VIRIAL FUNCTIONALS IN FLUID DYNAMICS

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Dedicated to V. I. Arnold

ABSTRACT. The aim of this paper is to show that functionals similar to
the 'virial' function of classical mechanics can be introduced for several
dynamical systems of fluid mechanics provided that those dynamical
systems can be described by Hamilton's principle of least action. The
main requirement to 'virials' is their increasing by virtue of equations
of motion. Applications of those functionals to hydrodynamic stability
theory are reviewed and further perspectives are discussed.

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1. INTRODUCTION

An extremely attractive but very difficult problem in fluid dynamics is to find a
regular way of constructing functionals which grow monotonically with time. Ex-
istence of such a functional would mean that an initially 'small' solution gradually
becomes 'large', which is closely related to such fundamental problems as instability
of fluid flows, magnetohydrodynamic dynamo, etc. Also it would mean a certain
irreversibility of fluid flows. A class of such functionals is known for long time, for
brevity we shall call them 'virials'. They represent an inner product between per-
turbations of velocities and displacements of material particles. First it appeared
(as a function, not a functional!) in classical mechanics in so called 'virial equa-
tion' and in the 'virial theorem' (see, e. g., [13]). Then it was extensively used as
Liapunov function in proofs of instability of finite-dimensional mechanical systems
by Liapunov [17] and Chetaev [11]. In continuous mechanics 'virial' was used in
astrophysical applications by Chandrasekhar [10] and others. The extensive use of
the 'virial equality' in a problem of fluid instability is due to Rumiantsev [19] who
considered instability of 'fluid + solid' equilibria with a free surface[1].

All earlier 'virial' results on instability of fluid flows had been obtained for the
linearized problems of an inviscid fluid. More recent papers have been concentrated

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[1]There are also terms 'virial function' and 'virial functional' which are sometimes used for cross-
products of a force and coordinates (see the 'Clausius virial' [13]), or velocities and coordinates
(see [10]).

on generalising the 'virial functional' to exact nonlinear problems, to viscous fluids and to some other fluid systems [8], [21], [22], [26], [27], [23], [9]. The first exact nonlinear result on instability of an equilibrium of an inviscid fluid with free surface was presented in [8], where a special class of fluid equilibria was considered. This result attracted the attention of V. I. Arnold, who considered it as a serious achievement. There were also applications of 'virial' to the problem of instability of special classes of fluid flows, which can be reduced to effective equilibria (see, e. g., [16], [12]).

At the same time, any rational reason, why monotonically increasing 'virial' functions or functionals should exist, is unknown. In the present paper, we discuss one possible general way of introduction 'virial functionals' for various infinite-dimensional systems of fluid mechanics. We can introduce the notion of 'virial' (and show that it is useful) for a fluid system as soon as we know the expression for its Lagrangian. Technically, our approach is based on the introduction of one-parameter families of vector fields. Variations of velocities and other fields correspond to derivatives with respect to that parameter. Our 'virial' appears in the equality for the second derivative of Lagrangian with respect to that parameter (or in other words in the equation for second variation of the action functional). It shows its close link to the solution of the well-known 'Jacobi equation for geodesic deviations' [6]. In the linear approximation, when a state of rest is taken as the basic solution, our 'virial' coincides with the conventional one.

We use the 'virial' in order to obtain results concerning *a priori* estimates of solutions and instabilities. Particular cases of an inviscid incompressible fluid (with and without free surface), stratified fluid, and ideal magnetohydrodynamics have been considered. Our approach recovers and unifies all known instability results obtained using 'virial' and, in particular, shows that the instability considered earlier by Arnold [5] can be recovered using the direct Lyapunov method with the 'geodesic virial' as the Lyapunov functional[2]. Since our approach has evident parallels in classical mechanics, it is instructive to draw a parallel with finite-dimensional results.

First, we introduce the 'virial' as it was used by Lyapunov [17] and Chetaev [11]. Consider $2N$-dimensional Hamiltonian system described by generalised coordinates $q = q(t) = (q_1(t), \ldots, q_N(t))$ and momenta $p = p(t) = (p_1(t), \ldots, p_N(t))$ which satisfy the equations

$$p_t = -H_q, \quad q_t = H_p, \tag{1.1}$$

where subscripts stay for partial or ordinary derivatives. The 'virial' function

$$V(t) = pq = \sum_{i=1}^{N} p_i q_i \tag{1.2}$$

is defined on trajectories $p(t)$, $q(t)$ in the phase space of the system. We have

$$V_t = p_t q + p q_t = p H_p - q H_q. \tag{1.3}$$

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[2]Arnold's results includes also algebraic instability of vortex flows, which is out of our consideration in this paper.

If we now assume that

$$H(p,\, q) = K(p) + \Pi(q) \tag{1.4}$$

where the kinetic energy $K(p)$ is a quadratic nonnegative function of p_i and the potential energy $\Pi(q)$ is quadratic in q_i, then (1.3) can be written as

$$V_t = 2(K - \Pi) = 4K - 2H. \tag{1.5}$$

The equation (1.5) is often referred to as the virial equality. Since of $H = H(t) = $ const on the trajectories and $K \geq 0$ the virial $V(t)$ (for negative full energy $H < 0$) represents at least a linearly growing function:

$$V(t) \geq V(0) + |H|t \tag{1.6}$$

This fact was employed by Lyapunov to prove *the converse Lagrange theorem* stating that an equilibrium $p = q = 0$ in the conditions of maximum or saddle point of potential energy Π is unstable. The negative sign of energy $H < 0$ in this situation can be naturally provided by an appropriate choice of initial data (say $K = 0$, $\Pi < 0$) [17], [11]. For this problem variables p and q involved in V serve as the finite perturbations (differences between a perturbed solution $p(t)$, $q(t)$ and the basic solution $(0,\, 0)$)[3].

In the present paper, we work with a somewhat different definition of the 'virial', which may be introduced as follows. Let us consider an arbitrary one-dimensional function[4] $q(t,\, \epsilon)$ of two scalar variables t and ϵ, and an arbitrary differentiable function $L(q,\, q_t)$ (subscript stays for partial derivative). Direct calculation yields the identity

$$L_\epsilon = (q_\epsilon p)_t - q_\epsilon(p_t - L_q) + p(q_{t\epsilon} - q_{\epsilon t}), \quad p \equiv L_{q_t}, \tag{1.7}$$

which we call 'the fundamental identity'. Next we accept that

$$q_{t\epsilon} = q_{\epsilon t} \tag{1.8}$$

and function L satisfies the condition:

$$p_t - L_q = 0 \tag{1.9}$$

First equation requires certain smoothness, second one introduces a dynamical system via its Lagrangian L. After those two suggestion are accepted, functions $q(t,\, \epsilon)$ describe one parametric family of solutions for the considered dynamical system. As the result the last two terms in (1.7) vanish and the identity reduces to

$$L_\epsilon = (pq_\epsilon)_t \tag{1.10}$$

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[3]However there is an important technical restriction: the virial equality in the form (1.5) is valid only if K is quadratic in p and Π is quadratic in q. Therefore (1.5), (1.6) for the cases of nonquadratic Hamiltonian are valid only in linear approximation. In the general case of arbitrary K and/or Π those equalities take a more complicated form (usually it represents an infinite series of subsequent variations[11]), but in the majority of situations the linear terms dominate within a small vicinity of an equilibria and the virial function still may be used as a Lyapunov function.

[4]One dimensional case is presented here only for the sake of simplicity, general N-dimensional situation can be treated in the same way.

which represents the fundamental identity taken on the family of trajectories of the dynamical system. Another form of the same equality is:

$$W_t = L_\epsilon, \quad W \equiv pq_\epsilon. \tag{1.11}$$

The differentiation of (1.11) with respect to ϵ and accepting of (1.11) produces an equality for $L_{\epsilon\epsilon}$ which can be presented as:

$$(p_\epsilon q_\epsilon + pq_{\epsilon\epsilon})_t = K_{\epsilon\epsilon} - \Pi_{\epsilon\epsilon}. \tag{1.12}$$

If we consider this equality at $\epsilon = 0$ and take the equilibrium $p(t, 0) = q_t(t, 0) = 0$ as the basic solution, then (1.12) immediately gives the linearized version of (1.4).

