THE WHITTAKER-KOTEL'NIKOV-SHANNON THEOREM, SPECTRAL TRANSLATES AND PLANCHEREL'S FORMULA

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ABSTRACT. The classical sampling or WKS theorem on reconstructing signals from uniformly spaced samples assumes the signals to be band-limited (*i.e.*, with spectrum in a bounded interval $[-W, W]$). This assumption was later weakened to a disjoint translates condition on the spectrum which led to an extension of the sampling theorem to multi-band signals with spectra in a union of finite intervals. In this paper the disjoint translates condition is replaced by a more natural 'null intersection' condition on spectral translates. This condition is shown to be equivalent to an analogue of Plancherel's isometric formula when the spectrum has finite measure. Thus to some extent multi-band sampling theory has a logical structure similar to classical Fourier analysis. The relationship between the null intersection condition and the isometric formula is illustrated by considering the consequences when the null intersection condition does not hold. In this case, the sampling representation cannot hold for any function, whereas the isometric formula can still hold for some functions.

1. INTRODUCTION

The Whittaker-Kotel'nikov-Shannon (WKS) theorem, also known as Shannon's sampling theorem, is a cornerstone of communication theory [30, 31]. Its importance is indicated by the number of names associated with its independent discovery (see [14] for further details of its rather tangled history). Shannon's name is most often associated with the theorem because he placed the result in a comprehensive theory of communication. He acknowledged earlier versions and cites [33] as a source of his principal theorem, due to E. T. Whittaker [32], which demonstrates a mathematical equivalence between certain continuous or *analogue* signals and discrete signals. In engineering applications, the theorem provides a theoretical basis for signal reconstruction from discrete samples, though the implementation is often indirect. From a mathematical point of view, the theorem can be regarded as an interpolation result and has stimulated a great deal of interest in this and related areas. Indeed the theorem falls naturally into the theory of square-integrable functions and is closely related to Paley-Wiener theory. A disjoint translates condition on what is essentially the support of the Fourier transform (or *spectrum*) of the signal underlies the WKS theorem and leads to a so-called *multi-band* signal theory (see §3).

In this paper, the role of the spectral translates condition is investigated and the logical structure of the multi-band theory compared to the classical theory of Fourier analysis. It is shown that spectral translates condition is equivalent to an analogue of Plancherel's

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isometric formula (Theorem 2). Consequences of this are that the usual translates condition can be replaced by an energy expression and that the standard uniform sampling theory has to involve a spectral translates condition.

To set notation, the Fourier transform $f^\wedge\colon \mathbb{R} \to \mathbb{C}$ of the square-integrable function $f\colon \mathbb{R} \to \mathbb{C}$ (*i.e.*, $f \in L^2(\mathbb{R})$) will be defined by

$$f^\wedge(\omega) = \int_{\mathbb{R}} f(t)e^{-2\pi it\omega}\,dt, \tag{1}$$

where $f^\wedge$ is defined as the usual limit in the mean if $f \notin L^1(\mathbb{R})$. The inverse Fourier transform of $\varphi \in L^2(\mathbb{R})$ is given by

$$\varphi^\vee(t) = \int_{\mathbb{R}} \varphi(\omega)e^{2\pi it\omega}\,d\omega \tag{2}$$

where $\varphi^\vee$ is defined as the usual limit in the mean if $\varphi \notin L^1(\mathbb{R})$. By the inversion theorem, each $f \in L^2(\mathbb{R})$ satisfies

$$f(t) = \int_{\mathbb{R}} f^\wedge(\omega)e^{2\pi it\omega}\,d\omega$$

almost everywhere, *i.e.*, $f^{\wedge\vee} = f$ almost everywhere, so that the Fourier transform $^\wedge$ is a unitary operator on $L^2(\mathbb{R})$ (more details are in [34, Chap. 16] and [29, Chap. 9]).

Plancherel's isometric formula

$$\int_{\mathbb{R}} |f(t)|^2\,dt := \|f\|^2 = \int_{\mathbb{R}} |f^\wedge(\omega)|^2\,d\omega := \|f^\wedge\|^2, \tag{3}$$

which expresses the fact that the Fourier transform is an isometry on $L^2(\mathbb{R})$, is one of the fundamental results in Fourier analysis [29]. The polarisation identity implies that formula (3) is equivalent to Parseval's formula [29, p. 189] that for $f, g \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} f(t)\overline{g(t)}\,dt = \int_{\mathbb{R}} f^\wedge(\omega)\overline{g^\wedge(\omega)}\,d\omega. \tag{4}$$

This in turn implies the convolution theorem that the Fourier transform of the convolution

$$(f * g)(t) = \int_{\mathbb{R}} f(u)g(t-u)\,du$$

of two functions f, g is the product of their Fourier transforms, *i.e.*,

$$(f * g)^\wedge = f^\wedge\, g^\wedge$$

[34, Vol. II, Chap. XVI, p. 252]. This can be seen to imply Plancherel's formula by considering the value of the inverse Fourier transform of the product of the Fourier transforms. (A different approach is taken in [29, Chap. 9] where the formulae of Plancherel and Parseval are deduced from the convolution theorem.) The nomenclature for the above results is by no means standard (for example (3) is called the Plancherel-Parseval formula by Zygmund [34, Chap. 16, p. 248]); we follow Rudin [29] in distinguishing between the Plancherel formula (3), which involves the norm and the Parseval formula (4), which involves the inner product.

