SIMULTANEOUS DIOPHANTINE APPROXIMATION ON THE CIRCLE AND HAUSDORFF DIMENSION

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The functional relations between the coordinates of points on a manifold make the study of Diophantine approximation on manifolds much harder than the classical theory in which the variables are independent. Nevertheless there has been considerable progress in the metric theory of Diophantine approximation on smooth manifolds. To describe this, some notation and terminology are needed. Given $\mathbf{x} \in \mathbb{R}^n$, let $|\mathbf{x}| = \max\{|x_1|, \ldots, |x_n|\}$ be the height of $\mathbf{x}$, let M be an m-dimensional manifold embedded in $\mathbb{R}^n$ and let $\psi\colon \mathbb{N} \to \mathbb{R}^+$ converge to 0 at infinity. For each $t \in \mathbb{R}$, let

$$\|t\| = \min\{|t-r| \colon r \in \mathbb{Z}\} = \operatorname{dist}(t, \mathbb{Z}),$$

the distance of t from $\mathbb{Z}$, and for each $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let

$$\|\mathbf{x}\| = \max\{\|x_1\|, \ldots, \|x_n\|\}.$$

Points in the set

$$\mathscr{S}(M;\psi) = \{\mathbf{x} \in M : \|q\mathbf{x}\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\},$$

are called simultaneously ψ-approximable. When $\psi(q) = q^{-v}$, we denote the set $\mathscr{S}(M;\psi)$ by $\mathscr{S}_v(M)$, so that

$$\mathscr{S}_v(M) = \{\mathbf{x} \in M : \|q\mathbf{x}\| < q^{-v} \text{ for infinitely many } q \in \mathbb{N}\}.$$

The manifold $M \subset \mathbb{R}^n$ is termed *extremal* if the Lebesgue measure of $\mathscr{S}_v(M)$ induced on M is 0 for some $v > 1/n$. There is an equivalent 'dual' definiton of extremality, arising from Khintchine's transference principle. Let

$$\mathscr{L}(M;\psi) = \{\mathbf{x} \in M : \|\mathbf{q}\cdot\mathbf{x}\| < \psi(|\mathbf{q}|) \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\} \tag{1}$$

and

$$\mathscr{L}_v(M) = \{\mathbf{x} \in M : \|\mathbf{q}\cdot\mathbf{x}\| < |\mathbf{q}|^{-v} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\}.$$

Then M is extremal if the induced measure of $\mathscr{L}_v(M)$ is 0 for $v > n$. Kleinbock and Margulis have shown in [12] that manifolds M embedded in $\mathbb{R}^n$ which are 'non-degenerate' almost everywhere on M (*i.e.*, everywhere except for a set of induced Lebesgue measure zero on M) and which are smooth (*i.e.*, with C^k local parametrisation functions for certain $k \in \mathbb{N}$) are extremal (in fact they showed that such manifolds are strongly extremal). Non-degeneracy can be regarded as a local 'non-flatness' condition; further details are in [12]. Note that by Dirichlet's theorem, $\mathscr{S}_v(M) = M$ when $v \leqslant 1/n$ and $\mathscr{L}_v(M) = M$ when $v \leqslant n$.

Hausdorff dimension provides a way of distinguishing between sets of measure 0 (see [10, 13] for further information). It is known for instance that when M is a C^3 planar curve with non-zero curvature everywhere except on a set of Hausdorff dimension zero, the Hausdorff dimension $\dim \mathscr{L}_v(M)$ of $\mathscr{L}_v(M)$ is given by

$$\dim \mathscr{L}_v(M) = \frac{3}{v+1}$$

for $v \geqslant 2$ [3]. It is also known that for m-dimensional C^3 manifolds M where $m \geqslant 2$ and for which two principal curvatures do not vanish (this condition reduces to Gaussian curvature for surfaces in 3 dimensions) except on a set of Hausdorff dimension $m-1$, the Hausdorff dimension of $\mathscr{L}(M;\psi)$ is $m - 1 + (n+1)/(\lambda+1)$ when ψ is monotonically decreasing and $\lambda \geqslant n$, where $\lambda = \liminf_{N\to\infty}(\log(1/\psi(N))/(\log N)$, the lower order of $1/\psi$ at infinity [6]. This dimension is a lower bound when M is extremal and C^1 [5].

