EXTREMAL MANIFOLDS AND HAUSDORFF DIMENSION

H. DICKINSON AND M. M. DODSON

ABSTRACT. The best possible lower bound for the Hausdorff dimension of the set of v-approximable points on a C^1 extremal manifold is obtained.

1. INTRODUCTION

The recent proof by D. Y. Kleinbock and G. A. Margulis [11] of Sprindžuk's conjecture for smooth 'non-degenerate' manifolds M means that the set $\mathscr{L}_v(M)$ of v-approximable points (this and other terminology is explained below) on M is of zero induced Lebesgue measure. This raises the question of its Hausdorff dimension. Bounds and indeed the exact dimension for manifolds satisfying a variety of arithmetic, geometric and analytic conditions are known [2, 3, 5, 7]. In this paper ubiquity is used to obtain a lower bound for the Hausdorff dimension of a set more general than $\mathscr{L}_v(M)$ for any extremal C^1 manifold M. Hitherto volume estimates which depend on curvature conditions were used to overcome a 'small denominators' problem. It turns out however that extremality when combined with Fatou's lemma is all that is needed. We begin with some notation.

Let $|x| = \max\{|x_1|, \ldots, |x_n|\}$ denote the supremum norm or height of the point $x = (x_1, \ldots, x_n)$ in n-dimensional Euclidean space $\mathbb{R}^n$ and let $|x|_2 = (x_1^2 + \cdots + x_n^2)^{1/2}$ denote its Euclidean norm. Throughout $\mathbf{q} = (q_1, \ldots, q_n)$ will be a vector in $\mathbb{Z}^n$ and $\mathbf{q} \cdot x = q_1x_1 + \cdots + q_nx_n$ will denote the usual inner product. For positive numbers a, b, we use the Vinogradov notation $a \ll b$ and $b \gg a$ if $a = O(b)$; if $a \ll b \ll a$, we write $a \asymp b$. A point $x \in \mathbb{R}^n$ which satisfies

$$\|\mathbf{q} \cdot x\| < |\mathbf{q}|^{-v} \tag{1}$$

for infinitely many $\mathbf{q} \in \mathbb{Z}^n$ will be called *v-approximable* ($\|x\|$ is the distance of the real number x from $\mathbb{Z}$). Let M be an m-dimensional manifold in $\mathbb{R}^n$. The set of v-approximable points in the manifold M will be denoted by $\mathscr{L}_v(M)$. The manifold M is called *extremal* if for any $v > n$, $\mathscr{L}_v(M)$ has Lebesgue measure 0. Equivalently, by Khintchine's transference principle, M is extremal if the set $\mathscr{S}_w(M)$ of points $x \in M$ which are simultaneously w-approximable, *i.e.,* for which

$$\|qx\| < |q|^{-w}$$

for infinitely many $q \in \mathbb{Z}$, is null (*i.e.,* of measure zero) when $w > 1/n$. Khintchine's theorem implies that the real line is extremal and the terminology reflects the fact that

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the order of approximation given by Dirichlet's theorem is unimprovable for almost all points on an extremal manifold [12, p. 66].

Let U be an open set in $\mathbb{R}^m$, where $m \leqslant n$. V. G. Sprindžuk conjectured that if the functions $\theta_j \colon U \to \mathbb{R}$, $j = 1, \dots, n$ are analytic and, together with 1, independent over $\mathbb{R}$, then the manifold

$$\{(\theta_1(u), \dots, \theta_n(u)) \colon u \in U\} = \theta(U) \subset \mathbb{R}^n$$

is extremal (see Conjecture H_1 in [19]). Manifolds satisfying a variety of additional or different analytic, geometric and number theoretic conditions have been shown to be extremal; references and further details can be found in [18, 19]; see also [4, 7, 9, 11, 20].

