On viscous and inviscid stability of magnetohydrodynamic shock waves

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Abstract

We study ...

1 Introduction

According to standard physical theory [?, ?], magnetohydrodynamics (MHD), i. e., the dynamics of compressible quasineutrally ionized fluids under the influence of electromagnetic fields, is governed by the system

$$\rho_t + \mathrm{div}\,(\rho u) = 0, \tag{1}$$

$$(\rho u)_t + \mathrm{div}\,(\rho u \otimes u - H \otimes H) + \nabla q = \nu \Delta u + (\lambda + \nu)\nabla \mathrm{div}\, u, \tag{2}$$

$$H_t - \nabla \times (u {\times} H) = \eta \Delta H, \tag{3}$$

$$\big(E + \tfrac{1}{2}H^2\big)_t + \mathrm{div}\,\big((E+p)u + H{\times}(u{\times}H)\big) = \mathrm{div}\,(\Sigma u) + \kappa\Delta\theta + \eta(H \times (\nabla \times H)) \tag{4}$$

of partial differential equations. In (1)–(4), $\rho > 0$ denotes density, $u \in \mathbb{R}^d$ fluid velocity, $H \in \mathbb{R}^d$ magnetic field, $p(\rho,\theta)$ pressure, $\theta > 0$ temperature, $q = p + \frac{1}{2}H^2$ total pressure, $E = \rho(e + \frac{1}{2}u^2)$ specific energy, $e = e(\rho,\theta)$ internal energy, and $\Sigma = \lambda \mathrm{div}\,(u)I + \nu\big(\nabla u + (\nabla u)^t\big)$ viscous stress tensor, λ, $\nu > 0$ are coefficients of viscosity and κ, $\eta > 0$ coefficients of heat conductivity and electrical resistivity. System (1)–(4) is supplemented by the divergent constraint

$$\mathrm{div}\, H = 0 \tag{5}$$

on the initial data $U|_{t=0} = (\rho, u, H, \theta)|_{t=0} = U_0$. In the case $(\lambda,\nu,\kappa,\eta) = (0,0,0,0)$, (1)–(4) is called the system of *ideal MHD*, otherwise *dissipative MHD*.

A discontinuous (piecewise constant) solution

$$U(t,x) = \begin{cases} \bar{U}^- & \text{for} \quad x\cdot N - st < 0, \\ \bar{U}^+ & \text{for} \quad x\cdot N - st > 0, \end{cases} \tag{6}$$

of the system of ideal MHD is called a (planar) *ideal MHD shock wave* if constant states $\bar{U}^{\pm} = (\bar{\rho}^{\pm}, \bar{u}^{\pm}, \bar{H}^{\pm}, \bar{\theta}^{\pm})$ satisfy the Rankine-Hugoniot conditions (see Sect. 2 and 3) and $\bar{u}^{-} \cdot N \neq s$, $\bar{\rho}^{-} \neq \bar{\rho}^{+}$. A smooth solution

$$\widetilde{U}(t,x) = \Phi(x \cdot N - st) \tag{7}$$

of the system of dissipative MHD satisfying

$$\Phi(\pm\infty) = \bar{U}^{\pm} \tag{8}$$

is called a corresponding (planar) *viscous MHD shock wave*, with a profile Φ. Hereafter we assume without loss of generality that $N = (1, 0^{d-1})$.

One calls an MHD shock wave *multidimensionally* (multi-D) *strongly unstable at the inviscid level* if its ideal version violates even the weak Kreiss-Lopatinski condition for local-in-time persistence [3, 8, 10, 13]. An MHD shock wave is called *one-dimensionally* (1D) *stable at the viscous level* with respect to $(\lambda, \nu, \kappa, \eta) \neq (0,0,0,0)$ if it has a viscous version (7) with respect to these values of the viscosity coefficients and this profile is time-asymptotically stable towards planar perturbations in the sense of Liu [9].

When thermal effects can be neglected, one alternatively considers *barotropic* MHD, i.e., system (1)–(3) for the same variables except for the temperature θ and with the same dissipation coefficients except for the heat conductivity κ. We use the abovementioned terminology analogously in the barotropic case. This paper establishes that there are MHD shock waves that are 1D stable at the viscous level while they are not multi-D stable even at the inviscid level.

More precisely, we show:

Theorem 1.1 *Fix $(\lambda, \nu) \in (0, \infty)^2$ arbitrarily and consider equations (1)–(3) of barotropic MHD with γ-law pressure,*

$$p = a\rho^{\gamma}, \tag{9}$$

where $\gamma \in [1, \infty)$ and a are given positive constants. Then there exist Lax type shock waves which are multi-D strongly unstable at the inviscid level, while they are 1D stable at the viscous level with respect to (λ, ν, η) for some $\eta > 0$.

Theorem 1.2 *Fix $(\lambda, \nu, \kappa, \eta) \in (0, \infty)^4$ arbitrarily and consider equations (1)–(4) of full MHD for a polytropic gas, i.e., with*

$$p = (\gamma - 1)\rho e, \quad e = c_v \theta, \tag{10}$$

where $\gamma \in (1, \infty)$ and c_v are given positive constants. Then there exist Lax type shock waves which are multi-D strongly unstable at the inviscid level, while they are 1D stable at the viscous level with respect to $(\lambda, \nu, \kappa, \eta)$.

Remarks ...

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(iii) Texier and Zumbrun [17] have considered galloping instabilities at the 1D viscous level, a situation corresponding to a pair of purely imaginary zeroes of the Evans function. Whether this situation can occur in MHD, does not presently seem to be known.

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2 One-dimensional viscous stability of barotropic MHD shock waves

2.1 Stability problem for viscous parallel MHD shock waves

Consider system (1)–(3) in 1D written in Lagrangian coordinates:

$$v_t - u_{1x} = 0, \qquad u_{1t} + q_x = \mu \left(\frac{u_{1x}}{v}\right)_x, \tag{11}$$

$$u_{jt} - H_1^* H_{jx} = \nu \left(\frac{u_{jx}}{v}\right)_x, \quad (vH_j)_t - H_1^* u_{jx} = \eta \left(\frac{H_{jx}}{v}\right)_x, \qquad j = 2, 3, \tag{12}$$

where $v = 1/\rho$ is the specific volume, $\mu = \lambda + 2\nu > 0$, and, in view of (3) and (5), $H_1 = H_1^*$ is a constant which is assumed without loss of generality to be positive, $H_1^* > 0$. In (11), where $q = p + (H_2^2 + H_3^2)/2$, the pressure is assumed to obey the γ-law (9), i.e., $p(v) = av^{-\gamma}$.

