j)\,+\,\sum_{j=1}^{K}r_j\,d\Psi_j^k\,,\quad k=\overline{1,n}.$$

Multiplying (6) from the left by the matrix S and adding to the result the sum $\sum_{j=1}^{K}\mathbf{R}_j\operatorname{div}\boldsymbol{\Psi}_j(\mathbf{U})$ we come to the symmetric system (7).

Note that for the MHD system (1) we do not need to proceed in this way because it can be trivially symmetrized by rewriting it in the nonconservative form

$$
\begin{aligned}
&\frac{1}{\rho c^2}\,\frac{dp}{dt}+\operatorname{div}\mathbf{v}=0\,, \qquad \rho\,\frac{d\mathbf{v}}{dt}-(\mathbf{H},\nabla\mathbf{H})+\nabla q=0\,,\\
&\frac{d\mathbf{H}}{dt}-(\mathbf{H},\nabla)\mathbf{v}+\mathbf{H}\operatorname{div}\mathbf{v}=0\,, \qquad \frac{dS}{dt}=0,
\end{aligned} \tag{8}
$$

where constraint (1) was taken into account, $d/dt=\partial_t+(\mathbf{v},\nabla)$, and c is the sound speed. System (8) is written in the symmetric form (7) for the vector $\mathbf{U}=(p,\mathbf{v},\mathbf{H},S)$. The hyperbolicity condition is $A_0>0$, i.e.,

$$
\rho>0\,,\quad c^2>0\,. \tag{9}
$$

Suppose now that for system (6) there exists a generalized Friedrichs symmetrizer $\mathbb{S}_1$ different from the *basic* symmetrizer

$$
\mathbb{S}_0=\left(\left(\frac{\partial\mathbf{Q}}{\partial\mathbf{U}}\right)^{\mathsf{T}},\left(\frac{\partial\mathbf{Q}}{\partial\mathbf{U}}\right)^{\mathsf{T}}\frac{\partial r_1}{\partial\mathbf{Q}},\ldots,\left(\frac{\partial\mathbf{Q}}{\partial\mathbf{U}}\right)^{\mathsf{T}}\frac{\partial r_K}{\partial\mathbf{Q}}\right)
$$

(we exclude the uninteresting case when $\mathbb{S}_1=\mathrm{const}\,\mathbb{S}_0$). Then, the symmetrizer $\mathbb{S}_2$ such that $\mathbb{S}_1=\mathbb{S}_2\circ\mathbb{S}_0$ is a *secondary* generalized Friedrichs symmetrizer for the symmetric system (7). Indeed, system (7) was already symmetric, but the result of the application of the secondary symmetrizer $\mathbb{S}_2$ is again a symmetric system. It is quite natural that the hyperbolicity condition for the resulting symmetric system can be more restrictive than that for system (7).

For the symmetric MHD system (8) a secondary generalized Friedrichs symmetrizer was proposed in [13] and has the form $\mathbb{S}_2=(\widetilde{S},\mathbf{R})$, with

$$
\widetilde{S}=\begin{pmatrix}
1 & \dfrac{\lambda H_1}{\rho c^2} & \dfrac{\lambda H_2}{\rho c^2} & \dfrac{\lambda H_3}{\rho c^2} & 0 & 0 & 0 & 0\\
\lambda H_1\rho & 1 & 0 & 0 & -\rho\lambda & 0 & 0 & 0\\
\lambda H_2\rho & 0 & 1 & 0 & 0 & -\rho\lambda & 0 & 0\\
\lambda H_3\rho & 0 & 0 & 1 & 0 & 0 & -\rho\lambda & 0\\
0 & -\lambda & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & -\lambda & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & -\lambda & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},\qquad
\mathbf{R}=-\lambda\begin{pmatrix}1\\0\\0\\0\\H_1\\H_2\\H_3\\0\end{pmatrix},
$$

and the function $\lambda=\lambda(\mathbf{U})$ is arbitrary. Applying $\mathbb{S}_2$ to system (8), we get the symmetric system