The logical equivalence of Plancherel's isometric formula, Parseval's inner product formula and the convolution theorem means that each of the three could be regarded as a starting point of the subsequent development of Fourier theory. The multi-band theory can be given a similar logical structure in which certain analogues of these three fundamental results are also equivalent. In addition, the reconstruction or representation formula (given in (8) below) when the spectrum has finite measure is equivalent to the null intersection condition (given in (9) below). Thus instead of the null translates intersection condition, one could start the multi-band theory with the energy of the signal being the same as the sum of the squares of the values of the samples (modulo a multiplicative constant of the sampling interval, see Theorem 2 (2.ii) below).

The multi-band theory analogue of the inversion formula is not straightforward as the notion of a unitary operator makes no sense. However, by introducing an auxiliary set, an inversion result can be obtained (see (14) below).

2. The WKS theorem

An analogue or continuous signal can be modelled by a function $f : \mathbb{R} \to \mathbb{C}$ (for real signals f, $f(\mathbb{R}) \to \mathbb{R}$). For physical reasons, it is natural to assume that f is square-integrable or has finite energy, *i.e.*, $\|f\|_{\mathbb{R}}^2 = \int_{\mathbb{R}} |f(t)|^2 dt < \infty$. The spectrum of f is given by the Fourier transform $f^\wedge \colon \mathbb{R} \to \mathbb{C}$. It is also usual to assume that vibrations in a physical system cannot be infinite and so the frequencies of analogue signals are taken to be bounded above. This assumption is based on the vibrations being perfectly sinusoidal but the validity of this model will not be pursued here (this model is discussed in [16, Chap. 17]). Signals with spectra confined to an interval $[-W, W]$ are called *band-limited* and have the representation

$$f(t) = \int_{-W}^{W} f^\wedge(\omega) e^{2\pi i t\omega}\, d\omega,$$

where W is the maximum frequency (strictly speaking W is the supremum of the frequencies, a technicality which is usually disregarded) and where f is analytic by the Paley-Wiener theorem [18, Theorem 7.2]. The WKS theorem gives the following formula for the reconstruction of a band-limited signal f with maximum frequency W in terms of the discrete values or samples $f(k/2W)$. For each $t \in \mathbb{R}$,

$$f(t) = \frac{1}{2W} \sum_{k\in\mathbb{Z}} f\left(\frac{k}{2W}\right) \frac{\sin 2\pi W(t - k/2W)}{\pi(t - k/2W)}. \tag{5}$$

The formula (5) corresponds to a sampling rate of $2W$ (often called the Nyquist rate, twice the maximum frequency of the signal). One interpretation of the theorem is that the output after passing a train of impulses (Dirac delta functions) weighted by successive sample values through an ideal low-pass filter is identical to the original signal. As perfect impulses cannot be generated in practice, the formula (5) is not often used directly. For example, in a common technique of digital-to-analogue conversion in signal processing, the weighted impulses are replaced by a step function (or more picturesquely by a 'sample-and-hold' function) in which the amplitude of each step is that of the initial 'instantaneous' sample value. The energy of the step function approximation is the same as that of the original signal (see §7). The step function is then smoothed with a low-pass filter to give an approximate reconstruction of the original signal.

As well as providing a theoretical justification for the Nyquist sampling rate (5), the WKS theorem and its extensions are also used in the error analysis of signal processing techniques (see for example [4]). Sampling theory was extended to cover the important class of *band pass* signals which have spectra confined to a symmetric pair of intervals $[-a, -a+W] \cup [a-W, a]$, where $a > W$ [21], and subsequently to multi-band signals with spectra made up of the union of a finite number of finite intervals [5, 23, 24]. The reconstruction of signals which are not band limited has been considered by Butzer and others using compactly supported kernels (see [11] and references cited there; a very general abstract approach is given in [1]).

Besides its significance in information theory and engineering [19], the theorem has attracted a great deal of mathematical interest (more details can be found in the survey articles [7, 8, 14] and the books [16, 18, 26, 27]). There is a close connection with interpolation theory, where instead of reconstructing a given function from a subset of its values, one seeks a function assuming given values at specified points. Indeed Whittaker's pioneering paper [32] constructs an analytic function with frequencies at most W which interpolates a given set of data. Naturally enough, there are also connections with Paley-Wiener theory and Fourier analysis (for example, (5) is equivalent to the Poisson summation formula and the WKS theorem also arises surprisingly often in other branches of mathematics [8, 17, 22, 28]). The history of the WKS theorem and its equivalence to Cauchy's theorem and other results from analysis are treated fully in [9].

3. A generalisation of the WKS theorem

The formula (5) can be proved using the observation that translates $[-W, W] + 2Wk$, $k \in \mathbb{Z}$, essentially tessellate $\mathbb{R}$. This observation leads to a multi-band generalisation of the WKS theorem to functions f with Fourier transforms which vanish outside a Lebesgue measurable set A (the Lebesgue measure of A will be denoted by $|A|$) that satisfies the disjoint translates condition

$$A \cap \left(A + \frac{k}{s}\right) = \emptyset, \quad k \in \mathbb{Z} \setminus \{0\}, \tag{6}$$

where here and henceforth in this paper $s \in (0, \infty)$. Note that the possibility of A containing unbounded frequencies is not excluded and we return to this point below. However, the condition (6) implies that for any A satisfying (6), $|A| \leq 1/s$ (see [12] and §4). As well, a more general version of the WKS theorem holds.