In the stronger Baker-Sprindžuk conjecture, the hypotheses on the manifold M are the same but the approximation function $|\mathbf{q}|^{-v}$ is replaced by a larger multiplicative anisotropic function. One asks if when $v > n$, the set of points $x \in M$ for which

$$\|\mathbf{q} \cdot x\| < \prod_{j=1}^{n} (|q_j| + 1)^{-v/n} \tag{2}$$

for infinitely many $\mathbf{q} \in \mathbb{Z}^n$ is relatively null, in which case M is said to be strongly extremal (Conjecture H_2 in [19]). Points satisfying (2) for infinitely many $\mathbf{q} \in \mathbb{Z}^n$ are called multiplicatively v-approximable. Transference principles allow simultaneous and multiplicative approximation forms of these conjectures (see [11, 18]). The conjecture H_2 was first proposed by A. Baker for the rational normal curve

$$\mathscr{V} = \{(t, t^2, \dots, t^n) \colon t \in \mathbb{R}\}$$

in [1, p 96] and proved for this case by V. I. Bernik [6].

J. Kubilius proved the parabola extremal in 1949 [13] and in 1964 W. M. Schmidt established the remarkable result that any C^3 planar curve with non-zero curvature almost everywhere is extremal [16]. About the same time, Sprindžuk [17] proved Mahler's conjecture, corresponding to the rational normal curve being extremal. Recently D. Y. Kleinbock and G. A. Margulis [11] have proved a result which implies not only Sprindžuk's conjecture H_1 but also the Baker-Sprindžuk conjecture H_2. They used ideas from dynamical systems, namely unipotent flows in homogeneous spaces of lattices and the correspondence between multiplicatively v-approximable points for $v > n$ and unbounded orbits in the space of lattices. Although at the moment their techniques do not yield non-trivial upper bounds for the Hausdorff dimension, they do give a partial Khintchine-type result and might open the way to further progress.

R. C. Baker [3] refined W. M. Schmidt's result [16] by showing that if the curvature of a C^3 planar curve vanishes only on a set with Hausdorff dimension 0, then for $v \geqslant 2$

$$\dim \mathscr{L}_v(M) = \frac{3}{v+1}.$$

Using the idea of regular systems, A. Baker and W. M. Schmidt [2] showed that $\dim \mathscr{L}_v(\mathscr{V}) \geqslant (n+1)/(v+1)$ for $v \geqslant n$. The complementary upper inequality was established by V. I. Bernik [5], giving

$$\dim \mathscr{L}_v(\mathscr{V}) = \frac{n+1}{v+1}$$

for $v \geqslant n$. For manifolds M with dimension $m \geqslant 2$ and satisfying a curvature condition that reduces to non-vanishing Gaussian curvature for surfaces in $\mathbb{R}^3$,

$$\dim \mathscr{L}_v(M) = m - 1 + \frac{n+1}{v+1}$$

for $v \geqslant n$ [7]. We shall use ubiquity [8] to obtain the best possible lower bound for the Hausdorff dimension of the more general set

$$\mathscr{L}(M;\psi) = \{x \in M \colon \|\mathbf{q} \cdot x\| < \psi(|\mathbf{q}|) \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\}$$

when M is a C^1 extremal manifold in $\mathbb{R}^n$ and the function $\psi \colon \mathbb{N} \to \mathbb{R}^+$ decreases. Note that when $\psi(q) = q^{-v}$, we write $\mathscr{L}_v(M)$ for $\mathscr{L}(M;\psi)$. For more information about Hausdorff dimension, see [10, 14].

2. Ubiquitous systems

Let U be a non-empty open subset of $\mathbb{R}^m$. Let

$$\mathscr{R} = \{R_j \subset U \colon j \in J\}$$

be a family of sets indexed by J; these sets will be called *resonant*. Suppose further that each $j \in J$ has a weight $\lfloor j \rfloor > 0$ and let $\rho \colon \mathbb{N} \to \mathbb{R}^+$ be a function converging to 0 at infinity. Suppose that for each sufficiently large positive integer N there exists a set $A(N) \subset U$ for which

$$\lim_{N\to\infty} |U \setminus A(N)| = 0. \tag{3}$$

Let

$$B(R_j;\delta) = \{u \in U \colon \operatorname{dist}(u, R_j) < \delta\}, \tag{4}$$

where $\operatorname{dist}(u,R) = \inf\{|u-r| \colon r \in R\}$. Let $H/2$ denote the hypercube H shrunk by $1/2$ and with the same centre.