$$
\mathcal{A}_0(\mathbf{U})\partial_t\mathbf{U}+\sum_{j=1}^{3}\mathcal{A}_j(\mathbf{U})\partial_j\mathbf{U}=0 \tag{10}
$$

$$
\left(=\widetilde{S}A_0\partial_t\mathbf{U}+\sum_{j=1}^{3}\widetilde{S}A_j\partial_j\mathbf{U}+\mathbf{R}\,\mathrm{div}\mathbf{H}\right),
$$

where

$$\mathcal{A}_0 = \widetilde{S}A_0 = \begin{pmatrix} \dfrac{1}{\rho c^2} & \dfrac{\lambda H_1}{c^2} & \dfrac{\lambda H_2}{c^2} & \dfrac{\lambda H_3}{c^2} & 0 & 0 & 0 & 0 \\ \dfrac{\lambda H_1}{c^2} & \rho & 0 & 0 & -\rho\lambda & 0 & 0 & 0 \\ \dfrac{\lambda H_2}{c^2} & 0 & \rho & 0 & 0 & -\rho\lambda & 0 & 0 \\ \dfrac{\lambda H_3}{c^2} & 0 & 0 & \rho & 0 & 0 & -\rho\lambda & 0 \\ 0 & -\rho\lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\rho\lambda & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\rho\lambda & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

for the concrete form of the matrices $\mathcal{A}_k$ we refer to [13]. Note that system (10) coincides with (8) if $\lambda = 0$. The symmetric system (10) is hyperbolic if $\mathcal{A}_0 > 0$ (this also guarantees that $\det \widetilde{S} \neq 0$). Direct calculations show that the last condition is satisfied if inequalities (9) hold together with the additional requirement

$$\rho\lambda^2 < \frac{1}{1 + (c_{\mathrm{A}}^2/c^2)}. \tag{11}$$

Of course, the hyperbolicity condition for system (10) is much more restrictive than the usual natural assumptions (9). It should be also noted that condition (11) guarantees the equivalence of systems (8) and (10) on smooth solutions provided that $\lambda(\mathbf{U})$ is a smooth function of $\mathbf{U}$, and the analogous assertion can also be proved for current-vortex sheet solutions.

The new symmetric form (10) for the MHD equations was guessed by considering the magnetoacoustics system and using for it a “compressible” counterpart of the so-called cross-helicity integral $d/dt\left(\int_{\mathbb{R}^3}(\mathbf{v},\mathbf{H})\,d\mathbf{x}\right) = 0$ taking place for incompressible MHD. This gives a new conserved integral for the linearized constant coefficient MHD equations and, respectively, a concrete form of the matrix $\mathcal{A}_0$, etc.

3 Linear stability of planar current-vortex sheets

To reduce the free boundary value problem (1), (3), (4) to that in fixed domains we make, as usual (see, e.g., [8]), the change of variables in $\mathbb{R}\times\mathbb{R}^3$: $\widetilde{t} = t$, $\widetilde{x}_1 = x_1 - f(t,\mathbf{y})$, $\widetilde{\mathbf{y}} = \mathbf{y}$. Then, $\widetilde{\mathbf{U}}(\widetilde{t},\widetilde{\mathbf{x}}) := \mathbf{U}(t,\mathbf{x})$ is a smooth vector-function for $\widetilde{\mathbf{x}} \in \mathbb{R}^3_\pm$, and problem (1), (3), (4) is reduced to the following problem (we omit tildes to simplify the notation):

$$L(\mathbf{U}, \nabla_{t,\mathrm{y}} f)\mathbf{U} = 0 \quad \text{in } [0,T]\times(\mathbb{R}^3_+\cup\mathbb{R}^3_-), \tag{12}$$

$$v_{\mathrm{N}}^{\pm} = f_t\,, \quad H_{\mathrm{N}}^{\pm} = 0\,, \quad [q] = 0 \quad \text{on } [0,T]\times\{x_1 = 0\}\times\mathbb{R}^2, \tag{13}$$

$$\mathbf{U}|_{t=0} = \mathbf{U}_0 \quad \text{in } \mathbb{R}^3_+ \cup \mathbb{R}^3_-, \qquad f|_{t=0} = f_0 \quad \text{in } \mathbb{R}^2. \tag{14}$$