Theorem 1. *Let A be a measurable subset of $\mathbb{R}$ and suppose that there exists an $s > 0$ such that (6) holds. Let $f \in L^2(\mathbb{R})$ be continuous and suppose that the Fourier transform*

$f^\wedge$ of f vanishes outside A. Then

$$\int_{\mathbb{R}} |f(t)|^2\,dt = s\sum_{k\in\mathbb{Z}} |f(sk)|^2 \tag{7}$$

and for each $t \in \mathbb{R}$,

$$f(t) = s\sum_{k\in\mathbb{Z}} f(sk)\chi_A^\vee(t - sk), \tag{8}$$

where the series converges absolutely and uniformly.

A proof of Theorem 1 and further references to earlier results are given in [12] where the approach is based on square integrable functions $\varphi\colon \mathbb{R} \to \mathbb{C}$ which vanish outside A. A more general result for stochastic processes was obtained earlier by S. P. Lloyd who showed that the existence of a support A for the spectral distribution which satisfied (6) was equivalent to the random variables of a stationary (in the wide sense) stochastic process being expressible as a linear combination of the sample values [25]; the formula (2) follows when $A = [-W, W]$. For a generalisation of Theorem 1 to abstract harmonic analysis, see [2, 3, 20]. Butzer and Gessinger note in [6] that the formula (7) in the case $A = [-W, W]$ implies the WKS theorem. Equally, one can first prove the formula (8) and then deduce (7), as in [30] or in the abstract harmonic analysis version [20]. When f is not continuous, convergence in norm can be deduced from an orthogonality argument [20]. A common approach is to use the fact that the set $\{e^{2\pi ik\omega/W}\colon k \in \mathbb{Z}\}$ is an orthogonal basis for $L^2([-W, W])$ and that

$$f^\wedge(\omega) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi ik\omega/W}\chi_{[-W,W]}(\omega)$$

] for almost all $\omega \in [-W, W]$. The result follows on taking the inverse Fourier transform and observing that $a_k = f(-k/2W)/2W$.

4. Spectral translates and the sets S_A and $L^2(A)$

From now on the set $A \subset \mathbb{R}$ will be assumed to have positive Lebesgue measure. The set A will be said to satisfy the *null (translates) intersection condition* if for each non-zero $k \in \mathbb{Z}$,

$$\left| A \cap \left(A + \frac{k}{s}\right)\right| = 0. \tag{9}$$

The questions under consideration involve integrals over the spectral domain, so that the condition (9) is essentially equivalent to the disjoint translate condition (6) but is more natural.

The translation $\omega \mapsto \omega + k/s$ on $\mathbb{R}$ can be regarded as an action by the additive group $\mathbb{Z}/s$ on $\mathbb{R}$. Orbits or cosets $\omega + \mathbb{Z}/s = \{\omega + k/s\colon k \in \mathbb{Z}\}$ are disjoint and the factor group $\mathbb{R}/(\mathbb{Z}/s)$ has $[0, 1/s)$ as a transversal or complete set of coset representatives (of course $[-1/2s, 1/2s)$ is also a transversal). The condition (9) is equivalent to the measure of the set of $a' \in A$ for which $a - a' \in \mathbb{Z}/s$ being zero.

Let $\nu\colon \mathbb{R} \to [0,1/s)$ be the projection map given by

$$\nu(x) = x + \frac{k_x}{s},$$

where k_x is the unique integer such that $x+k_x/s \in [0,1/s)$. Then $\nu(\omega)$ is the unique element in the orbit $\mathbb{Z}/s+\omega$ which falls in $[0,1/s)$, so that ν is well defined, $\nu(\mathbb{R}) = [0,1/s)$ and $|\nu(A)| \le 1/s$.

Assume for the rest of this section that A satisfies (9). Then the restriction map $\nu|_A\colon A \to [0,1/s)$ is one-one almost always since for $a,a' \in A$, $\nu(a) = \nu(a')$ implies that $a-a' \in \mathbb{Z}/s$. Moreover, the function $\nu|_A$ is translation by an integer multiple of $1/s$ and so $d\nu|_A(a)/da = 1$ almost always, whence

$$\frac{1}{s} = |\nu(\mathbb{R})| \ge |\nu(A)| := \int_{\nu(A)} dx = \int_A (\nu|_A)'(a)da = \int_A da = |A|. \tag{10}$$

The set of functions $\varphi \in L^2(\mathbb{R})$ with φ vanishing almost everywhere outside A will be called $L^2(A)$. Let

$$S_A = \{f \in L^2(\mathbb{R})\colon f^\wedge(\omega) = 0 \text{ for almost all } \omega \notin A\} = \{f \in L^2(\mathbb{R})\colon f^\wedge \in L^2(A)\}.$$

Then S_A is a linear space over $\mathbb{C}$ and indeed is a Hilbert sub-space of $L^2(\mathbb{R})$. Moreover, for each $\varphi \in L^2(A)$, the induced map $\widetilde{\nu}\colon L^2(A) \to L^2([0,1/s))$, given by

$$\widetilde{\nu}(\varphi)(x) = \varphi \circ (\nu|_A)^{-1}(x)$$

for almost all $x \in [0,1]$, also preserves norms since

$$\int_0^{1/s} |\widetilde{\nu}(\varphi)|^2 = \int_0^{1/s} |\varphi \circ (\nu|_A)^{-1}(x)|^2 dx = \int_A |\varphi|^2$$

and so $L^2(A)$ can be embedded isometrically in the Hilbert space $L^2([0,1/s])$. This embedding is the reason why results in sampling theory are a mixture of Fourier series and transforms.