Not much, however, is known about the Hausdorff dimension of the set $\mathscr{S}(M;\psi)$, although a Khintchine-type theorem holds for 2-convex C^3 manifolds [8] and an asymptotic formula (which is a stronger version of the Khintchine-type theorem) holds under pretty restrictive curvature conditions [9]. Only estimates for the Hausdorff dimension of the set $\mathscr{S}_v(M)$ are known (see [4] for further details and references). Y. V. Melnichuk [14] studied Diophantine approximation on the unit circle $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and obtained estimates for the Hausdorff dimension of the set

$$\begin{aligned}\mathscr{S}_v(\mathbb{S}^1) &= \{\mathbf{x} \in \mathbb{S}^1 \colon \|q\mathbf{x}\| < q^{-v} \text{ for infinitely many } q \in \mathbb{N}\}\\ &= \{(x,y) \in \mathbb{S}^1 : \max\{\|qx\|, \|qy\|\} < q^{-v} \text{ for infinitely many } q \in \mathbb{N}\}.\end{aligned}$$

He obtained estimates for the case $1/2 < v \leqslant 1$ using exponential sums. By applying Khintchine's Transference Principle to R. C. Baker's general result [3], these can be improved to

$$\frac{3(1-v)}{1+v} \leqslant \dim \mathscr{S}_v(\mathbb{S}^1) \leqslant \frac{3}{2(1+v)}.$$

In the case when $v > 1$, Melnichuk used Pythagorean triples to show that

$$\frac{1}{2(1+v)} \leqslant \dim \mathscr{S}_v(\mathbb{S}^1) \leqslant \frac{1}{1+v}. \tag{2}$$

(Non-zero integers p, r, q satisfying $p^2 + r^2 = q^2$ are called a Pythagorean triple [1].) The usual covering argument gives the upper bound and he used regular systems [2] and Dirichlet's theorem for the lower bound. In this paper we use ubiquity to establish the lower bound $1/(v+1)$ in (2), thus obtaining the exact Hausdorff dimension for $v > 1$.

Theorem 1. *For $v > 1$*

$$\dim \mathscr{S}_v(\mathbb{S}^1) = \frac{1}{v+1}.$$

As far as we know, this is the first reasonably complete non-trivial result for the Hausdorff dimension of the set $\mathscr{S}_v(M)$ for a smooth manifold M in $\mathbb{R}^n$ when v is larger than the extremal value of $1/n$. Melnichuk's paper [14] is in Russian and is not readily available so his proof for the upper bound will be included for completeness.

1. Proof of Theorem 1

The key to the theorem is that rational points $(p/q, r/q)$ which approximate a point (x,y) on the unit circle sufficiently closely must be on the circle.

Lemma 1. *Suppose the point $(x,y) \in \mathbb{S}^1$ satisfies the inequalities*

$$|qx - p|,\ |qy - r| = o(1/q)$$

for $p, q, r \in \mathbb{Z}$, $q \geqslant 1$. Then for q sufficiently large, the point $(p/q, r/q) \in \mathbb{S}^1$.

Proof. Let $(x, y) \in \mathbb{S}^1$ so that $x^2 + y^2 = 1$. Assume that there exist $p, r \in \mathbb{Z}$, $q \in \mathbb{N}$ such that $qx - p = \varepsilon$ and $qy - r = \varepsilon'$ for some $\varepsilon, \varepsilon' \in \mathbb{R}$ with $|\varepsilon|, |\varepsilon'| = o(1/q)$. Then $q^2(1 - x^2) = (r + \varepsilon')^2$, $q^2x^2 = (p + \varepsilon)^2$ and $|p| = O(q)$, $|r| = O(q)$. On rearranging it is readily verifed that

$$q^2 = p^2 + r^2 + 2p\varepsilon + 2r\varepsilon' + \varepsilon^2 + \varepsilon'^2.$$

But

$$|q^2 - p^2 - r^2| = |2p\varepsilon + 2r\varepsilon' + \varepsilon^2 + \varepsilon'^2| = O(q)o(1/q) = o(1) < 1$$

for q large enough. As $p, q, r \in \mathbb{Z}$, this implies that $q^2 = r^2 + p^2$, whence the point $(p/q, r/q)$ lies on the circle. $\square$

