UNIVERSITY OF YORK

BA, BSc and MMath Examinations 2002
MATHEMATICS
Complex Analysis

Time Allowed: $1\frac{1}{2}$ hours.

Answer <u>three</u> questions.
Standard calculators will be provided.

1. (a) Let $a, b, z \in \mathbb{C}$, $a \neq b$, $k \in (0, \infty)$ and suppose that

$$\left|\frac{z-a}{z-b}\right| = k.$$

Show that the locus of the point z is a circle or a straight line.

(b) Suppose $\lambda \in \mathbb{C}$ and $r \in (0, \infty)$, $r \neq |\lambda|$. Let

$$w(z) = \frac{r(z-\lambda)}{r^2 - \overline{\lambda} z}.$$

Prove that when $|w(z)| = 1$, $|z| = r$.

2. Let $f(z)$ be defined on an open set $A \subseteq \mathbb{C}$ say and let $a \in A$.

(a) Define the derivative $f'(a)$ of the complex function. Let $c \in \mathbb{C}$ and n be a non-negative integer. Show that when $f(z) = cz^n$,

$$f'(a) = n\, ca^{n-1},$$

stating clearly any assumptions made.

(b) Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at the point $z = (x, y) \in A$. Prove that the functions $u\colon A \to \mathbb{R}$, $v\colon A \to \mathbb{R}$ satisfy the Cauchy-Riemann equations.

(c) Explain what it means to say that f is a *regular* function on A. Find a regular function $f\colon \mathbb{C} \to \mathbb{C}$ with real part $u\colon \mathbb{C} \to \mathbb{R}$ given by

$$u(x, y) = 2(x+1)y.$$

3. Define a *path* in the complex plane. Let f be a complex valued function defined in a region of the complex plane.

(a) Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise continuously differentiable path and let f be continuous. Define the path integral

$$\int_\gamma f(z)dz \tag{1}$$

in terms of the Riemann integral (which need not be defined but those properties used should be stated). Now suppose $f(z) = F'(z)$ for some continuous function F. Prove that

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

and deduce that if γ is a closed path, the integral vanishes (you may quote any relevant theorems in your proof). State but do not prove Cauchy's theorem.

(b) Evaluate the integral (1) when (i) $\gamma \colon [0, 2\pi] \to \mathbb{C} \colon t \mapsto e^{it}$ and $f(z) = cz^n, n \in \mathbb{Z}$.

4. Let f be a complex valued function defined in a region A of the complex plane. Explain what is meant by the statement that f has a *simple pole* at a. Also explain what a *residue* of f is. State but do not prove Cauchy's residue theorem.

Use the theory of residues to evaluate the integral,

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta},$$

where $a > 1$, justifying your arguments where necessary. Hence or otherwise evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2}.$$

1. *Solution (a) bookwork, (b) unseen bookwork*

(a) Since

$$\left|\frac{z-a}{z-b}\right| = k,$$

we have $|z-a|^2 = k^2|z-b|^2$, *i.e.*,

$$(z-a)(\overline{z}-\overline{a}) = z\overline{z} - z\overline{a} - a\overline{z} + a\overline{a} = k^2(z\overline{z} - z\overline{b} - b\overline{z} + b\overline{b}),$$

i.e.,

$$(1-k^2)z\overline{z} - z(\overline{a} - k^2\overline{b}) - \overline{z}(a - k^2 b) + |a|^2 - k^2|b|^2 = 0.$$

When $k \neq 1$, this can be written as

$$z\overline{z} - z\overline{c} - \overline{z}c + |c|^2 - \rho^2 = |z-c|^2 - \rho^2 = 0,$$

which is the equation of a circle centred at $c = (a-k^2b)/(1-k^2)$ and with radius

$$\begin{aligned}
\rho^2 &= c\overline{c} - \frac{|a|^2 - k^2|b|^2}{1-k^2} \\
&= \left(\frac{a-k^2b}{1-k^2}\right)\left(\frac{\overline{a}-k^2\overline{b}}{1-k^2}\right) - \frac{|a|^2 - k^2|b|^2 - k^2|a|^2 + k^4|b|^2}{(1-k^2)^2} \\
&= \frac{|a|^2 - k^2a\overline{b} - k^2b\overline{a} + k^4b\overline{b} - |a|^2 + k^2|b|^2 + k^2|a|^2 - k^4|b|^2}{(1-k^2)^2} \\
&= \frac{k^2(a\overline{a} + b\overline{b} - a\overline{b} - \overline{a}b)}{(1-k^2)^2} = \frac{k^2((a-b)(\overline{a}-\overline{b})}{(1-k^2)^2} = \frac{k^2|a-b|^2}{(1-k^2)^2}
\end{aligned}$$

When $k = 1$, it is the equation of a straight line – the perpendicular bisector of the segment joining a to b.