If $|A| = 1/s$, then $|\nu(A)| = 1/s$ and $\nu|_A$ is a bijection for almost all $a \in A$, whence the map $\widetilde{\nu}$ is a Hilbert space isometric isomorphism and invertible. In the case of the WKS theorem (5), $A = [-W,W]$, $s = 1/2W$ and the map reduces to the identity. Any real square summable sequence $(a_{k/2W}\colon k \in \mathbb{Z})$ can be interpolated by an analytic function with frequencies at most W.

Since A satisfies (9), the measure of A is finite (indeed $|A| \le 1/s$) and it follows that $\varphi \in L^1(\mathbb{R})$ and that $\varphi^\wedge$ is continuous, *i.e.*, each $f \in S_A$ is continuous. Thus S_A a suitable model for analogue signals. In particular, the characteristic function χ_A of A is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and its inverse Fourier transform $\chi_A^\vee$, given by

$$\chi_A^\vee(t) = \int_{\mathbb{R}} e^{2\pi i\omega t}\chi_A(\omega)d\omega = \int_A e^{2\pi i\omega t}d\omega,$$

is continuous and lies in S_A. Write $A_k = \{\omega \in A\colon k_\omega = k\}$ for the (measureable) k-level set of frequencies, so that $A = \cup_{k\in\mathbb{Z}} A_k$. Then

$$\begin{aligned}\chi_A^\vee(t) &= \sum_{k\in\mathbb{Z}} \int_{A_k} e^{2\pi i\omega t} d\omega = \sum_{k\in\mathbb{Z}} \int_{\nu(A_k)} e^{2\pi i(\omega + k_\omega/s)t} (\nu|_A)'(\omega) d\omega \\ &= \sum_{k\in\mathbb{Z}} \int_{\nu(A_k)} e^{2\pi i(\omega+k/s)t} d\omega = \sum_{k\in\mathbb{Z}} e^{2\pi i k t/s} \int_{\nu(A_k)} e^{2\pi i\omega t} d\omega,\end{aligned}$$

where $\nu(A_k) \subseteq [0, 1/s)$. Note that for each $k' \in \mathbb{Z}$,

$$\chi_A^\vee(sk') = \int_A e^{2\pi i\omega sk'} d\omega = \int_{\nu(A)} e^{2\pi i(\omega+k_\omega/s)sk'} (\nu|_A)'(\omega) d\omega,$$

so that when $|A| = 1/s$ and $\nu(A) = [0, 1/s)$ in measure,

$$\chi_A^\vee(sk) = \int_0^{1/s} e^{2\pi i\omega sk} d\omega = \chi_{[0,1/s)}^\vee(sk) = \frac{1}{s}\,\delta_{0k}. \tag{11}$$

Although a set A satisfying (9) can be unbounded, by (10) its Lebesgue measure is at most $1/s$. For example, the symmetric set

$$A = \bigcup_{n\in\mathbb{N}} A_n, \tag{12}$$

where

$$A_n = \left[-n + 1/2 + 2^{-n-1}, -n + 1/2 + 2^{-n}\right] \cup \left[n - 1/2 - 2^{-n}, n - 1/2 - 2^{-n-1}\right],$$

satisfies $|A \cap \{t \in \mathbb{R}\colon |t| \geq N\}| = 2^{-N}$ for each $N \in \mathbb{N}$, has measure 1 and its intersection with the translate $A + k$ by any non-zero $k \in \mathbb{Z}$ (corresponding to $s = 1$) is null. This observation can be used to construct functions with unbounded spectrum. For example, the inverse Fourier transform $\chi_A^\vee$ of the characteristic function χ_A on A is given by

$$\begin{aligned}\chi_A^\vee(t) &= \frac{2}{\pi t} \sum_{n\in\mathbb{N}} \cos\pi(2n - 1 - 3\,2^{-n-1})t\,\sin 2^{-n-1}\pi t \\ &= \sum_{n\in\mathbb{N}} \frac{\cos\pi(2n-1-3\,2^{-(n+1)})t}{2^n} \operatorname{sinc}\pi 2^{-(n+1)}t,\end{aligned}$$

in which each summand contains the factor $(\cos\pi nt)^2 - 1$, so that $\chi_A^\vee(0) = 1$ and $\chi_A^\vee(k) = 0$ for non-zero $k \in \mathbb{Z}$. Note that this also follows from (11). It can be seen that $\|\chi_A^\vee\| = 1 = |\operatorname{supp}\chi_A|$ and $\operatorname{ess\,sup} A = \infty$, so that Paley-Wiener theory, although closely related, does not offer a suitable framework.

From the point of view of signals, the norm can be regarded as the square root of the energy and the preservation of norm can be interpreted as conservation of energy between the original signal and its values (samples) on $s\mathbb{Z}$. The factor s is the time interval between samples and so keeps the dimensions of the two sides of (7) consistent.

5. Plancherel-type formulae and the general WKS theorem

From now on, A will not be assumed to have finite measure, though in fact the measure will always turn out to be finite. Thus we now take the set S_A to be defined by

$$S_A = \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \colon f^\wedge(\omega) = 0 \text{ for almost all } \omega \notin A\} \tag{13}$$

and the functions $f \colon \mathbb{R} \to \mathbb{C}$ we consider will be restricted to being continuous. This conforms to S_A being a model for analogue signals for which A does not necessarily satisfy (9) and it also avoids the problem of indeterminate values on sets of measure zero.