When p, r are coprime and p is even, the Pythagorean triples (p, r, q) are generated by positive coprime integers a, b of the following form:

$$p = 2ab, \quad r = a^2 - b^2, \quad q = a^2 + b^2 \tag{3}$$

(see [11]). Let $V(\theta; Q)$ be a sector in the disc of radius $Q^{1/2}$ subtending an angle θ at the origin. The number of lattice points (a, b) of opposite parity (*i.e.*, a, b are not both even or both odd) in $V(\theta; Q)$ is comparable to θQ (the number is $\theta Q(1 + o(1))/4$). By using the Möbius function, it can be seen that the number $\mathscr{N}(V(\theta; Q))$ of such lattice points (a, b) for which a and b are also coprime is comparable to θQ. Indeed

$$\mathscr{N}(V(\theta; Q)) = 3\theta Q(1 + o(1))/2\pi^2 \asymp \theta Q, \tag{4}$$

where for positive a, b, $a \asymp b$ indicates $a = O(b)$ and $b = O(a)$. The points $(p/q, r/q)$, $(r/q, p/q)$, where p and r are coprime to q, in $\mathbb{S}^1 \cap [0, 1]^2$ are in one-to-one correspondence with lattice points (a, b) where a, b are coprime positive integers of opposite parity with $a > b$ and satisfying (3).

1.1. The upper bound.

Let

$$S_v = \left\{u \in [-1, 1) \colon \|q(u, (1 - u^2)^{1/2})\|\} < q^{-v} \text{ for infinitely many } q \in \mathbb{N}\right\}.$$

Any cover of $\mathscr{S}_v(\mathbb{S}^1)$ by translates of squares Δ_i with sides parallel to the axes can be converted by projection to a cover of S_v by intervals I_i with s-length $\sum_i \ell(I_i)^s = \sum_i \ell(\Delta_i)^s$, where $\ell(I)$ is the length of I and $\ell(\Delta)$ is the side length of Δ. It follows that

$$\dim S_v = \dim \mathscr{S}_v(\mathbb{S}^1). \tag{5}$$

It is more convenient to work with S_v which can be expressed as

$$S_v = \{u \in [-1, 1] \colon u \in B(q) \text{ for infinitely many } q \in \mathbb{N}\},$$

where

$$\begin{aligned} B(q) &= \{u \in [-1, 1] \colon \|q(u, \sqrt{1 - u^2})\| < q^{-v}\} \\ &\subset \bigcup_{p=-q}^{q} \left\{u \in \left(\frac{p}{q} - \frac{1}{q^{v+1}}, \frac{p}{q} + \frac{1}{q^{v+1}}\right) \colon \left|\sqrt{1 - u^2} - \frac{r}{q}\right| < \frac{1}{q^{v+1}} \text{ for some } r \in \mathbb{Z}\right\}. \end{aligned}$$

The number of positive rationals p/q with $q \leqslant Q$ such that $p, r = \sqrt{q^2 - p^2}, q$ is a Pythagorean triple is $\ll Q$ ($\ll$ is the usual Vinogradov notation used instead of $O(Q)$). Taking symmetries into account, the number of distinct rational points $(p/q, r/q)$ in their lowest terms, with $2^t \leqslant q \leqslant 2^{t+1}$ lying on the circle $\mathbb{S}^1$ is

$$\ll \mathscr{N}(2^{t+1}) - \mathscr{N}(2^t) \ll 2^t,$$

where $\mathscr{N}(N)$ is the number of distinct rational points $(p/q, r/q)$ in their lowest terms, with $q \leqslant N$ lying on $\mathbb{S}^1$.

Since the length of the interval $(p/q - 1/q^{v+1}, p/q + 1/q^{v+1})$ is $2/q^{v+1} \ll (2^t)^{-v-1}$ and since p, q can be taken to be coprime without loss of generality, S_v has a cover of intervals with s-length

$$\ll \sum_{t=0}^{\infty} 2^{-t(v+1)s} \left(\mathscr{N}(2^{t+1}) - \mathscr{N}(2^t) \right) \ll \sum_{t=0}^{\infty} 2^{-t(v+1)s+t} < \infty$$

when $s > 1/(v+1)$. Hence by (5),

$$\dim \mathscr{S}_v(\mathbb{S}^1) = \dim S_v \leqslant 1/(v+1) \tag{6}$$

for $v > 1$.