Another interesting way of presenting (1.12) is

$$\tfrac{1}{2}(A\phi_t)_t = A\psi + B\phi \tag{1.13}$$

where $\phi \equiv q_\epsilon^2$, $\psi \equiv q_{\epsilon t}^2$, $A \equiv L_{q_t q_t}$, $B \equiv L_{qq} - L_{qq_t t}$. This equality is valid by virtue of exact equations (1.9) and holds for any family of trajectories of the system. Let us accept $A > 0$ (the Legendre condition) and use (1.4), which means $B = -\Pi_{qq}$. As the next step we consider the equilibrium $q(t, 0) = q_t(t, 0) = 0$ as the basic solution at $\epsilon = 0$. In such situation the exponential growth of $\phi = q_\epsilon^2$ follows immediately from (1.13) if $\Pi_{qq} < 0$ (the local maximum of potential energy). One can say that function ϕ plays a part of 'virial' in this consideration. Some related nonlinear results also can be obtained, however we are not going to develop this problem here. Our target is to illustrate only the way of introducing our 'virials' W and ϕ and correspondent 'virial equalities' (1.11), (1.12), (1.13).

Close connection between (1.13) the Jacobi equation is evident. Indeed the same equality can be obtained directly from the exact equation of motion (1.9) after its differentiation with respect to ϵ (which gives us the well known Jacobi equation for geodesic deviations) and subsequent multiplying by q_ϵ. Therefore one can call W and ϕ 'geodesic virials'.

The main difference between the properties of 'virial' V and 'geodesic virials' W, ϕ is now clear. The first one formally operates with the finite perturbations of p and q, but actually is valid only for a linearized system (when both p and q are infinitely small, or the Hamiltonian represents a quadratic form), while the latter two explicitly deal with the values p_ϵ, q_ϵ, and the virial equality is valid for the exact equations of motion.

In this paper, we shall derive equalities similar to (1.12), (1.13) for a number of dynamical systems of fluid mechanics and employ them to obtain various sufficient conditions for instability. The construction of our virial functionals has been performed in several steps which follow to the scheme described above. First we introduce certain identities (called the fundamental identities) defined for two two-parameters families of vector fields. Then we require that flows defined by these vector fields commute, so that the vector fields must satisfy a compatibility condition. Finally, we treat one of the vector fields as a one-parameter family of solutions of the governing equations. Then the geodesic virial equalities are obtained by differentiating the fundamental equality (taken on a family of solutions) with respect to the parameter. The outline of the paper is as follows. In Section 2, we introduce a virial functional for general steady or unsteady flows of an inviscid

incompressible fluid in a fixed domain and use this functional to obtain sufficient conditions for instability. Section 3 is devoted to the extension of the results of Section 2 to free-surface flows, stratified flows (in Boussinesq approximation) and magnetohydrodynamic flows. Section 4 contains the discussion of the results and conclusions.

2. Virial for inviscid incompressible fluid

2.1. Fundamental identity. Let $\mathcal{D}$ be a fixed simply-connected three-dimensional domain with smooth boundary $\partial \mathcal{D}$ and let $\boldsymbol{u}(\boldsymbol{x}, t, \epsilon)$ and $\boldsymbol{f}(\boldsymbol{x}, t, \epsilon)$ be two families of smooth vector fields in $\mathcal{D}$ depending on scalar parameters t and ϵ. We assume that $\boldsymbol{u}(\boldsymbol{x}, t, \epsilon)$ and $\boldsymbol{f}(\boldsymbol{x}, t, \epsilon)$ are divergence-free ($\boldsymbol{\nabla} \cdot \boldsymbol{u} = \boldsymbol{\nabla} \cdot \boldsymbol{f} = 0$ in $\mathcal{D}$) and tangent to the boundary ($\boldsymbol{n} \cdot \boldsymbol{u} = \boldsymbol{n} \cdot \boldsymbol{f} = 0$ at $\partial \mathcal{D}$ where $\boldsymbol{n}$ is the unit outward normal on $\partial \mathcal{D}$), but are otherwise arbitrary.

Let us introduce two operators of differentiation:

$$D_t \equiv \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}, \quad D_\epsilon \equiv \frac{\partial}{\partial \epsilon} + \boldsymbol{f} \cdot \boldsymbol{\nabla}.$$

Consider now the function

$$\mathcal{L}(\boldsymbol{x}, t, \epsilon) \equiv \frac{\boldsymbol{u}^2}{2}.$$

Differentiation of $\mathcal{L}$ with respect to D_ϵ and some manipulations result in the identity

$$D_\epsilon \mathcal{L} = D_t(\boldsymbol{f} \cdot \boldsymbol{u}) + \boldsymbol{u} \cdot (D_\epsilon \boldsymbol{u} - D_t \boldsymbol{f}) + \boldsymbol{\nabla} \cdot (p\boldsymbol{f}) - \boldsymbol{f} \cdot (D_t \boldsymbol{u} + \boldsymbol{\nabla} p), \qquad (2.1)$$

where $p(\boldsymbol{x}, t, \epsilon)$ is an arbitrary scalar function. Equation (2.1) which we call the fundamental identity is valid for arbitrary vector fields $\boldsymbol{u}(\boldsymbol{x}, t, \epsilon)$ and $\boldsymbol{f}(\boldsymbol{x}, t, \epsilon)$.

This identity can be used in two different ways. Firstly, it can be employed to derive the Euler equations from Hamilton's principle of the least action. Secondly, it can be used to obtain the virial equality.

2.2. Trajectories of fluid particles and compatibility condition. Let us now assume that domain $\mathcal{D}$ is filled with a continuous medium with the dependence between Eulerian $\boldsymbol{x}$ and Lagrangian $\boldsymbol{a}$ coordinates given as $\boldsymbol{x}(\boldsymbol{a}, t, \epsilon)$. Let us also identify vector fields $\boldsymbol{u}(\boldsymbol{x}, t, \epsilon)$ and $\boldsymbol{f}(\boldsymbol{x}, t, \epsilon)$ from the previous subsection with partial derivatives with respect to t and ϵ correspondingly:

$$\boldsymbol{x}_t(\boldsymbol{a}, t, \epsilon) = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{a}, t, \epsilon), t, \epsilon), \quad \boldsymbol{x}_\epsilon(\boldsymbol{a}, t, \epsilon) = \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{a}, t, \epsilon), t, \epsilon) \qquad (2.2)$$

The compatibility condition for the fields $\boldsymbol{u}(\boldsymbol{x}, t, \epsilon)$ and $\boldsymbol{f}(\boldsymbol{x}, t, \epsilon)$ follows from the requirement that partial derivatives with respect to t and ϵ (at fixed $\boldsymbol{a}$) commute, i. e. $\boldsymbol{x}_{t\epsilon} = \boldsymbol{x}_{\epsilon t}$, in Eulerian coordinates it yields the relation

$$D_\epsilon \boldsymbol{u} = D_t \boldsymbol{f} \qquad (2.3)$$

or, equivalently,

$$\boldsymbol{f}_t + [\boldsymbol{f}, \boldsymbol{u}] = \boldsymbol{u}_\epsilon \qquad (2.4)$$

where

$$[\boldsymbol{f}, \boldsymbol{u}] \equiv (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{f} - (\boldsymbol{f} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla} \times (\boldsymbol{f} \times \boldsymbol{u})$$

is the commutator of the (solenoidal) vector fields $\boldsymbol{u}$ and $\boldsymbol{f}$.

The compatibility condition (2.3) means that one term in the fundamental identity (2.1) vanishes.

An important consequence of (2.3)/(2.4) is that all invariant operators associated with the fields $\boldsymbol{u}(\boldsymbol{x},\,t,\,\epsilon)$ and $\boldsymbol{f}(\boldsymbol{x},\,t,\,\epsilon)$ commute. In particular, the operators D_t and D_ϵ commute, i. e.

$$D_t\,D_\epsilon = D_\epsilon\,D_t. \tag{2.5}$$

2.3. Two applications of the fundamental identity.

2.3.1. *Hamilton's principle.* We have assumed that $\boldsymbol{u}(\boldsymbol{x},\,t,\,\epsilon)$ and $\boldsymbol{f}(\boldsymbol{x},\,t,\,\epsilon)$ are divergence-free in $\mathcal{D}$, tangent to $\partial\mathcal{D}$ and satisfy the compatibility condition (2.3)/(2.4). Integration of (2.1) over $\mathcal{D}$ and over t from t_1 to t_2, yields

$$\frac{d}{d\epsilon}\int_{t_1}^{t_2}\int_{\mathcal{D}}\mathcal{L}\,dV\,dt = \int_{t_1}^{t_2}\int_{\mathcal{D}}D_\epsilon\mathcal{L}\,dV\,dt$$
$$= \int_{t_1}^{t_2}\int_{\mathcal{D}}(-D_t\boldsymbol{u}-\boldsymbol{\nabla}p)\cdot\boldsymbol{f}\,dV\,dt + \int_{\mathcal{D}}\boldsymbol{f}\cdot\boldsymbol{u}\,dV\,\bigg|_{t=t_1}^{t=t_2}. \tag{2.6}$$