We will show that if $f \in S_A$, then the formula (7), the representation (8), and the null translate intersection condition (9) are equivalent. As a first step, a general set of equivalences is proved for the set S_A, linking analogues of Plancherel's isometric formula, Parseval's formula and the convolution theorem with the null translates intersection condition. These analogues can be regarded as 'hybrids' of the theorems for Fourier series and transform. The Fourier series aspect derives from $\mathbb{R}$ being essentially tessellated by translates of the spectrum A, so that in a sense sampling theory is dual to classical Fourier series. Moreover the construction of the 1-periodic auxiliary function F, defined in (17) below, is based on the translates condition (9). In the case $s = 1/|A|$, there is a similarity with Fourier series, in the sense that for each $\varphi \in L^2(A)$,

$$\lim_{N\to\infty} \int_A \left| \varphi(\omega) - s \sum_{k=-N}^{N} \varphi^\wedge(sk) e^{2\pi i sk\omega} \right|^2 d\omega = 0 \tag{14}$$

(see [12, Theorem 5]).

Theorem 2. *Let A be a Lebesgue measurable subset of $\mathbb{R}$ and $s > 0$. Then the following are equivalent.*

(2.i) *For each non-zero integer k*

$$\left| A \cap \left(A + \frac{k}{s} \right) \right| = 0.$$

(2.ii) *Each $f \in S_A$ satisfies*

$$||f||^2_{\mathbb{R}} = \int_{\mathbb{R}} |f(t)|^2 \, dt = s \sum_{k\in\mathbb{Z}} |f(sk)|^2.$$

(2.iii) *Each $f, g \in S_A$ satisfies*

$$\int_{\mathbb{R}} f(t)\overline{g(t)} \, dt = s \sum_{k\in\mathbb{Z}} f(sk)\overline{g(sk)}.$$

(2.iv) *Each $f, g \in S_A$ satisfies*

$$(f *_{\mathbb{R}} g)(t) = s(f *_{s\mathbb{Z}} g)(t) = s \sum_{k\in\mathbb{Z}} f(sk) g(t - sk), \tag{15}$$

where the series converges to $f * g$ absolutely and uniformly.

The proof is in stages and follows the treatment in [34, Chap. 16] in which Plancherel's theorem is the precursor to Parseval's theorem and the convolution theorem. Of course, that (2.i) implies (2.ii) is not new and the equivalence of (2.ii), (2.iii) and (2.iv) is not surprising as S_A is a sub-space of the Hilbert space $L^2(\mathbb{R})$. However, the result that (2.ii) implies (2.i) does not appear to have been noticed. Butzer *et al.* have discussed (2.iv) in terms of a discretisation of convolution for the band-limited case [10]. By (9), (2.i) implies that $A| \le 1/s$ but no explicit use is made of this in Theorem 2.

5.1. **Hybrid Plancherel and Parseval formulae.** The propositions which follow are modifications of known results (see [12] and references therein). The only change is that the intersection of the translates has measure zero instead being of empty but the proofs are included for completeness.

Proposition 1. *Suppose that for some* $s > 0$,

$$\left| A \cap \left(A + \frac{k}{s} \right) \right| = 0 \tag{16}$$

for each nonzero integer k. *Then* $|A| \le 1/s$ *and for each* $f \in S_A$,

$$||f||_{\mathbb{R}}^2 = s \sum_{k \in \mathbb{Z}} |f(sk)|^2 = \|f^\wedge\|_{\mathbb{R}}^2.$$

Proof. By (10), $|A| \le 1/s$. Also by (16), the function $F \colon \mathbb{R} \to \mathbb{R}$ given by

$$F(\omega) = \sum_{k \in \mathbb{Z}} f^\wedge(\omega + k/s) \tag{17}$$

is well defined and absolutely convergent for almost all $\omega \in A$ (the sum collapses to at most one term $f^\wedge(\omega + k_\omega/s)$ say) and has period $1/s$. Let

$$F_N(\omega) = \sum_{k=-N}^{N} f^\wedge(\omega + \frac{k}{s}) \text{ and } \Phi(\omega) = \sum_{k \in \mathbb{Z}} |f^\wedge(\omega + \frac{k}{s})|^2.$$

Then $\lim_{N\to\infty} F_N = F$, and by (16),

$$\begin{aligned}
\int_0^{1/s} |F_N(\omega)|^2 d\omega &= \int_0^{1/s} \sum_{k=-N}^{N} \sum_{k'=-N}^{N} f^\wedge(\omega + \frac{k}{s}) \overline{f^\wedge(\omega + \frac{k'}{s})} d\omega \\
&= \int_0^{1/s} \sum_{k=-N}^{N} |f^\wedge(\omega + \frac{k}{s})|^2 d\omega = \sum_{k=-N}^{N} \int_0^{1/s} |f^\wedge(\omega + \frac{k}{s})|^2 d\omega \\
&= \int_{-N/s}^{(N+1)/s} |f^\wedge(\omega')|^2 d\omega' \le \|f\|_{\mathbb{R}}^2 < \infty,
\end{aligned}$$

since $f, f^\wedge \in L^2(\mathbb{R})$. Indeed

$$\lim_{N\to\infty} \int_0^{1/s} |F_N(\omega)|^2 d\omega = \|f\|_{\mathbb{R}}^2. \tag{18}$$

