SHANNON'S SAMPLING THEOREM, INCONGRUENT RESUIDUE CLASSES AND PLANCHEREL'S THEOREM

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ABSTRACT. Sampling theory for multi-band signals is shown to have a logical structure similar to that of Fourier analysis.

Dedicated to Michel Mendès France for his 65th birthday

1. INTRODUCTION

In the rich and varied landscape of mathematics, Fourier analysis occupies a grand and formal garden (see Figure 1). There is a corner of this garden, which borders interpolation theory and Paley-Wiener theory, called sampling theory. This is not concerned with statistics but with information theory, more precisely with Shannon's second or *sampling* theorem. Sometimes known as the Whittaker-Kotel'nikov-Shannon theorem, it is a cornerstone of communication theory [20, 21], providing a mathematical basis for an equivalence between *analogue* (or continuous) signals and discrete signals.

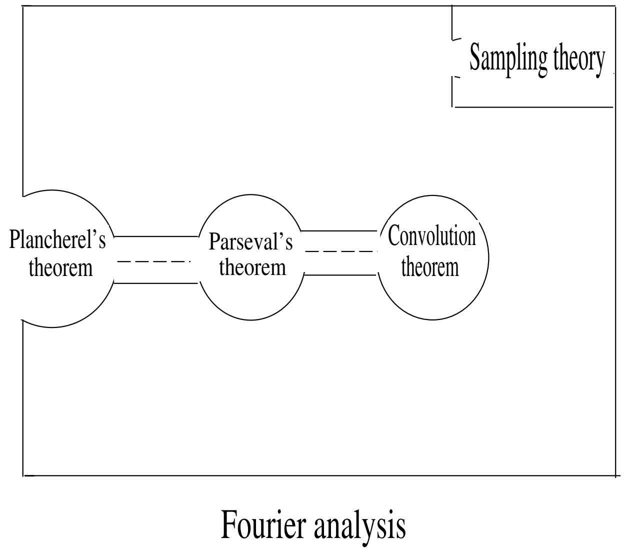

FIGURE 1. The three fundamental results in Fourier analysis with Plancherel's theorem at the 'entrance'

1991 *Mathematics Subject Classification.* 94A05, 42A99.
Key words and phrases. Shannon's sampling theorem, Plancherel's theorem.

In applications, the main purpose of Shannon's sampling theorem is to reconstruct a signal from discrete samples. The theorem falls naturally into the theory of square-integrable functions and is closely related to Paley-Wiener theory. A disjoint translates condition on the Fourier transform of the signal underlies Shannon's sampling theorem and leads to a more general theory sometimes called multiband signal theory (see §3).

Plancherel's theorem is one of the fundamental results in Fourier analysis and implies Parseval's theorem which in turn implies the convolution theorem. These three results are however, logically equivalent and each could be regarded as the starting point of the theory of Fourier analysis. The multiband theory has a similar logical structure in which hybrid versions of the above three theorems, together with the null intersection condition (see (5) below) and a reconstruction or representation formula (see (4) below), are also equivalent. This is proved for trigonometric polynomials which are a finite dimensional counterpart of continuous signals.

2. Shannon's sampling theorem

An analogue or continuous signal can be modelled by a function $f : \mathbb{R} \to \mathbb{C}$ (when f is a real signal, $f(\mathbb{R}) \subset \mathbb{R}$). For physical reasons, it is assumed that f is square-integrable or has finite energy, *i.e.*, $\|f\|_{\mathbb{R}}^2 = \int_{\mathbb{R}} |f(t)|^2 dt < \infty$. The spectrum of f is given by the Fourier transform $f^\wedge : \mathbb{R} \to \mathbb{C}$, where

$$f^\wedge(\gamma) = \int_{\mathbb{R}} f(t) e^{-2\pi i t\gamma}\, dt$$

and where $f^\wedge$ is defined as a limit if $f \notin L^1(\mathbb{R})$. It is usual to assume that vibrations in a physical system cannot be infinite and so the frequencies of analogue signals are taken to be bounded above. Signals with bounded spectra are called *band-limited* and have the representation

$$f(t) = \int_{-W}^{W} f^\wedge(\gamma) e^{2\pi i t\gamma}\, d\gamma,$$

where W is the maximum frequency (strictly speaking W is the supremum of the frequencies, a technicality which is usually disregarded). The sampling theorem gives the following formula for the reconstruction of a band-limited signal f with maximum frequency W in terms of the discrete values or samples $f(k/2W)$. For each $t \in \mathbb{R}$,

$$f(t) = \frac{1}{2W} \sum_{k \in \mathbb{Z}} f\left(\frac{k}{2W}\right) \frac{\sin 2\pi W(t - k/2W)}{\pi(t - k/2W)}. \tag{1}$$

