DYNAMICAL SYSTEMS AND APPLICATIONS

These notes consist of some handouts for the course together with solutions to some of the exercises. There are no guarantees as to correctness, so the calculations should be checked carefully!

Texts. There are many but the course was based mainly on

S. H. Strogatz ***, *Nonlinear Dynamics and Chaos*, Westview Press, 1994 (S7.38 STR).
with
G. F. Simmons, *Differential equations with applications and historical notes*, International Series in Pure and Applied Mathematics. McGraw-Hill, 1972 (S7.38 SIM)
for background and differential equations.

One dimensional systems

The one dimensional system is given by the first order differential equation

$$\dot{x} = \frac{dx}{dt} = f(x),$$

where f is a smooth real valued function of the real variable t (time). The solution is called a (one-dimensional) trajectory. We shall usually take the variable t (or time) to start at $t = 0$, in which case the *initial* value $x_0 = x(0)$ is the value of the trajectory $x(t)$ when $t = 0$. The equilibrium points x^* are the zeros of f, *i.e.*, $f^{-1}(0)$. There are 3 types of equilibrium point, stable (attracting), unstable (repelling), half-stable. The *potential* $V(x)$ of the system at the point x is defined by

$$V(x) = -\int_{x_0}^{x} f(u)du.$$

Thus

$$f(x) = -\frac{dV(x)}{dx}$$

and by the chain rule,

$$\begin{aligned} \dot{V}(x(t)) &= \frac{dV(x(t))}{dt} = \frac{dV(x)}{dx}\frac{dx(t)}{dt} = \frac{dV(x)}{dx}\dot{x}(t) \\ &= -f(x)f(x) = -f(x)^2 < 0 \end{aligned}$$

for $x \neq x^*$, thus the potential decreases along trajectories.

Example Let $\dot{x} = a - x^2$, where a is a real parameter. Find the equilibrium points x^* and obtain the trajectories for $a < 0$, $a = 0$ and $a > 0$. Discuss the nature of the equilibrium points briefly. Sketch the phase portraits and the bifurcation diagram. Find the potential V for the system.

Answer: Equilibrium points: $f(x) = a - x^2$, $f(x^*) = 0$ for $x^* = \pm\sqrt{a} \in \mathbb{R}$ iff $a \geq 0$. For $a > 0$, 0 is a stable (attracting) equilibrium point and a is an unstable (repelling) equilibrium point. When $a = 0$, 0 is half-stable.

Trajectories:

$$\dot{x} = a - x^2, i.e., \ \frac{dx}{dt} = a - x^2, i.e., \ \frac{dx}{a - x^2} = dt.$$

When $a < 0$,

$$t = \int_{x_0}^{x(t)} \frac{du}{a - u^2} = -\left[\frac{1}{\sqrt{-a}}\tan^{-1}\frac{u}{\sqrt{-a}}\right]_{x_0}^{x} = \frac{-1}{\sqrt{-a}}\left(\tan^{-1}\frac{x}{\sqrt{-a}}\tan^{-1}\frac{x_0}{\sqrt{-a}}\right),$$

where $x_0 = x(0)$, *i.e.*,

$$x(t) = -\sqrt{-a}\tan(\sqrt{-a}\,t) + x_0.$$

Thus as $t \to \pi/2$, $x(t) \to -\infty$.
When $a = 0$,

$$\frac{dx}{dt} = -x^2, i.e., \ -\frac{dx}{x^2} = dt, \text{ whence } \frac{1}{x} - \frac{1}{x_0} = t, \ x = \frac{x_0}{x_0 t + 1}$$

and $x \to 0$ (like t^{-1}) as $t \to \infty$.

When $a > 0$,

$$\frac{dx}{dt} = a - x^2, i.e., \; dt = \frac{dx}{a - x^2} = \frac{dx}{(\sqrt{a} - x)(\sqrt{a} + x)},$$

or for $|x_0| < \sqrt{a}$

$$t = \frac{1}{2} \int_{x_0}^{x} \left(\frac{1}{\sqrt{a} - u} + \frac{1}{\sqrt{a} + u} \right) du = \log \frac{\sqrt{a} + x}{\sqrt{a} - x} R_0,$$

where $R_0 = (\sqrt{a} - x_0)/(\sqrt{a} + x_0)$, and $x(t) \to \sqrt{a}-$ as $t \to \infty$. For $x_0 > \sqrt{a}$,

$$dt = -\frac{dx}{x^2 - a} = \frac{dx}{(x - \sqrt{a})(x + \sqrt{a})} = \frac{1}{2\sqrt{a}} \left(\frac{dx}{x - \sqrt{a}} + \frac{dx}{x + \sqrt{a}} \right),$$

so that

$$\begin{aligned} -t &= \frac{1}{2\sqrt{a}} \left(\int_{x_0}^{x} \frac{dx}{x - \sqrt{a}} - \frac{dx}{x + \sqrt{a}} \right) = \frac{1}{2\sqrt{a}} \left[\log(x - \sqrt{a}) - \log(x + \sqrt{a}) \right]_{x_0}^{x} \\ &= \frac{1}{2\sqrt{a}} \left(\log \frac{x - \sqrt{a}}{x + \sqrt{a}} - \log \frac{x_0 - \sqrt{a}}{x_0 + \sqrt{a}} \right) \end{aligned}$$

whence

$$\frac{(x - \sqrt{a})}{(x + \sqrt{a})} \frac{(x_0 + \sqrt{a})}{(x_0 - \sqrt{a})} = e^{-2\sqrt{a}t}$$

so when $x_0 > 0$ and $t \to \infty$, $x(t) \to \sqrt{a}+$; when $x_0 > 0$ and $t \to -1/x_0$, $x(t) \to -\infty$.

$x_0 < \sqrt{a}$

Stability for $a > 0$

$f'(x^*) = -2x^*$ so $f'(\sqrt{a}) = -2\sqrt{a}$, $f'(-\sqrt{a}) = 2\sqrt{a}$.

$V(x)$ has local minimum and maximum at zeros $-\sqrt{a}$ and $\sqrt{a}$ of f respectively, corresponding to stable and unstable equilibrium points respectively. When $a = 0$, equilibrium points coalesce to give one half stable equilibrium point. The bifurcation diagram:

Potential:

$$V(x) = -\int_{x_0}^{x} (a - u^2) du + V(x_0) = ax - \frac{x^3}{3} + ax_0 + \frac{x_0^3}{3}$$

where x_0 is any point in $\mathbb{R}$. For convenience can take $x_0 = 0$, so $V(x) = ax - x^3/3$.