Consider the action functional S and the Lagrangian L given as

$$S(\epsilon) = \int_{t_1}^{t_2}L\,dt, \quad L(\epsilon,\,t) = \int_{\mathcal{D}}\mathcal{L}\,dV = \int_{\mathcal{D}}\frac{\boldsymbol{u}^2}{2}dV. \tag{2.7}$$

By definition,

$$\delta S \equiv \epsilon S_\epsilon(0), \tag{2.8}$$

Assuming that the vector field $\delta\boldsymbol{x}(\boldsymbol{x},\,t) \equiv \epsilon\boldsymbol{f}(\boldsymbol{x},\,t,\,0)$ vanishes at the ends of time interval

$$\delta\boldsymbol{x}(\boldsymbol{x},\,t_1) = \delta\boldsymbol{x}(\boldsymbol{x},\,t_2) = 0 \tag{2.9}$$

we find from (2.6) and (2.7) that

$$\delta S = \int_{t_1}^{t_2}\int_{\mathcal{D}}(-D_t\boldsymbol{u}-\boldsymbol{\nabla}p)\cdot\delta\boldsymbol{x}\,dV\,dt, \tag{2.10}$$

The Hamilton principle states that the actual motion of the system corresponds to a critical point of S, i. e. $\delta S = 0$. Since, by its definition, $\delta\boldsymbol{x}$ is an arbitrary vector field that is solenoidal and tangent to the boundary, the condition $\delta S = 0$ is equivalent to the equation

$$D_t\boldsymbol{u} = -\nabla p, \tag{2.11}$$

with a function p which is determined by the conditions

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in} \quad \mathcal{D}, \qquad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \partial\mathcal{D}. \tag{2.12}$$

Equations (2.11), (2.12) represent the Euler equations governing the dynamics of an inviscid incompressible fluid in a fixed domain $\mathcal{D}$. Thus, we have recovered that the fundamental equality (2.1) underlays the model of an ideal incompressible fluid. It also gives a rational interpretation to the Newcomb's form of Hamilton's principle for the Euler equations (see [20]), where the constraints for variations equivalent to the compatibility condition (2.3) (taken at the point $\epsilon = 0$) were accepted using a physical reasoning.

2.3.2. *Virial equality.* Now we assume that, in addition to the compatibility condition (2.3), the field $\boldsymbol{u}(\boldsymbol{x}, t, \epsilon)$ satisfies the Euler equations (2.11) and (2.12) for all ϵ. In other words, $\boldsymbol{u}(\boldsymbol{x}, t, \epsilon)$ represents a one parametric family of solutions of the Euler equations (depending on parameter ϵ). In this case, the fundamental identity (2.1) reduces to

$$D_t(\boldsymbol{f} \cdot \boldsymbol{u}) = D_\epsilon \mathcal{L} - \boldsymbol{\nabla} \cdot (p\boldsymbol{f}). \tag{2.13}$$

Integrating (2.13) over $\mathcal{D}$, we obtain

$$\mathcal{W}_t = L_\epsilon = E_\epsilon, \quad \mathcal{W} \equiv \int_\mathcal{D} \boldsymbol{f} \cdot \boldsymbol{u} \, dV. \tag{2.14}$$

Note that in the considered case the Lagrangian L coincides with the energy E. The conservation of the energy

$$E(\epsilon) = \tfrac{1}{2} \int_\mathcal{D} \boldsymbol{u}^2 \, dV \tag{2.15}$$

means that E does not depend on time t, but still depend on ϵ. Then the first derivative $E_\epsilon(\epsilon)$ (as well as derivatives of any other orders) is also conserved. Integration of (2.14) yields

$$\mathcal{W}(t, \epsilon) = \mathcal{W}(0, \epsilon) + E_\epsilon(\epsilon) \, t. \tag{2.16}$$

Equation (2.16) implies that $|\boldsymbol{f}|_{L_2}$ grows at least linearly with time. Indeed, the Cauchy–Buniakowski–Schwarz inequality gives

$$\mathcal{W}^2(t, \epsilon) = \left(\int_\mathcal{D} \boldsymbol{f} \cdot \boldsymbol{u} \, dV \right)^2 \leq \int_\mathcal{D} \boldsymbol{u}^2 \, dV \int_\mathcal{D} \boldsymbol{f}^2 \, dV = 2E \, |\boldsymbol{f}|_{L_2}^2.$$

It follows that

$$|\boldsymbol{f}|_{L_2} \geq \frac{1}{\sqrt{2E}} \, |\mathcal{W}(0, \epsilon) + E_\epsilon(\epsilon) \, t|. \tag{2.17}$$

Inequality (2.17) gives us an *a priori* estimate which is valid for arbitrary one-parameter family of solutions of the Euler equations: it shows that L_2-norm of $\boldsymbol{f}$ grows at least linearly with time provided that $E_\epsilon(\epsilon) \neq 0$. Inequality (2.17) corresponds to the most fundamental property of an inviscid fluid: perturbations, once they are 'imposed' at the initial instant of time, will exist forever and result in at least a linear increase of infinitesimal separation of distance (in L_2-norm) between two flows. One should notice that (2.17) represents an exact result, obtained without using of any approximations. Some further discussions related to that question are presented below in the Section 2.4 concerning hydrodynamic stability.

The differentiation of (2.14) with respect ϵ gives

$$\mathcal{W}_{t\epsilon} = \frac{\partial}{\partial t} \int_\mathcal{D} (D_\epsilon \boldsymbol{f} \cdot \boldsymbol{u} + \boldsymbol{f} \cdot D_\epsilon \boldsymbol{u}) \, dV = \int_\mathcal{D} \left((D_\epsilon \boldsymbol{u})^2 + \boldsymbol{u} \cdot D_\epsilon^2 \boldsymbol{u} \right) dV = L_{\epsilon\epsilon}.$$

With the help of (2.3), (2.11), and (2.12), this equation can be rewritten as

$$\frac{\partial}{\partial t} \int_\mathcal{D} \boldsymbol{f} \cdot D_\epsilon \boldsymbol{u} \, dV = \int_\mathcal{D} \left((D_\epsilon \boldsymbol{u})^2 - D_\epsilon \boldsymbol{f} \cdot D_t \boldsymbol{u} \right) dV$$

$$= \int_\mathcal{D} \left((D_t \boldsymbol{f})^2 - \boldsymbol{f} \cdot (\boldsymbol{f} \cdot \boldsymbol{\nabla}) \boldsymbol{\nabla} p \right) dV. \tag{2.18}$$

Next, we note that

$$\int_{\mathcal{D}} \boldsymbol{f} \cdot D_\epsilon \boldsymbol{u} \, dV = \int_{\mathcal{D}} \boldsymbol{f} \cdot D_t \boldsymbol{f} \, dV = \frac{\partial}{\partial t} \int_{\mathcal{D}} \frac{\boldsymbol{f}^2}{2} \, dV. \tag{2.19}$$

Substituting this into (2.18), we obtain

$$\frac{\partial^2}{\partial t^2} \int_{\mathcal{D}} \boldsymbol{f}^2 \, dV = 2 \int_{\mathcal{D}} \left((D_t \boldsymbol{f})^2 - \boldsymbol{f} \cdot (\boldsymbol{f} \cdot \boldsymbol{\nabla}) \boldsymbol{\nabla} p \right) dV. \tag{2.20}$$

Equation (2.20) holds for any family of solutions $\boldsymbol{u}(\boldsymbol{x},\, t,\, \epsilon)$ of the Euler equations. The associated equation for the field $\boldsymbol{f}(\boldsymbol{x},\, t,\, \epsilon)$ is obtained by differentiating the Euler equations (2.11), (2.12) with respect to ϵ and applying the compatibility condition. This yields

$$D_t^2 \boldsymbol{f} = -\boldsymbol{\nabla} p_\epsilon - (\boldsymbol{f} \cdot \boldsymbol{\nabla}) \nabla p, \tag{2.21}$$

where function p_ϵ is determined by the conditions

$$\nabla \cdot \boldsymbol{f} = 0 \quad \text{in} \quad \mathcal{D}, \qquad \boldsymbol{f} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial \mathcal{D}. \tag{2.22}$$

Equations (2.18), (2.20) may be viewed as forms of 'geodesic virial' equality, which has been successfully used to prove the converse Lagrange theorem on the stability of equilibria of various continuum systems. The rest of the paper is devoted to the derivation of equalities of this type for various dynamical systems of continuum mechanics and their application to stability analysis. In the next subsection, we shall show that the equality (2.18) leads to certain results on the stability of steady and unsteady flows of an inviscid incompressible fluid.