Suppose that there exists a constant $d \in [0,m]$ such that given any hypercube $H \subset U$ with sidelength $\ell(H) = \rho(N)$ and $H/2$ meeting $A(N)$, there exists a $j \in J$ with $\lfloor j \rfloor \leqslant N$ such that for all $\delta \in (0,\, \rho(N)]$,

$$|H \cap B(R_j;\delta)| \gg \delta^{m-d}\,\ell(H)^d. \tag{5}$$

Suppose further that given any other hypercube H' in U with $\ell(H') \leqslant \rho(N)$,

$$|H' \cap H \cap B(R_j;\delta)| \ll \delta^{m-d}\,\ell(H')^d. \tag{6}$$

Then the pair $(\mathscr{R}, \lfloor\,.\,\rfloor)$ is called a *ubiquitous system with respect to* ρ.

In the one-dimensional case and when the resonant sets consist of points, ubiquitous and regular systems are virtually equivalent and essentially differ only in their formulation [15].

3. Hausdorff dimension

The distribution of the resonant sets in ubiquitous systems allows the determination of a general lower bound for the lim-sup set

$$\Lambda(\mathscr{R};\psi) = \{u \in U\colon\ \operatorname{dist}(u, R_j) < \psi(\lfloor j \rfloor) \text{ for infinitely many } j \in J\},$$

where $\psi\colon \mathbb{N} \to \mathbb{R}^+$ is a decreasing function [8].

Theorem 1. *Suppose $(\mathscr{R}, \lfloor\,.\,\rfloor)$ is ubiquitous with respect to $\rho\colon \mathbb{N} \to \mathbb{R}^+$ and that $\psi\colon \mathbb{N} \to \mathbb{R}^+$ is a decreasing function satisfying $\psi(N) \leqslant \rho(N)$ for N sufficiently large. Then*

$$\dim \Lambda(\mathscr{R};\psi) \geqslant d + \gamma(m-d)$$

where $\gamma = \limsup_{N\to\infty} (\log \rho(N))/(\log \psi(N)) \leqslant 1$.

The hypothesis that $\psi(N) \leqslant \rho(N)$ for N sufficiently large implies that $\gamma \leqslant 1$.

We now apply Theorem 1 to Diophantine approximation on a manifold. The lower order $\lambda(f)$ of the function $f\colon \mathbb{N} \to \mathbb{R}^+$ is defined by

$$\lambda(f) = \liminf_{N\to\infty} \frac{\log f(N)}{\log N}.$$

Theorem 2. *Let M be an m-dimensional C^1 extremal manifold embedded in $\mathbb{R}^n$. Let $\psi\colon \mathbb{N} \to \mathbb{R}^+$ be decreasing with the lower order of $1/\psi$ denoted by λ. Then for $\lambda \geqslant n$*

$$\dim \mathscr{L}(M;\psi) \geqslant m - 1 + (n+1)/(\lambda+1).$$

Since $\dim \mathscr{L}(M\cap V;\psi) \leqslant \dim \mathscr{L}(M;\psi)$, it suffices to consider the open subset $M \cap V$ of M, where V is a suitable open set in $\mathbb{R}^n$. We assume without loss of generality that $M\cap V \subset [-1,1]^n$ and that V is sufficiently small. Let $\theta\colon U \to M\cap V$ be the local parametrisation where the domain U is a sufficiently small hypercube in $[-1,1]^m$. We write $M_U = M \cap V = \theta(U)$. Each point $x \in M_U$ can be written $x = \theta(u)$ for some $u \in U$.