Classifying planar linear systems

Second order ODE's. The solution of the first order homogeneous differential equation (or 1-dimensional system) $\dot{x} = ax$, $x(0) = x_0$ was found by trying

$$x(t) = x_0 e^{\lambda t}, \text{ where } x(0) = x_0.$$

The same method works for the second order homogeneous differential equation

$$a\ddot{x} + b\dot{x} + c = 0. \tag{1}$$

Substituting the trial function $x(t) = x_0 e^{\lambda t}$ yields

$$a\lambda^2 + b\lambda + c = 0.$$

If the roots are distinct, the two solutions λ_1, λ_2 for the exponent are given by

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \; \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Since sums and scalar multiples of solutions of (1) is also a solution, the general solution is

$$x(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t},$$

where the constants α_1, α_2 are determined by the two conditions on the solution, usually initial values of $x(t)$ ($x(0) = x_0$) and $\dot{x}(t)$ ($\dot{x}(0) = v_0$). If the eigenvalues are complex, they occur as conjugate pairs (since the coefficents on the LHS of (1) are real) and so are of the form

$$\lambda_1 = \frac{-b + i\sqrt{-\Delta}}{2a}, \ \lambda_2 = \frac{-b - i\sqrt{-\Delta}}{2a},$$

where $\Delta = b^2 - 4ac$, the discriminant, whence the general solution is of the form

$$\begin{aligned} x(t) &= \alpha_1 e^{-bt/2a} e^{i\sqrt{|\Delta|}/2a} + \alpha_2 e^{-bt/2a} e^{-i\sqrt{|\Delta|}/2a} \\ &= \alpha_1' e^{-bt/2a} \cos\left(\sqrt{\Delta}t/2a\right) + \alpha_2' e^{-bt/2a} \sin\left(\sqrt{\Delta}t/2a\right). \end{aligned}$$

Planar linear systems. In the simple cases discussed earlier, the matrices were diagonal and the solutions were of the form $x(t) = x_0 e^{\lambda t}$, $y(t) = y_0 e^{\lambda t}$. Of course the origin $(0,0)$ is the only equilibrium point as before. And as before the x- and y-axes play a special role and indeed are invariant in the sense that the flow does not leave them and the flow in phase spase is organised about them. By analogy in the general case, we seek vectors $\mathbf{u}$ (or directions) in the plane for which solutions of

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \ \mathbf{x}(0) = \mathbf{x}_0, \tag{2}$$

take the simple form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{u} \tag{3}$$

(note that the initial value $\mathbf{x}(0) = \mathbf{u}$ is not arbitrary). On trying (3), we get

$$\dot{\mathbf{x}}(t) = \lambda e^{\lambda t}\mathbf{u} = e^{\lambda t} A\mathbf{u}.$$

Thus for (2) to hold, we need to find a $\mathbf{u} \in \mathbb{R}^2$ and a scalar λ such that

$$A\mathbf{u} = \lambda\mathbf{u}, \textit{i.e.}, (A - \lambda 1)\mathbf{u} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is the familiar eigenvalue problem. The (non-zero) eigenvector of the matrix A exists if for some (eigenvalue) λ, the determinant $(a-\lambda)(d-\lambda) - bc$ of $(A - \lambda 1)$ vanishes. Thus the desired eigenvectors are the roots of the (characteristic) quadratic

$$\lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \operatorname{tr} A\lambda + \det A = (\lambda - \lambda_1)(\lambda - \lambda_2) = 0, \tag{4}$$

say

$$\lambda_1 = \frac{-\operatorname{tr} A + \sqrt{\Delta}}{2}, \ \lambda_2 = \frac{-\operatorname{tr} A - \sqrt{\Delta}}{2},$$

where the discriminant $\Delta = (\operatorname{tr} A)^2 - 4\det A$ and

$$\mathbf{u}_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix},$$

are the eigenvectors.

Discriminant $\Delta \neq 0$.

(1) If $\Delta > 0$, the eigenvectors are linearly independent (this is more than just not the same, they are not parallel). In this case the general solution (linear combination of two particular solutions) is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix} = c_1 e^{\lambda_1 t}\mathbf{u}_1 + c_2 e^{\lambda_2 t}\mathbf{u}_2$$

and since the eigenvectors are linearly independent, c_1, c_2 can chosen so that $\mathbf{x}(0) = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{x}_0$. Thus the initial condition in (2) is solved as well. In fact a general result on the existence and uniqueness of solutions gurantees that this is the *only* solution with this initial value.

(2) If $\Delta < 0$, the eigenvalues are complex:

$$\lambda_1 = \frac{-\operatorname{tr} A + i\sqrt{|\Delta|}}{2}, \ \lambda_2 = \frac{-\operatorname{tr} A - i\sqrt{|\Delta|}}{2}$$

and the real general solution is

$$\mathbf{x}(t) = c_1 e^{(-\operatorname{tr} A)t/2} \cos(\sqrt{|\Delta|}t/2)\mathbf{u}_1 + c_2 e^{(-\operatorname{tr} A)t/2} \sin(\sqrt{|\Delta|}t/2)\mathbf{u}_2.$$

Thus the phase portrait is an expanding spiral when $\operatorname{tr} A < 0$ (equilibrium point is unstable) or contracting when $\operatorname{tr} A > 0$ (stable) or a centre when $\operatorname{tr} A = 0$ (neutral).

Discriminant $\Delta = 0$. The eigenvalues are equal, $\lambda_1 = \lambda_2 = \lambda$ say and the eigenvectors

$$\mathbf{u}_1 = \begin{pmatrix} b \\ \lambda - a \end{pmatrix} \quad \text{or } \mathbf{u}_1 = \begin{pmatrix} \lambda - d \\ c \end{pmatrix} \tag{5}$$

are given by the solutions of the corresponding pair of linear homogeneous equations

$$\begin{aligned} (a-\lambda)u + bv &= 0 \\ cu + (d-\lambda) &= 0 \end{aligned}$$

If there are 2 linearly independent eigenvectors, then $b = c = 0$, $a = d = \lambda$ and

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

This can also be seen by observing that the general solution is

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{x}_0 = e^{\lambda t}(c_1\mathbf{u}_1 + c_2\mathbf{u}_2),$$

so that

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) = \lambda e^{\lambda t}\mathbf{x}_0 = \lambda\mathbf{x}(t) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\mathbf{x}(t).$$

If there is only one (non-zero) linearly independent eigenvector, then $(b, c) \neq (0, 0)$ and A is a *shear*. If $b \neq 0, c = 0$, then $a = d = \lambda$ and

$$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}.$$

Classification Diagram. The various kinds of behaviour of the trajectories near the equilibrium point in the Classification Diagram be captured in a very useful diagram (see Strogatz, p 137, Fig 5.2.8 or Simmons,).