2.4. Perturbations and stability of inviscid flows. Let

$$\boldsymbol{u}_0(\boldsymbol{x},\, t) \equiv \boldsymbol{u}(\boldsymbol{x},\, t,\, 0), \quad p_0(\boldsymbol{x},\, t) \equiv p(\boldsymbol{x},\, t,\, 0) \tag{2.23}$$

describe the basic flow whose stability is investigated. In this situation the velocity field $\boldsymbol{u}(\boldsymbol{x},\, t,\, \epsilon)$ with a fixed $\epsilon \neq 0$ corresponds to a perturbed flow and the difference $\boldsymbol{u}(\boldsymbol{x},\, 0,\, \epsilon) - \boldsymbol{u}_0(\boldsymbol{x},\, 0)$ represents the initial perturbation.

The field of Lagrangian displacements $\boldsymbol{\xi}$ is defined as the difference between the 'disturbed' and 'undisturbed' position of a particle with the same Lagrangian coordinate $\boldsymbol{a}$

$$\boldsymbol{\xi}(\boldsymbol{a},\, t,\, \epsilon) \equiv \boldsymbol{x}(\boldsymbol{a},\, t,\, \epsilon) - \boldsymbol{x}(\boldsymbol{a},\, t,\, 0) = \int_0^\epsilon \boldsymbol{f}(\boldsymbol{a},\, t,\, \mu) \, d\mu \tag{2.24}$$

where the fields $\boldsymbol{f}$ and $\boldsymbol{u}$ are linked by the compatibility condition (2.3).

Assuming that ϵ is small and using Taylor's formula, we obtain

$$\boldsymbol{u}(\boldsymbol{x},\, t,\, \epsilon) = \boldsymbol{u}_0(\boldsymbol{x},\, t) + \epsilon \, \boldsymbol{u}_\epsilon(\boldsymbol{x},\, t,\, 0) + o(\epsilon). \tag{2.25}$$

$$\boldsymbol{\xi}(\boldsymbol{x},\, t,\, \epsilon) = \epsilon \boldsymbol{f}(\boldsymbol{x},\, t,\, 0) + o(\epsilon). \tag{2.26}$$

Evidently,

$$\tilde{\boldsymbol{u}}(\boldsymbol{x},\, t) \equiv \epsilon \, \boldsymbol{u}_\epsilon(\boldsymbol{x},\, t,\, 0), \quad \tilde{p}(\boldsymbol{x},\, t) \equiv \epsilon \, p_\epsilon(\boldsymbol{x},\, t,\, 0), \quad \tilde{\boldsymbol{\xi}}(\boldsymbol{x},\, t) \equiv \epsilon \, \boldsymbol{f}(\boldsymbol{x},\, t,\, 0) \tag{2.27}$$

can be identified with the infinitesimal perturbation of velocity, pressure, and the Lagrangian displacement, which satisfy to the linearized equations

$$\tilde{\boldsymbol{u}}_t = -(\tilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla}) \boldsymbol{u}_0 - (\boldsymbol{u}_0 \cdot \boldsymbol{\nabla}) \tilde{\boldsymbol{u}} - \boldsymbol{\nabla} \tilde{p}. \tag{2.28}$$

and linearized compatibility conditions

$$\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{\xi}}_t + [\tilde{\boldsymbol{\xi}}, \, \boldsymbol{u}_0]. \tag{2.29}$$

The equation governing the evolution of $\tilde{\boldsymbol{\xi}}(\boldsymbol{x}, t)$ is obtained from (2.21) by putting $\epsilon = 0$:

$$D_{0t}^2 \tilde{\boldsymbol{\xi}} = -\boldsymbol{\nabla}\tilde{p} - (\tilde{\boldsymbol{\xi}} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}p_0, \tag{2.30}$$

where

$$D_{0t} = \frac{\partial}{\partial t} + \boldsymbol{u}_0 \cdot \boldsymbol{\nabla},$$

and the function $\tilde{p}(\boldsymbol{x}, t)$ is determined by the conditions

$$\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{\xi}} = 0 \quad \text{in} \quad \mathcal{D}, \qquad \tilde{\boldsymbol{\xi}} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\mathcal{D}. \tag{2.31}$$

As was mentioned already, equation (2.30) represents the Jacobi equation for the action functional S (2.7). The Jacobi equation is closely connected with the linearized Euler equations (2.28). It can be shown that the linearized Euler equations (2.28), supplemented with (2.29), lead to (2.30). This means that, given any solution $\tilde{\boldsymbol{\xi}}(\boldsymbol{x}, t)$ of (2.30), we can obtain the corresponding solution of the linearized Euler equations (2.28) using (2.29). This correspondence between solutions of (2.28) and (2.30) is however not one-to-one. Namely, more than one field $\tilde{\boldsymbol{\xi}}(\boldsymbol{x}, t)$ produce the same Eulerian velocity perturbation. In fact, suppose that $\tilde{\boldsymbol{\xi}}^\star(\boldsymbol{x}, t) = \tilde{\boldsymbol{\xi}}(\boldsymbol{x}, t) + c\boldsymbol{\eta}(\boldsymbol{x}, t)$ where $\tilde{\boldsymbol{\xi}}(\boldsymbol{x}, t)$ is a solution of (2.30), c is a real constant, and the vector field $\boldsymbol{\eta}(\boldsymbol{x}, t)$ is divergence-free, tangent to the boundary $\partial\mathcal{D}$, and satisfies the equation $\boldsymbol{\eta}_t + [\boldsymbol{\eta}, \, \boldsymbol{u}_0] = 0$. Then, $\tilde{\boldsymbol{\xi}}^\star(\boldsymbol{x}, t)$ is also the solution of (2.30), and this new solution corresponds to the same velocity perturbation.

The linearized version of (2.20) is given by

$$\frac{d^2}{dt^2} \int_{\mathcal{D}} \tilde{\boldsymbol{\xi}}^2 \, dV = 2 \int_{\mathcal{D}} \left(D_{0t}\tilde{\boldsymbol{\xi}} \cdot D_{0t}\tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\xi}} \cdot (\tilde{\boldsymbol{\xi}} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}p_0 \right) dV. \tag{2.32}$$

Let

$$M = \int_{\mathcal{D}} \tilde{\boldsymbol{\xi}}^2 \, dV, \quad T = \frac{1}{2} \int_{\mathcal{D}} (D_{0t}\tilde{\boldsymbol{\xi}})^2 \, dV,$$

$$I = \frac{1}{2} \int_{\mathcal{D}} \tilde{\boldsymbol{\xi}} \cdot (\tilde{\boldsymbol{\xi}} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}p_0 \, dV = \frac{1}{2} \int_{\mathcal{D}} H_{ik}\xi_i\xi_k \, dV. \tag{2.33}$$

With this notation, (2.32) can be written as

$$\ddot{M} = 4(T - I). \tag{2.34}$$

The sign of I is determined by the Hessian of the pressure in the basic flow

$$H_{ik} \equiv \frac{\partial^2 p_0}{\partial x_i \partial x_k}.$$

First we assume that the Hessian of the pressure is strictly negative definite in $\mathcal{D}$, i.e.

$$H_{ik}\tilde{\xi}_i\tilde{\xi}_k \leq -\gamma\,\tilde{\boldsymbol{\xi}}^2 \tag{2.35}$$

for some positive γ. Then,

$$I \leq -\frac{\gamma M}{2} \quad \Rightarrow \quad \ddot{M} = 4(T - I) \geq 2\gamma M. \tag{2.36}$$

Let $\lambda = \sqrt{2\gamma}$. Since

$$\dddot{M} - 2\gamma M = \ddot{M} - \lambda^2 M = e^{\lambda t} \frac{d}{dt}\left(e^{-2\lambda t}\frac{d}{dt}\left(e^{\lambda t}M\right)\right) \geq 0\,,$$

we have

$$\frac{d}{dt}\left(e^{\lambda t}M\right) \geq e^{2\lambda t}(\dot{M}(0) + \lambda M(0)),$$

whence

$$M(t) \geq M(0)e^{-\lambda t} + \frac{\dot{M}(0) + \lambda M(0)}{\lambda}\,\text{sh}\,\lambda t. \tag{2.37}$$

This inequality shows that *any inviscid incompressible flow satisfying the condition* (2.35) *is exponentially unstable* in the linear approximation.

The irrotational flow near a stagnation point with the velocity field

$$\boldsymbol{u}_0(\boldsymbol{x},\,t) = (x,\,y,\,-2z), \quad \boldsymbol{x} = (x,\,y,\,z) \tag{2.38}$$

is an example of a flow satisfying the condition (2.35). In this case, $H_{ik} = \text{diag}(-1,\,-1,\,-4)$, so that $H_{ik}\tilde{\xi}_i\tilde{\xi}_k \leq -\tilde{\boldsymbol{\xi}}^2$. The class of the exponentially unstable flows with the velocity field being a linear function of coordinates is broader than example (2.38).