The formula (1) corresponds to a sampling rate of $2W$ (often called the Nyquist rate, twice the maximum frequency of the signal[1]).

As well as its significance in information theory and engineering [14], the theorem has a long history and has has been proved many times, for example by E. Borel, E. T. Whittaker, V. A. Kotel'nikov, K. S. Krishnan, H. Raabe, I. Someya to name a selection. However, Shannon was the first to appreciate the significance of the result in his comprehensive and definitive theory of communication (more details can be found in the survey articles [5, 7, 11] and the books [12, 18]). A full treatment of the history of the sampling theorem and its equivalence to Cauchy's theorem and other results from analysis is in [8], see also [13]. Whittaker proved the result from the point of view of interpolation theory [22] and essentially established formula (2) for an analytic function f (which he called the Cardinal function or series). This assumed given values at regular intervals and was free of 'rapid oscillations'. As well as the connection with interpolation theory, there are also connections with Paley-Wiener theory and Fourier analysis (for example (1) is equivalent to the Poisson summation formula and the result also arises surprisingly often in other branches of mathematics [7, 16, 19]).

3. A generalisation of Shannon's sampling theorem

The formula (1) can be proved using the observation that translates $[-W, W] + 2Wk$, $k \in \mathbb{Z}$, of the support $[-W, W]$ of the Fourier transform $f^\wedge$ of f essentially tessellate $\mathbb{R}$. This leads to a multiband generalisation of the sampling theorem to functions f with Fourier transforms which vanish outside a measurable set A that satisfies the disjoint translates condition

$$A \cap \left(A + \frac{k}{s} \right) = \emptyset, \quad k \in \mathbb{Z} \setminus \{0\}. \tag{2}$$

This condition implies that the Lebesgue measure $|A|$ of A is at most $1/s$ [9] (see §4). For such functions f a more general version of Shannon's sampling theorem holds.

Theorem 1. *Let A be a measurable subset of $\mathbb{R}$ and suppose that there exists an $s > 0$ such that (2) holds. Let $f \in L^2(\mathbb{R})$ be continuous and suppose that the Fourier transform $f^\wedge$ of f vanishes outside A. Then*

$$\int_{\mathbb{R}} |f(t)|^2 \, dt = s \sum_{k \in \mathbb{Z}} |f(sk)|^2 \tag{3}$$

and for each $t \in \mathbb{R}$,

$$f(t) = s \sum_{k \in \mathbb{Z}} f(sk) \chi_A^\vee(t - sk), \tag{4}$$

[1]That a signal can be reconstructed from samples taken at this rate was referred to by Shannon as 'a fact which is common knowledge in the communication art'.

where the series converges absolutely and uniformly.

A proof of this theorem and further references to earlier results are given in [9] where the approach is based on square integrable functions $\varphi\colon \mathbb{R} \to \mathbb{C}$ which vanish outside A. A more general result for stochastic processes was obtained by S. P. Lloyd who showed that the disjoint translates condition was equivalent to a random variable being determined by a linear combination of the sample values [17]. For a generalisation of Theorem 1 to abstract harmonic analysis, see [1, 15]. P. L. Butzer and A. Gessinger note in [6] that the formula (3) with $A = (-W, W)$ implies the Shannon's sampling theorem. Equally, one can first prove the formula (4) and then deduce (3), as in [20] or in the abstract harmonic analysis version [15]. When f is not continuous, convergence in norm can be deduced from an orthogonality argument [15]. A common approach is to use the fact that the set $\{e^{\pi ikt/W}\colon k \in \mathbb{Z}\}$ is an orthogonal basis for $L^2(-W, W)$ and that

$$f^\wedge(\gamma) = \sum_{k\in\mathbb{Z}} a_k e^{\pi ik/W} \chi_{(-W,W)}(\gamma)$$

for almost all $\gamma \in (-W, W)$. The result follows on taking the inverse Fourier transform and observing that $a_k = f(k/2W)/2W$. The form of (4), however, suggests convolution. This observation is the basis of the parallel between the logical structures of Sampling Theory and Fourier analysis.