Nonlinear systems

We consider the general dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \;\; \mathbf{x}(0) = \mathbf{x}_0, \tag{6}$$

where $\mathbf{f}$ is *smooth*[1] in a domain D[2]. Thus for each point $\mathbf{x} \in \mathbb{R}^2$, there is a vector $\dot{\mathbf{x}}$. This gives a *vector field* on $\mathbb{R}^2$.

Theorem 1. *Suppose* $\mathbf{f}$ *is a smooth function on a domain* D*. Let* $\mathbf{x}_0 \in D$*. Then there exists a* $\tau > 0$ *such that* (6) *has a unique solution* $\mathbf{x}(t)$ *for* $t \in (-\tau, \tau)$*.*

Corollary 1. *The phase space* $\{\mathbf{x} \colon \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})\}$ *fills out* $\mathbb{R}^n$*.*

For any point $\mathbf{x}_0 \in \mathbb{R}^n$ can be an initial value.

Corollary 2. *Trajectories do not intersect.*

For otherwise trajectories $\mathbf{x}(t)$ (solutions) would not be unique.

Thus, although potentially very complicated, phase portraits (which are a picture of the vector field) have 'combed' appearence, except at equilibrium points ('crowns'). There are important implications for $n = 2$, namely the Poincaré-Bendixson theorem, but for $n \geq 3$ much more complex geometry can occur.

Equilibrium points and linearisation. There is an extraordinary variety of phase portraits, even in the plane. In general it is unusual to be able to find trajectories for nonlinear systems analytically and even when it is possible, they are often too complicated to analyse and they give little insight. Luckily, for many nonlinear systems, the structure around equilibrium points is the same as the linearised approximation. Thus we can determine the *qualitative* behaviour of the system from some of the properties of f. This approach has its limitations and classifies up to *homeomorphism*[3], from this point of view a coffee cup is equivalent to a doughnut.

Terminology. These different kinds of behaviour call for some terminology. Recall that $\mathbf{x}^*$ is an equilibrium (fixed) point of a system. We will only consider the vicinity of $\mathbf{x}^*$.

(1) If there exists a $\delta > 0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*$ [4] when $|\mathbf{x}(0) - \mathbf{x}^*|_2 < \delta$, then $\mathbf{x}^*$ is an *attracting* equilibrium point. Thus $\mathbf{x}(t)$ can leave a neighbourhood of $\mathbf{x}^*$ in the short run but eventually (after time t_0) stays within it.

   Examples: $\lambda_1 \leq \lambda_2 < 0$, λ's complex with strictly negative real part.

(2) If for each $\varepsilon > 0$, there exists a $\delta > 0$ such that when $|\mathbf{x}(0) - \mathbf{x}^*|_2 < \delta$, $|\mathbf{x}(t) - \mathbf{x}^*|_2 < \varepsilon$ for *all* $t \geq 0$, then $\mathbf{x}^*$ is a *Liapunov stable* equilibrium point. Thus $\mathbf{x}(t)$ stays close to $\mathbf{x}^*$ for all positive time.

   Examples: $\lambda_1 < 0, \lambda_2 = 0$ (spine – non-attracting), node.

(3) If $\mathbf{x}^*$ is both attracting and Liapunov stable, it is *asymptotically stable.*

   Examples: $\lambda_1 \leq \lambda_2 < 0$, λ's complex with strictly negative real part.

(4) If $\mathbf{x}^*$ is neither attracting nor Liapunov stable, it is *unstable.*

   Examples: $\lambda_1 \leq 0 < \lambda_2$, λ's complex with strictly positive real part.

Main features.

(1) **Equilibrium points** $\mathbf{x}^*$. Satisfy $\mathbf{f}(\mathbf{x}^*) = 0$ and correspond to equlibria.
(2) **Closed orbits**. Associated with centres and correspond to periodic solutions (there exists τ such that $\mathbf{x}(t+\tau) = \mathbf{x}(t)$ for all t). The phase portrait can be unstable, in sense that can get either continuous family of closed orbits or an isolated closed orbit.
(3) Structure of trajectories influenced by nearby equilibrium points and closed orbits.
(4) Attracting (stable) and repelling (unstable) effects of equilibrium points and closed orbits.

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[1] The function $\mathbf{f}$ is smooth if it is sufficiently differentiable. Usually f will be taken to be C^2, *i.e.*, the n^2 second order derivatives

$$\frac{\partial^2 \mathbf{f}}{\partial x_1^2}, \frac{\partial^2 \mathbf{f}}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 \mathbf{f}}{\partial x_n^2}$$

are continuous.

[2] A *domain* D is an open connected set in $\mathbb{R}^n$.

[3] cts 1-1 onto maps with cts inverses

[4] Given $\varepsilon > 0$, there exists a $t_0 = t_0(\varepsilon)$ such that $|\mathbf{x}(t) - \mathbf{x}^*|_2 < \varepsilon$ for all $t \geq t_0$.

Linearisation near equilibrium points. The derivative $f'(x^*)$ played a key role in organising the behaviour of the trajectories near an equilibrium point. The 2-dimensional theory goes in an analogous way and we work in real variables so that we can use Taylor's theorem[5]. Thus instead of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, we consider

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y), \end{aligned}$$

where f, g are smooth, $\mathbf{x}^* = (x^*, y^*)$ is an equilibrium point, so that

$$f(x^*, y^*) = g(x^*, y^*) = 0.$$

Let (h, k) be a small perturbation of $(x^*, y^*))$ and write

$$x = x^* + h, \; y = y^* + k.$$

To investigate whether the perturbation grows or shrinks, we investigate $\dot{h}$ and $\dot{k}$. First since (x^*, y^*) is a constant vector and since $f(x^*, y^*) = 0$,

$$\begin{aligned} \dot{h} &= \dot{x} = f(x^* + h, y^* + k) \\ &= f(x^*, y^* + k) + h\frac{\partial f(x^*, y^* + k)}{\partial x} + O(h^2) \\ &= f(x^*, y^*) + h\frac{\partial f(x^*, y^*)}{\partial x} + k\frac{\partial f(x^*, y^*)}{\partial y} + O(h^2, hk, k^2) \\ &= h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + O(h^2, hk, k^2). \end{aligned}$$

Similarly,

$$\dot{y} = h\frac{\partial g}{\partial x} + k\frac{\partial g}{\partial y} + O(h^2, hk, k^2).$$

Collecting terms, the evolution of the perturbation (h, k) can be expressed in matrix form:

$$\begin{aligned} \dot{\mathbf{h}} &= \begin{pmatrix} \dot{h} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} + O(h^2, k^2) \\ &= J\mathbf{h} + O(|\mathbf{h}|^2). \end{aligned}$$

The matrix $J = J(\mathbf{f}, \mathbf{x}^*)$ is the *Jacobian* of $\mathbf{f} = (f, g)$ at $\mathbf{x}^*$ and is the 'linear approximation' of $\mathbf{f}$ in the vicinity of $\mathbf{x}^*$. For small h, k, it is plausible that the error term $O(h^2, k^2)$ can be ignored and just the so-called *linearised* (and familiar) system for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

$$\dot{\mathbf{h}} = \begin{pmatrix} \dot{h} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = J\,\mathbf{h}$$

considered.