Consider now a viscous MHD shock wave (7) for the case when the magnetic field is parallel to $N = (1, 0, 0)$, i.e., $\widetilde{H} = (H_1^*, 0, 0)$. Such viscous shocks and corresponding ideal MHD shocks (see Sect. 3) are called *parallel*. The end states (8) should satisfy the Rankine-Hugoniot conditions

$$s(\bar{v}^+ - \bar{v}^-) = \bar{u}_1^- - \bar{u}_1^+, \qquad s(\bar{u}_1^+ - \bar{u}_1^-) = p(\bar{v}^+) - p(\bar{v}^-), \tag{13}$$

$$\bar{u}_j^+ = \bar{u}_j^- \quad (j = 2, 3), \qquad \bar{H}_1^+ = \bar{H}_1^- = \bar{H}_1^*, \tag{14}$$

and we assume that the constants $\bar{v}^\pm$, $\bar{u}_1^\pm$ satisfy the well-known compressibility conditions

$$0 < \bar{v}^+ < \bar{v}^-, \quad 0 < \bar{u}_1^- < \bar{u}_1^+. \tag{15}$$

Because of (14) we can choose a reference frame in which

$$\bar{u}_2^+ = \bar{u}_2^- = 0, \quad \bar{u}_3^+ = \bar{u}_3^- = 0. \tag{16}$$

Then, without loss of generality we suppose that the velocity is also parallel to N. That is, we consider a profile

$$\Phi = (\tilde{v}, \tilde{u}, \widetilde{H}) = (\mathcal{V}, \mathcal{U}, 0, 0, H_1^*, 0, 0). \tag{17}$$

Since H_1^* is a constant we have roughly speaking a gas dynamic viscous profile because the "tangential" part of the profile is trivial, $(\tilde{u}_2, \tilde{u}_3, \widetilde{H}_2, \widetilde{H}_2) = (0,0,0,0)$. Hence, the existence of smooth traveling wave solutions (17) is guaranteed by conditions (13) and (15). By introducing the new variable $\xi = x - st$, we get the following equations (see, e.g., [11]) for the solutions $(\mathcal{V}, \mathcal{U})(\xi)$ of (11):

$$\mathcal{V}' = \frac{\mathcal{V}}{s\mu}\left(\bar{p}^+ - p(\mathcal{V}) + s^2(\bar{v}^+ - \mathcal{V})\right), \qquad \mathcal{U} + s\mathcal{V} = \bar{u}_1^+ + s\bar{v}^+ = \bar{u}_1^- + s\bar{v}^-, \tag{18}$$

where $\mathcal{V}' = d\mathcal{V}/d\xi$ and $\bar{p}^\pm := p(\bar{v}^\pm)$. In view of (15), $\mathcal{V}' > 0$.

As is known, the Lax conditions

$$0 < \sqrt{\gamma\bar{p}^-/\bar{v}^-} < s < \sqrt{\gamma\bar{p}^+/\bar{v}^+} \tag{19}$$

for gas dynamic shocks are equivalent to (15). Since for $H_2 = H_3 = 0$ system (11) coincides with the barotropic Navier-Stokes equations, then a viscous parallel MHD shock is *fast*, i.e., a corresponding ideal shock wave is 1-shock (see also Sect. 3), if inequalities (19) hold together with

$$s > \frac{H_1^*}{\sqrt{\bar{v}^+}}. \tag{20}$$

Analogously, a viscous parallel MHD shock is *slow*, i.e., a corresponding ideal shock wave is 3-shock, if inequalities (19) hold together with

$$s < \frac{H_1^*}{\sqrt{\bar{v}^-}}. \tag{21}$$

Rewriting (11), (12) in the new variables (t, ξ) and linearizing the resulting equations about profile (17) we obtain the separate system (see [11])

$$\begin{aligned} &v_t - sv_\xi - u_\xi = 0, \\ &u_t - su_\xi - \left(\frac{f(\mathcal{V})}{\mathcal{V}^{\gamma+1}}v\right)_\xi = \mu\left(\frac{u_\xi}{\mathcal{V}}\right)_\xi, \end{aligned} \tag{22}$$

with

$$f(\mathcal{V}) = a(\gamma - 1) + (\bar{p}_+ + s^2\bar{v}^+)\mathcal{V}^\gamma - s^2\mathcal{V}^{\gamma+1},$$

for perturbations (v, u) of $(\mathcal{V}, \mathcal{U})$ and the system

$$\begin{aligned} &w_t - sw_\xi - H_1^* h_\xi = \nu\left(\frac{w_\xi}{\mathcal{V}}\right)_\xi, \\ &(\mathcal{V}h)_t - s(\mathcal{V}h)_\xi - H_1^* w_\xi = \eta\left(\frac{h_\xi}{\mathcal{V}}\right)_\xi \end{aligned} \tag{23}$$

for perturbations (w, h) of $(0, 0)$. System (23) depends on the gas dynamical part (22) only by means of the profile $\mathcal{V}(\xi)$.

It is clear that a viscous parallel MHD shock wave is linearly stable, i.e., the perturbations (v, u, w, h) are bounded in time, if and only if a corresponding gas dynamic shock is linearly stable and the Cauchy problem for system (23) has only time-bounded solutions (w, h). The nonlinear stability of viscous shock waves in barotropic gas dynamics with respect to zero-mass perturbations was shown by Matsumura and Nishihara [11], provided that some sufficient stability condition holds. In particular, this condition (see (29) below) holds for small-amplitude (weak) shocks or if the adiabatic exponent γ is closed to 1. Of course, the result in [11] implies a weaker linear result, i.e., the same shock waves are linearly stable under zero-mass perturbations $(v, u) = (\phi_\xi, \psi_\xi)$. Clearly, linearized zero-mass stability implies spectral stability (see, e.g., discussion in [19]). At the same time, as follows from the more recent work of Zumbrun and collaborators (see [20] and references therein), spectral stability implies time-asymptotic orbital (nonlinear) stability [9].