Using the results of Wiles or Falting, it can be shown that when $k \geqslant 3$ and Γ_k is the curve $\{(x, y)\colon x^k + y^k = 1\}$, the set $\mathscr{S}_v(\Gamma_k)$ is finite for $v > k-1$ and is Γ_k when $v \leqslant 1/2$. Hence

$$\dim \mathscr{S}_v(\Gamma_k) \begin{cases} = 1, & v \leqslant 1/2, \\ \leqslant 3/2(v+1), & 1/2 < v < k-1, \\ = 0, & v \geqslant k-1. \end{cases}$$

1.2. The lower bound. This will be proved using the modification [15] of ubiquitous systems introduced in [7]. The definition we now give is simpler than the general one in [7] but is equivalent for affine spaces and in particular for intervals with resonant sets consisting of points.

Let Ω be a non-empty open set in Euclidean space, let R a measurable subset of Ω and let $|R|$ be the Lebesgue measure of R. For each $\delta > 0$, let

$$B(R; \delta) = \{\mathbf{x} \in \Omega : \operatorname{dist}(\mathbf{x}, R) \leqslant \delta\},$$

where $\operatorname{dist}(\mathbf{x}, R) = \inf\{|\mathbf{x} - \mathbf{y}|\colon \mathbf{y} \in R\}$. Let $\mathscr{R} = \{R_j : j \in J\}$ be a family of affine subsets of Ω, where J is an index set with each $j \in J$ having a weight $\lfloor j \rfloor > 0$ and each R_j having the same dimension $\dim \mathscr{R}$ and codimension $\operatorname{codim} \mathscr{R} = \dim \Omega - \dim \mathscr{R}$. The family $\mathscr{R}$ is said to be *ubiquitous with respect to a function* $\rho : \mathbb{N} \to \mathbb{R}^+$ if for any open subset $A \subset \Omega$ and all $N \in \mathbb{N}$, there exists a constant $c > 0$ such that

$$\left| A \cap \left(\bigcup_{\lfloor j \rfloor = 1}^{N} B(R_j, \rho(N)) \right) \right| \geqslant c|A|. \tag{7}$$

Given a function $\Psi\colon \mathbb{N} \to \mathbb{R}^+$, denote by $\Lambda(\mathscr{R};\Psi)$ the lim-sup set of points in Ω which lie in infinitely many $B(R_j;\Psi(\lfloor j\rfloor))$, *i.e.*,

$$\Lambda(\mathscr{R};\Psi) = \bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty}\bigcup_{\lfloor j\rfloor=k} B(R_j;\Psi(\lfloor j\rfloor)).$$

The following lower bound for the Hausdorff dimension of $\Lambda(\mathscr{R};\Psi)$ is proved in [7].

Theorem 2. *Let $\Psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonically decreasing function and suppose that the family $\mathscr{R}$ is ubiquitous with respect to a function $\rho\colon \mathbb{N} \to \mathbb{R}^+$, where $\rho(N)$ tends to 0 monotonically as N tends to infinity. Then*

$$\dim \Lambda(\mathscr{R};\Psi) \geqslant \dim \mathscr{R} + \gamma \operatorname{codim} \mathscr{R},$$

where

$$\gamma = \max\left\{1, \limsup_{N\to\infty} \frac{\log \rho(N)}{\log \Psi(N)}\right\}.$$

The relationship between the lattice points (a, b) and the corresponding points $p/q \in \mathbb{S}^1$ has to be discussed further.

Lemma 2. *Let A be an open subset of $[\sqrt{15}/8, \sqrt{3}/2]$ and let Q be sufficiently large. Then there exists a constant $K > 0$ independent of A, Q such that*

$$\left| A \cap \bigcup_{p/q} \left(\frac{p}{q} - \frac{K}{Q}, \frac{p}{q} + \frac{K}{Q}\right)\right| \geqslant K\,|A|,$$

where here and elsewhere the union is over rationals p/q with $p, \sqrt{q^2 - p^2}, q$ a Pythagorean triple.