The validity of neglecting nonlinear terms. Providing the fixed point $\mathbf{x}^*$ under consideration is not close to one of the borderline cases in the Classification Diagram, the structure of the linearised system is essentially the same as the original system. That is, a node, a saddle or a spiral in the linearised system correspond respectively to a node, a saddle or a spiral in the nonlinear system.

Centres lie on the x-axis ($trA = 0$) of the Classification Diagram and clearly, the structure changes dramatically with a small perturbation of the trace. Similarly other such cases are sensitive to small changes and the linearised system tells us only that the original nonlinear system is not robust.

Examples. From Strogatz 6.3.1, 6.3.2.

(1)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x + x^3 \\ -2y \end{pmatrix}$$

(2)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y + ax(x^2 + y^2) \\ x + ay(x^2 + y^2) \end{pmatrix}$$

[5] if f is C^2, then f' is C^1 and $f'(a + h) = f'(a) + hf''(a) + O(h^2)$.

Robust or structurally stable equilibrium points.

- **Repellers** (sources): each eigenvalue λ has positive real part ($\Re\lambda > 0$).
- **Attractors** (sinks): each eigenvalue λ has negative real part ($\Re\lambda < 0$).
- **Saddles**: eigenvalues real, non-zero and of opposite sign ($\lambda_1\lambda_2 < 0$)

Real part of neither eigenvalue vanishes (equilibrium point *hyperbolic*); analogue of $f'(x^*) \neq 0$. This extends to higher dimensions and by the Hartman-Grobman theorem, the stability type of a hyperbolic equilibrium point is preserved by the linearised form (*i.e.*, they are *topologically equivalent* – structures are homeomorphic, trajectories map to trajectories preserving direction).

Delicate or structurally unstable equilibrium points.

- Centres : each eigenvalue pure imaginary ($0 \neq \lambda \in i\mathbb{R}$).
- Higher order and non-isolated equilibrium points: at least one eigenvalue vanishes.

At least one eigenvalue satisfies $\Re\lambda = 0$; analogue of $f'(x^*) = 0$.

Lotka-Volterra equations (i): rabbits and foxes. [Simmons §39] Consider an idealised isolated environment inhabited by rabbits and foxes, with a supply of grass that is always adequate. The foxes reduce the numbers of rabbits until there are too few to support the foxes, whose numbers then also drop. This allows the rabbit numbers to recover, in turning allowing fox numbers to increase also. And so it goes. As a first approximation, in the absence of foxes, the rate of increase $\dot{x}$ in the number $x = x(t)$ (not necessarily in $\mathbb{N}$!) of rabbits is proportional to x (carrying capacity is infinite). On the other hand, because too many foxes will reduce the number of rabbits, the number of foxes is modelled (crudely) by $\dot{y} = -cy$, $c > 0$ (no carrying capacity term y^2). A natural assumption is that the number of conclusive encounters of rabbits with foxes is proportional to xy, where y is the number of foxes. This is assumed to have a positive effect on fox numbers y, $\dot{y} = -cy + dxy$, and a negative one for the rabbit, giving $\dot{x} = x(a - by)$ for some positive constants a, b. Combining these we get that such a predator-prey system can be modelled by

$$\begin{aligned} \dot{x} &= x(a - by) \\ \dot{y} &= y(-c + dx) \end{aligned}$$

The equilibrium points are $\mathbf{x}^* = (0, 0)$ and $\mathbf{x}^* = (c/d, a/b)$. The Jacobian

$$J(x, y) = \begin{pmatrix} a - by & -bx \\ dy & -c + dx \end{pmatrix}.$$

Hence

- $$J(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}, \quad \text{eigenvalues } a,\ -c \text{ of opposite sign, saddle, eigenvector's } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $$J\left(\frac{c}{d}, \frac{a}{b}\right) = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{pmatrix} : \ \operatorname{tr} A = 0, \det A > 0, \text{ eigenvalues purely imaginary, centre}$$

  The numbers $c/d, a/b$ are called *equilibrium populations.* The 'linearised' centre is made up of a family of ellipses

  $$\frac{(x - c/d)^2}{b^2c} + \frac{(y - a/b)}{ad^2} = K.$$

Thus near the equilibrium populations, the two populations oscillate.

Lotka-Volterra equations (ii): sheep and rabbits. [Strogatz §6.4] Sheep and rabbits *competing* for same resource. Here we introduce the ideas of *growth rate* r, *carrying capacity* K and the *logistic equation*. For an isolated population

$$\dot{x} = rx\left(1 - \frac{x}{K}\right).$$

Let $x(t) = \#$ rabbits and $y(t) = \#$ sheep. We will take $r = 3$, $rK = 1$ for rabbits and and $r = 2$, $rK = 1$ for sheep. Then combining the two populations, the system is

$$\begin{aligned} \dot{x} &= x(3 - x - 2y) \\ \dot{y} &= y(2 - x - y). \end{aligned}$$

This 'toy' model gives useful insights. The equilibrium points (where $\dot{\mathbf{x}} = (0,0)$) are $\mathbf{x}^* = (0,0), (0,2), (3,0), (1,1)$ and the Jacobian is

$$J(x,y) = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}$$

- $$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} : \text{ eigenpairs } (3, (1,0)); (2, (0,1))\text{: unstable (repelling) node}$$

- $$J(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} : \text{ eigenpairs: } (-1, (1,-2)); (-2, (0,1))\text{: stable (attracting) node}$$

- $$J(3,0) = \begin{pmatrix} -3 & 6 \\ 0 & -1 \end{pmatrix} : \text{ eigenpairs } (-1, (3,-1)); (-3, (1,0))\text{: stable (attracting) node}$$

- $$J(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} : \text{ eigenpairs } (-1+\sqrt{2}, (\sqrt{2},-1)); (-1-\sqrt{2}, (\sqrt{2},1))\text{: saddle}$$