Moreover for almost all ω,

$$|F_N(\omega)|^2 \leq \sum_{k=-N}^{N} |f^\wedge(\omega + \frac{k}{s})|^2 \leq \sum_{k\in\mathbb{Z}} |f^\wedge(\omega + \frac{k}{s})|^2 = \Phi(\omega) = |f^\wedge(\omega + \frac{k_\omega}{s})|^2$$

by (16); and since $\sum_{k\neq k_\omega} \int_0^{1/s} |f^\wedge(\omega + k/s)|^2 d\omega = 0$,

$$\begin{aligned}\int_0^{1/s} \Phi(\omega)d\omega &= \int_0^{1/s} \sum_{k\in\mathbb{Z}} |f^\wedge(\omega + \frac{k}{s})|^2 d\omega = \int_0^{1/s} |f^\wedge(\omega + \frac{k_\omega}{s})|^2 d\omega \\ &= \sum_{k\in\mathbb{Z}} \int_0^{1/s} |f^\wedge(\omega + \frac{k}{s})|^2 d\omega = \int_{\mathbb{R}} |f^\wedge(\omega')|^2 d\omega' = \|f\|_{\mathbb{R}}^2.\end{aligned}$$

Thus $\Phi \in L^1([0, 1/s])$. But $|F_N(\omega)| \leq \Phi(\omega)$ for $\omega \in [0, 1/s]$, whence by (18) and Lebesgue's dominated convergence theorem,

$$\|f\|_{\mathbb{R}}^2 = \lim_{N\to\infty} \int_0^{1/s} |F_N|^2 = \int_0^{1/s} \lim_{N\to\infty} |F_N|^2 = \int_0^{1/s} |F|^2$$

and $F \in L^2([0, 1/s])$ and hence $F \in L^1([0, 1/s])$. Now for each $m \in \mathbb{Z}$,

$$\begin{aligned}F^\wedge(m) &= s\int_0^{1/s} F(\omega)e^{-2\pi i m s\omega} d\omega = s\sum_{k\in\mathbb{Z}} \int_0^{1/s} f^\wedge(\omega + \frac{k}{s})e^{-2\pi i m s\omega} d\omega \\ &= s\sum_{k\in\mathbb{Z}} \int_{k/s}^{(k+1)/s} f^\wedge(\omega)e^{-2\pi i m s(\omega - k/s)} d\omega = s\int_{\mathbb{R}} f^\wedge(\omega)e^{-2\pi i m s\omega} d\omega \\ &= sf^{\wedge\wedge}(sm).\end{aligned}$$

But f is continuous, so that the Fourier coefficients of F are $sf(-sm)$, $m \in \mathbb{Z}$, and F has a Fourier series representation

$$F(\omega) \sim s\sum_{k\in\mathbb{Z}} f(-sk)e^{2\pi i s k\omega} = s\sum_{k\in\mathbb{Z}} f(sk)e^{-2\pi i s k\omega}.$$

Hence by Plancherel's formula for Fourier series,

$$\int_0^{1/s} |F(\omega)|^2 d\omega = s\sum_{k\in\mathbb{Z}} |f(sk)|^2$$

and the proposition is proved. □

The conclusion of Proposition 1 can be regarded as a hybrid of Plancherel's formula for Fourier integrals and series.

Proposition 2. *The equation*

$$\int_{\mathbb{R}} |f(t)|^2 dt = s\sum_{k\in\mathbb{Z}} |f(sk)|^2$$

holds for $f, g \in S_A$ if and only if

$$\int_{\mathbb{R}} f\bar{g} = s\sum_{k\in\mathbb{Z}} f(sk)\overline{g(sk)}$$

holds for $f, g \in S_A$.

Proof. The polarisation identity that for any $a, b \in \mathbb{C}$,

$$4a\bar{b} = |a+b|^2 - |a-b|^2 + i|a+ib|^2 - i|a-ib|^2 \tag{19}$$

is applied in the usual way (S_A is linear over $\mathbb{C}$). By (19) and (7),

$$\begin{aligned}4\int_{\mathbb{R}} f\bar{g} &= \int_{\mathbb{R}} |f+g|^2 - \int_{\mathbb{R}} |f-g|^2 + i\int_{\mathbb{R}} |f+ig|^2 - i\int_{\mathbb{R}} |f-ig|^2 \\ &= s\left(\sum_{s\mathbb{Z}} |f+g|^2 - \sum_{s\mathbb{Z}} |f-g|^2 + i\sum_{s\mathbb{Z}} |f+ig|^2 - i\sum_{s\mathbb{Z}} |f-ig|^2\right) \\ &= s\left(\sum_{s\mathbb{Z}} \left(|f+g|^2 - |f-g|^2 + i|f+ig|^2 - i|f-ig|^2\right)\right) = 4s\sum_{k\in\mathbb{Z}} f(sk)\bar{g}(sk),\end{aligned}$$

where the sum $\sum_{s\mathbb{Z}}$ is the sum over sk, $k \in \mathbb{Z}$. The reverse conclusion follows on putting $g = f$. □

The next step is to pass from a hybrid Parseval formula to a hybrid convolution result.

Proposition 3. *Suppose that* $f, g \in S_A$ *and*

$$\int_{\mathbb{R}} f\overline{g} = s\sum_{k\in\mathbb{Z}} f(sk)\overline{g(sk)}. \tag{20}$$

Then for each $t \in \mathbb{R}$,

$$(f *_{\mathbb{R}} g)(t) = s(f *_{s\mathbb{Z}} g)(t) = s\sum_{k\in\mathbb{Z}} f(sk)g(t-sk), \tag{21}$$

where the series converges absolutely and uniformly to $f *_{\mathbb{R}} g$.