The condition (2.35) is rather restrictive. For example it excludes flows with constant pressure where the Hessian is zero at least at one point. Let us relax inequality (2.35) and assume that the Hessian is negative semi-definite in $\mathcal{D}$, i.e.

$$H_{ik}\tilde{\xi}_i\tilde{\xi}_k \leq 0.$$

Then $I \leq 0$ and (2.34) yields:

$$\ddot{M} \geq 4T. \tag{2.39}$$

Since, by the Cauchy–Buniakowski–Schwarz inequality,

$$\dot{M}^2 = 4\left(\int_{\mathcal{D}}\tilde{\boldsymbol{\xi}}\cdot D_{0t}\tilde{\boldsymbol{\xi}}\,dV\right)^2 \leq 4\int_{\mathcal{D}}\tilde{\boldsymbol{\xi}}^2\,dV\int_{\mathcal{D}}\left(D_{0t}\tilde{\boldsymbol{\xi}}\right)^2\,dV = 8MT, \tag{2.40}$$

we obtain

$$\frac{\ddot{M}}{M} - \frac{\dot{M}^2}{2M^2} \geq 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\dot{M}}{M^{1/2}}\right) \geq 0. \tag{2.41}$$

Hence,

$$\frac{\dot{M}}{M^{1/2}} \geq \frac{\dot{M}}{M^{1/2}}\bigg|_{t=0} \equiv 2\lambda. \tag{2.42}$$

Since the initial values for $\tilde{\boldsymbol{\xi}}$ and $\tilde{\boldsymbol{\xi}}_t$ can be specified independently, we choose them in such a way that $\lambda > 0$. With this choice,

$$M^{1/2}(t) \geq M(0)^{1/2} + \lambda\,t \quad \Leftrightarrow \quad |\tilde{\boldsymbol{\xi}}(t)|_{L_2} \geq |\tilde{\boldsymbol{\xi}}(0)|_{L_2} + \lambda\,t. \tag{2.43}$$

and there is at least linear growth (in time) of Lagrangian displacements of fluid particles in the L_2 norm.

Note that inequality (2.43) is complementary to the linearized version of (2.17) and allows us to analyze the structure of growing perturbations. To analyze correspondence with (2.17), let us denote:

$$W_0(t) = \epsilon\mathcal{W}(t,\,0) = \int_{\mathcal{D}}\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{u}_0\,dV \equiv \langle\tilde{\boldsymbol{\xi}},\,\boldsymbol{u}_0\rangle. \tag{2.44}$$

Then the linearized version of (2.16) (that was used to obtain (2.17)) can be written as

$$W_0(t) = W_0(0) + E_\epsilon(0)\, t$$

or, equivalently,

$$\langle \tilde{\boldsymbol{\xi}},\, \boldsymbol{u}_0 \rangle(t) = \langle \tilde{\boldsymbol{\xi}},\, \boldsymbol{u}_0 \rangle(0) + E_\epsilon(0)\, t. \tag{2.45}$$

where

$$E_\epsilon(0) = \langle \boldsymbol{u}_0,\, D_{0t}\tilde{\boldsymbol{\xi}} \rangle = \text{const}. \tag{2.46}$$

Let us decompose $\boldsymbol{\xi}$ into 'parallel' and 'orthogonal' to $\boldsymbol{u}_0$ components (in L_2):

$$\tilde{\boldsymbol{\xi}} = \tilde{\boldsymbol{\xi}}^{\parallel} + \tilde{\boldsymbol{\xi}}^{\perp}, \quad \tilde{\boldsymbol{\xi}}^{\parallel} = \frac{\langle \tilde{\boldsymbol{\xi}},\, \boldsymbol{u}_0 \rangle}{\langle \boldsymbol{u}_0,\, \boldsymbol{u}_0 \rangle}\, \boldsymbol{u}_0 \equiv \alpha(t)\boldsymbol{u}_0(\boldsymbol{x},\, t) \tag{2.47}$$

According to (2.45) the numerator of (2.47) represents a linear function of time, so $\alpha(t) = c_0 + c_1 t$, with arbitrary constants c_0 and c_1. It is a remarkable fact that

$$\tilde{\boldsymbol{\xi}}^{\parallel} = \alpha\boldsymbol{u}_0, \quad \tilde{p} = \alpha p_{0t} + 2\alpha_t p_0, \quad \alpha = c_0 + c_1 t \tag{2.48}$$

represents an exact solution of the Jacobi equation (2.30), (2.31). Because of the linearity of this equation, the 'orthogonal' component $\boldsymbol{\xi}^{\perp}$ is also a solution. Now one can see, that for $\boldsymbol{\xi} = \boldsymbol{\xi}^{\perp}$ all terms in (2.45) are identically zero, so the *a priori* estimate (2.17) does not produce any result. However the estimate (2.43) still works, and shows the linear growth of $\boldsymbol{\xi} = \boldsymbol{\xi}^{\perp}$ in L_2-norm.

Solutions (2.48) are sometimes called 'trivial' since the Lagrangian displacements are parallel to the main velocity. The corresponding perturbation of the velocity field takes a form:

$$\tilde{\boldsymbol{u}} = \alpha_t\boldsymbol{u}_0 + \alpha\boldsymbol{u}_{0t} \tag{2.49}$$

It is clear that the linear growth of $W_0(t)$ corresponds to that 'trivial' solution. In the special case of steady basic field $\boldsymbol{u}_0(\boldsymbol{x})$ the existence of the 'trivial' solutions does not imply any deviations of fluid particles from steady streamlines and this is the reason why the correspondent grows may be not considered as 'real' physical instability. The most interesting point here is the existence of the split (2.47) which allow us to eliminate the linear 'trivial' perturbations.

Thus, we may conclude that *if in a steady or unsteady flow the Hessian of the pressure is negative semi-definite at every point of the flow domain, then this flow is unstable and the growth rate is given by* (2.43). For example, any steady plane-parallel flow has constant pressure, and therefore does satisfy our sufficient condition for instability. A similar result was obtained earlier in [18].

3. Free-surface flows, stratified flows and MHD flows

3.1. Governing equations for free-surface flows. Let us return to the exact problems. As in Section 2, we consider a one-parametric family of fields $\boldsymbol{u}(\boldsymbol{x},\, t,\, \epsilon)$ and $\boldsymbol{f}(\boldsymbol{x},\, t,\, \epsilon)$ that satisfy the compatibility condition (2.3). Let $\mathcal{D}(t,\, \epsilon)$ be a three-dimensional domain containing an inviscid incompressible fluid. The boundary of the domain $\partial\mathcal{D}(t,\, \epsilon)$ consists of two parts: the fixed boundary Σ and the free boundary $S(t,\, \epsilon)$, i.e. $\partial\mathcal{D} = \Sigma \cup S$. We assume that $\boldsymbol{u}(\boldsymbol{x},\, t,\, \epsilon)$ and $\boldsymbol{f}(\boldsymbol{x},\, t,\, \epsilon)$

are defined in the domain $\mathcal{D}(t,\,\epsilon)$, and both are divergence-free and satisfy to the boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n}=0,\qquad \boldsymbol{f}\cdot\boldsymbol{n}=0\quad\text{on }\Sigma; \tag{3.1}$$

$$\boldsymbol{u}\cdot\boldsymbol{n}=v_n,\quad \boldsymbol{f}\cdot\boldsymbol{n}=w_n\quad\text{on }S(t). \tag{3.2}$$

The free surface is defined by the equation

$$F(\boldsymbol{x},\,t,\,\epsilon)=0,$$

and

$$D_tF=0\quad\text{and}\quad D_\epsilon F=0\quad\text{at }F(\boldsymbol{x},\,t,\,\epsilon)=0, \tag{3.3}$$

so that

$$v_n=-F_t/|\boldsymbol{\nabla}F|,\quad w_n=-F_\epsilon/|\boldsymbol{\nabla}F|\quad\text{at }F(\boldsymbol{x},\,t,\,\epsilon)=0. \tag{3.4}$$

Consider now the density of the Lagrangian, given by

$$\mathcal{L}(\boldsymbol{x},\,t,\,\epsilon)\equiv\frac{\boldsymbol{u}^2}{2}-\Phi$$

where $\Phi(\boldsymbol{x})$ is a given potential of an external body force. Differentiating $\mathcal{L}$ with respect to ϵ and using the compatibility condition, we obtain

$$D_\epsilon\mathcal{L}=D_t(\boldsymbol{f}\cdot\boldsymbol{u})+\boldsymbol{\nabla}\cdot(p\boldsymbol{f})-\boldsymbol{f}\cdot(D_t\boldsymbol{u}+\boldsymbol{\nabla}(p+\Phi)), \tag{3.5}$$

where $p(\boldsymbol{x},\,t,\,\epsilon)$ is an arbitrary function. Equation (3.5) is valid for arbitrary vector fields $\boldsymbol{u}(\boldsymbol{x},\,t,\,\epsilon)$ and $\boldsymbol{f}(\boldsymbol{x},\,t,\,\epsilon)$ satisfying the compatibility condition (2.3) and is a generalization of the fundamental identity of Section 2.