Here

$$L = A_0(\mathbf{U})\partial_t + A_\nu(\mathbf{U},\mathbf{F})\partial_1 + A_2(\mathbf{U})\partial_2 + A_3(\mathbf{U})\partial_3;$$

$$A_\nu = A_1(\mathbf{U}) - f_t A_0(\mathbf{U}) - f_{x_2}A_2(\mathbf{U}) - f_{x_3}A_3(\mathbf{U}) \quad \text{is the } \textit{boundary matrix},$$

Since the boundary matrix A_ν is singular at $x_1 = 0$ (see [13]), compressible current-vortex sheets are *characteristic* discontinuities.

Let $(\widehat{\mathbf{U}}(t,\mathbf{x}), \hat{f}(t,\mathbf{y}))$ be a given vector-function, where $\widehat{\mathbf{U}} = (\hat{p}, \hat{\mathbf{v}}, \widehat{\mathbf{H}}, \widehat{S})$ is supposed to be smooth for $\mathbf{x} \in \mathbb{R}^3_\pm$. Then the linearization of (12)–(14) results in a variable coefficients problem for determining small perturbations $(\delta\mathbf{U}, \delta f)$ (below we drop δ). The interior equations for this problem contain first-order terms for f. To avoid this difficulty we make the change of unknowns (see [1])

$$\bar{\mathbf{U}} = \mathbf{U} - f\widehat{\mathbf{U}}_{x_1}. \tag{15}$$

In terms of the "good unknown" (15) (below we omit bars) and after dropping the zero-order term $f\partial_1\{L(\widehat{\mathbf{U}},\nabla_{t,\mathrm{y}}f)\widehat{\mathbf{U}}\}$ in the interior equations (as was recommended in [1]) we get the following linear problem

$$L(\widehat{\mathbf{U}},\nabla_{t,\mathrm{y}}\hat{f})\mathbf{U} + \widehat{C}\mathbf{U} = \mathbf{f} \quad \text{in } [0,T]\times(\mathbb{R}^3_+ \cup \mathbb{R}^3_-), \tag{16}$$

$$\begin{pmatrix} f_t + \hat{v}_2^+ f_{x_2} + \hat{v}_3^+ f_{x_3} - (\hat{v}_{\mathrm{N}})^+_{x_1} f - v_{\mathrm{N}}^+ \\ f_t + \hat{v}_2^- f_{x_2} + \hat{v}_3^- f_{x_3} - (\hat{v}_{\mathrm{N}})^-_{x_1} f - v_{\mathrm{N}}^- \\ \widehat{H}_2^+ f_{x_2} + \widehat{H}_3^+ f_{x_3} - (\widehat{H}_{\mathrm{N}})^+_{x_1} f - H_{\mathrm{N}}^+ \\ \widehat{H}_2^- f_{x_2} + \widehat{H}_3^- f_{x_3} - (\widehat{H}_{\mathrm{N}})^-_{x_1} f - H_{\mathrm{N}}^- \\ [q] + f[\hat{q}_{x_1}] \end{pmatrix} = \mathbf{g} \qquad \text{if} \quad x_1 = 0\,. \tag{17}$$