Since the manifold M is C^1, by shrinking and closing U if necessary, we can assume that the geodesic distance between two points x, x' on M_U is comparable with $|x - x'|$ and that θ is bi-Lipschitz on U. Hence we can assume that the Hausdorff dimension of $\mathscr{L}(M_U;\psi)$ and that of

$$L(\psi) = \{u \in U\colon \|\mathbf{q}\cdot\theta(u)\| < \psi(|\mathbf{q}|) \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\}$$

are the same [10]. We write L_v for $L(\psi)$ when $\psi(r) = r^{-v}$; thus

$$L_v = \{u \in U\colon \|\mathbf{q}\cdot\theta(u)\| < |\mathbf{q}|^{-v} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\}.$$

By the inverse function theorem, we can also assume that M_U is the graph of a C^1 (Monge) ordinate function $\varphi\colon U \to \mathbb{R}^k$, where $k = n - m$, so that

$$M_U = \{\theta(u)\colon u \in U\} = \{(u, \varphi(u))\colon u \in U\}$$

and $\theta = 1_U \times \varphi$. The corresponding local chart $h\colon M_U \to U$ is the restriction to M_U of the projection $\mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$. Moreover by shrinking and closing U again if necessary, we can assume $|\partial\varphi_j/\partial u_i| \leqslant K_{ij} < \infty$ for each $u \in U$, $i = 1,\ldots,m$, $j = 1,\ldots,k$. Indeed given $\delta > 0$, we can choose U so that for any $u \in U$,

$$K_{ij} - \delta \leqslant |\partial\varphi_j(u)/\partial u_i| \leqslant K_{ij}.$$

Thus we can assume that the change in the direction of a vector along any geodesic in M_U is small.

It follows that M_U is 'not close to orthogonal' to the plane $\mathbb{R}^m \times \{0\}$, $0 = (0, \dots, 0) \in \mathbb{R}^k$, as indicated in Figure 1. More precisely, for each $\theta(u)$ in M_U, the angle ϑ say between any vector t in the tangent space $T_{\theta(u)}M_U$ and the plane $\mathbb{R}^m \times \{0\}$ satisfies $\cos\vartheta \geqslant c$ for some constant $c > 0$ (*i.e.,* in the Vinogradov notation $\cos\vartheta \gg 1$). Thus for any $\theta(u)$ in M_U the plane $\mathbb{R}^m \times \{0\}$ is 'not close to being orthogonal' to $T_{\theta(u)}M_U$. In other words, the normal space $T_{\theta(u)}M_U^{\perp}$ is not 'close to being parallel' to $\mathbb{R}^m \times \{0\}$.

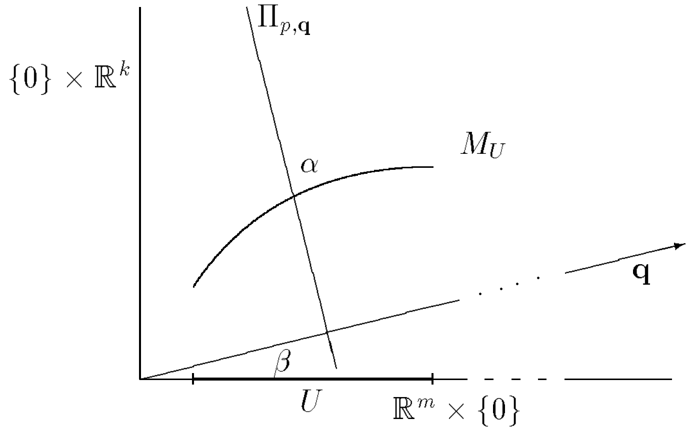

FIGURE 1. The manifold M_U and a resonant set $\Pi_{p,\mathbf{q}}$

Since M is extremal, $\mathscr{L}_v(M_U) = \theta(L_v)$ is null for $v > n$ in the induced measure on M and, since θ is bi-Lipschitz on U, the set L_v is null in $\mathbb{R}^m$ when $v > n$. To obtain a lower bound for the Hausdorff dimension of $\mathscr{L}_v(M)$ or equivalently for L_v, it suffices to find a sequence of suitable sets $A(N) \subset U$ which approximate U in measure and which satisfy the intersection conditions (5) and (6) above. Using the geometry of numbers, integer vectors $\mathbf{q}$ will be chosen so that the hyperplanes