4. Spectral translates

The set A considered here need not be an interval but any measurable set such that distinct translates have null intersection, *i.e.*, for each non-zero $k \in \mathbb{Z}$,

$$\left| A \cap \left(A + \frac{k}{s} \right) \right| = 0. \tag{5}$$

The possibility of A containing unbounded frequencies is not excluded. For this reason, the Paley-Wiener setting, although closely related, is not suitable.

The translation $\gamma \mapsto \gamma + k/s$ can be regarded as an action by the additive group $\mathbb{Z}/s$ on $\mathbb{R}$. Orbits or cosets $\gamma + \mathbb{Z}/s = \{\gamma + k/s\colon \mathbb{Z}\}$ are disjoint and the factor group $\mathbb{R}/(\mathbb{Z}/s)$ has $[0, 1/s)$ as a complete set of coset representatives or a transversal. Let $\nu\colon \mathbb{R} \to [0, 1/s)$ be the projection map given by

$$\nu(a) = a + \frac{k_a}{s},$$

where k_a is the unique integer such that $a + k_a/s \in [0, 1/s)$. Then $\nu(\gamma)$ is the unique element in the orbit $\mathbb{Z}/s + \gamma$ which falls in $[0, 1/s)$. Thus ν is well defined, $\nu(\mathbb{R}) = [0, 1/s)$ and $|\nu(A)| \leqslant 1/s$.

The restriction map $\nu|_A \colon A \to [0, 1/s)$ is one-one almost always by (5) since $\nu(a) = \nu(a')$ implies that $a - a' \in \mathbb{Z}/s$. The function $\nu|_A$ is translation by an integer multiple of $1/s$ and so

$$|A| = |\nu(A)| \leqslant \frac{1}{s}. \tag{6}$$

5. Plancherel-type theorems and the general sampling theorem

When the set A satisfies (5) for some $s > 0$, the map $\iota \colon A \to [0, 1)$ given by

$$\iota(a) = \{sa\},$$

where $\{x\} = x - [x]$ is the fractional part of x ($[x]$ is the integer part of x), is an injection which preserves inner products for almost all $a \in A$. If $|A| = 1/s$, then ι is a bijection for almost all $a \in A$. Thus the set (which will be denoted $L^2(A)$) of $\varphi \in L^2(\mathbb{R})$ with φ vanishing almost everywhere outside A is isomorphic to $L^2([0, 1))$. In the case of Shannon's theorem, the isomorphism is simply a rescaling. When $|A| \leqslant 1/s$, one can consider an auxiliary set $\tilde{A} \supseteq A$ which satisfies (5) with $|\tilde{A}| = 1/s$ and $\iota \colon \tilde{A} \to [0, 1)$ an isomorphism almost everywhere. The natural embedding of $L^2(A)$ ($\subseteq L^2(\tilde{A})$) into $L^2([0, 1))$ is why results in sampling theory are a mixture of Fourier series and transforms.

Plancherel's theorem asserts that the Fourier transform is an isometry on $L^2(\mathbb{R})$; (3) also implies that the map $\rho \colon L^2(\mathbb{R}) \to \ell^2(s\mathbb{Z})$ given by

$$\rho(f) = s^{1/2} f|_{s\mathbb{Z}},$$

where $f|_{s\mathbb{Z}}$ is the restriction of $f \colon \mathbb{R} \to \mathbb{C}$ to $s\mathbb{Z}$, is also an isometry. From the point of view of signals, the norm can be regarded as energy and the preservation of norm can be interpreted as conservation of energy between the original signal and its values (samples) on $s\mathbb{Z}$. The factor s is the time interval between samples and so keeps the dimensions of the two sides of (3) consistent.