Conservative systems. The motivating example is a unit mass particle at position $x \in \mathbb{R}$, acted on by a force $F(x)$, where F is continuous and does *not* depend on t or $\dot{x}$ (thus no friction or damping). By Newton's laws, the trajectory (motion) of the particle is given by the (2nd order ordinary) differential equation

$$\ddot{x}(t) = F(x). \tag{7}$$

For example, for the particle falling under gravity $F(x) = -g$ or at the end of a horizontal spring $F(x) = -\lambda x$ (Hooke's Law). The potential energy $V(x)$ of the system (7) at x is defined by

$$V(x) = -\int_{x_0}^{x} F(u)du,$$

so that

$$\frac{dV}{dx} = -F(x).$$

(Note that since F is continuous, $V(x)$ always exists and involves an arbitrary constant.) Then

$$\ddot{x} = -\frac{dV}{dx}$$

and so

$$\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = \dot{x}\ddot{x} + \frac{dV}{dx}\frac{dx}{dt} = \dot{x}\ddot{x} + \frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + V\right) = 0.$$

Hence the quantity $\dot{x}^2/2 + V$ does not change with time. The constant is called the *energy* of the system (7) and is usually denoted by E:

$$E = \frac{\dot{x}^2}{2} + V.$$

More formally, a *conserved quantity* of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{8}$$

is a continuous real valued function E which is *non-constant* on open sets[6] which is nevertheless constant on trajectories (solutions) $\mathbf{x}(t)$ of the system (8)[7]. The system (8) is said to be *conservative* if it possesses a conserved quantity. Thus the 1st order system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= F(x) \end{aligned}$$

derived from (7) will be conservative. The trajectories are level curves

$$\{\mathbf{x}(t) : E(\mathbf{x}(t)) = E(\mathbf{x}(0))\}$$

of *constant* energy and are closed. Conservative systems are *not* the same as *gradient* systems.

A consequence A conservative system has no attracting or repelling points. For if there were an equilibrium point $\mathbf{x}^*$, then for any point $\mathbf{x}$ in the basin of attraction of $\mathbf{x}^*$, $E(\mathbf{x}) = E(\mathbf{x}^*)$, whence $E(\mathbf{x})$ is constant on any open set in the basin, contradicting the requirement that E be non-constant on open sets.

Example: Consider 2nd order, 1 dimensional system

$$\ddot{x} = -\frac{dV}{dx}.$$

Convert to 2nd order 1 dimensional system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{dV}{dx}. \end{aligned}$$

Let potential energy

$$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}.$$

Put $E(0,0) = 0$. What is E for some other orbits? For what values of E is the trajectory a closed curve? [The phase portrait is a 'crossed-eyed monster'.]

Trajectories that begin and end at the *same* (resp *different*) equilibrium points are called *homoclinic* (resp. *heteroclinic*).

Curves in a phase portrait where either $\dot{\mathbf{x}} = 0$ or $\dot{\mathbf{y}} = 0$ are called *Nullclines*.

Recall that trajectories which meet at an equilibrium point fall into different types:

- stable manifolds
- unstable manifolds.

Just one type meets at a node, both types meet at a saddle.

Reversible systems. The pendulum described in cylindrical phase space and using time reversal symmetric energy surface. These systems contain closed orbits [Strogatz, §6.6].

Gradient systems. [Strogatz, p. 199] The smooth planar first order system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = (f(x,y), g(x,y))$$

is a *gradient* system if there exists a smooth scalar *potential* fuction $V(\mathbf{x})\,(\in \mathbb{R})$ such that

$$\dot{x} = f(x,y) = -\frac{\partial V}{\partial x},\ \dot{y} = g(x,y) = -\frac{\partial V}{\partial y},$$

i.e., if

$$\dot{\mathbf{x}} = -\nabla V = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}\right).$$

The trajectories $\mathbf{x}(t)$ of a gradient system cross the equipotential (level) curves

$$L_c = \{\mathbf{x} \colon V(\mathbf{x}) = c\}$$

of V orthogonally, *except* at equilibrium points. For the normal to L_c is given by ∇V and the tangent to the trajectory $\mathbf{x}(t)$ is given by $\dot{\mathbf{x}}(t)$. Since for $\mathbf{x}$ not an equilibrium point, the non-zero vector

$$\nabla V(\mathbf{x}) = (\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}) = \dot{\mathbf{x}}$$

[6]an open set in the plane is a union of open discs

[7]being non-constant on trajectories excludes the trivial solution $\mathbf{x}(t) = 0$ for all t

the normal and tangent are parallel, whence L_c and $\mathbf{x}(t)$ meet orthogonally.

Motivating example [Note that this is a 2nd order 3-dimensional system] The force of gravitational attraction on a planet of mass m by a sun of mass M is given (in polar coordinates) by

$$\ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r},$$

where G is the universal constant of gravitation and $r = \|\mathbf{r}\|$. The gravitational potential $V(\mathbf{r})$ is given by

$$V(\mathbf{r}) = -\frac{GM}{r}$$

(the potential when the planet is at ∞ is 0 and decreases as the planet approaches the sun).

Corollary 3. *Closed orbits are impossible in a planar gradient system* $\dot{\mathbf{x}} = -\nabla V$.

Suppose there were a closed orbit C, with period T. Then

$$[V(\mathbf{x}(t))]_C = V(\mathbf{x}(T)) - V(\mathbf{x}(0)) = 0$$

($V(\mathbf{x})$ scalar function). But also

$$\begin{aligned} [V(\mathbf{x}(t))]_C &= \int_0^T \frac{dV}{dt}\,dt = \int_0^T \left(\frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}\right) dt = \int_0^T (\nabla V \cdot \dot{\mathbf{x}})\,dt \\ &= -\int_0^T (\dot{\mathbf{x}} \cdot \dot{\mathbf{x}})dt = -\int_0^T \|\mathbf{x}\|^2 dt < 0 \end{aligned}$$

by continuity – a contradiction.

NB Do not confuse with the opposite conclusion for *conservative* systems.

Liapunov functions. Conservative systems have an energy E which is constant on trajectories. This idea can be weakened to energy-like functions which decrease along trajectories (increasing time) and which are called *Liapunov* functions L; this however prevents closed orbits. More formally, suppose the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

has an equilibrium point at $\mathbf{x}^*$. A real valued positive definite smooth function $L(\mathbf{x})$ such that$L(\mathbf{x}^*) = 0$ and

(1) $L(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$
(2) $\dot{L} < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$.