$$\Pi_{p,\mathbf{q}} = \{x \in \mathbb{R}^n : \mathbf{q} \cdot x = p\}$$

associated with the resonant sets $R_{p,\mathbf{q}}$, defined below in (11), are not close to being parallel or tangential to M_U (see Figure 1). This condition is stronger than $\Pi_{p,\mathbf{q}}$ being transversal to M_U.

Let $\eta > 0$ be arbitrary and $N \in \mathbb{N}$ be sufficiently large. By Minkowski's linear forms theorem, given $u \in U$ there exist $\mathbf{q} = \mathbf{q}(u) = (q_1, \dots, q_n) \in \mathbb{Z}^n$ with $1 \leqslant |\mathbf{q}| \leqslant N$ and $p = p(u) \in \mathbb{Z}$ such that

$$\left.\begin{array}{rcl} |\mathbf{q} \cdot \theta(u) - p| & \leqslant & N^{-n+k\eta}(\log N)^k \\ |q_i| & \leqslant & N, \ i = 1, \dots, m \\ |q_{m+j}| & \leqslant & N^{1-\eta}(\log N)^{-1}, \ j = 1, \dots, k. \end{array}\right\} \tag{7}$$

Hence for each $N = 1, 2, \dots$, the set U can be written

$$U = A(N) \cup S(N) \cup E(N), \tag{8}$$

where $E(N)=\{u\in U\colon \operatorname{dist}(u,\partial U)\leqslant 1/N\}$ (∂U is the boundary of U),

$$S(N)=\{u\in U\colon 1\leqslant|\mathbf{q}|<N^{1-\eta}\text{ for some }\mathbf{q}\text{ satisfying (7)}\}$$

and

$$A(N)=U\setminus(E(N)\cup S(N)),$$

so that $A(N)$ consists of points $u\in U\setminus E(N)$ for which there exist $\mathbf{q}\in\mathbb{Z}^n$ and $p\in\mathbb{Z}$ satisfying (7) and

$$N^{1-\eta}\leqslant|\mathbf{q}|\leqslant N. \tag{9}$$

Thus each $u\in A(N)$ is at least $1/N$ from ∂U (in the supremum metric) and there exists a large vector $\mathbf{q}\in\mathbb{Z}^n$ and an integer p satisfying (7).

The measure of $E(N)$ converges to 0 as $N\to\infty$ since

$$|E(N)|=|\{u\in U\colon \operatorname{dist}(u,\partial U)\leqslant 1/N\}|\ll \ell(U)^m-\left(\ell(U)-\frac{1}{N}\right)^m\ll N^{-1}.$$

The vector $\mathbf{q}=\mathbf{q}(u)\in\mathbb{Z}^n$ can be written

$$\mathbf{q}=(q_1,\dots,q_m,0,\dots,0)+(0,\dots,0,q_{m+1},\dots,q_n)=\mathbf{q}'+\mathbf{q}''$$

say where $\mathbf{q}'\in\mathbb{R}^m\times\{(0,\dots,0)\}$, and $\mathbf{q}''\in\{(0,\dots,0)\}\times\mathbb{R}^k$. Since N is large enough, for each $u\in A(N)$, the vector $\mathbf{q}$ is close to being parallel to $\mathbf{q}'$. Indeed the angle β which $\mathbf{q}$ makes with $\mathbb{R}^m\times\{0\}$ satisfies

$$\cos\beta=\frac{\mathbf{q}}{|\mathbf{q}|_2}\cdot\frac{\mathbf{q}'}{|\mathbf{q}'|_2}\geqslant 1-\frac{q_{m+1}^2+\cdots+q_n^2}{|\mathbf{q}|_2^2}=1-O\left(\frac{1}{\log N}\right)^2$$

by (7) and (9). Hence the hyperplane $\Pi_{p,\mathbf{q}}$ which is normal to $\mathbf{q}$ meets M_U not close to tangentially. This implies that $\Pi_{p,\mathbf{q}}\cap M_U$ is a connected $(m-1)$-dimensional submanifold of M_U.