3.2. Hamilton's principle for free surface flows. Consider the action functional given by

$$S(\epsilon)=\int_{t_1}^{t_2}L\,dt,\quad L=\int_{\mathcal{D}}\mathcal{L}dV=\int_{\mathcal{D}}\left(\frac{\boldsymbol{u}^2}{2}-\Phi\right)dV. \tag{3.6}$$

Differentiating (3.6) with respect to ϵ and using (3.5), we find that

$$S_\epsilon=\int_{t_1}^{t_2}\int_{\mathcal{D}}(-D_t\boldsymbol{u}-\boldsymbol{\nabla}(p+\Phi))\cdot\boldsymbol{f}\,dV\,dt$$

$$+\int_{t_1}^{t_2}\int_S(\boldsymbol{f}\cdot\boldsymbol{n})p\,dS\,dt+\left(\int_{\mathcal{D}}\boldsymbol{f}\cdot\boldsymbol{u}\,dV\right)\Big|_{t=t_1}^{t=t_1}. \tag{3.7}$$

Putting $\epsilon=0$ and assuming that the vector field $\boldsymbol{f}(\boldsymbol{x},\,t,\,0)$ vanishes at $t=t_1$ and $t=t_2$, we find from (3.7) that

$$\delta S\equiv\epsilon S_\epsilon(0)$$

$$=\int_{t_1}^{t_2}\int_{\mathcal{D}}(-D_t\boldsymbol{u}-\boldsymbol{\nabla}p-\boldsymbol{\nabla}\Phi)\cdot\boldsymbol{\xi}\,dV\,dt+\int_{t_1}^{t_2}\int_S(\boldsymbol{\xi}\cdot\boldsymbol{n})p\,dS\,dt, \tag{3.8}$$

where $\boldsymbol{\xi}(\boldsymbol{x},\,t)\equiv\epsilon\boldsymbol{f}(\boldsymbol{x},\,t,\,0)$. At a critical point, $\delta S=0$. Since, by definition, $\boldsymbol{\xi}$ represents an arbitrary vector field (which is solenoidal and tangent to the fixed part Σ of the boundary $\partial\mathcal{D}$), the condition $\delta S=0$ leads to the Euler equation

$$D_t\boldsymbol{u}=-\boldsymbol{\nabla}p-\boldsymbol{\nabla}\Phi\quad\text{in }\mathcal{D}(t), \tag{3.9}$$

and the dynamic boundary condition $p = 0$ at the free boundary $S(t)$.

3.3. Virial equality for free-surface flows. Now we assume that, in addition to the compatibility condition, the field $\boldsymbol{u}(\boldsymbol{x},\,t,\,\epsilon)$ satisfies (3.1)–(3.3) for all ϵ in a neighbourhood of $\epsilon = 0$. In this case, the fundamental identity (3.5) reduces to the form:

$$D_t(\boldsymbol{f} \cdot \boldsymbol{u}) = D_\epsilon \mathcal{L} - \boldsymbol{\nabla} \cdot (p\boldsymbol{f}). \tag{3.10}$$

Integrating (3.10) over $\mathcal{D}$, we obtain

$$\frac{d}{dt} \int_{\mathcal{D}} \boldsymbol{f} \cdot \boldsymbol{u}\, dV = \int_{\mathcal{D}} \left(D_\epsilon \left(\frac{\boldsymbol{u}^2}{2} - \Phi \right) - \boldsymbol{\nabla} \cdot (p\boldsymbol{f}) \right) dV. \tag{3.11}$$

The differentiation of it with respect to ϵ gives

$$\frac{d}{dt} \int_{\mathcal{D}} (D_\epsilon \boldsymbol{f} \cdot \boldsymbol{u} + \boldsymbol{f} \cdot D_\epsilon \boldsymbol{u})\, dV = \int_{\mathcal{D}} \left((D_\epsilon \boldsymbol{u})^2 + \boldsymbol{u} \cdot D_\epsilon^2 \boldsymbol{u} - D_\epsilon^2 \Phi - D_\epsilon \boldsymbol{\nabla} \cdot (p\boldsymbol{f}) \right) dV.$$

Using the compatibility condition (2.3) and (3.9), we obtain

$$\frac{d}{dt} \int_{\mathcal{D}} \boldsymbol{f} \cdot D_\epsilon \boldsymbol{u}\, dV = \int_{\mathcal{D}} \left((D_t \boldsymbol{f})^2 - \boldsymbol{f} \cdot (\boldsymbol{f} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}(p + \Phi) \right) dV - \int_{S} p_\epsilon (\boldsymbol{f} \cdot \boldsymbol{n})\, dS. \tag{3.12}$$

In addition to (3.1)–(3.3) let us assume that the pressure $p(\boldsymbol{x},\,t,\,\epsilon)$ for all ϵ satisfies to boundary condition:

$$p(\boldsymbol{x},\,t,\,\epsilon) = 0 \quad \text{at } S(t,\,\epsilon) \quad \Rightarrow \quad D_\epsilon p = 0 \quad \text{at } S(t,\,\epsilon). \tag{3.13}$$

Hence, (3.12) takes the form

$$\frac{d}{dt} \int_{\mathcal{D}} \boldsymbol{f} \cdot D_\epsilon \boldsymbol{u}\, dV = \int_{\mathcal{D}} \left((D_t \boldsymbol{f})^2 - \boldsymbol{f} \cdot (\boldsymbol{f} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}(p + \Phi) \right) dV$$
$$+ \int_{S} (\boldsymbol{f} \cdot \boldsymbol{n})(\boldsymbol{f} \cdot \boldsymbol{\nabla}p)\, dS. \tag{3.14}$$

This relation represents a generalisation of equality (2.18). Below we use this equation to obtain sufficient conditions for instability of free-surface flows.

3.4. Stability of free-surface flows. Let, in the basic state whose stability is investigated, the fluid occupy the domain $\mathcal{D}_0(t) = \mathcal{D}(t,\,0)$ with the free boundary $S_0(t) = S(t,\,0)$ defined by the equation

$$F_0(\boldsymbol{x},\,t) = 0 \tag{3.15}$$

where $F_0(\boldsymbol{x},\,t) = F_0(\boldsymbol{x},\,t,\,0)$. And let

$$\boldsymbol{u}_0(\boldsymbol{x},\,t) \equiv \boldsymbol{u}(\boldsymbol{x},\,t,\,0), \quad p_0(\boldsymbol{x},\,t) \equiv p(\boldsymbol{x},\,t,\,0) \tag{3.16}$$

be the velocity and the pressure in the basic state. The velocity field $\boldsymbol{u}(\boldsymbol{x},\,t,\,\epsilon)$ with some fixed $\epsilon \neq 0$ corresponds to the perturbed flow. Assuming that ϵ is small and using Taylor's formula, we obtain

$$\boldsymbol{u}(\boldsymbol{x},\,t,\,\epsilon) = \boldsymbol{u}_0(\boldsymbol{x},\,t) + \epsilon\, \boldsymbol{u}_\epsilon(\boldsymbol{x},\,t,\,0) + o(\epsilon)$$
$$= \boldsymbol{u}_0(\boldsymbol{x},\,t) + (\tilde{\boldsymbol{\xi}}_t + [\tilde{\boldsymbol{\xi}},\, \boldsymbol{u}_0]) + o(\epsilon) \tag{3.17}$$

where, as in Section 2,

$$\tilde{\boldsymbol{\xi}}(\boldsymbol{x},\,t) \equiv \epsilon\, \boldsymbol{f}(\boldsymbol{x},\,t,\,0). \tag{3.18}$$

The equation for $\tilde{\boldsymbol{\xi}}(\boldsymbol{x},\,t)$, given by

$$D_{0t}^2\tilde{\boldsymbol{\xi}} = -\boldsymbol{\nabla}\tilde{p} - (\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}(p_0 + \Phi), \qquad (3.19)$$

is the same as (2.30) except for the presence of Φ. Boundary conditions for (3.19) are

$$\tilde{F} + \tilde{\boldsymbol{\xi}}\cdot\boldsymbol{\nabla}F_0 = 0 \quad\text{and}\quad \tilde{p} + \tilde{\boldsymbol{\xi}}\cdot\boldsymbol{\nabla}p_0 = 0 \quad\text{at } F_0(\boldsymbol{x},\,t) = 0, \qquad (3.20)$$

where $\eta(\boldsymbol{x},\,t)$ describes the perturbation of the free surface and defined by the equation

$$F(\boldsymbol{x},\,t,\,\epsilon) = F_0(\boldsymbol{x},\,t) + \epsilon\tilde{F}(\boldsymbol{x},\,t) + o(\epsilon).$$