((2) implies that all trajectories flow 'downhill' to the minimum $\mathbf{x}^*$ of V.) Then $\mathbf{x}^*$ is globally asymptotically stable: for all initial conditons $\mathbf{x}(0)$, $\mathbf{x}(t) \to \mathbf{x}^*$ at $t \to \infty$. In particular, the system has no closed orbits as all trajectories move monotonically down the graph (surface) of $V(\mathbf{x})$ to minimum $\mathbf{x}^*$. (Jordan and Smith 1987).

No systematic procedure to construct Liapunov functions – trial and error.

Index Theory. The index of a closed (positively oriented) curve C measures the 'winding' of the vector field $(\mathbf{x}, \mathbf{f}(\mathbf{x}))$ on C. First the angle the vector $\mathbf{f}(\mathbf{x})$ makes with the x-axis can be measured by means of the formula

$$\tan\theta = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{g(x,y)}{f(x,y)},$$

providing $\mathbf{f}(\mathbf{x}) \neq 0$ for $\mathbf{x}$ on C. The number of times the vector $\mathbf{f}(\mathbf{x}) = \dot{\mathbf{x}}$ on C turns around as it describes the curve, starting at $\theta(0)$ and finishing at $\theta(T)$, is given by

$$\frac{1}{2\pi}[\theta]_C = \frac{1}{2\pi}(\theta(T) - \theta(0)) = \frac{1}{2\pi}\Delta\theta \in \mathbb{Z}.$$

The curve C can transformed continuously by a sufficiently small amount to a new curve C' without passing through any equilibrium points (since they are a closed set $\mathbf{f}^{-1}(0)$) and without altering the vector field very much. Thus I_C can be regarded as an integer valued, continuous function of C. As such I_C cannot change value in an open connected set and so is locally constant. Thus, if the curve C is transformed continuously to a new C' without passing through any equilibrium points, then $I_{C'} = I_C$.

Corollary 4. *If* C *does not enclose an equilibrium point, then* $I_C = 0$.

For the curve C can be shrunk to one, say Σ, which is so small that $I_\Sigma = 0 = I_C$.

Index of an equilibrium point. Let C be *any* closed curve enclosing just *one* isolated equilibrium point $\mathbf{x}^*$. Then I_C is the index of $\mathbf{x}^*$ and is well defined (the same for any such curve C). Thus we can define the *index of an isolated equilibrium point* x^*:

$$I(\mathbf{x}^*) := I_C.$$

Theorem 2. *The index of*

$$\begin{cases} \textit{stable node} \\ \textit{unstable saddle} \\ \textit{saddle} \end{cases} = \begin{cases} +1 \\ +1 \\ -1 \end{cases}$$

Theorem 3. *Suppose a closed curve C encloses n isolated equilibrium points $x_1^*, \ldots, x_n^*$. Then*

$$I_C = I(x_1^*) + \cdots + I(x_n^*)$$

Proof. Consider the figure:

□

Lemma 1. *The index of a closed orbit (trajectory) is* $+1$

For $\mathbf{f}(\mathbf{x}(t)) = \dot{\mathbf{x}}(t)$ is tangent to the curve (trajectory) $\mathbf{x}(t)$ and goes round just once.

Application Closed orbits are impossible in 'two-competitor system' model (rabbits v sheep).

Limit cycles. A closed orbit or trajectory associated with a centre $\mathbf{x}^*$ is not isolated in the sense that there is a continuum of closed orbits about $\mathbf{x}^*$. A *limit cycle* is an *isolated* closed orbit.

Example [Strogatz, §7] The differential equation

$$\dot{r} = r(1 - r^2), \ \dot{\theta} = 1$$

has solution for $r(t) < 1$ given by

$$\frac{r(t)^2}{1 - r(t)^2} R_0 = e^{2t}, \ \theta = \theta_0.$$

On $\mathbb{S}^1$, *i.e.*, for $r = 1$, $r(t) = 1$ for all t.

Poincaré-Bendixson theorem and order. This covers closed orbits, including limit cycles.

Theorem 4. *Suppose*

(1) Ω *is a closed bounded region*
(2) *the system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *is smooth in an open set containing* Ω,
(3) Ω *contains no equilibrium points,*
(4) *there is a trajectory* $\mathbf{x}(t)$ *which does not leave* Ω *for all* $t \geq 0$.

Then the trajectory is either a closed orbit or it spirals towards one as $t \to \infty$, *so that* Ω *always contains a closed orbit.*

This is one of the fundamental results in dynamical systems and shows that even though there more variety in the plane than in the real line, the plane is still pretty limited. A trajectory confined to a closed bounded region with no equilibrium points is either a loop or approached a loop. In dimensions of three or greater, the theorm no longer applies and the behaviour of trajectories can be extaordinary complicated, with trajectories wandering around for ever without ever eventually settling down. Extreme sensitivity to initial conditions and fractal sets make their appearance and with it, chaos.

Bifurcation

The boundaries between the regions in the Classification Diagram are associated with the vanishing of higher order terms (or *degeneracy*) in the (multivariate) Taylor's series expansion for the function $\mathbf{f}(\mathbf{x})$. The degeneracy (or vanishing of $\det' s$ or eigenvectors) of higher terms corresponds to the coalescence of local minima or maxima and so to the disappearance of stability or instability.

Saddle-Node Bifurcation. To make this clear, first consider the one-dimensional case where the function $f(x) = f_a(x) = f(a, x) = a - x^2$ has a parameter a which affects the character of the equilibrium points (zeros of f_a):

$$\dot{x} = a - x^2$$

A bifurcation occurs at $a = 0$, where $f'(0) = f''(0) = 0$. The stable and unstable equilibrium points in the vector field (phase line) shown in (a) ($a < 0$) coalesce at $a = 0$ to give a half-stable point which disappears for $a > 0$. This can be represented in a *bifurcation diagram.*

Fold bifurcation: $a = \pm\sqrt{x^*}$

Bifurcation diagrams

$$\{(x, a) \colon f(x, a) = 0\} = f^{-1}(0)$$

will be curves (in 2 dimensions) and not graphs in general.

Ex. Calculate time spent in a neighbourhood of the origin for $a > 0$.

This behaviour occurs when in the Taylor's series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + O(x^3)$$

for f at 0, $f'(0) = 0$, $f''(0) \neq 0$. Thus a fold bifurcation will occur when a 'parabolic' part of the graph of $f(x, a)$ in the $x, \dot{x}$ plane cuts the x-axis. The formula

$$r = \pm\sqrt{x^*}$$

will hold approximately in the vicinity of x^*. In general the variables might not be simple but the logarithmic form

$$\log a = \frac{1}{2}\log x^* + C$$

is sometimes more convenient.