Quite recently, the unconditional spectral stability of barotropic gas dynamic shock waves was shown in [1, 16] by numerical Evans function calculations for $\gamma \in [1, 2.5]$. Thus, having in hand stability results for barotropic gas dynamic shock waves (either analytical ones in [11] or numerical ones in [1, 16]), to establish the nonlinear stability of viscous parallel MHD shock waves one needs to study system (23) for zero-mass perturbations

$$(w, \mathcal{V}h) = (\omega_\xi, \alpha_\xi). \tag{24}$$

By substituting (24) into (23) and integrating the result with respect to ξ, one gets

$$\begin{aligned} \omega_t - s\omega_\xi - \frac{H_1^*}{\mathcal{V}}\alpha_\xi &= \frac{\nu}{\mathcal{V}}\omega_{\xi\xi}, \\ \alpha_t - s\alpha_\xi - H_1^*\omega_\xi &= \frac{\eta}{\mathcal{V}}\left(\frac{\alpha_\xi}{\mathcal{V}}\right)_\xi. \end{aligned} \tag{25}$$

2.2 Sufficient stability conditions

Let $\delta > 0$ is a constant. By multiplying (25) by $(\mathcal{V}^{\delta+1}\omega, \mathcal{V}^\delta\alpha)$ and integrating the result over the domain $[0, t] \times \mathbb{R}$ we obtain the energy identity

$$\frac{1}{2}\int_{\mathbb{R}} \mathcal{V}^\delta(\mathcal{V}\omega^2 + \alpha^2)d\xi\Big|_0^t + \int_0^t\int_{\mathbb{R}} \mathcal{V}^{\delta-2}\left(\tfrac{1}{2}\mathcal{V}\mathcal{V}'\mathcal{A}y \cdot y + \nu\mathcal{V}^2\omega_\xi^2 + \alpha_\xi^2\right)d\xi dt = 0, \tag{26}$$

with $y = (\omega, \alpha)$ and

$$\mathcal{A} = \begin{pmatrix} (\delta+1)s\mathcal{V} - \delta\left(\dfrac{\nu}{\mu}\right)a_1 & \delta H_1^* \\ \delta H_1^* & \delta s - (\delta - 1)\left(\dfrac{\eta}{\mu\bar{v}^+}\right)a_2 \end{pmatrix},$$

where

$$a_1 = \frac{\delta\left(\bar{p}^+ - p(\mathcal{V}) + s^2(\bar{v}^+ - \mathcal{V})\right) - \mathcal{V}\left(s^2 + p'(\mathcal{V})\right)}{s},$$

$$a_2 = \bar{v}^+ \frac{(\delta-2)\left(\bar{p}^+ - p(\mathcal{V}) + s^2(\bar{v}^+ - \mathcal{V})\right) - \mathcal{V}\left(s^2 + p'(\mathcal{V})\right)}{s\mathcal{V}^2}.$$

While deriving (26) we used relations like

$$-\mathcal{V}^\delta \omega_{\xi\xi}\omega = -\mathcal{V}^\delta(\omega_\xi\omega)_\xi + \mathcal{V}^\delta\omega_\xi^2 = -(\mathcal{V}^\delta\omega_\xi\omega)_\xi + \tfrac{1}{2}\delta\mathcal{V}^{\delta-1}\mathcal{V}'(\omega^2)_\xi + \mathcal{V}^\delta\omega_\xi^2$$
$$= \left(\tfrac{1}{2}\delta\mathcal{V}^{\delta-1}\mathcal{V}'\omega_\xi^2 - \mathcal{V}^\delta\omega_\xi\omega\right)_\xi - \tfrac{1}{2}\delta\mathcal{V}^{\delta-2}\left((\delta-1)(\mathcal{V}')^2 + \mathcal{V}\mathcal{V}''\right)\omega^2 + \mathcal{V}\omega_\xi^2,$$

integration by parts, and (18), that yields

$$\mathcal{V}'' = \frac{\mathcal{V}'}{s\mu}\left\{\bar{p}^+ - p(\mathcal{V}) + s^2(\bar{v}^+ - \mathcal{V}) - \mathcal{V}(s^2 + p'(\mathcal{V}))\right\}.$$

Since $\mathcal{V}' > 0$, it follows from (26) that the viscous parallel MHD shock wave is 1D stable if

$$\mathcal{A} > 0 \tag{27}$$

(provided that a corresponding gas dynamic shock wave is stable). Inequality (27) can give a number of sufficient stability conditions. In particular, for the case $\nu/\mu \ll 1$ inequality (27) with $\delta = 1$ holds if

$$2s^2\bar{v}^+ > (H_1^*)^2. \tag{28}$$

In view of (20), the sufficient stability condition (28) for viscous parallel MHD shock waves for $\nu/\mu \ll 1$ is always satisfied for fast shocks, i.e., fast parallel MHD shock waves are 1D stable for $\nu/\mu \ll 1$. This result is purely analytical provided that a corresponding gas dynamic shock wave satisfies the sufficient stability condition of Matsumura and Nishihara [11]:

$$2(\gamma-1)(1 + s^2/p'(\bar{v}^+)) < \gamma - 1 + (s^2/p'(\bar{v}^+))^2. \tag{29}$$

Otherwise, we can refer to the numerical results in [1, 16] assuming only that $\gamma \in [1, 2.5]$ (hereafter, we assume by default that $\gamma \in [1, 2.5]$).

Let us now δ is a small parameter, $\delta \ll 1$. Consider the case $\eta/(\mu\bar{v}^+) \ll 1$, more precisely, let the dimensionless value $\eta/(\mu\bar{v}^+) = \mathcal{O}(\delta^2)$. Then, condition (27) holds for $\delta \ll 1$. Indeed, the diagonal elements of $\mathcal{A}$ are positive (they are $s\mathcal{V} + \mathcal{O}(\delta)$ and $\delta s + \mathcal{O}(\delta^2)$), and $\det\mathcal{A} = \delta s^2\mathcal{V} + \mathcal{O}(\delta^2)$ is positive too. We collect the results in the following theorem.

Theorem 2.1 *For $\nu/\mu \ll 1$ fast viscous parallel shock waves in barotropic MHD are 1D stable, and corresponding slow shock waves are 1D stable provided condition (28) is satisfied. For $\eta/(\mu\bar{v}^+) \ll 1$ fast and slow viscous parallel shock waves in barotropic MHD are 1D stable.*

3 Multidimensional inviscid instability of slow barotropic MHD shocks in the high-magnetic field limit

We now go on to the multi-D stability analysis of shock discontinuties in barotropic MHD (with γ-law pressure). The present goal is to show the strong instability of slow MHD shock waves in the high-magnetic field limit. For full MHD this was shown in [2] (see also [3]) and here we closely follow the arguments of [2, 3]. Since in the previous section we considered viscous profiles only for parallel MHD shocks, here we also concentrate mainly on the case of parallel magnetic field making brief remarks about the general case of nonparallel shocks.

3.3 Free boundary problem for nonplanar MHD shocks

It is known [3, 4] that, as for gas dynamic shock waves, the linearized stability problem for parallel MHD shocks (fast or slow ones) has a symmetry along the directions tangential to the shock front. Therefore, without loss of generality we can consider the 2D case $x = (x_1, x_1) \in \mathbb{R}^2$ and assume that the fluid velocity and the magnetic field are 2D vector fields: $u = (u_1, u_2) \in \mathbb{R}^2$, $H = (H_1, H_2) \in \mathbb{R}^2$. Unlike the previous section, the first component H_1 of the magnetic field is now not a constant and satisfies the divergent constraint (5).