From now on the set $A \subset \mathbb{R}$ will be assumed to have positive Lebesgue measure. The functions $f \colon \mathbb{R} \to \mathbb{C}$ will be restricted to being continuous, as otherwise nothing can be said about values on sets of measure zero. The inverse Fourier transform of $\varphi \in L^2(\mathbb{R})$ will be denoted by $\varphi^\vee$, *i.e.*,

$$\varphi^\vee(t) = \int_{\mathbb{R}} \varphi(\gamma) e^{2\pi i \gamma t} d\gamma, \tag{7}$$

where if $\varphi \notin L^1(\mathbb{R})$, $\varphi^\vee$ is defined as the usual limit. If $|A| < \infty$, then the characteristic function χ_A of A is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and its inverse

Fourier transform $\chi_A^\vee$, given by

$$\chi_A^\vee(t) = \int_{\mathbb{R}} e^{2\pi i\gamma t}\chi_A(\gamma)d\gamma = \int_A e^{2\pi i\gamma t}d\gamma, \tag{8}$$

is continuous. Let

$$S_A = \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R})\colon\ f^\wedge(\gamma) = 0 \text{ for almost all } \gamma \notin A\} \tag{9}$$

Theorem 2. *Let A be a Lebesgue measurable subset of $\mathbb{R}$ and s be a positive real number. Then the following are equivalent.*

(I) *For each non-zero integer k*

$$\left|A \cap \left(A + \frac{k}{s}\right)\right| = 0. \tag{10}$$

(II) *for each $f \in S_A$,*

$$||f||_{\mathbb{R}}^2 = \int_{\mathbb{R}} |f(t)|^2\,dt = s\sum_{k\in\mathbb{Z}} |f(sk)|^2. \tag{11}$$

(III) *each $f, g \in S_A$ satisfies*

$$\int_{\mathbb{R}} f(t)\overline{g(t)}\,dt = s\sum_{k\in\mathbb{Z}} f(sk)\overline{g(sk)}. \tag{12}$$

(IV) *each $f, g \in S_A$ satisfies*

$$(f *_{\mathbb{R}} g)(t) = s(f *_{s\mathbb{Z}} g)(t) = s\sum_{k\in\mathbb{Z}} f(sk)g(t-sk), \tag{13}$$

where the series converges to $f * g$ absolutely and uniformly.

Note that (II) can regarded as hybrid transform version of the classical Plancherel theorem since by (11),

$$\int_{\mathbb{R}} |f^\wedge(\gamma))|^2\,d\gamma = s\sum_{k\in\mathbb{Z}} |f(sk)|^2.$$

and similarly (III) and (IV) are versions of the Parseval and convolution theorems.

The proof that (I) implies (II) relies on going a little deeper and considering the set $\tilde{A}$ which contains A and has Lebesgue measure $1/s$, to obtain the orthogonality necessary for (11). The proof that (II) implies (III) is on the same lines as the standard treatment in functional analysis in which Plancherel's theorem is regarded as fundamental and the precursor to Parseval's theorem and the convolution theorem (they are of course all equivalent). Recall that $f \in L^2(\mathbb{R})$ is assumed to be continuous and that $f^\wedge \in L^2(\mathbb{R})$ vanishes almost everywhere outside A. By (10), (I) implies that $|A| \leqslant 1/s$ but no explicit use is made of this in Theorem 2.

5.1. **The reconstruction formula and the translates condition.** The equivalence of the reconstruction formula and the translates condition follows from Theorem 2; the estimate $|A| \leqslant 1/s$ implied by (10) is used explicitly. Lloyd has proved a similar but more general result for stochastic processes [17, Theorem 1]. The necessity of the translates condition is related to the phenomenon of 'aliasing' [4].

Theorem 3. *Let A be a measureable subset of $\mathbb{R}$ and $s \in \mathbb{R}^+$. Then the following are equivalent.*

(I) *For each non-zero integer k*

$$\left| A \cap \left(A + \frac{k}{s} \right) \right| = 0. \tag{14}$$

(II) $|A| \leqslant 1/s$ *and for each $f \in S_A$,*

$$||f||_{\mathbb{R}}^2 = \int_{\mathbb{R}} |f(t)|^2 \, dt = s \sum_{k \in \mathbb{Z}} |f(sk)|^2. \tag{15}$$

(III) $\chi_A^\vee \in S_A$ *and each $f \in S_A$ is given by*

$$f(t) = s \sum_{k \in \mathbb{Z}} f(sk) \chi_A^\vee (t - sk), \tag{16}$$

where the series converges to f absolutely and uniformly.