Here $v_{\mathrm{N}} = (\mathbf{v}, \widehat{\mathbf{N}})$, $(\hat{v}_{\mathrm{N}})^\pm_{x_1} = (\hat{v}_{\mathrm{N}})_{x_1}|_{x_1=\pm 0}$, $q = p + (\widehat{\mathbf{H}}, \mathbf{H})$, etc.; $\widehat{\mathbf{N}} = (1, -\hat{f}_{x_2}, -\hat{f}_{x_2})$; the matrix $\widehat{C}$ depends on $\widehat{\mathbf{U}}$, $\widehat{\mathbf{U}}_t$, $\nabla\widehat{\mathbf{U}}$, $\nabla_{t,\mathrm{y}}\hat{f}$ and can be explicitly written out (see [13]). We introduced the source terms $\mathbf{f}(t,\mathbf{x}) = \mathbf{f}^\pm(t,\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n_\pm$ and $\mathbf{g}(t,\mathbf{y})$ to make the interior equations and the boundary conditions inhomogeneous because this is needed to attack the nonlinear problem. Note that if we make suitable assumptions for $\mathbf{f}$, then we can prove that the third and the fourth boundary conditions in (17) with zero source terms as well as the divergent constraint $\partial_1 H_{\mathrm{N}} + \partial_2 H_2 + \partial_3 H_3 = 0$ are just the restrictions on the initial data for problem (16), (17). This is necessary to have a correct number of boundary conditions (the property of maximality) and to use the divergent constraint for getting a priori estimates for problem (16), (17).[2]

[2]It follows from the divergent constraint that H_{N} is one of the "noncharacteristic" unknowns. The other ones are q and v_{N}. If we however omit zero-order terms for f in (17) (that is actually inadmissible if we want to use a priori estimates for Nash-Moser iterations), then the divergent constraint just gives us an additional regularity for H_{N} but we do not really need it for getting a priori estimates.

Let us first freeze the coefficients in (16), (17), omit zero-order terms, and consider the case of homogeneous interior equations and boundary conditions. Then we have the problem that is the result of the linearization of (12), (13) with respect to the piecewise constant solution $\widehat{\mathbf{U}} = \widehat{\mathbf{U}}^{\pm}$ for $x_1 \gtrless 0$ for the planar current-vortex sheet $x_1 = 0$ (without loss of generality we take $\hat{f} = 0$). Since the constant coefficients linearized problem for compressible current-vortex sheets is of independent interest in connection with astrophysical applications mentioned in Sect. 1, we do not introduce in it artificial source terms. At the same time, the a priori estimates proved in [13] for this problem (see just below) can be easily generalized to the case of inhomogeneous problem.

We can show that current-vortex sheets cannot be *uniformly* stable [13].

Lemma 1. *For the constant coefficients linearized problem for planar current-vortex sheets (see (16), (17) with frozen coefficients), the uniform Kreiss-Lopatinski condition is never satisfied.*

Since the linearized problem for current-vortex sheets is a hyperbolic problem with characteristic boundary, there appears a loss of control on derivatives in the normal (x_1-)direction. Therefore, in the theorem below we use the following "nonsymmetric" Sobolev norm:

$$\|\mathbf{U}(t)\|^2_{\widetilde{H}^1(\mathbb{R}^3_+\cup\mathbb{R}^3_-)} = \sum_{\pm}\sum_{|\alpha|\le 1,\ \alpha_1=0}\Big(\|\partial^\alpha \mathbf{U}(t)\|^2_{L_2(\mathbb{R}^3_\pm)} + \|\partial_1 \mathbf{U}_{\rm n}(t)\|^2_{L_2(\mathbb{R}^3_\pm)}\Big)\,,$$

where $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$, $\mathbf{U}_{\rm n} = (q, v_1, H_1)$.

Theorem 1. *Let* $\widehat{\mathbf{H}}^+ \times \widehat{\mathbf{H}}^- \neq 0$ *and either* $[\hat{\mathbf{v}}] = 0$ *(current-sheet) or (5) is satisfied for* $\widehat{\mathbf{U}}$. *Then, for the constant coefficients linearized problem for planar current-vortex sheets the Lopatinski condition is satisfied and the a priori estimates*

$$\|\mathbf{U}(t)\|_{\widetilde{H}^1(\mathbb{R}^3_+\cup\mathbb{R}^3_-)} \le C\|\mathbf{U}_0\|_{\widetilde{H}^1(\mathbb{R}^3_+\cup\mathbb{R}^3_-)}\,, \tag{18}$$

$$\|f(t)\|_{H^1(\mathbb{R}^2)} \le \|f_0\|_{L_2(\mathbb{R}^2)} + C\|\mathbf{U}_0\|_{\widetilde{H}^1(\mathbb{R}^3_+\cup\mathbb{R}^3_-)} \tag{19}$$

hold for any $t \in [0,T]$, *where* $T > 0$, $C = C(T) = \mathrm{const} > 0$.