For (1)–(3) the Rankine-Hugoniot conditions

$$[\mathfrak{j}] = 0, \quad [H_n] = 0, \quad \mathfrak{j}\,[u_n] + [q] = 0, \quad \mathfrak{j}\,[u_\tau] = H_{\rm n}[H_\tau], \quad H_n[u_\tau] = \mathfrak{j}\,[vH_\tau] \tag{30}$$

should be satisfied at each point of the shock front

$$\Gamma(t) = \{x_1 - \varphi(t, x_2) = 0\} \tag{31}$$

that is assumed to be a smooth hypersurface in $[0, T] \times \mathbb{R}^2$, where $[g] = g^+|_\Gamma - g^-|_\Gamma$ denotes the jump of g, with $g^\pm := g$ in $\Omega^\pm(t) = \{x_1 \gtrless \varphi(t, x_2)\}$,

$$\mathfrak{j} = \rho(u_{\rm n} - \varphi_t), \quad u_n = u_1 - \varphi_{x_2} u_2, \quad H_n = H_1 - \varphi_{x_2} H_2,$$

$$u_\tau = \varphi_{x_2} u_1 + u_2, \quad H_\tau = \varphi_{x_2} H_1 + H_2, \quad H_n|_\Gamma := H_n^\pm|_\Gamma, \quad \mathfrak{j} := \mathfrak{j}^\pm|_\Gamma.$$

3.4 Jump conditions for planar parallel MHD shocks

For planar discontinuities $\varphi(t, x_2) = st$, see (6) with $N = (1, 0)$. Since now we work in Eulerian coordinates we can assume without loss of generality that $s = 0$. Consider a piecewise constant solution (6) of (1)–(3), (30):

$$\bar{U}^\pm = (\bar{\rho}^\pm, \bar{u}^\pm, \bar{H}^\pm), \quad x_1 \gtrless 0. \tag{32}$$

We are interested in parallel MHD shocks, i.e.,

$$\bar{H}_2^+ = \bar{H}_2^- = 0. \tag{33}$$

It follows from (30) that, cf. (14),

$$\bar{H}_1^+ = \bar{H}_1^- = H_1^*, \qquad \bar{u}_2^+ = \bar{u}_2^- \tag{34}$$

and

$$\frac{\bar{\rho}^+}{\bar{\rho}^-} = \frac{\bar{u}_1^-}{\bar{u}_1^+} := R, \tag{35}$$

Without loss of generality we again have (16) and assume that $H_1^* > 0$. Moreover, we again consider compressive shocks, i.e., inequalities (15) are satisfied with $\bar{v}^\pm = 1/\bar{\rho}^\pm$.

In view of (15), (30), (33), and (34), the dimensionless parameter R should satisfy the inequality

$$R > 1, \tag{36}$$

and the non-zero constants $\bar{\rho}^\pm$ and $\bar{u}_1^\pm$ are related by

$$\bar{\rho}^+\bar{u}_1^+(\bar{u}_1^+ - \bar{u}_1^-) + a\left((\bar{\rho}^+)^\gamma - (\bar{\rho}^-)^\gamma\right) = 0. \tag{37}$$

Introducing the downstream Mach number

$$M := M_+ = \frac{\bar{u}_1^+}{c_+}$$

where the square of the downstream sound velocity $c_+^2 = p'(\bar{\rho}^+) = a\gamma(\bar{\rho}^+)^{\gamma-1}$, we rewrite (37) as

$$M^2 = \frac{R^\gamma - 1}{\gamma R^\gamma (R-1)}. \tag{38}$$

Omitting standard calculations (see also Remark 3.1 below), we assert that the parallel MHD shock is *fast* shock (1-shock) if

$$q < M < 1 \tag{39}$$

and it is *slow* shock (3-shock) if

$$M < 1, \quad q > M\sqrt{R}, \tag{40}$$

where

$$q^2 = \frac{(H_1^*)^2}{\gamma a (\bar{\rho}^+)^\gamma}$$

is the ratio between the magnetic and fluid pressures behind the shock. In view of (38), the condition $M < 1$ in (39) and (40) is equivalent to (36). Note also that in terms of the upstream Mach number $M_- = \bar{u}_1^-/c_- = MR^{(\gamma+1)/2}$ the condition $M < 1$ is $M_- > 1$.[1]

Remark 3.1 Using (13) and (35), we can express the shock speed s from Sect. 2 in Eulerian coordinates:

$$s^2 = \left(\frac{\bar{u}_1^- - \bar{u}_1^+}{\bar{v}^+ - \bar{v}^-}\right)^2 = \left(\frac{\bar{u}_1^+}{\bar{v}^+}\right)^2 = \left(\frac{c_+}{\bar{v}^+}\right)^2 M^2 = -p'(\bar{v}^+)M^2 = -p'(\bar{v}^-)M_-^2. \tag{41}$$

Noting $q^2 = \bar{v}^+(H_1^*)^2/c_+^2$, it follows from (41) that (19) and (20) are equivalent to (39), and (19) and (21) are equivalent to (40).

[1] Below, unlike, for example, [1, 16], we mainly work with the downstream Mach number M.

3.5 Stability problem for slow parallel MHD shocks

We straighten, as usual (see, e.g., [3, 10, 13]), the unknown front (31), i.e., the unknowns $U^{\pm} = (\rho^{\pm}, u^{\pm}, H^{\pm})$ being smooth in $\Omega^{\pm}(t)$ are replaced by the functions $U^{\pm}(t, \pm x_1 + \varphi(t, x_2), x_2)$, that are smooth in the fixed domain $x_1 > 0$. Linearizing (1)–(3) and (30) written in terms of the straightened variables about the piecewise constant solution (32) we get a constant coefficients stability problem for small perturbations $(\delta\rho^{\pm}, \delta u^{\pm}, \delta H^{\pm})$ and $\delta\varphi$. For the forthcoming normal modes analysis in the high-magnetic field limit, $q \gg 1$, it is convenient to reduce this problem to a dimensionless form by introducing the following scaled values:

$$t' = \frac{t\bar{u}_1^+}{l}, \quad x' = \frac{x}{l}, \quad \rho^{\pm} = \frac{\delta\rho^{\pm}}{\bar{\rho}^+}, \quad u^{\pm} = \frac{\delta u^{\pm}}{\bar{u}_1^+}, \quad H^{\pm} = \frac{\delta H^{\pm}}{\sqrt{\gamma a(\bar{\rho}^+)^{\gamma}}}, \quad \varphi = \frac{\delta\varphi}{l},$$