Pitchfork Bifurcation. There are two types:

Supercritical. Normal form

$$\dot{x} = ax - x^3 = x(a - x^2)$$

Supercritical pitchfork bifurcation diagram

Subcritical. Normal form

$$\dot{x} = ax + x^3$$

Subcritical pitchfork bifurcation diagram

Dimensionless equations. It is often convenient to remove dimensions from equations. Use dimensional analysis

Imperfect Bifurcations and Catastrophes. Most real systems don't exhibit perfect symmetry and including imperfections can have an important influence on the nature of the solutions. For example, introduce an *imperfection parameter* h into the normal form for the Supercritical pitchfork bifurcation, as follows

$$\dot{x} = h + ax - x^3. \tag{9}$$

The roots of the cubic depend on the discriminant Δ of the coefficients:

$$\Delta = h^2 - \frac{4}{27}a^3 \begin{cases} < 0 & \text{3 real roots} \\ = 0 & \text{2 real roots} \\ > 0 & \text{1 real root} \end{cases}$$

Bifurcation diagram: semi-cubical parabola given by $3^3h^2 - 2^2a^3 = 0$

The root information is encoded in the Riemann-Hugoniot surface:

Riemann-Hugoniot surface associated with roots of (9)

Two dimensional saddle node and pitchfork bifurcations. The eigenvectors are real. The two dimensional saddle node bifurcation is given by the (decoupled) system

$$\dot{x} = a - x^2,\ \dot{y} = -y.$$

Supercritical pitchfork:

$$\dot{x} = ax - x^3,\ \dot{y} = -y$$

The bifurcation diagrams essentially correspond to the one-dimensional case and the bifurcations are associated with an eigenvalue vanishing [Strogatz, §8.1].

The Hopf bifurcation. The eigenvalues λ_1, λ_2 of the Jacobian $J(\mathbf{x}^*)$ are the roots of a real quadratic, given by

$$\lambda_1 = \frac{-b + i\sqrt{-\Delta}}{2a},\ \lambda_2 = \frac{-b - i\sqrt{-\Delta}}{2a},$$

and are either both real or complex conjugates ($\bar{\lambda_1} = \lambda_2$). The dependence of the eigenvalues on the parameters are the key to bifurcation. For a stable equilibrium point, the eigenvalues the real parts of the λ_1, λ_2 must be negative. For stability to be lost as the parameters vary, one of the eigenvalues must cross into the positive half plane. When the eigenvalues are real (and negative), they must pass through the origin and hence vanish for some value of the parameters. The cases when this happens are the saddle node and pitchfork bifurcations.

When the eigenvalues are complex conjugates, they can acquire positive real part without passing through the origin.

An idealised example To illustrate this consider the supercritical bifurcation exhibited by the simple 3-parameter system (in polar co-ordinates, the natural choice for centres)

$$\begin{aligned} \dot{r} &= ar - r^3 \\ \dot{\theta} &= \omega + br^2, \end{aligned}$$

where a controls the stability of the equilibrium point at 0, ω is constant angular velocity component and b governs the angular velocity in terms of r. When $a < 0$, there is a stable spiral at 0 (with sense depending on sign of ω). When $a = 0$, 0 is a (weakly) stable spiral ($r(t) \to 0$ like $t^{-1/2}$). When $a > 0$, there is an unstable spiral at 0 with a stable limit cycle at $r = \sqrt{a}$.

To analyse the behaviour of the eigenvalues through the bifurcation, switch to cartesian co-ordinates

$$\begin{aligned} \dot{x} &= (a - (x^2 + y^2))x - (\omega + b(x^2 + y^2))y = bx - \omega y + O(x^3, y^3) \\ \dot{y} &= (a - (x^2 + y^2))y + (\omega + b(x^2 + y^2))x = \omega x + ay + O(x^3, y^3) \end{aligned}$$

(best for computing Jacobians) with Jacobian

$$J(0,0) = \begin{pmatrix} a & -\omega \\ \omega & a \end{pmatrix}$$

has eigenvalues $\lambda = a \pm i\omega$. These have negative real part when $a < 0$, zero real part when $a = 0$ and positive real part when $a > 0$, *i.e.*, the eigenvalues cross the imaginary axis as a increases from negative to positive values.

For a more general supercritical situation, when the eigenvalue $\lambda = \lambda(a) = \Re\lambda(a) + i\Im\lambda(a)$, and critical value is $\Re\lambda(a_c)$, radius of limit cycle for parameter a not too much greater that a_c is approximately $\sqrt{a - a_c}$, frequency of cycle is $\Im\lambda(a) = \Im\lambda(a_c) + O(a - a_c)$.

Subcritical Hopf bifurcation Consider the system

$$\begin{aligned} \dot{r} &= ar + r^3 - r^5 \\ \dot{\theta} &= \omega + br^2, \end{aligned}$$

Because of the sign change, in the bifurcation, trajectories are forced away from the origin. For $a < 0$, there is an attracting (stable) limit cycle, a stable equilibrium point at origin and between them a repelling (unstable) limit cycle. At $a = 0$, the unstable cycle swallows the origin, resulting in an unstable equilibrium point. For $a > 0$, there is one stable limit cycle, so that trajectories move awy from the origin. On reversing the process, *hysteresis* occurs.

Finally, there is a *degenerate* Hopf bifurcation, *e.g.*, the damped pendulum with differential equation $\ddot{x} + a\dot{x} + \sin x = 0$.

Distinguishing between the different kinds of Hopf bifurcation can be tricky when the Hessian $H(x,y)$ of $\mathbf{f}(\mathbf{x}) = (f(x,y), g(x,y))$, given by

$$H(x,y) = \begin{pmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x \partial y \\ \partial^2 g/\partial x \partial y & \partial^2 g/\partial y^2 \end{pmatrix},$$

is not invertible ($\det H = 0$). but computers can be used in practice.

Quasiperiodicity The natural configuration space for systems of the form

$$\begin{aligned}\dot{\theta}_1 &= f_1(\theta_1, \theta_2)\\ \dot{\theta}_2 &= f_2(\theta_1, \theta_2),\end{aligned}$$

where f_1, f_2 are periodic (2π) in both arguments is the *torus* $\mathbb{T} := \mathbb{S}^1 \times \mathbb{S}^1$ which can be represented as the unit square $[0,1]^2$ with opposite sides identified.

The discrete and continuous. Notation: for any $x \in \mathbb{R}$, the *integer part* $[x]$ is the greatest integer at most x (the graph looks like a staircase); the *fractional* part $\{x\} := x - [x]$. The graph is 1-periodic; there should be no confusion with the singleton set $\{x\}$. The name is, however, unfortunate since the fractional part can be irrational and *non-integer* would be better.