It follows from (3.14) taken at $\epsilon = 0$ that

$$\frac{d^2}{dt^2}\int_{\mathcal{D}_0}\tilde{\boldsymbol{\xi}}^2\,dV = 2\int_{\mathcal{D}_0}\left((D_{0t}\tilde{\boldsymbol{\xi}})^2 - \tilde{\boldsymbol{\xi}}\cdot(\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}(p_0 + \Phi)\right)dV$$

$$+ 2\int_{S_0}(\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{n})(\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{\nabla}p_0)\,dS. \quad (3.21)$$

This can be written as

$$\ddot{M} = 4(T - I - \Pi). \qquad (3.22)$$

where (cf. (2.34))

$$M = \int_{\mathcal{D}_0}\tilde{\boldsymbol{\xi}}^2\,dV\,, \qquad T = \tfrac{1}{2}\int_{\mathcal{D}_0}(D_{0t}\tilde{\boldsymbol{\xi}})^2\,dV,$$

$$I = \tfrac{1}{2}\int_{\mathcal{D}_0}\tilde{\boldsymbol{\xi}}\cdot(\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}(p_0 + \Phi)\,dV\,, \qquad \Pi = -\tfrac{1}{2}\int_{\mathcal{S}_0}(\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{n})^2(\boldsymbol{n}\cdot\boldsymbol{\nabla}p_0)\,dS\,. \tag{3.23}$$

Sufficient conditions for instability, similar to the conditions formulated in Section 2, can be easily obtained using the same procedure. The result can be formulated as follows.

If there exists a real number λ such that, for all $\tilde{\boldsymbol{\xi}}$,

$$\frac{\partial^2(p_0 + \Phi)}{\partial x_i\partial x_k}\,\tilde{\xi}_i\tilde{\xi}_k \le -\lambda^2\,\tilde{\boldsymbol{\xi}}^2$$

everywhere in $\mathcal{D}$ and if $\boldsymbol{n}\cdot\boldsymbol{\nabla}p_0 \ge 0$ on S_0, then the considered basic state is exponentially unstable.

If we require that the Hessian of $p_0 + \Phi$ is negative semi-definite, then the basic state is still unstable, but we can guarantee only linear (with time) growth of perturbations.

The most transparent stability result can be obtained for hydrostatic equilibria of the fluid with free surface. To prove the instability in this case, we employ the technique developed in [21] (see also [26], [23]). We consider the stability of the equilibrium state, given by

$$\boldsymbol{u}_0 = 0, \quad p_0(\boldsymbol{x}) = -\Phi(\boldsymbol{x}) + C \quad\text{in } \mathcal{D}, \qquad (3.24)$$

where C is a constant such that

$$\Phi(\boldsymbol{x}) = C \quad\text{at}\quad F_0(\boldsymbol{x}) = 0. \qquad (3.25)$$

For the basic state (3.24), (3.25), equations (3.22), (3.23) simplify to

$$\dot{W} = 2(T - \Pi) \qquad (3.26)$$

where

$$W \equiv \dot{M}/2 = \int_{\mathcal{D}_0} \tilde{\boldsymbol{\xi}} \cdot \tilde{\boldsymbol{\xi}}_t \, dV, \tag{3.27}$$

M is the same as in (3.23) and where

$$T = \tfrac{1}{2} \int_{\mathcal{D}_0} \tilde{\boldsymbol{\xi}}_t^2 \, dV, \quad \Pi = \tfrac{1}{2} \int_{\mathcal{S}_0} (\tilde{\boldsymbol{\xi}} \cdot \boldsymbol{n})^2 (\boldsymbol{n} \cdot \boldsymbol{\nabla}\Phi) \, dS. \tag{3.28}$$

The equality (3.26) is usually called the virial equality and the functional W (3.27) is called the virial.

Since the basic state considered is time-independent, (3.19), (3.20) conserve the integral

$$\tilde{E} = T + \Pi. \tag{3.29}$$

[It can be shown that $\tilde{E}$ is the second variation of the total energy evaluated at the equilibrium (3.24), (3.25), i.e. $\tilde{E} = \tfrac{1}{2}\epsilon^2 E''(0)$.] In view of (3.27) and (3.29), equation (3.26) can be written as

$$\ddot{M} = 8T - 4\tilde{E}. \tag{3.30}$$

Now we assume that the 'potential energy' Π can be negative for some displacement field $\tilde{\boldsymbol{\xi}}(\boldsymbol{x}, 0)$. Then by choosing $\tilde{\boldsymbol{\xi}}_t(\boldsymbol{x}, 0)$ such that $T < |\Pi|$ at $t = 0$, we obtain that $\tilde{E}|_{t=0} < 0$. Since the 'total energy' $\tilde{E}$ is conserved by (3.19), (3.20), it remains negative for all $t > 0$. For perturbations with such initial data, it follows from (3.30) that

$$\ddot{M} > 8T. \tag{3.31}$$

Using inequality (2.40), we obtain (cf. (2.41))

$$\frac{\ddot{M}}{M} - \frac{8T}{M} > 0 \quad \Rightarrow \quad \frac{\ddot{M}}{M} - \frac{\dot{M}^2}{M^2} > 0 \quad \Rightarrow \quad \frac{d}{dt}\Big(\frac{\dot{M}}{M}\Big) > 0. \tag{3.32}$$

Therefore,

$$\frac{\dot{M}}{M} > \frac{\dot{M}}{M}\bigg|_{t=0} = 2\lambda. \tag{3.33}$$

Since the initial values for $\tilde{\boldsymbol{\xi}}$ and $\tilde{\boldsymbol{\xi}}_t$ can be specified independently, we choose them in such a way that $\lambda > 0$. With this choice,

$$M(t) > M(0)e^{2\lambda t} \quad \Leftrightarrow \quad |\tilde{\boldsymbol{\xi}}(t)|_{L_2} > |\tilde{\boldsymbol{\xi}}(0)|_{L_2} e^{\lambda t}. \tag{3.34}$$

This inequality gives us the lower bound for $|\tilde{\boldsymbol{\xi}}(t)|_{L_2}$ that guarantees exponential growth of perturbations provided that Π can take negative values. It can be shown that Π is the second variation of the exact potential energy of the fluid in the external body force with the potential Φ. Therefore, if at an equilibrium Π can be negative, then this equilibrium corresponds to a maximum or saddle point of the exact potential energy. Thus, our result can be formulated as the converse Lagrange theorem: *if an equilibrium does not correspond to a local minimum of the potential energy then this equilibrium is unstable.*

3.5. Stratified flows. In the Boussinesq approximation, the governing equations for flows of an incompressible, inviscid, stratified (inhomogeneous in density) fluid contained in a three-dimensional domain $\mathcal{D}$ with fixed rigid boundary $\partial\mathcal{D}$ are

$$D_t\boldsymbol{u} = -\boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\Phi, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \quad D_t\rho = 0, \tag{3.35}$$

where ρ is the density and $\Phi(\boldsymbol{x})$ is the potential of the external body force. Boundary condition for (3.35) is the condition of no normal flow through the rigid boundary:

$$\boldsymbol{n}\cdot\boldsymbol{u} = 0 \quad \text{on } \partial\mathcal{D} \tag{3.36}$$

($\boldsymbol{n}$ is the unit outward normal on $\partial\mathcal{D}$).

3.5.1. *Fundamental identity and compatibility conditions.* The density of the Lagrangian (defined on a one-parameter family of stratified flows with the velocity $\boldsymbol{u}(\boldsymbol{x}, t, \epsilon)$ and the density $\rho(\boldsymbol{x}, t, \epsilon)$) is given by

$$\mathcal{L} = \frac{\boldsymbol{u}^2}{2} - \rho\Phi. \tag{3.37}$$

Applying the operator D_ϵ to this equation, we obtain the identity

$$D_\epsilon\mathcal{L} = D_t\left(\boldsymbol{f}\cdot\boldsymbol{u}\right) + \boldsymbol{u}\cdot(D_\epsilon\boldsymbol{u} - D_t\boldsymbol{f}) - \Phi D_\epsilon\rho + \boldsymbol{\nabla}\cdot(p\boldsymbol{f}) - \boldsymbol{f}\cdot(D_t\boldsymbol{u} + \boldsymbol{\nabla}p + \rho\boldsymbol{\nabla}\Phi), \tag{3.38}$$

where $p(\boldsymbol{x}, t, \epsilon)$ is an arbitrary function. Equation (3.38) is an analog of the fundamental identity (2.1).