where l is a typical length. After dropping the primes, taking into account (33) and (16), and eliminating by cross differentiation the front φ from the boundary conditions, the stability problem for the scaled perturbations $U^{\pm} = (\rho^{\pm}, u^{\pm}, H^{\pm})$ has the form:

$$\begin{cases} L_+\rho^+ + (u_1^+)_{x_1} + (u_2^+)_{x_2} = 0, \qquad M^2 L_+ u_1^+ + \rho^+_{x_1} = 0, \\ M^2 L_+ u_2^+ + \rho^+_{x_2} + q(H_1^+)_{x_2} - q(H_2^+)_{x_1} = 0, \\ L_+ H_1^+ + q(u_2^+)_{x_2} = 0, \quad L_+ H_2^+ - q(u_2^+)_{x_1} = 0, \\ R L_-\rho^- - (u_1^-)_{x_1} + (u_2^-)_{x_2} = 0, \qquad M^2 R^{\gamma-2} L_- u_1^- - \rho^-_{x_1} = 0, \\ M^2 R^{\gamma-2} L_- u_2^- + \rho^-_{x_2} + qR^{\gamma-1}(H_1^-)_{x_2} + qR^{\gamma-1}(H_2^-)_{x_1} = 0, \\ L_- H_1^- + q(u_2^-)_{x_2} = 0, \quad L_- H_2^- + q(u_2^-)_{x_1} = 0 \qquad \text{for} \quad x_1 > 0, \end{cases} \tag{42}$$

$$\begin{cases} u_1^+ + b_1\rho^+ - u_1^- - b_2\rho^- = 0, \\ (u_2^+)_t + b_3\rho^+_{x_2} - (u_2^-)_t + b_4\rho^-_{x_2} + b_5(u_1^-)_{x_2} = 0, \\ H_1^+ - H_1^- = 0, \qquad H_2^+ - RH_2^- - qu_2^+ + qu_2^- = 0 \qquad \text{at} \quad x_1 = 0, \end{cases} \tag{43}$$

where

$$L_+ = \partial_t + \partial_{x_1}, \quad L_- = \partial_t - R\partial_{x_1}, \quad b_1 = \frac{1+M^2}{2M^2}, \quad b_2 = \frac{1}{2}\left(R^2 + \frac{1}{M^2R^{\gamma-1}}\right),$$

$$b_3 = \frac{R(1-M^2)}{2(M^2-q^2)}, \quad b_4 = -\frac{RM^2}{2(M^2-q^2)}\left(R^2 - 2R + \frac{1}{M^2R^{\gamma-1}}\right), \quad b_5 = \frac{M^2(1-R)}{M^2-q^2}.$$

If we multiply the last two equations in (42) by $R^{\gamma-1}$, equations (42) form the linear symmetric hyperbolic system

$$A_0 W_t + A_1 W_{x_1} + A_2 W_{x_2} = 0 \tag{44}$$

for the vector $W = (U^+, U^-) \in \mathbb{R}^{10}$ with block-diagonal matrices $A_\alpha = \mathrm{diag}(A_\alpha^+, A_\alpha^-)$, where the symmetric constant coefficients matrices $A_\alpha^{\pm}$ can be easily written down.

3.6 Normal modes analysis

For Lax MHD shocks (fast or slow) system (44) has four outgoing characteristic modes. In particular, for slow parallel shock waves the Lax conditions (40) imply that the matrix $(A_0^+)^{-1}A_1^+$ has three positive eigenvalues whereas the matrix $(A_0^-)^{-1}A_1^-$ has one positive eigenvalue. Then, in view of Hersh's lemma [15], the equation

$$\det(\tau A_0^+ + \xi A_1^+ + i\omega A_2^+) = 0 \tag{45}$$

has three roots ξ and the equation

$$\det(\tau A_0^- + \xi A_1^- + i\omega A_2^-) = 0 \tag{46}$$

has one root ξ with $\Re\xi < 0$ for all τ with $\Re\tau > 0$ and for all real ω. One can show that slow parallel MHD shocks in a γ-law gas are 1D stable (we omit simple calculations). Therefore, without loss of generality we assume below that the wave number $\omega = 1$.

To show the strong instability of slow parallel shocks in the high-magnetic field limit, i.e., to prove the ill-posedness of problem (42), (43) for $q \gg 1$ we look for its solutions in the form

$$W = \sum_{k=1}^{4} W_k \exp(\tau t + \xi_k x_1 + i x_2), \tag{47}$$

where $\xi = \xi_j$ $(j = 1, 2, 3)$ are the roots of (45) with $\Re\xi_j < 0$ for $\Re\tau > 0$ (to avoid so-called glancing modes [8] we assume that they are different for a given τ), and $\xi = \xi_4$ is the unique root of (46) with $\Re\xi_4 < 0$ for $\Re\tau > 0$. It is clear that $W_j = (U_j^+, 0)$ $(j = 1, 2, 3)$ and $W_4 = (0, U_4^-)$, where the constant vectors U_j^+ and U_4^- satisfy the algebraic equations

$$(\tau A_0^+ + \xi_j A_1^+ + i A_2^+)U_j^+ = 0, \quad j = 1, 2, 3, \tag{48}$$

$$(\tau A_0^- + \xi_4 A_1^- + i A_2^-)U_4^- = 0 \tag{49}$$

(they are the eigenvectors of the matrices $(A_1^\pm)^{-1}(\tau A_0^\pm + i\omega A_2^\pm)$).

The dispersion relation (45) explicitly reads

$$\widehat{L}_+ \left\{ \varepsilon^2 M^2 \widehat{L}_+^2 (M^2 \widehat{L}_+^2 - \xi^2 + 1) + (1 - \xi^2)(M^2 \widehat{L}_+^2 - \xi^2) \right\} = 0, \tag{50}$$

where $\widehat{L}_+ = \tau + \xi$ and $\varepsilon = 1/q$. We easily find the root

$$\xi_1 = -\tau \tag{51}$$

that solves $\widehat{L}_+ = 0$. It follows from (50) and (51) that $\xi = \xi_2$ and $\xi = \xi_3$ solve the equation

$$\varepsilon^2 M^2 \widehat{L}_+^2 (M^2 \widehat{L}_+^2 - \xi^2 + 1) + (1 - \xi^2)(M^2 \widehat{L}_+^2 - \xi^2) = 0. \tag{52}$$

Since $\widehat{L}_- \neq 0$ for $\Re\xi < 0$ and $\Re\tau > 0$, where $\widehat{L}_- = \tau - R\xi$, the dispersion relation (46) for $\xi = \xi_4$ implies

$$\varepsilon^2 M^2 \widehat{L}_-^2 (M^2 R^{\gamma-1} \widehat{L}_-^2 - \xi^2 + 1) + R(1-\xi^2)(M^2 R^{\gamma-1} \widehat{L}_-^2 - \xi^2) = 0. \tag{53}$$