The case when $\widehat{\mathbf{H}}^+ \times \widehat{\mathbf{H}}^- = 0$, $\widehat{\mathbf{H}}^{\pm} \times [\hat{\mathbf{v}}] = 0$ *and either*

$$|[\hat{\mathbf{v}}]| < \max\big\{\max\{\gamma^+,\gamma^-\}\,,\, 2\min\{\gamma^+,\gamma^-\}\big\}\,, \qquad \gamma^{\pm} = \frac{\hat{c}^{\pm}\hat{c}^{\pm}_{\rm A}}{\sqrt{(\hat{c}^{\pm})^2 + (\hat{c}^{\pm}_{\rm A})^2}}\,,$$

or $[\hat{\mathbf{v}}] = 0$ *corresponds to the transition to violent instability, and for the function* $f(t,\mathbf{x}')$ *we have the weaker estimate*

$$\|f(t)\|_{L_2(\mathbb{R}^2)} \le \|f_0\|_{L_2(\mathbb{R}^2)} + C\|\mathbf{U}_0\|_{\widetilde{H}^1(\mathbb{R}^3_+\cup\mathbb{R}^3_-)}\,.$$

For the detailed proof of Theorem 1 we refer to [13]. The proof is based on the use of system (10). For a certain choice of the constants $\lambda^{\pm} = \lambda(\widehat{\mathbf{U}}^{\pm})$ the boundary conditions are *dissipative* (even conservative). The hyperbolicity condition (11) for the linearized system (10) for $x_1 \gtrless 0$ for chosen $\lambda^{\pm}$ gives the sufficient stability condition (5). Note also that the process of getting the a priori estimates (18), (19) can be formalized by introducing the notations of *dissipative p-symmetrizers* [14]. In fact, for the constant coefficients problem the dissipative (but *not* strictly dissipative [14]) 0-symmetrizer $\mathbb{S} = \left\{ \tilde{S}(\widehat{\mathbf{U}}^+), \tilde{S}(\widehat{\mathbf{U}}^-), \mathbf{R}(\widehat{\mathbf{U}}^+), \mathbf{R}(\widehat{\mathbf{U}}^-) \right\}$ (cf. (10)) has been constructed.

4 The variable coefficients analysis

The main difficulties in the variable coefficients analysis are connected with zero-order terms for f in the boundary conditions (17). Moreover, for constant coefficients we could work in usual Sobolev spaces. In the variable coefficients analysis we have to require a little bit more regularity for solutions. In fact, the natural functional setting is provided by the anisotropic weighted Sobolev spaces H^m_* (see [11, 12] and references therein).

The function space $H^m_*(\Omega)$ (in our case $\Omega = \mathbb{R}^3_+ \cup \mathbb{R}^3_-$) is defined as follows:

$$H^m_*(\Omega) := \left\{ u \in L_2(\Omega)\ :\ |\quad \partial^\alpha_* \partial^k_1 u \in L_2(\Omega) \quad \text{if} \quad |\alpha| + 2k \leq m \right\},$$

where $\partial^\alpha_* = (\sigma(x_1)\partial_1)^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$, and $\sigma(x_1) \in C^\infty(\mathbb{R}_+) \cap C^\infty(\mathbb{R}_-)$ is a monotone increasing function for $x_1 > 0$ and monotone decreasing for $x_1 < 0$ such that $\sigma(x_1) = |x_1|$ in a neighborhood of the origin and $\sigma(x_1) = 1$ for $|x_1|$ large enough. The space $H^m_*(\Omega)$ is normed by

$$\|u\|^2_{m,*} = \sum_{\pm} \sum_{|\alpha|+2k \leq m} \|\partial^\alpha_* \partial^k_1 u\|^2_{L_2(\mathbb{R}^3_\pm)} .$$