Proof. Let $W(\theta;Q) = V(\theta;Q) \setminus V(\theta;Q/2)$, so that $W(\theta;Q)$ is the set of points (x, y) in $[0, Q^{1/2}]^2$ satisfying

$$Q/2 < x^2 + y^2 \leqslant Q,\ \ \theta' \leqslant \tan^{-1} y/x \leqslant \theta' + \theta$$

for some $\theta' \in [0, \pi/2 - \theta]$. Thus $W(\theta;Q)$ is a sector of an annulus subtending an angle θ with area

$$\theta(Q - Q/4) \asymp \theta Q.$$

Hence the number $\mathscr{N}(W(\theta;Q))$ of lattice points $(a, b) \in \mathbb{Z}^2 \cap W(\theta;Q)$, where a, b are of opposite parity and coprime, is comparable to $\theta\, Q$; in fact, by (4) above,

$$\mathscr{N}(W(\theta;Q)) = \frac{3\theta Q}{8\pi^2}(1 + o(1)) > \frac{\theta Q}{32}. \tag{8}$$

The set A is a countable union of disjoint open sets (α_r, β_r). Choose, as we may, $r_0 \in \mathbb{N}$ so that $|\bigcup_{r=1}^{r_0} (\alpha_r, \beta_r)| \geqslant |A|/2$. Consider one of these intervals (α, β) say. The interval (α, β) is the projection of the arc on $\mathbb{S}^1$ with endpoints $(\alpha, \sqrt{1 - \alpha^2})$ and $(\beta, \sqrt{1 - \beta^2})$. The angle θ subtended by the arc is comparable to $\beta - \alpha$, *i.e.*,

$$\theta \asymp \beta - \alpha. \tag{9}$$

As the lattice points (a,b) are co-prime, each lies on a unique line through the origin of gradient b/a. The function $f:[\sqrt{3},\sqrt{15}]\to[\sqrt{15}/8,\sqrt{3}/2]$ given by

$$f(x)=\frac{2x}{1+x^2}$$

takes b/a to $2ab/(a^2+b^2)=p/q$, where $p,\sqrt{q^2-p^2},q$ is a Pythagorean triple. The function f is one-to-one since its derivative

$$f'(x)=\frac{2(1-x^2)}{(1+x^2)^2}<0$$

for all $x\in[\sqrt{3},\sqrt{15}]$. It follows that

$$\mathscr{N}(W(\theta;Q))\asymp\nu(\alpha,\beta),$$

the number of $p/q\in(\alpha,\beta)$ with $Q/2<q\leqslant Q$ and $p,\sqrt{q^2-p^2},q$ a Pythagorean triple. But by the above, (8) and (9),

$$\nu(\alpha,\beta)\asymp\mathscr{N}(W(\theta;Q))\asymp\theta\,Q\asymp(\beta-\alpha)Q\geqslant c_1Q(\beta-\alpha), \tag{10}$$

where $c_1>0$ is independent of α, β and Q.

Also the function f is bi–Lipschitz on the interval $[\sqrt{3},\sqrt{15}]$, so that

$$|f(x)-f(x')|\asymp|x-x'|$$

for $x,x'\in[\sqrt{3},\sqrt{15}]$. Thus when $a/b\neq a'/b'$,

$$\left|f\left(\frac{b}{a}\right)-f\left(\frac{b'}{a'}\right)\right|=\left|\frac{2ab}{a^2+b^2}-\frac{2a'b'}{a'^2+b'^2}\right|\asymp\left|\frac{b}{a}-\frac{b'}{a'}\right|\geqslant\frac{1}{aa'}\geqslant\frac{1}{Q}$$

since $a,a'\leqslant Q^{1/2}$. Moreover if $p/q\neq p'/q'$, then $a/b\neq a'/b'$ and

$$\left|\frac{p}{q}-\frac{p'}{q'}\right|=\left|f\left(\frac{b}{a}\right)-f\left(\frac{b'}{a'}\right)\right|\geqslant\frac{c_2}{Q} \tag{11}$$

for some constant $c_2>0$. Now the length of the interval $(p/q-c_2/Q,p/q+c_2/Q)$ is $2c_2/Q$ and so by (11),

$$\left|(\alpha,\beta)\cap\bigcup_{p/q}\left(\frac{p}{q}-\frac{c_2}{Q},\frac{p}{q}+\frac{c_2}{Q}\right)\right|\geqslant\nu(\alpha,\beta)\frac{c_2}{Q}\geqslant c_1c_2(\beta-\alpha)$$