We assume that

$$\tau = \tau_0 + \tau_1 \varepsilon + \tau_2 \varepsilon^2 + \dots$$

for $\varepsilon \ll 1$. Then, by expanding ξ into series in ε we find appropriate roots of (52) and (53) for $\Re\tau > 0$ and for a fixed Mach number (or a fixed parameter R):

$$\begin{cases} \xi_2 = -1 + \varepsilon^2 \dfrac{M^4(\tau_0 - 1)^4}{2(M^2(\tau_0-1)^2 - 1)} + \mathcal{O}(\varepsilon^3), & \\ \xi_3 = -\dfrac{M\tau_0}{1+M} - \varepsilon \dfrac{M\tau_1}{1+M} + \mathcal{O}(\varepsilon^2) & \text{if} \quad \tau_0 \neq 1 + \dfrac{1}{M}, \end{cases}$$

$$\xi_{2,3} = -1 + \varepsilon \frac{-M\tau_1 \pm \sqrt{M^2\tau_1^2 - 1 - M}}{2(1+M)} + \mathcal{O}(\varepsilon^2) \qquad \text{if} \quad \tau_0 = 1 + \frac{1}{M}, \tag{54}$$

$$\xi_4 = -1 + \varepsilon^2 \frac{M^4 R^{\gamma-2}(\tau_0 + R)^4}{2(M^2 R^{\gamma-1}(\tau_0+R)^2 - 1)} + \mathcal{O}(\varepsilon^3).$$

The denominator appearing in the expression for ξ_4 vanishes only for negative τ_0. We also assume that $\tau \neq 1$ because $\tau = 1$ is a glancing mode at which $\xi_1 = \xi_2$.

Remark 3.2 The equation (45) written for nonparallel shocks coincides with the analogous dispersion relation in the non-barotropic case [3]. For nonparallel shocks [3]

$$\xi_3 = -\frac{M_0 l \tau_0 + im}{l(1+M_0)} + \mathcal{O}(\varepsilon) \quad \text{if} \quad \tau_0 \neq \tau_0^*, \tag{55}$$

with

$$\tau_0^* = 1 + \frac{l - im}{l M_0}, \quad l = \cos\beta, \quad m = \sin\beta, \quad M_0 = \frac{\bar{u}_1}{c_s^+},$$

where β is the angle of inclination of the magnetic field to the planar shock front (for parallel shocks $\beta = 0$), M_0 is the downstream slow Mach number (for parallel shocks $M_0 = M$), and c_s^+ is the downstream slow magnetosonic velocity (for barotropic MHD see, e.g., [14]). Note that the condition $\tau_0 \neq \tau_0^*$ implies $\xi_2|_{\varepsilon=0} \neq \xi_3|_{\varepsilon=0}$ (for parallel shocks $\tau_0^* = 1 + 1/M$).

From (48) and (49), taking into account (51), we find the eigenvectors

$$U_1^+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tau \\ -i \end{pmatrix} C_1, \qquad U_k^+ = \begin{pmatrix} -iM^2 L_k^2 \\ i\xi_k L_k \\ L_k \ell_k \\ -iq\ell_k \\ q\xi_k \ell_k \end{pmatrix} C_k, \quad U_4^- = \begin{pmatrix} -iM^2 R^{\gamma-2} L_4^2 \\ i\xi_4 L_4 \\ L_4 \ell_4 \\ -iq\ell_4 \\ q\xi_4 \ell_4 \end{pmatrix} C_4, \tag{56}$$

where $k = 2, 3$,

$$L_j = \tau + \xi_j \quad (j = 1, 2, 3), \quad L_4 = \tau - R\xi_4, \quad \ell_k = M^2 L_k^2 - \xi_k^2, \quad \ell_4 = M^2 R^{\gamma-1} L_4^2 - \xi_4^2,$$

and C_α ($\alpha = \overline{1,4}$) are constants. We now substitute (47), (56) into the boundary conditions (43). Omitting simple calculations, from the last two boundary conditions in (43) we derive $(\tau^2 - 1)C_1 = 0$. Since by assumption $\tau \neq 1$ and we are not interested in the stable mode $\tau = -1$ one gets $C_1 = 0$, and the last two conditions in (43) written in terms of the constants C_2, C_3, and C_4 are linearly dependent. Then from the first three boundary conditions in (43) we obtain the following linear system for the constant vector $X = (C_2, C_3, C_4)$:

$$\mathcal{L}X = 0,$$

where

$$\mathcal{L} = \begin{pmatrix} (\xi_2 - b_1 M^2 L_2)L_2 & (\xi_3 - b_1 M^2 L_3)L_3 & (b_2 M^2 R^{\gamma-2} L_4 - \xi_4)L_4 \\ (\tau\ell_2 + b_3 M^2 L_2)L_2 & (\tau\ell_3 + b_3 M^2 L_3)L_3 & (b_4 M^2 R^{\gamma-2} L_4 - b_5 \xi_4 L_4 - \tau\ell_4)L_4 \\ \ell_2 & \ell_3 & -\ell_4 \end{pmatrix},$$

In fact, $\Delta(\tau) = \det \mathcal{L}$ is a reduced Lopatinski determinant [8].

Since $b_k = \mathcal{O}(\varepsilon^2)$ for $k = 3, 4, 5$ and, in view of (54), $l_3|_{\varepsilon=0} = 0$ we have

$$\Delta(\tau) = \left\{(b_1 M^2 L_3 - \xi_3)L_3\right\}|_{\varepsilon=0} \det \left.\begin{pmatrix} \tau\ell_2 L_2 & -\tau\ell_4 L_4 \\ \ell_2 & -\ell_4 \end{pmatrix}\right|_{\varepsilon=0} + \mathcal{O}(\varepsilon).$$

That is,

$$\Delta = \Delta_0 + \Delta_1\varepsilon + \Delta_2\varepsilon^2 + \dots,$$

where

$$\Delta_0(\tau_0) = \frac{1}{2}(R+1)\tau_0^3 \left(M^2(\tau_0 - 1)^2 - 1\right)\left(M^2 R^{\gamma-1}(\tau_0 + R)^2 - 1\right)$$

and τ_0, τ_1, τ_2, $\dots$ are found recursively from the equations $\Delta_0 = 0$, $\Delta_1 = 0$, $\Delta_2 = 0$, $\dots$. The condition $\Re\tau_0 > 0$ implies $\Re\tau > 0$. The equation $\Delta_0 = 0$ has only one root with $\Re\tau_0 > 0$, and this root $\tau_0 = 1 + 1/M$, that solves the equation $\ell_2|_{\varepsilon=0} = 0$, can potentially correspond to a glancing mode $\tau_{\rm g} = \tau_0 + \mathcal{O}(\varepsilon)$ at which $\xi_2 = \xi_3$, see (54). In fact, it does correspond to the glancing mode $\tau_{\rm g}$ that is the root of the equation $\Delta(\tau) = 0$ because the first two columns of the matrix $\mathcal{L}$ coincide for $\xi_2 = \xi_3$. But, one can show that the glancing mode is not a root of the equation $\Delta = 0$ when Δ is properly determined for $\xi_2 = \xi_3$.