So far we have only made the assumption that $\boldsymbol{u}$ and $\boldsymbol{f}$ are divergence-free and tangent to the boundary. Now we assume that fields $\boldsymbol{u}$ and $\boldsymbol{f}$ satisfy the compatibility condition (2.3) and that the function $\rho(\boldsymbol{x}, t, \epsilon)$ satisfies the condition

$$D_\epsilon\rho = 0, \tag{3.39}$$

which we shall also call the compatibility condition. In a certain sense, the condition (3.39) is dual to the equation

$$D_t\rho = 0. \tag{3.40}$$

As a consequence of (2.3), the operators D_t and D_ϵ commute. Therefore, if (3.39) is satisfied at $t = 0$, then it is satisfied for all t by virtue of (3.40) and, conversely, if (3.40) is satisfied at $\epsilon = 0$, then it is satisfied for all ϵ by virtue of (3.39).

3.5.2. *Virial equality and stability of stratified flows.* With the compatibility conditions (2.3) and (3.39), the fundamental identity (3.38) reduces to

$$D_\epsilon\mathcal{L} = D_t(\boldsymbol{f}\cdot\boldsymbol{u}) + \boldsymbol{\nabla}\cdot(p\boldsymbol{f}) - \boldsymbol{f}\cdot(D_t\boldsymbol{u} + \boldsymbol{\nabla}p + \rho\boldsymbol{\nabla}\Phi). \tag{3.41}$$

As before, (3.41) leads to Hamilton's principle and to the generalised virial inequality. The latter has the form

$$\frac{d^2}{dt^2}\int_\mathcal{D}\boldsymbol{f}\cdot D_\epsilon\boldsymbol{u}\,dV = \int_\mathcal{D}\left((D_t\boldsymbol{f})^2 - \boldsymbol{f}\cdot(\boldsymbol{f}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}p - \rho\boldsymbol{f}\cdot(\boldsymbol{f}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\Phi\right)dV. \tag{3.42}$$

Using the same procedure as that used for free-surface flows, we can (i) obtain sufficient conditions for instability of steady or unsteady stratified flows and (ii) prove the converse Lagrange theorem for hydrostatic equilibria of stratified fluid (see, e. g., [21]).

3.6. Magnetohydrodynamic flows. In this subsection we show that the same technique can be applied to magnetohydrodynamic (MHD) flows of an inviscid, incompressible, perfectly conducting fluid.

We consider MHD flows in a fixed three-dimensional domain $\mathcal{D}$ with perfectly conducting boundary $\partial\mathcal{D}$. The governing equations are

$$\begin{aligned} D_t\boldsymbol{u} = -\boldsymbol{\nabla}p + \operatorname{curl}\boldsymbol{h}\times\boldsymbol{h}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \\ D_t\boldsymbol{h} = (\boldsymbol{h}\cdot\boldsymbol{\nabla})\boldsymbol{u}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{h} = 0, \end{aligned} \tag{3.43}$$

where $\boldsymbol{h}$ is the magnetic field Boundary condition for (3.43) are given by

$$\boldsymbol{n}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{n}\cdot\boldsymbol{h} = 0 \quad \text{on } \partial\mathcal{D}. \tag{3.44}$$

3.6.1. *Fundamental identity and compatibility conditions.* The density of the Lagrangian (defined on an one-parameter family of MHD flows with the velocity $\boldsymbol{u}(\boldsymbol{x},\,t,\,\epsilon)$ and the magnetic field $\boldsymbol{h}(\boldsymbol{x},\,t,\,\epsilon)$) is given by

$$\mathcal{L} = \frac{\boldsymbol{u}^2}{2} - \frac{\boldsymbol{h}^2}{2}. \tag{3.45}$$

Differentiating this with respect to ϵ, we obtain the fundamental identity

$$D_\epsilon\mathcal{L} = D_t(\boldsymbol{f}\cdot\boldsymbol{u}) + \boldsymbol{u}\cdot(D_\epsilon\boldsymbol{u} - D_t\boldsymbol{f}) - \boldsymbol{h}\cdot(D_\epsilon\boldsymbol{h} - (\boldsymbol{h}\cdot\boldsymbol{\nabla})\boldsymbol{f})$$
$$- \boldsymbol{f}\cdot(D_t\boldsymbol{u} - \operatorname{curl}\boldsymbol{h}\times\boldsymbol{h} + \boldsymbol{\nabla}p) + \boldsymbol{\nabla}\cdot\left(\boldsymbol{f}\left(p + \frac{\boldsymbol{h}^2}{2}\right) - \boldsymbol{h}\left(\boldsymbol{f}\cdot\boldsymbol{h}\right)\right). \tag{3.46}$$

Now we assume that fields $\boldsymbol{u}$ and $\boldsymbol{f}$ satisfy the compatibility condition (2.3) and that the magnetic field $\boldsymbol{h}(\boldsymbol{x},\,t,\,\epsilon)$ satisfies the equation

$$D_\epsilon\boldsymbol{h} = (\boldsymbol{h}\cdot\boldsymbol{\nabla})\boldsymbol{f} \tag{3.47}$$

(the compatibility condition for $\boldsymbol{h}$). Equation (3.47) is dual to the equation

$$D_t\boldsymbol{h} = (\boldsymbol{h}\cdot\boldsymbol{\nabla})\boldsymbol{u} \tag{3.48}$$

(that governs the evolution of the magnetic fields with t) in the sense that if (3.47) is satisfied at $t = 0$, then it is satisfied for all t by virtue of (3.48) and, conversely, if (3.48) is satisfied at $\epsilon = 0$, then it holds for all ϵ by virtue of (3.47). This is a consequence of the compatibility condition (2.3) for the fields $\boldsymbol{u}$ and $\boldsymbol{f}$.

3.6.2. *Virial equality for MHD flows.* Using the compatibility conditions (2.3) and (3.47), we can rewrite the fundamental identity (3.46) as

$$D_\epsilon\mathcal{L} = D_t(\boldsymbol{f}\cdot\boldsymbol{u}) - \boldsymbol{f}\cdot(D_t\boldsymbol{u} - \operatorname{curl}\boldsymbol{h}\times\boldsymbol{h} + \boldsymbol{\nabla}p)$$
$$+ \boldsymbol{\nabla}\cdot\left(\boldsymbol{f}\left(p + \frac{\boldsymbol{h}^2}{2}\right) - \boldsymbol{h}\left(\boldsymbol{f}\cdot\boldsymbol{h}\right)\right). \tag{3.49}$$

Equation (3.49) leads to Hamilton's principle (see [20]) and to the virial equality:

$$\frac{d^2}{dt^2}\int_\mathcal{D}\boldsymbol{f}\cdot D_\epsilon\boldsymbol{u}\,dV = \int_\mathcal{D}\left((D_t\boldsymbol{f})^2 - \boldsymbol{f}\cdot(\boldsymbol{f}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\left(p + \frac{\boldsymbol{h}^2}{2}\right) - ((\boldsymbol{h}\cdot\boldsymbol{\nabla})\boldsymbol{f})^2\right)dV. \tag{3.50}$$

Now the same arguments as those used for free-surface flows lead to (i) sufficient conditions for instability of steady or unsteady MHD flows and (ii) a proof of the converse Lagrange theorem for magnetostatic equilibria (see [14]).

4. Conclusion

We have presented a general scheme of obtaining sufficient conditions for fluid instability by the direct Lyapunov method. The main idea of the direct Lyapunov method is to construct a functional which grows with time by virtue of equations of motion. In this paper we have used the 'virial' as a Lyapunov functional and described a regular procedure of constructing this functional. The virial functionals are proved to be useful in obtaining both sufficient conditions for instability and in estimating of the growth rate.

We have considered only conservative systems (inviscid fluids). The effect of dissipation on the stability of equilibria has been studied in [22], [27], [14], [24], where the converse Lagrange theorem has been proved for a number of dissipative systems of fluid mechanics. All the 'viscous' results have been obtain in linear approximation.

Much remains to be done in the applications of virial. We can identify two major unsolved problems. First of all, most of the results on instability in fluid mechanics are essentially linear. Even in the simplest case of hydrostatic equilibria there are only a few results on genuine nonlinear instability (see [8], [15]). The second unsolved problem is the conversion of the energy stability theorems such as, for example, Arnold's theorem on the stability of steady plane curvilinear flows of an inviscid fluid [1], [3]. Although, as it is shown in this paper, we can generalise virial to the case of *steady* basic flow rather than an *equilibrium*, the corresponding virial equality yields only quite restrictive sufficient conditions for instability. The third major problem is to find a Lyapunov functional for the cases of nonmonotonic growth of perturbations in the correspondent linearized spectral problem.

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