On replacing N by $N^{1/(1-\eta)}$ in (7), it can be seen that the set $S(N^{1/(1-\eta)})$ is contained in the set of points $u\in U$ for which there exist $p,\mathbf{q}$ satisfying

$$|\mathbf{q}\cdot\theta(u)-p|<N^{-(n-k\eta)/(1-\eta)}(\log N)^k(1-\eta)^{-k}$$

with $1\leqslant|\mathbf{q}|\leqslant N$. Moreover $S(N^{1/(1-\eta)})$ is also a subset of

$$T_\delta(N)=\{u\in U\colon |\mathbf{q}\cdot\theta(u)-p|<N^{-n-\delta}\text{ for some }\mathbf{q}\in\mathbb{Z}^n,p\in\mathbb{Z},1\leqslant|\mathbf{q}|\leqslant N\},$$

where $0<\delta<\eta(n-k)/(1-\eta)$.

Lemma 1. *For any $\delta>0$,*

$$\limsup_{N\to\infty}T_\delta(N)=\bigcap_{k=1}^{\infty}\bigcup_{N=k}^{\infty}T_\delta(N)\subseteq L_{n+\delta}.$$

Proof. Let $u \in \cap_{k=1}^{\infty} \cup_{N=k}^{\infty} T_\delta(N)$. Then $u \in T_\delta(N_j)$ for an infinite sub-sequence N_j, $j = 1, 2, \ldots$. Hence for each $j = 1, 2, \ldots$, there exist $\mathbf{q}^{(j)} \in \mathbb{Z}^n$ with $1 \leqslant |\mathbf{q}^{(j)}| \leqslant N_j$ and $p^{(j)} \in \mathbb{Z}$ such that

$$|\mathbf{q}^{(j)} \cdot \theta(u) - p^{(j)}| < N_j^{-n-\delta}.$$

Suppose there are only finitely many different $\mathbf{q}^{(j)}$ for which the last displayed inequality holds and let

$$\min\{|\mathbf{q}^{(j)} \cdot \theta(u) - p^{(j)}| \colon j \in \mathbb{N}\} = c$$

say. If $c > 0$, then choosing j so that $N_j^{-n-\delta} < c$ gives a contradiction. If $c = 0$, then for each $r \in \mathbb{N}$, $r \leqslant |r\mathbf{q}^{(j)}| \leqslant rN_j$ and

$$|(r\mathbf{q}^{(j)}) \cdot \theta(u) - (rp^{(j)})| = 0 < (rN_j)^{-n-\delta}.$$

Thus there are infinitely many solutions, contradicting the supposition that there exist only a finite number of different $\mathbf{q}^{(j)}$. But $1 \leqslant |\mathbf{q}^{(j)}| \leqslant N_j$, whence

$$|\mathbf{q}^{(j)} \cdot \theta(u) - p^{(j)}| < |\mathbf{q}^{(j)}|^{-n-\delta}$$

holds for infinitely many j. Thus $u \in L_{n+\delta}$. $\square$

By Fatou's lemma, for any $\delta > 0$,

$$\limsup_{N\to\infty} |T_\delta(N)| \leqslant |\limsup_{N\to\infty} T_\delta(N)| \leqslant |L_{n+\delta}| = 0$$

since M is extremal. Thus $\lim_{N\to\infty} |T_\delta(N)| = 0$. But when $0 < \delta < \eta(n-k)/(1-\eta)$, $T_\delta(N) \supseteq S(N^{1/(1-\eta)})$ and so