Thus, we have to take $\tau_0 = 0$. Analyzing the structure of the matrix $\mathcal{L}$ for $\tau_0 = 0$, we find that

$$\Delta = c\tau_1^3\varepsilon^3 + \mathcal{O}(\varepsilon^4),$$

where c is a nonzero constant. Therefore, $\tau_1 = 0$. Analyzing again the structure of the matrix $\mathcal{L}$ for $\tau_0 = \tau_1 = 0$, one can see that

$$\Delta = \Delta_6\varepsilon^6 + \mathcal{O}(\varepsilon^6), \qquad \Delta_6(\tau_2) = \det \mathcal{M}, \qquad \mathcal{M} =$$

$$\begin{pmatrix} \bigl(-1 - b_1M^2L_2^{(0)}\bigr)L_2^{(0)} & \bigl(\xi_3^{(2)} - b_1M^2L_3^{(2)}\bigr)L_3^{(2)} & (b_2M^2R^{\gamma-2}L_4^{(0)} + 1)L_4^{(0)} \\ \bigl(\tau_2\ell_2^{(0)} + b_3^{(2)}M^2L_2^{(0)}\bigr)L_2^{(0)} & b_3^{(2)}M^2\bigl(L_3^{(2)}\bigr)^2 & \bigl(b_4^{(2)}M^2R^{\gamma-2}L_4^{(0)} + b_5^{(2)}L_4^{(0)} - \tau_2\ell_4^{(0)}\bigr)L_4^{(0)} \\ \ell_2^{(0)} & 0 & -\ell_4^{(0)} \end{pmatrix},$$

where $(\cdot) = (\cdot)^{(0)} + (\cdot)^{(1)}\varepsilon + (\cdot)^{(2)}\varepsilon^2 + \dots$, in particular,

$$\xi_3^{(2)} = -\frac{M\tau_2}{1+M}, \quad L_2^{(0)} = -1, \quad L_3^{(2)} = \frac{\tau_2}{1+M}, \quad L_4^{(0)} = R, \quad \ell_2^{(0)} = M^2 - 1, \quad \ell_4^{(0)} = M^2R^{\gamma+1} - 1,$$

$$b_3^{(2)} = \frac{R(M^2-1)}{2}, \quad b_4^{(2)} = \frac{RM^2}{2}\left(R^2 - 2R + \frac{1}{M^2R^{\gamma-1}}\right), \quad b_5^{(2)} = M^2(R-1).$$

After some algebra one gets

$$\Delta_6(\tau_2) = -\frac{1}{2}(1-M^2)\tau_2^2\left\{(1+R)(M_-^2 - 1)\tau_2 - RM^2\sigma\right\}, \tag{57}$$

where

$$\sigma = \frac{M_-^2 - 1}{1+M} + R(R-1) + \frac{1}{2}RM_-^2\left(R^2 - 2R + \frac{R^2}{M_-^2}\right) - \frac{R(1-M)}{2(1+M)}(M_-^2 + 3).$$

Recall that M_- is the upstream Mach number and $M_-^2 = M^2R^{\gamma+1} > 1$. In (57) it is assumed that $1 - M^2 = \mathcal{O}(1)$, i.e., the Mach number is fixed while the parameter q is being taken to be sufficiently large.[2]

Taking into account (38), we can consider σ as a function of R, where $R > 1$, see (36). One can show that $\sigma(1) = 0$ and $\sigma(+\infty) = +\infty$. Then, at least for sufficiently strong shocks (when R is large enough) $\sigma > 0$ for all $\gamma \geq 1$. A more delicate analysis (we omit calculations) shows that $\sigma > 0$ for all $\gamma \in [1, 2]$ (the case $\gamma > 2$ needs numerical study of the function $\sigma(R)$). That is, the root $\tau_2 = RM^2\sigma/((1+R)(M_-^2 - 1))$ of the equation $\Delta_6(\tau_2) = 0$ is positive, and we have found an unstable root $\tau = \tau_2\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ of the equation $\Delta(\tau) = 0$ with $\Re\tau > 0$.

Remark 3.3 Actually, normal modes analysis for nonparallel shocks being technically involved is principally simplier than that for parallel shocks studied above. The structure of the Lopatinski determinant for nonparallel slow shocks in barotropic MHD is internally analogous to that in full MHD. For nonparallel shocks (i.e., for $\beta > 0$, see Remark 3.2)

$$\Delta = \Delta_{-4}q^4 + \ldots + \Delta_{-1}q + \widetilde{\Delta}_0 + \widetilde{\Delta}_1\varepsilon + \dots,$$

[2]If vice-versa q is fixed but $M \to 1$, then our arguments do not work even for large q. The limit $M \to 1$ corresponds to extremely weak shocks that are known to be uniformly stable [12].

where $\Delta_{-k}|_{\beta=0} = 0$, $k = \overline{1,4}$, $\widetilde{\Delta}_i|_{\beta=0} = \Delta_i$, $i = 0, 1, 2, 3, \ldots$. As in the non-barotropic case [2, 3], for nonparallel shocks there is an unstable root of the equation $\Delta_{-4}(\tau_0) = 0$ that is

$$\tau_0 = \hat{\tau}_0 = 1 + \frac{1}{M_0 l \sqrt{2l}} \left(\sqrt{l+1} - \frac{im}{\sqrt{l+1}} \right),$$

where $\hat{\tau}_0|_{\beta=0} = 1 + 1/M$. But $\hat{\tau}_0 \neq \tau_0^*$ for $\beta > 0$, cf. (55), whereas $\hat{\tau}_0 = \tau_0^*$ for $\beta = 0$. Thus, for nonparallel shocks we do not have any difficulties connected with glancing modes! The root $\tau = \hat{\tau}_0 + \mathcal{O}(\varepsilon)$ is a genuine (not fictitious) unstable root of the Lopatinski determinant for all $\beta > 0$. Note that the non-barotropic MHD shocks studied in [2] were assumed by default to be nonparallel. But, reasoning as above, for parallel slow shocks in full MHD one can also exhibit an unstable root $\tau = \mathcal{O}(\epsilon^2)$.