For solutions of problem (16), (17) we use also the norm

$$|||\mathbf{U}(t)|||^2_m = |||\mathbf{U}(t)|||^2_{m,*} + |||\partial_1 \mathbf{U}_{\rm n}(t)|||^2_{m-1,*} ,$$

where $|||(\cdot)(t)|||^2_{m,*} = \sum_{j=0}^k \|\partial^j_t (\cdot)(t)\|^2_{k-j,*}$; $\mathbf{U}_{\rm n} = (q, v_{\rm N}, H_{\rm N})$ is the "noncharacteristic" unknown. Here we use the notations from [11].

For the basic state $(\widehat{\mathbf{U}}, \hat{f})$, we assume that there exists a constant $K > 0$ such that

$$\|\widehat{\mathbf{U}}\|_{W^2_\infty(\Omega_T)} + \|\hat{f}\|_{W^3_\infty(\partial\Omega_T)} + \|\partial_1 \widehat{\mathbf{U}}^{\pm}\|_{W^2_\infty(\partial\Omega_T)} \leq K ,$$

where $\Omega_T = [0, T] \times \Omega$, $\partial\Omega_T = [0, T] \times \mathbb{R}^2$. We also suppose that the basic state satisfies the Rankine-Hugoniot conditions, the divergent constraint, and the hyperbolicity condition (9). We are now in a position to formulate the main result from [13] but unlike [13] we formulate it for the inhomogeneous problem ($\mathbf{f} \neq 0$, $\mathbf{g} \neq 0$).

Theorem 2. *Let the basic state* $(\widehat{\mathbf{U}}, \hat{f})$ *satisfies all the assumptions above. Let also there exists a positive constant* δ *such that*

$$\inf_{\partial\Omega_T} |\hat{\mathbf{h}}^+ \times \hat{\mathbf{h}}^-| \geq \delta > 0 \tag{20}$$

and the condition

$$r^{\pm}(t,\mathbf{y}) < b^{\pm}(t,\mathbf{y}) \tag{21}$$

holds for all $t \in [0,T]$ *at each point* $\mathbf{y} \in \mathbb{R}^2$ *such that* $\hat{\mathbf{u}}^+(t,\mathbf{y}) \neq \hat{\mathbf{u}}^-(t,\mathbf{y})$, *where*

$$\hat{\mathbf{h}} = (\widehat{H}_{\mathrm{N}}, \widehat{H}_2, \widehat{H}_3)\,, \quad \hat{\mathbf{u}} = (\hat{v}_{\mathrm{N}} - \hat{f}_t, \hat{v}_2, \hat{v}_3)\,, \quad r^{\pm}(t,\mathbf{x}) = \sqrt{\hat{\rho}^{\pm}\left(1 + \frac{(\hat{c}^{\pm}_{\mathrm{A}})^2}{(\hat{c}^{\pm})^2}\right)}\,,$$

$$b^{\pm}(t,\mathbf{x}') = \frac{|\hat{\mathbf{h}}^{\pm}|\,|\sin(\varphi^+ - \varphi^-)|}{|[\hat{\mathbf{u}}]|\,|\sin\varphi^{\mp}|}\,, \quad \cos\varphi^{\pm}(t,\mathbf{x}') = \frac{([\hat{\mathbf{u}}], \hat{\mathbf{h}}^{\pm})}{|[\hat{\mathbf{u}}]|\,|\hat{\mathbf{h}}^{\pm}|}\,.$$

Then, for problem (16), (17) the a priori estimate

$$\begin{aligned} &|||\mathbf{U}(t)|||_1 + \|f\|_{H^1(\partial\Omega_T)} \\ &\leq C\left\{\|\mathbf{f}\|_{H^1(\Omega_T)} + \|\mathbf{g}\|_{H^2(\partial\Omega_T)} + |||\mathbf{U}_0|||_1 + \|f_0\|_{H^1(\mathbb{R}^2)}\right\} \end{aligned} \tag{22}$$

holds for any $t \in [0,T]$. *Here* $C = C(T,K)$ *is a positive constant independent of the data* $(\mathbf{U}_0, f_0, \mathbf{f}, \mathbf{g})$.