Since $|A| \leqslant 1/s$ implies that

$$\int_{\mathbb{R}} \chi_A = \int_{\mathbb{R}} |\chi_A|^2 = |A| \leqslant 1/s,$$

(III) follows from (II) and so $\chi_A \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, whence $\chi_A^\vee$ is continuous and $\chi_A^\vee \in L^2(\mathbb{R})$, *i.e.*, $\chi_A^\vee \in S_A$. The representation (4) now follows from the hybrid convolution theorem applied applied to the equation

$$f^\wedge = f^\wedge \chi_A$$

which holds amost everywhere to give

$$f(t) = (f^\wedge \chi_A)^\vee(t) = (f *_{\mathbb{R}} \chi_A^\vee)(t) = s \sum_{k \in \mathbb{Z}} f(sk) \chi_A^\vee (t - sk).$$

6. Trigonometric polynomials

Trigonometric polynomials are the finite dimensional counterpart of square integrable functions. The version of Shannon's sampling theorem for such functions is as follows.

Theorem 4. *Let $f \to \mathbb{R} \to \mathbb{C}$ be 1-periodic and let*

$$f(t) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi n t}$$

where $c_n = 0$ *for* $|n| > 0$. *Then*

$$f(t) = \frac{1}{2N+1}\sum_{n=-N}^{N} f(\frac{n}{2N+1})\frac{\sin((2N+1)\pi(t-\frac{n}{2N+1}))}{\sin\pi(t-\frac{n}{2N+1})}.$$

The counterpart of Theorem 3 is the more general form of Theorem 4, which will be stated and proved to illustrate the arguments without analytic technicalities.

Theorem 5. *Let* p, q *be positive integers, let* $J = \{j_1, \dots, j_p\} \subset \mathbb{Z}$ *and let*

$$f(t) = \sum_{j\in J} c_j e^{2\pi ijt} = \sum_{k=1}^{p} c_{j_k} e^{2\pi ij_k t}, \tag{17}$$

where $c_j = 0$ *for* $j \notin J$. *Then the following are equivalent:*

(I) *For each non-zero integer* n,

$$J \cap (J + nq) = \emptyset. \tag{18}$$

(II)

$$f(t) = \frac{1}{q}\sum_{m=0}^{q-1} f(\frac{m}{q})\chi_J^\vee(t-\frac{m}{q}), \tag{19}$$

where the inverse Fourier transform $\chi_J^\vee$ *of* χ_J *is given by*

$$\chi_J^\vee(t) = \sum_{j\in J} e^{2\pi ijt}.$$

Proof. First (I) implies (II). For given $m \in \mathbb{Z}$, $0 \leqslant m \leqslant q-1$, consider

$$f(m/q) = \sum_{j\in J} c_j e^{2\pi jm/q} = \sum_{k=1}^{p} c_{j_k} e^{2\pi j_k m/q}.$$

Write

$$\mathbf{f} = (f(1/q), \dots, f((q-1)/q)), \quad \mathbf{c} = (c_1, \dots, c_k),$$

$M = \left(e^{2\pi ij_k l}\right)$ and $M^* = \overline{M}' = \left(e^{-2\pi il j_k}\right)$. Then

$$\mathbf{f}M^* = \mathbf{c}MM^* = \mathbf{c}\left(\sum_{m=0}^{q-1} e^{2\pi i(j_k-j_l)m/q}\right) = q\mathbf{c}1_{p\times p},$$

where $1_{p\times p}$ is the $p\times p$ unit matrix. But by (18), when $k \neq l$,

$$j_k \not\equiv j_l \ (\mathrm{mod} q)$$

and the formula in (II) follows on substituting for c_j in (17) and using the definition of $\chi_J^\vee$.