Thus, we have proved the following theorem (for the non-barotropic case and nonparallel slow shocks it was proved in [2], see also [3]).

Theorem 3.1 *For any fixed downstream Mach number $M < 1$ there exists a positive number q_* such that the slow barotropic MHD shock is violently unstable for all $q > q_*$, where the parameter q measures the competition between the magnetic and fluid pressures behind the shock.*

It is worth to note that, as follows from the numerical results of [4] where the roots of the Lopatinski determinant for MHD shocks in a polytropic gas were being sought numerically for $\gamma = 5/3$, slow shocks in full MHD are unstable in a wide range of the parameter q. That is, the number q_* from the counterpart of Theorem 3.1 for full MHD [2, 3] is actually not extremely large (except the case of weak shocks). It is clear that the same should be true for the barotropic case.

3.7 Proof of Theorem 1.1

The proof of Theorem 1.1 follows from Theorems 2.1 and 3.1. Indeed, fix $(\lambda, \nu) \in (0, \infty)^2$ arbitrarily. Then, it follows from Theorem 2.1 that there exists some (possibly rather small) $\eta^* > 0$ that slow parallel shock waves in barotropic MHD are 1D stable at the viscous level for all $\eta \in (0, \eta^*)$. At the same time, Theorem 3.1 implies that these shock waves are multi-D strongly unstable at the inviscid level for all $q > q^*$, with some (possibly rather large) $q^* > 0$.

4 Viscous and inviscid stability of shock waves in full MHD

To prove a counterpart of Theorem 1.1 for shock waves in full MHD one needs only to repeat arguments of Sect. 2 making appropriate brief remarks. We first make these remarks and then prove Theorem 1.2 by using a scaling property of full MHD found in [5]. It is worth to note that this argument is inapplicable for barotropic MHD.

For the case of full MHD, equations (11) and (12) and the Rankine-Hugoniot conditions (13), (14) (for *parallel* shocks) are complemented by

$$\left(e + \tfrac{1}{2}u^2 + \tfrac{1}{2}v(H_2^2 + H_3^2)\right) + \left(qu_1 - H_1^*(H_2u_2 + H_3u_3)_x\right.$$
$$= \left(\frac{\mu u_1 u_{1x} + \nu(u_2u_{2x} + u_3u_{3x}) + \kappa\theta_x + \eta(B_2B_{2x} + B_3B_{3x})}{v}\right)_x,$$

and

$$s\left(c_v(\bar\theta^+ - \bar\theta^-) + \tfrac{1}{2}((\bar u_1^+)^2 - (\bar u_1^-)^2)\right) = p(\bar v^+, \bar\theta^+)\bar u_1^+ - p(\bar v^-, \bar\theta^-)\bar u_1^-$$

respectively, where $q = p + (H_2^2 + H_3^2)/2$, $p(v, \theta) = (\gamma - 1)c_v\theta/v$, and $e = c_v\theta$, cf. (10). For parallel shocks the profile is

$$\Phi = (\mathcal{V}, \mathcal{U}, 0, 0, H_1^*, 0, 0, \Theta),$$

where $\mathcal{V}$, $\mathcal{U}$, and Θ are determined by (18) (with $\bar p^\pm = p(\bar v^\pm, \bar\theta^\pm)$) and (see [6])

$$\Theta' = \frac{s\mathcal{V}}{\kappa}\left\{(\bar v^+ - \mathcal{V})\left(\bar p^+ + \tfrac{1}{2}(\bar v^+ - \mathcal{V})\right) - c_v(\Theta - \bar\theta^+)\right\}.$$

One can show [7] that there exists a positive constant C independent on γ such that for $\xi \in \mathbb{R}$

$$|\Theta'/\mathcal{V}'| \le C(\gamma - 1). \tag{58}$$

At last, we note that for full MHD the Lax conditions (19), (20), and (21) remain the same.

In full MHD the study of viscous 1D stability of parallel shock waves is also reduced to the analysis of system (25). We can follow the arguments of Sect. 2 giving the energy identity (26) with $p'(\mathcal{V})$ substituted for

$$p_v(\mathcal{V}, \Theta) + (\Theta'/\mathcal{V}')p_\theta(\mathcal{V}, \Theta) = \frac{(\gamma - 1)c_v}{\mathcal{V}}(\Theta'/\mathcal{V}' - \Theta/\mathcal{V}).$$

The 1D stability of viscous gas dynamic shock waves was shown in [7] provided that $(\gamma - 1)(\bar v^+ - \bar v^-) \ll 1$. This condition does not exclude the case of strong shocks when γ is close to 1. Then, taking into account (58), we get a counterpart of Theorem 2.1 for full MHD for the case when γ is close to 1 (or under the hypothesis that viscous shock waves in full gas dynamics are always stable in a polytropic gas[3]). The multi-D ideal strong instability of slow (nonparallel) shock waves in the high-magnetic field limit was proved in [2], and the instability of corresponding parallel shocks can be shown by arguments analogous to those in Sect. 3 (see Remark 3.3). Thus, we have proved a counterpart of Theorem 1.1 for full MHD.

Remark 4.1 A careful comparison of the sufficient stability condition (28) for slow shocks in full MHD with the numerical results of [4] for $\gamma = 5/3$ can give us a counterpart of Theorem 1.1 (for full MHD) when the viscosity coefficients $(\lambda, \eta) \in (0, \infty)^2$ are arbitrarily fixed and shocks are stable

[3]This hypothesis is quite reasonable in the light of the results in [1, 16].

for some $\nu > 0$ (under the hypothesis that corresponding viscous shock waves in gas dynamics are stable). In Eulerian coordinates (see Remark 3.1) condition (28) becomes $q^2 < 2M^2$. Recalling the Lax conditions (40), we have

$$M^2 R < q^2 < 2M^2. \tag{59}$$

Note that for parallel shocks in a polytropic gas (see [18]), $R = ((\gamma - 1)M^2 + 2)/(\gamma + 1)M^2$. Numerical Lopatinski determinant calculations in [4] show that slow parallel MHD shocks[4] are strongly unstable in a wide range of the parameter q. Rewriting the dimensionless parameters used in [4] in terms of M and q, we can see that there are strongly unstable slow parallel MHD shocks satisfying conditions (59) (we omit detailed calculations).

We prove Theorem 1.2 by using an interesting effect of scalings in [5] on the dissipation coefficients ...

5 Appendix on gas dynamics

The equations of *full* gas dynamics and those of *barotropic* gas dynamics ...

[4]In the notations of [4] they correspond to the case $\varphi_1 = 0$.

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