by (10). Since $A = \bigcup_{r=1}^{\infty}(\alpha_r, \beta_r)$ and the (α_r, β_r) are disjoint,

$$\begin{aligned}
\left| A \cap \bigcup_{p/q} \left(\frac{p}{q} - \frac{c_2}{Q}, \frac{p}{q} + \frac{c_2}{Q} \right) \right| &= \left| \bigcup_{r=1}^{\infty} (\alpha_r, \beta_r) \cap \bigcup_{p/q} \left(\frac{p}{q} - \frac{c_2}{Q}, \frac{p}{q} + \frac{c_2}{Q} \right) \right| \\
&= \sum_{r=1}^{\infty} \left| (\alpha_r, \beta_r) \cap \bigcup_{p/q} \left(\frac{p}{q} - \frac{c_2}{Q}, \frac{p}{q} + \frac{c_2}{Q} \right) \right| \\
&\geqslant \sum_{r=1}^{r_0} \left| (\alpha_r, \beta_r) \cap \bigcup_{p/q} \left(\frac{p}{q} - \frac{c_2}{Q}, \frac{p}{q} + \frac{c_2}{Q} \right) \right| \\
&\geqslant \sum_{r=1}^{r_0} c_1 c_2 (\beta_r - \alpha_r) \geqslant c_1 c_2 \frac{|A|}{2} = K|A|,
\end{aligned}$$

where $K = c_1 c_2 / 2$ is independent of Q, A, as claimed. □

Lemma 3. *The set of points $p/q \in [\sqrt{15}/8, \sqrt{3}/2]$ where $p, \sqrt{q^2 - p^2}, q$ is a Pythagorean triple is ubiquitous with respect to the function $\rho\colon \mathbb{N} \to \mathbb{R}^+$ given by $\rho(N) = c_2/N$.*

Proof. Let A be an open subset of $[\sqrt{15}/8, \sqrt{3}/2]$. Take the resonant points R_j to be p/q with $\lfloor j \rfloor = q$. Then $\dim R_j = \dim \mathscr{R} = 0$, $\operatorname{codim} \mathscr{R} = 1$,

$$B(R_j; \rho(\lfloor j \rfloor)) = \left(\frac{p}{q} - \frac{c_2}{q}, \frac{p}{q} + \frac{c_2}{q} \right)$$

and

$$\left| A \cap \bigcup_{\lfloor j \rfloor = 1}^{Q} B(R_j; \rho(Q)) \right| \geqslant \left| A \cap \bigcup_{p/q} \left(\frac{p}{q} - \frac{c_2}{Q}, \frac{p}{q} + \frac{c_2}{Q} \right) \right| \geqslant K|A|,$$

which satisfies the ubiquity condition (7), whence the family $\{p/q\}$ is ubiquitous with respect to $\rho\colon N \mapsto c_2/N$. □

Let $\Psi(q) = q^{-v-1}$. Then $\Lambda(\mathscr{R}; \Psi)$ is the set of points $u \in [\sqrt{15}/8, \sqrt{3}/2]$ such that

$$\max \left\{ \left| u - \frac{p}{q} \right|, \left| \sqrt{1 - u^2} - \frac{r}{q} \right| \right\} < q^{-v-1}$$

holds for infinitely many $p, q \in \mathbb{N}, r \in \mathbb{Z}$. It is readily verified that

$$\Lambda(\mathscr{R}; \Psi) \subset S_v,$$

whence $\dim S_v \geqslant \dim \Lambda(\mathscr{R}; \Psi)$. But by Theorem 2,

$$\dim \Lambda(\mathscr{R}, \Psi) \geqslant 0 + \limsup_{N \to \infty} \frac{\log N^{-1}}{\log N^{-1-v}} = \frac{1}{v+1}.$$

It follows from (5) and (6) that the dimension of $\mathscr{S}_v(\mathbb{S}^1)$ is $1/(1+v)$ and Theorem 2 is proved.

Lemma 1 could equally be used in conjunction with regular systems to obtain the desired lower bound. In [14], the function corresponding to ρ arises from Dirichlet's theorem on simultaneous Diophantine approximation and its value at N is comparable to $N^{1/2}$.

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