Next (II) implies (I). By definition, $f^\wedge$ is given by

$$f^\wedge(k) = \int_0^1 f(t)e^{-2\pi ikt}\,dt = c_k.$$

Each $r \in \mathbb{Z}$ can be expressed as $r = p + nq$, for unique $n \in \mathbb{Z}$ and p, $0 \leqslant p < q$. Then since f satisfies (19) and since the Fourier transform of $f(t-c)$ is $f^\wedge(k)e^{-2\pi ick}$, $k \in \mathbb{Z}$, it follows that

$$f^\wedge(k) = \frac{1}{q}\sum_{m=0}^{q-1} f(m/q)\chi_J(k)e^{-2\pi ikm/q}. \tag{20}$$

Take $f = \chi_J^\vee$ so that for each $t \in \mathbb{R}$ and $k \in \mathbb{Z}$,

$$f(t) = \chi_J^\vee(t) = \sum_{j\in J} e^{2\pi ijt} = \sum_{r\in\mathbb{Z}} \chi_J(r)e^{2\pi ijt} \text{ and } f^\wedge(k) = \chi_J(k).$$

Then by (20) and writing $k = l + qn$,

$$\begin{aligned}
\chi_J(k) &= \frac{1}{q}\sum_{m=0}^{q-1} \chi_J^\vee(m/q)\chi_J(k)e^{-2\pi imk/q}\\
&= \frac{1}{q}\chi_J(k)\sum_{m=0}^{q-1}\left(\sum_{r\in\mathbb{Z}}\chi_J(r)e^{2\pi irm/q}\right)e^{-2\pi iml/q}\\
&= \frac{1}{q}\chi_J(k)\sum_{m=0}^{q-1}\sum_{p=0}^{q-1}\sum_{n\in\mathbb{Z}}\chi_J(p+nq)e^{2\pi i(p+nq-l)m/q}\\
&= \frac{1}{q}\chi_J(k)\sum_{m=0}^{q-1}\sum_{p=0}^{q-1}\sum_{n\in\mathbb{Z}}\chi_J(p+nq)e^{2\pi i(p-l)m/q}\\
&= \chi_J(k)\sum_{n\in\mathbb{Z}}\chi_J(l+nq).
\end{aligned}$$

Now for $k \in J$, $\chi_J(k) = 1$. Suppose $k_0 = l_0 + qn_0 \in J$. Then

$$\begin{aligned}
1 = \chi_J(k_0) &= \sum_{n\in\mathbb{Z}}\chi_J(l_0+nq) = \sum_{n\in\mathbb{Z}}\chi_J(k_0+(n-n_0q))\\
&= \sum_{n'\in\mathbb{Z}}\chi_J(k_0+n'q) = \chi_J(k_0) + \sum_{n'\neq 0}\chi_J(k_0+n'q),
\end{aligned}$$

whence for $n \neq 0$, $\chi_J(k_0+nq) = 0$. Thus $J\cap(J+nq) = \emptyset$ for $n \neq 0$. $\square$

Of course the coefficients c_j could be determined by solving the q linear equations in p ($\leqslant q$) unknowns but the equal spacing of the samples m/q and the incongruence condition allows the use of the simpler Hermitian conjugate instead of solving a system of linear equations. Theorem 4 follows by putting $J = \{-N, \ldots, N\}$ and $p = q = 2N + 1$.

To prove the hybrid trigonometric polynomial analogue of Plancherel's theorem, further consideration must be given to the set J which has to be extended to a complete set $\widetilde{J}$ of residues mod q, so that $|\widetilde{J}| = q$. Since $J \subseteq \widetilde{J}$, the Fourier transform $f^\wedge$ of the 1-periodic function f

certainly also vanishes outside $\widetilde{J}$. Consider

$$f(t)=\sum_{j\in\widetilde{J}} c_j e^{2\pi i\widetilde{j}t}=\sum_{k=1}^{q} c_{j_k} e^{2\pi i\widetilde{j}_k t},$$

where $\widetilde{j}_k, k=0,\dots,q-1$ runs over a complete set of residues modulo q and $M=(e^{2\pi i\widetilde{j}_k m/q})$ is a $q\times q$ matrix. The above argument and the Riesz-Fischer theorem give

$$\begin{aligned}\int_0^1 |f(t)|^2dt &= \sum_{\widetilde{j}\in\widetilde{J}} |c_j|^2=|\mathbf{c}|^2=\frac{1}{q^2}|\mathbf{f}M^*|^2\\ &=\frac{1}{q^2}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1} f(m/q)\overline{f(n/q)}\sum_{k=0}^{q-1} e^{2\pi i(m-n)\widetilde{j}_k}\\ &=\frac{1}{q}\sum_{m=0}^{q-1}|f(m/q)|^2.\end{aligned}$$

It is implicit in Theorem 2 and 3 above that if the sampling theorem holds for any function $f\in S_A$ with Fourier Transform supported throughout A, then the null measure translates condition holds and the sampling theorem holds for each $f\in S_A$. However, there exist sets A with $|A|\leqslant 1/s$ such that $f\in S_A$ satisfies (11) but A does not satisfy (10) (see [2]).