Could also use geometry or observe that LHS is modulus of a Möbius transformation which preserves circles and straight lines since Möbius transformations decompose into compositions of translations, scalar multiplication and inversions, all of which preserve circles and straight lines. 15 Marks.

(b) $r \in (0, \infty)$, $\lambda \in \mathbb{C}$, $|\lambda| \neq r$ and

$$w(z) = \frac{r(z-\lambda)}{r^2 - \overline{\lambda} z}.$$

When $\lambda = 0$, $w(z) = z/r$, so that when $|w(z)| = 1$, $|z| = r$. Otherwise write

$$w(z) = \frac{r(z-\lambda)}{r^2 - \overline{\lambda} z} = \frac{-r}{\overline{\lambda}} \frac{z-\lambda}{z - r^2/\overline{\lambda}}$$

Then $|w(z)| = 1$ implies

$$\left| \frac{z-\lambda}{z - r^2/\overline{\lambda}} \right| = 1. \frac{|\overline{\lambda}|}{r} = \frac{|\lambda|}{r} = k$$

and

$$a = \lambda,\ b = \frac{r^2}{\overline{\lambda}}.$$

Since $|\lambda| \neq r$, $k \neq 1$.
By the above, the locus of z is a circle with centre

$$c = \frac{a - k^2 b}{1-k^2} = \left(\lambda - \left(\frac{|\lambda|}{r} \right)^2 \frac{r^2}{\overline{\lambda}} \right) (1-k^2)^{-1}$$
$$= \left(\lambda - \frac{\lambda \overline{\lambda}}{r^2} \frac{r^2}{\overline{\lambda}} \right) (1-k^2)^{-1} = 0.$$

The radius is given by

$$\rho^2 = \frac{k^2 |a-b|^2}{(1-k^2)^2} = \frac{k^2 |\lambda - r^4/\overline{\lambda}|^2}{(1-k^2)^2}$$
$$= \frac{k^2 r^4 |\lambda \overline{\lambda} r^{-2} - 1|^2}{|\overline{\lambda}|^2 (1-k^2)^2} = \frac{k^2 r^4 |k^2 - 1|^2}{|\overline{\lambda}|^2 (1-k^2)^2}$$
$$= \frac{k^2 r^4}{|\lambda|^2} = r^2,$$

as required.

15 Marks.

Total: 30 Marks

2. *Solution (a) simple exercise, (b) bookwork, (c) unseen standard exercise*

(a) The derivative $f'(a) = \lim_{z\to a} \frac{f(z)-f(a)}{z-a}$, where $z \to a$ in any manner. When $f(z) = cz^n$,

$$\begin{aligned} f'(a) &= \lim_{z\to a} \frac{cz^n - ca^n}{z-a} \\ &= c \lim_{z\to a} \frac{(z-a)(z^{n-1} + z^{n-2}a + \cdots + za^{n-2} + a^{n-1})}{z-a} \\ &= c \lim_{z\to a}(z^{n-1} + z^{n-2}a + \ \ldots \ + za^{n-2} + a^{n-1}) = nca^{n-1}. \end{aligned}$$

5 Marks.

(b) The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since $f(z) = u(x,y)+iv(x,y)$ is differentiable in A, $f'(z)$ exists for each $z = (x,y) \in \mathbb{R}^2$, *i.e.*,

$$f'(z) = \lim_{z'\to z} \frac{f(z') - f(z)}{z' - z} = \lim_{\zeta\to 0} \frac{f(z+\zeta) - f(z)}{\zeta}$$

where $\zeta = z' - z$ and the limit of the difference quotient

$$(f(z+\zeta) - f(z))/\zeta$$

is independent of the way ζ approaches 0. Suppose $\zeta = (h,k) = h + ik \to 0$ through real values, (i.e., $k = 0$ and $h \to 0$). Then

$$\begin{aligned} \frac{f(z+\zeta) - f(z)}{\zeta} &= \frac{f(z+h) - f(z)}{h} \\ &= \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \end{aligned}$$

and

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{h\to 0} \frac{v(x+h,y) - v(x,y)}{h} \\ &= \frac{\partial u(x,y)}{\partial x} + i\frac{\partial v(x,y)}{\partial x} \end{aligned}$$