$$\lim_{N\to\infty} |S(N)| = \lim_{N\to\infty} |S(N^{1/(1-\eta)})| = 0.$$

Applying this and the estimate for $|E(N)|$ above to (8), it follows that

$$|U \setminus A(N)| \leqslant |E(N)| + |S(N)| \to 0 \tag{10}$$

as $N \to \infty$ and $A(N)$ satisfies (3). The resonant sets in U are now chosen to be

$$R_{p,\mathbf{q}} = \{u \in U \colon \mathbf{q} \cdot \theta(u) = p\} = h(\Pi_{p,\mathbf{q}} \cap M_U) \tag{11}$$

where $\mathbf{q}$ and p are given by (7). Thus d, the dimension of $R_{p,\mathbf{q}}$, is $m-1$.

For each $u \in A(N)$ there exists a pair $(p, \mathbf{q})$ satisfying (7) and $N^{1-\eta} \leqslant |\mathbf{q}| \leqslant N$. For N sufficiently large, the hyperplane $\Pi_{p,\mathbf{q}}$ will be far from tangential to M_U. Because of this and θ being bi-Lipschitz,

$$\operatorname{dist}(u, R_{p,\mathbf{q}}) \asymp \operatorname{dist}(\theta(u), \theta(R_{p,\mathbf{q}})) \asymp \frac{|\mathbf{q} \cdot \theta(u) - p|}{|\mathbf{q}|_2 |\cos\varpi|},$$

where ϖ is the angle between the tangent plane $T_{\theta(u)}M_U$ and $\mathbf{q}$. Since $\Pi_{p,\mathbf{q}}$ meets M_U not close to tangentially, $\cos\varpi \asymp 1$ and so for any $u \in U$,

$$\operatorname{dist}(u, R_{p,\mathbf{q}}) \asymp \frac{|\mathbf{q} \cdot \theta(u) - p|}{|\mathbf{q}|_2}.$$

It follows from this and (7) that there are positive c_*, c^* such that

$$c_* \,|\mathbf{q}\cdot\theta(u) - p|\,|\mathbf{q}|^{-1} \leqslant \operatorname{dist}(u, R_{p,\mathbf{q}}) \leqslant c^*\, N^{-n-1+\eta(k+1)}(\log N)^k. \tag{12}$$

Let

$$\rho(N) = 4c^* N^{-n-1+(k+1)\eta}(\log N)^k. \tag{13}$$

We now show that the other ubiquity properties (5) and (6) hold for the family $\mathscr{R}$ of resonant sets $\{R_{p,\mathbf{q}}\}$ where $\lfloor(p,\mathbf{q})\rfloor = |\mathbf{q}|$ and $\rho\colon \mathbb{N} \to \mathbb{R}^+$ is given by (13). Let H be a hypercube with $\ell(H) = \rho(N)$. The choice of $\mathbf{q}$ which ensures that $\Pi_{p,\mathbf{q}}$ meets M_U not close to tangentially together with the choice of ρ imply that if in addition $u \in H/4$, then by (12), there exist $p, \mathbf{q}$ such that $\operatorname{dist}(u, R_{p,\mathbf{q}}) \leqslant \ell(H)/4$. Hence the resonant set $R_{p,\mathbf{q}}$ meets the hypercube H substantially and

$$|H \cap B(R_{p,\mathbf{q}};\delta)| \gg \ell(H)^{m-1}\delta,$$

where $B(R_{p,\mathbf{q}};\delta)$ is given by (4), as required for (5) to hold.