For the detailed proof of Theorem 2 we refer to [13]. Inequality (21) appearing in this theorem is the analogue of the stability condition (5) for variable coefficients.[3]

5 The a priori tame estimate

We now just formulate the recent result concerning an a priori *tame* estimate for problem (16), (17). We consider the case of zero initial data for this problem that is usual assumption, and postpone the case of non-zero initial data to the nonlinear analysis (construction of a so-called approximate solution, etc.).

Following [11], we define the space

$$\mathfrak{L}^2_T(H^m_*) = \bigcap_{k=0}^{m} H^k([0,T], H^{m-k}_*)$$

equipped with the norm $[u]^2_{m,*,T} = \int_0^T |||u(t)|||^2_{m,*}dt$.

[3] Unlike [13] we assume that the stability condition for variable coefficients is satisfied only on the boundary at each point of the nonplanar current-vortex sheet. This is possible thanks to the use of a kind of cut-off function for $\lambda(\mathbf{U})$, i.e., roughly speaking, the hyperbolicity condition (11) can be imposed only on the boundary.

Theorem 3. *Let $T > 0$ and m is an even number, $m \geq 6$. Assume that the basic state $(\widehat{\mathbf{U}}, \hat{f}) \in \mathfrak{L}^2_T(H^{m+4}_*(\Omega_T)) \times H^{m+4}(\partial\Omega_T)$ satisfies the hyperbolicity condition (9), the Rankine-Hugoniot conditions (3), the divergent constraint* $\operatorname{div} \hat{\mathbf{h}} = 0$*, the assumption (20), the stability condition (21), and*

$$[\widehat{\mathbf{U}}]_{10,*,T} + \|\hat{f}\|_{H^{10}(\partial\Omega_T)} \leq K,$$

where $K > 0$ is a constant. Assume also that the data $(\mathbf{f}, \mathbf{g}) \in \mathfrak{L}^2_T(H^m_*(\Omega_T)) \times H^{m+1}(\partial\Omega_T)$ vanish in the past. Then there exists a positive constant K_0, that does not depend on m and T, and there exists a constant $C = C(K_0) > 0$ such that, if $K \leq K_0$, then there exists a unique solution $(\mathbf{U}, f) \in \mathfrak{L}^2_T(H^m_*(\Omega_T)) \times H^m(\partial\Omega_T)$ to problem (16), (17) that vanishes in the past and obeys the following a priori tame estimate for T small enough:

$$\begin{aligned} [\mathbf{U}]_{m,*,T} + \|f\|_{H^m(\partial\Omega_T)} \leq Ce^{CT}\Big\{ & [\mathbf{f}]_{m,*,T} + \|\mathbf{g}\|_{H^{m+1}(\partial\Omega_T)} \\ & + \big([\mathbf{f}]_{6,*,T} + \|\mathbf{g}\|_{H^7(\partial\Omega_T)}\big)\big([\widehat{\mathbf{U}}]_{m+4,*,T} + \|\hat{f}\|_{H^{m+4}(\partial\Omega_T)}\big)\Big\}. \end{aligned} \tag{23}$$

The proof of Theorem 3 is based on the use of Moser-type inequalities following from the Gagliardo-Nirenberg inequality for H^m_* presented in [1].[4] The tame estimate (23) is the basic tool to prove the local-in-time existence of "stable" current-vortex sheet solutions to the nonlinear MHD equations by the Nash-Moser method. The proof has been recently finished [15].

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[4] The Gagliardo-Nirenberg inequality for H^m_* is valid if m is even. This is why we have to assume that m is even. Note also that to work globally in x_1 and to avoid assumptions about compact support we use in the proof the reduction to fixed domains proposed in [9] (see (4.1.2)) which differs from that in the begining of Sect. 3.

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