Lemma 2.

$$\{\{nx\} : n \in \mathbb{Z}\} \textit{ is } \begin{cases} \textit{finite} & \textit{if } x \in \mathbb{Q} \\ \textit{infinite} & \textit{if } x \notin \mathbb{Q} \end{cases}$$

Kronecker's theorem The one-dimensional case:

Theorem 5. *The set* $S = \{\{nx\} : n \in \mathbb{N}\}$ *is dense in* $[0,1]$, *i.e.,*

$$\overline{S} = \overline{\{\{nx\} : n \in \mathbb{N}\}} = [0,1].$$

Proof. Since the set $S(x) = \{\{nx\} : n \in \mathbb{N}\}$ is infinite and bounded (lies in $[0,1]$), $S(x)$ has a limit point l (Bolzano-Weierstrass theorem), *i.e.*, there is a subsequence of points $\{nx\}$ converging to l. Hence given any $\varepsilon > 0$, there are infinitely many pairs of points $P_n = \{nx\}$, $P_{n+r} = \{(n+r)x\}$ with

$$|\{nx\} - \{(n+r)x\}| = |\{rx\}| < \varepsilon.$$

Now take any point $P_m = \{mx\}$ and add the vector $\{(n+r)x\} - \{nx\}$ of length less than ε. Then the fractional part of the resulting point is $\{(m+r)x\}$ and lies in $S(x)$. Adding the vector (which has length less than ε) to $\{x\}$ successively we get a chain of intervals of equal length less than ε with end points in $S(x)$ and which cover $[0,1]$. Hence each point in $[0,1]$ is at most ε from a point in $S(x)$. $\square$

Dirichlet's theorem shows that ε can be taken to be $1/n$ and $l = 0$ or 1. In the quantitative form of Kronecker's theorem, $\varepsilon = 1/n$ and l can be *any* point in $[0,1]$. The higher dimensional case. This comes in two versions: first, the discrete, proved much as above.

Theorem 6. *Suppose* $\theta_1, \theta_2, \ldots, \theta_k, 1$ *are linearly independent and* $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in [0,1]^k$. *Then for any* $\varepsilon, N > 0$, *there exist integers* $n > N$ *and* p_1, p_2 *such that*

$$\max_{j=1,\ldots,k} \{|n\theta_j - p_j - \alpha_j|\} < \varepsilon,$$

i.e.,

$$\overline{S(\theta)} = \overline{\{\{n(\theta_1, \ldots, \theta_k)\} : n \in \mathbb{N}\}} = [0,1]^k.$$

And in an equivalent continuous or *quasiperiodic* flow version (note the weaker hypothesis and weaker conclusion):

Theorem 7. *Suppose* $\theta_1, \ldots, \theta_k$ *are linearly independent. Then for any* $(\alpha_1, \ldots, \alpha_k) \in [0,1]^k$ *and any* $\varepsilon, T > 0$, *there exist a real* $t > T$ *and* $p_1, \ldots, p_k \in \mathbb{Z}$ *such that*

$$\max_{j=1,\ldots,k} \{|t\theta_j - p_j - \alpha_j|\} < \varepsilon$$

i.e.,

$$\overline{\{\{t(\theta_1, \ldots, \theta_k)\} : t \in \mathbb{R}\}} = [0,1]^k.$$

Note that if these theorems hold for $(\alpha_1, \ldots, \alpha_k)$ with $(\theta_1, \ldots, \theta_k)$ and for $(\beta_1, \ldots, \beta_k)$ with $(\theta_1, \ldots, \theta_k)$, then it holds for $(\alpha_1 + \beta_1, \ldots, \alpha_k + \beta_k)$ with $(\theta_1, \ldots, \theta_k)$, since when $p\alpha - \alpha$ and $q\alpha - \beta$ are nearly integers, so is $(p+q)\theta - (\alpha + \beta)$. Also can take $0 < \theta_j < 1$ by relabelling the variables suitably. We'll sketch why Theorem 6 implies Theorem 7.

If $\theta_1, \ldots, \theta_k, \theta_k$ are linearly independent, then so are

$$\frac{\theta_1}{\theta_k}, \ldots, \frac{\theta_{k-1}}{\theta_k}, 1.$$

Hence by the $k-1$-dimensional case of Theorem 6 with $N = T$, there exist integers $n > N$, $p_1, \ldots, p_{k-1} \in \mathbb{Z}$ such that

$$\max_{j=1,\ldots,k-1} \left\{ \left| \frac{n\theta_j}{\theta_k} - p_j - \alpha_j \right| \right\} < \varepsilon.$$

Letting $t = n/\theta_k > n > T$ we get

$$\max_{j=1,\ldots,k-1} \{ |t\theta_j - p_j - \alpha_j| \} < \varepsilon$$

and $|t\theta_k - n - 0| = 0 < \varepsilon$, the result for $\alpha_1, \ldots, \alpha_{k-1}, 0$. To deal with the case $0, \ldots, 0, \alpha_k$, consider the linearly independent set

$$1, \frac{\theta_2}{\theta_1}, \ldots, \frac{\theta_k}{\theta_1}$$

and apply Theorem 6 to get

$$\max_{j=2,\ldots,k} \left\{ \left| \frac{n\theta_j}{\theta_k} - p_j - \alpha_j \right| \right\} < \varepsilon,$$

take $t = n/\theta_1$ and $p_1 - n$.

For details, see Hardy and Wright, *An Introduction to the Theory of Numbers*, Chapter 23.

The oscillator [Strogatz, §4.3, §8.6] The oscillator governed by the DE

$$\dot{\theta} = \omega - a \sin\theta,$$

where $a \geq 0$ and $\omega > 0$, has equilibrium points θ^* for $a > \omega$, given by the two solutions of

$$\sin\theta^* = \frac{\omega}{a}.$$

The equilibrium points are stable for $-a\cos\theta^* < 0$, *i.e.*, $\cos\theta^* > 0$ and unstable for $\cos\theta^* < 0$. Can be used for the coupled oscillator

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 + K_1 \sin(\theta_2 - \theta_1) \\ \dot{\theta}_2 &= \omega_2 + K_2 \sin(\theta_1 - \theta_2) \end{aligned}$$

by putting $\varphi = \theta_1 - \theta_2$, the phase difference, $\omega = \omega_1 - \omega_2$, $K = K_1 + K_2$ and considering

$$\dot{\theta} = \dot{\theta}_1 - \dot{\theta}_2 = \omega - K \sin\varphi.$$

MERRY WINTER SOLSTICE AND A HAPPY XMAS