It also follows that $\Pi_{p,\mathbf{q}}$ meets M_U in a connected $(m-1)$-dimensional submanifold, so that any hypercube H' with $\ell(H') \leqslant \rho(N)$ satisfies

$$|H' \cap H \cap B(R_{p,\mathbf{q}};\delta)| \ll \ell(H')^{m-1}\min\{\delta, \ell(H')\} \ll \ell(H')^{m-1}\delta$$

and (6) holds. Thus the family $\mathscr{R} = \{R_{p,\mathbf{q}} \colon \mathbf{q} \in \mathbb{Z}^n \setminus \{0\}, p \in \mathbb{Z}\}$ is ubiquitous in U with respect to ρ. Hence by Theorem 1, for any decreasing function $\widetilde{\psi}\colon \mathbb{N} \to \mathbb{R}^+$,

$$\dim \Lambda(\mathscr{R}, \widetilde{\psi}) \geqslant m - 1 + \gamma,$$

where $\Lambda(\mathscr{R}, \widetilde{\psi})$ is the set of points u in U satisfying

$$\operatorname{dist}(u, R_{p,\mathbf{q}}) < \widetilde{\psi}(\lfloor(p,\mathbf{q})\rfloor) = \widetilde{\psi}(|\mathbf{q}|)$$

for infinitely many $p, \mathbf{q}$ and where $\gamma = \limsup_{N\to\infty}(\log\rho(N))/(\log\widetilde{\psi}(N))$.

Choose $\widetilde{\psi}(r) = c_*\, r^{-1}\psi(r)$. Then by (12), $\operatorname{dist}(u, R_{p,\mathbf{q}}) < \widetilde{\psi}(|\mathbf{q}|)$ implies that $|\mathbf{q}\cdot\theta(u) - p| < \psi(|\mathbf{q}|)$. Therefore $u \in \Lambda(\mathscr{R}, \widetilde{\psi})$ implies that for infinitely many $p, \mathbf{q}$,

$$|\mathbf{q}\cdot\theta(u) - p| < \psi(|\mathbf{q}|)$$

and so $\Lambda(\mathscr{R}, \widetilde{\psi}) \subset L(\psi)$. Thus

$$\dim L(\psi) \geqslant \dim \Lambda(\mathbb{R}, \widetilde{\psi}) \geqslant m - 1 + \gamma,$$

where by (13)

$$\gamma = \limsup_{N\to\infty} \frac{\log\rho(N)}{\log(c_* N^{-1}\psi(N))} = \frac{n + 1 - \eta(k+1)}{\lambda + 1},$$

where λ is the lower order of $1/\psi$. Since η is an arbitrary positive number and U is a parametrisation domain, it follows that

$$\dim \mathscr{L}(M\,;\psi) \geqslant \dim L(\psi) \geqslant m - 1 + \frac{n+1}{\lambda+1} \tag{14}$$

and Theorem 2 is proved.

By [11], a C^r m-dimensional manifold embedded in $\mathbb{R}^n$ and ℓ-nondegenerate for some $\ell \leq r$ almost everywhere is extremal ($\theta(U)$ is ℓ-nondegenerate if $\mathbb{R}^n$ is spanned by the partial derivatives of θ up to order ℓ). Hence (14) holds for such manifolds with $r \geq 1$ and so in particular for manifolds with at least one principal curvature non-zero almost everywhere. If M is not extremal, then $\dim \mathscr{L}_w(M) = m$ for some $w > n$ and hence $\dim \mathscr{L}_v(M) = m$ for $v \leqslant w$.

Obtaining an upper bound for the Hausdorff dimension of $\mathscr{L}(M;\psi)$ involves estimating large contributions from near-tangential resonant sets $R_{p,\mathbf{q}}$ and is much more difficult. The upper bound for $\mathscr{L}(M;\psi)$ has been shown to be $m-1+(n+1)/(v+1)$ for $v \geqslant n$ when M is C^3, of dimension $m \geqslant 2$ and has at least two principal curvatures non-zero everywhere except on a set of Hausdorff dimension at most $m-1$ [7], so that the lower bound in Theorem 2 is best possible. It is likely that this is the Hausdorff dimension when at least one principal curvature is non-zero everywhere except on a set of Hausdorff dimension at most $m-1$. Determining the Hausdorff dimension in the case of simultaneous Diophantine approximation seems harder and much less is known.

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Department of Mathematics, University of York, York YO1 5DD, England

email: hd3@york.ac.uk, mmd1@york.ac.uk