Next suppose $\zeta = (h, k) = h + ik \to 0$ through imaginary values (i.e., $h = 0$). Then

$$f'(z) = \lim_{k\to 0} \frac{u(x, y+k) + iv(x, y+k) - u(x, y) - iv(x, y)}{ik}$$
$$= \lim_{k\to 0} \frac{u(x, y+k) - u(x, y)}{ik} + i \lim_{k\to 0} \frac{v(x, y+k) - v(x, y)}{ik}$$
$$= \frac{\partial u(x, y)}{i\partial y} + \frac{\partial v(x, y)}{\partial y}.$$

Hence

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

and equating real and imaginary parts,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ or } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which are the Cauchy-Riemann equations. **15 Marks.**

(c) A regular (or holomorphic or analytic) function f is differentiable at each point in an open set. **2 Marks.**

$2(x+1)y = 2xy + 2 = u(x, y)$ and

$$\frac{\partial u}{\partial x} = 2y = \frac{\partial v}{\partial y} \Rightarrow v(x, y) = y^2 + C(x)$$

Therefore $v(x, y) = y^2 + C(x)$ and $\frac{\partial u}{\partial y} = 2x + 2 = -\frac{\partial v}{\partial x} \Rightarrow v(x, y) = -x^2 - 2x + D(y)$, whence

$$v(x, y) = y^2 + C(x) = -x^2 - 2x + D(y),$$

so that $C'(x) = -2x - 2$ and so $C(x) = -x^2 - 2x + C$, C a constant, and similarly, $D(y) = y^2 + D$, D a constant. On substituting, $v(x, y) = y^2 - x^2 - 2x + C = -x^2 + y^2 - 2x + D$, whence $C = D$ and $v(x, y) = -2x - x^2 + y^2 + C$. Thus

$$f(z) = u(x, y) + iv(x, y) = 2xy + 2y + i(y^2 - x^2 - 2x) + C$$
$$= -i(x^2 - y^2 - i2xy) - 2i(x + iy) = -i(x + iy)^2 - 2i(x + iy)$$
$$= \frac{1}{i}(z^2 + 2z) + C$$

8 Marks.

Total: 30 Marks

3. *Solution: bookwork (a) bookwork (b) bookwork and exercise covered in lectures*

A *path*, usually denoted by γ, in the complex plane is a continuous function $\gamma\colon [a,b] \to \mathbb{C}\colon t \mapsto \gamma(t)$. For $\alpha, \beta \in \mathbb{C}$, the path γ goes from α to β if $\gamma(a) = \alpha, \gamma(b) = \beta$. The image $C = \gamma([a,b])$ of the path is also used to denote the path. 5 Marks.

(a) A path γ is continuously differentiable if the derivative γ' is continuous. The path γ is piece-wise continuously differentiable if γ consists of a finite number of continuously differentiable subpaths $\gamma_1, \ldots, \gamma_k$. This means that γ' is cts except at finitely many points. Also $f(\gamma(t)) = (f \circ \gamma)(t)$ is continuous (composition of two continuous funtions). Hence the real and imaginary parts $U(t), V(t)$ say are also cts except at finitely many points, so that the (real) Riemann integrals $\int_a^b U(t)dt$, $\int_a^b V(t))dt$ exist. Hence the integral

$$\begin{aligned}\int_a^b f(\gamma(t))\,\gamma(t)\,dt &= \int_a^b (U(t) + iV(t))dt \\ &= \int_a^b U(t)dt + i\int_a^b V(t)dt,\end{aligned}$$

exists. This defines the path integral of f along γ, *i.e.*,

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_a^b (f \circ \gamma)(t))\,\gamma'(t)\,dt.$$

Since $f(z) = F'(z)$, where F is cts,

$$\begin{aligned}\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt &= \int_a^b F'(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b (F \circ \gamma)'(t))\,dt = [F(\gamma(t))]_a^b \\ &= F(\gamma(b)) - F(\gamma(a)) = 0\end{aligned}$$

if $\gamma(b) = \gamma(a)$, *i.e.*, if γ is a closed path. 10 Marks.