7. Conclusion

Instead of the usual spectral translates conditions (2) or (5) in frequency space, the hybrid Plancherel theorem (15) can be regarded as a basis for the general sampling theorem, allowing sampling theory to be given a logical structure similar to that of Fourier analysis, with the hybrid versions of Plancherel's theorem, Parseval's theorem and the convolution theorem at the centre (note that the theorems must hold for *all* functions in S_A). Thus in Sampling Theory, each of the theorems in Figure 1 is replaced by its hybrid version, so that the theory begins with the hybrid Plancherel theorem which by Theorem 2 leads to and from the hybrid Parseval theorem and the hybrid convolution theorem. This leads by Theorem 3 to (but not directly back from) the translates condition and the general sampling theorem. The translates condition leads back to the hybrid Plancherel theorem (15). In addition Shannon's theorem together with the finiteness of the measure of A imply the translates condition, which in turn implies the hybrid Plancherel theorem. The conventional view of Sampling Theory would place the translates condition at the entrance of the garden.

As has been pointed out, providing the sampling interval s satisfies (5), the formula (3) can be interpreted as the energy of the original signal being conserved in the samples (the information is conserved, in the sense that the original signal can be recovered – in principle – from the samples). The step function approximation $\sigma\colon \mathbb{R} \to \mathbb{C}$ to f is given by

$$\sigma(t) = \sum_{k\in\mathbb{Z}} f(sk)\chi_{[sk,s(k+1))}(t).$$

It is immediate that its energy

$$\|\sigma\|_{\mathbb{R}}^2 = \int_{\mathbb{R}} |\sigma(t)|^2 dt = s\sum_{k\in\mathbb{Z}} |f(sk)|^2$$

and hence, providing s satisfies (5),

$$\int_{\mathbb{R}} |\sigma(t)|^2 dt = s\sum_{k\in\mathbb{Z}} |f(sk)|^2 = \int_{\mathbb{R}} |f(t)|^2 dt.$$

Thus the energy of original signal is also conserved, although the support of $\sigma^{\wedge} = \mathbb{R}$ and so contains A. Moreover the information is also preserved, again in the sense that the original signal can be recovered in principle from the values of σ. However, conservation of energy alone is not sufficient to allow the complete reconstruction of the signal using the sampling formula (4).

Shannon's theorem can be viewed as the result of passing a train of impulses (Dirac delta functions) weighted by successive sample values through an ideal low-pass filter. In practice, however, perfect impulses and ideal filters do not exist and the theorem cannot be implemented; thus the formula (1) is not often used directly. For example, digital-to-analogue conversion in signal processing replaces the weighted impulses indicated by the theorem by a step function in which successive steps have amplitude of successive sample values. The step function approximation σ is used in passing from a digital signal to an analogue signal $\sigma_A = ((\sigma^{\wedge})\chi_A)^{\vee}$, produced by filtering out all frequencies except those in A (see [3]). This reduces the energy of the signal but when A is an interval symmetric about the origin, Shannon's sampling theorem can be used to show that that the error is negligble when $1/s$ exceeds the Nyquist rate for band limited functions.

Placing the sampling theorem in the abstract harmonic analysis setting reveals more clearly the underlying group structure, the reciprocal character of the discrete sampling subgroup and the discrete group of translates, as well as the relationship between the support of the Fourier transform and the sampling interval [1, 10].

8. Acknowledgements

Michel Mendès France is a gifted mathematician of great humanity and abilities, with a gentle wit and wide interests (in more than mathematics). It is a privilege to know him and be a friend. I am grateful to Michael Beaty, Chris Fewster, Rowland Higgins and Jonathan Partington for many helpful discussions.

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