Proof. Note that since $f, g \in L^2(|R)$, Parseval's formula (4) holds and $\int_{\mathbb{R}} f\overline{g} = \int_{\mathbb{R}} f^\wedge\overline{g^\wedge}$. Given any each real t, let $\psi_t(\omega) = \overline{g^\wedge(\omega)}e^{-2\pi i\omega t}$. Then for each t, $\psi_t^\vee \in S_A$. By the convolution theorem and the hypothesis (20), for each $t \in \mathbb{R}$,

$$(f^\wedge g^\wedge)^\vee(t) = \int_{\mathbb{R}} f^\wedge(\omega)\overline{\psi_t(\omega)}d\omega = \int_{\mathbb{R}} f(u)\overline{\psi_t^\vee(u)}du = s\sum_{k\in\mathbb{Z}} f(sk)\overline{\psi_t^\vee(sk)}.$$

But

$$\begin{aligned}\overline{\psi_t^\vee(sk)} &= \overline{\int_{\mathbb{R}} \overline{g^\wedge(\omega)e^{2\pi it\omega}}e^{2\pi isk\omega}d\omega} = \int_{\mathbb{R}} g^\wedge(\omega)e^{2\pi it\omega}e^{-2\pi isk\omega}d\omega = \int_{\mathbb{R}} g^\wedge(\omega)e^{2\pi i(t-sk)\omega}d\omega \\ &= g^{\wedge\vee}(t-sk) = g(t-sk)\end{aligned}$$

and hence (21) follows.

The convergence of the series is absolute and uniform. For

$$s\sum_{k\in\mathbb{Z}}|f(sk)g(t-sk)| \le \big(s\sum_{k\in\mathbb{Z}}|f(sk)|^2\, s\sum_{k\in\mathbb{Z}}|g(t-sk)|^2\big)^{1/2}.$$

Now from (20) with $g=f$ and as $f\in S_A$, $s\sum_{k\in\mathbb{Z}}|f(sk)|^2=\|f\|_{\mathbb{R}}^2$; and

$$s\sum_{k\in\mathbb{Z}}|g(t-sk)|^2 = s\sum_{k\in\mathbb{Z}}|\overline{\psi_t^\vee(sk)}|^2 = \|\psi_t\|_{\mathbb{R}}^2 = \|g\|_{\mathbb{R}}^2, \tag{22}$$

on putting $f=g$ in (20). Hence

$$\sum_{k\in\mathbb{Z}}|f(sk)g(t-sk)| \le \frac{1}{s}\|f\|_{\mathbb{R}}\|g\|_{\mathbb{R}}.$$

To establish uniform convergence, note that by Proposition 1, given $\varepsilon>0$, there exists an $N_0=N_0(\varepsilon)$ such that for $N>N_0$, $\sum_{|k|>N}|f(sk)|^2<\varepsilon$. Now for each $t\in\mathbb{R}$,

$$\begin{aligned}\left|f*_{\mathbb{R}}g(t)-s\sum_{k=-N}^{N}f(sk)g(t-sk)\right| &= \left|s\sum_{|k|>N}f(sk)g(t-sk)\right|\\ &\le \left(s\sum_{|k|>N}|f(sk)|^2 s\sum_{|k|>N}|g(t-sk)|^2\right)^{1/2}\\ &< (s\varepsilon)^{1/2}\|g\|_{\mathbb{R}}\end{aligned}$$

by (22) and the convergence is uniform. □

The following hybrid convolution result for suitable $f\in S_A$ implies that the intersections of distinct translates of A are null.

Proposition 4. *Let $f=\varphi^\vee$, where $\varphi\colon A\to\mathbb{C}$ is given by $\varphi(\omega)=\chi_A(\omega)\,e^{-\omega^2}$. Suppose that*

$$(f*_{\mathbb{R}}f)(t) = s(f*_{s\mathbb{Z}}f)(t) = s\sum_{k\in\mathbb{Z}}f(sk)f(t-sk),$$

where the series converges absolutely. Then for each non-zero integer k

$$\left|A\cap\left(A+\frac{k}{s}\right)\right| = 0.$$

Proof. The function $f\colon\mathbb{R}\to\mathbb{C}$ given by $f(t)=\varphi^\vee(t)$, where $\varphi\colon\mathbb{R}\to\mathbb{C}$ is given by

$$\varphi(\omega)=e^{-\omega^2}\chi_A(\omega),$$

is in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. Hence $\varphi^\vee$ is continuous, $f\in S_A$ and $f^\wedge=\varphi$ almost everywhere. Therefore, by the convolution theorem,

$$(f*_{\mathbb{R}}f)^\wedge(\omega)=\varphi(\omega)^2=e^{-2\omega^2}\chi_A(\omega)$$

almost everywhere. But by hypothesis,

$$(f *_{\mathbb{R}} f)(t) = s \sum_{k\in\mathbb{Z}} f(sk) f(t - sk)$$

and so since the convergence is absolute,

$$(f^\wedge(\omega))^2 = e^{-2\omega^2}\chi_A(\omega) = s\sum_{k\in\mathbb{Z}} f(sk)e^{-2\pi isk\omega} f^\wedge(\omega) = s\sum_{k\in\mathbb{Z}} f(sk)e^{-2\pi isk\omega}e^{-\omega^2}\chi_A(\omega)$$