Cauchy's theorem: Let f be regular inside and on the simple closed curve $C = \gamma([a,b])$. Then

$$\int_C f(z)dz = \int_\gamma f(z)dz = 0.$$

5 Marks.

(b) $\gamma\colon [0,2\pi] \to \mathbb{S}^1\colon t \mapsto e^{it}$ and $f(z) = cz^n, n \in \mathbb{Z}$.
$\mathbb{S}^1 = \gamma([0,2\pi]) = \{e^{it}\colon t \in [0,2\pi]\} = \{z \in \mathbb{C}\colon |z| = 1\}$

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\ &= \int_{\mathbb{S}^1} z^n dz = \int_0^{2\pi} c(e^{it})^n ie^{it}dt \\ &= ic\int_0^{2\pi} (e^{i(n+1)t})^n\, dt \\ &= ic\left[\frac{e^{i(n+1)t}}{n+1}\right]_0^{2\pi} = ic[1-1] = 0,\end{aligned}$$

providing $n \neq -1$.
When $n = -1$,

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma \frac{c}{z}\, dz = \int_0^{2\pi} c\frac{1}{e^{it}}ie^{it}dt \\ &= ic\int_0^{2\pi} e^{-it+it}\, dt = ic\int_0^{2\pi}\, dt \\ &= 2\pi ic.\end{aligned}$$

When n is a non-negative integer, then $c\int_{\mathbb{S}^1} z^n\, dz = 0$ by Cauchy's theorem but for $n \in \mathbb{Z}$, $n \neq -1$, the integral vanishes as a consequence of the above since

$$z^n = \frac{1}{n+1}(z^{n+1})'.$$

10 Marks.

Total: 30 Marks

4. *Solution : bookwork (a) unseen modification of simple standard application covered in lectures, (b) unseen standard exercise with a ending*

The complex valued function f has a pole of order n at $a \in A$ if f has an isolated singularity at a, i.e., if for some $c_n \neq 0$ and some function g regular at a, f can be written

$$f(z) = \frac{c_n}{(z-a)^n}\, g(z) \;\left(= \sum_{k=-n}^{\infty} c_k (z-a)^k\right).$$

When $n = 1$ the pole is simple. **[5 Marks.]**

(a) To evaluate the integral

$$I = I(a) = \int_0^{2\pi} \frac{d\theta}{a + \cos\theta},$$

where $a > 1$, put $z = e^{i\theta}$. Then

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + \frac{1}{z}),$$

$dz = ie^{i\theta}d\theta = izd\theta$ and the integral

$$I = \frac{1}{i}\int_{|z|=1} \frac{dz}{z(a + \frac{1}{2}(z + \frac{1}{z}))} = -2i\int_{|z|=1} \frac{dz}{z^2 + 2az + 1}$$

Denominator of integrand

$$= z^2 + 2az + 1 = (z - a + \sqrt{a^2-1})(z - a - \sqrt{a^2-1})$$

The product of the two (real) roots is

$$(-a + \sqrt{a^2-1})(-a - \sqrt{a^2-1}) = 1$$

so only the root

$$\alpha = -a + \sqrt{a^2-1} \in D^2,$$

where D^2 is the unit disc $\{z : |z| \leq 1\}$, i.e., α is inside the circle $S^1 = \{z : |z| = 1\}$ and the conjugate root $\beta = -a - \sqrt{a^2-1}$ is outside D^2. Hence the integrand $f(z) = \frac{1}{z^2+2az+1}$ has one simple pole inside D^2 at $z = \alpha = -2 + \sqrt{a^2-1}$ with residue

$$\begin{aligned}\operatorname{res}\{f;\alpha\} &= \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{(z-\alpha)}{z^2+4z+1} \\ &= \lim_{z\to\alpha}\frac{(z-\alpha)}{(z-\alpha)(z-\beta)} = \frac{1}{\alpha-\beta} \\ &= \frac{1}{\sqrt{a^2-1}}\end{aligned}$$

But by the Residue Theorem

$$\int_{|z|=1}\frac{dz}{z^2+2az+1} = -2i\int_{|z|=1}\frac{dz}{(z-\alpha)(z-\beta)} = 2\pi i(-2i)\operatorname{res}\{f;\alpha\}$$

whence

$$I = I(a) = \frac{4\pi}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}$$

15 Marks.

To evaluate the integral

$$\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2},$$

differentiate $I(a)$ wrt a: then

$$I'(a) = -2\pi\frac{1}{2}(a^2-1)^{-3/2}2a = -2\pi a(a^2-1)^{-3/2}.$$

But by differentiating under the integral sign (OK since range finite and $a > 1$)

$$I'(a) = -\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2},$$

whence

$$\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}.$$

Alternatively could consider

$$J = \int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2}\int_{|z|=1}\frac{dz}{(z^2+2az+1)^2} = -2i\int_{|z|=1}\frac{dz}{(z-\alpha)^2(z-\beta)^2}$$

and the Residue Theorem corresponding to a double pole at α.

10 Marks.

Total: 30 Marks