for almost all $\omega \in A$, *i.e.*, for almost all $\omega \in A$,

$$s\sum_{k\in\mathbb{Z}} f(sk)e^{-2\pi isk\omega} = e^{-\omega^2}. \tag{23}$$

The $(1/s)$-periodic function $\Phi\colon \mathbb{R} \to \mathbb{C}$ given by

$$\Phi(\omega) = \sum_{k\in\mathbb{Z}} e^{-(\omega+k/s)^2}\chi_A(\omega + k/s) = \sum_{k\in\mathbb{Z}} \varphi(\omega + k/s).$$

is absolutely convergent and in $L^1([0, 1/s])$, with Fourier coefficient

$$\begin{aligned}\Phi^\wedge(m) &= s\int_0^{1/s} \sum_{k\in\mathbb{Z}} \varphi(\omega + k/s)e^{-2\pi ism\omega} d\omega \\ &= s\sum_{k\in\mathbb{Z}} \int_0^{1/s} \varphi(\omega + k/s)e^{-2\pi ism\omega} d\omega = s\int_{\mathbb{R}} \varphi(\omega)e^{-2\pi ism\omega} d\omega \\ &= s\varphi^\wedge(sm) = sf(-sm)\end{aligned}$$

since $\varphi^\vee(t) = f(t)$. Thus the Fourier series representation of Φ is

$$\Phi(\omega) \sim s\sum_{k\in\mathbb{Z}} f(-sk)e^{2\pi isk\omega} = s\sum_{k\in\mathbb{Z}} f(sk)e^{-2\pi isk\omega}$$

for almost all $\omega \in \mathbb{R}$ by (23). Hence, for almost all $\omega \in A$,

$$\Phi(\omega) = \sum_{k\in\mathbb{Z}} e^{-(\omega+k/s)^2}\chi_A(\omega + k/s) = s\sum_{k\in\mathbb{Z}} f(sk)e^{-2\pi isk\omega} = e^{-\omega^2}$$

and since $\chi_A(\omega) = 1$ for each $\omega \in A$, it follows that for almost all $\omega \in A$, $\chi_A(\omega + k/s) = 0$ when $k \neq 0$. □

Combining Propositions 1, 2, 3 and 4 proves Theorem 2. The proofs of these propositions can be generalised to locally compact abelian groups by using coset decomposition into transversals of a quotient group [2].

5.2. **The reconstruction formula and the translates condition.** The equivalence of the reconstruction formula and the translates condition follows from Theorem 2 and now explicit use is made of the estimate $|A| \leq 1/s$ implied by (9) (see below also). Lloyd [25, Theorem 1] has proved a similar but more general result for stochastic processes for which there exists a support for the spectral distribution satisfying a disjoint translates condition corresponding to (6) and equivalent to (9). The necessity of the translates condition is related to the phenomenon of 'aliasing', where spectral overlaps lead to a pointwise error in the representation [16, Chap. 11].

Theorem 3. *Let A be a measureable subset of $\mathbb{R}$ and $s > 0$. Then the following are equivalent.*

(3.i) *For each non-zero integer k*

$$\left| A \cap \left(A + \frac{k}{s} \right) \right| = 0.$$

(3.ii) $|A| \leq 1/s$ *and for each $f \in S_A$,*

$$||f||_{\mathbb{R}}^2 = \int_{\mathbb{R}} |f(t)|^2 \, dt = s \sum_{k \in \mathbb{Z}} |f(sk)|^2.$$

(3.iii) $\chi_A^\vee \in S_A$ *and each $f \in S_A$ is given by*

$$f(t) = s \sum_{k \in \mathbb{Z}} f(sk) \chi_A^\vee(t - sk),$$

where the series converges to f absolutely and uniformly.

Proof. We show that each assertion implies the next in cyclic order.

(3.i) implies (3.ii) by Proposition 1.

(3.ii) implies (3.iii) since $|A| \leq 1/s$ implies that

$$\int_{\mathbb{R}} \chi_A = \int_{\mathbb{R}} |\chi_A|^2 = |A| \leq 1/s,$$

χ_A is in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, whence $\chi_A^\vee$ is continuous and $\chi_A^\vee \in L^2(\mathbb{R})$, *i.e.*, $\chi_A^\vee \in S_A$. The representation in (3.iii) now follows from Proposition 2, Proposition 3 and the equation $f^\wedge = f^\wedge \chi_A$, which is valid almost everywhere, applied to f and to $g = \chi_A^\vee$ to give

$$f(t) = (f^\wedge \chi_A)^\vee(t) = (f *_{\mathbb{R}} \chi_A^\vee)(t) = s \sum_{k \in \mathbb{Z}} f(sk) \chi_A^\vee(t - sk).$$

(3.iii) implies (3.i) by Theorem 2 but the hypothesis $\chi_A^\vee \in S_A$ allows a simpler proof which is now sketched. Let $f = \chi_A^\vee$. Since $\chi_A^\vee \in S_A$,

$$\chi_A^\vee(t) = s \sum_{k \in \mathbb{Z}} \chi_A^\vee(sk) \chi_A^\vee(t - sk)$$

and so for almost all $\omega \in \mathbb{R}$,

$$\chi_A(\omega) = s \sum_{k \in \mathbb{Z}} \chi_A^\vee(sk) e^{2\pi i s k \omega} \chi_A(\omega) = \sum_{k \in \mathbb{Z}} \chi_A(\omega + k/s) \chi_A(\omega),$$

whence for almost all $\omega \in A$,

$$1 = \chi_A(\omega) = \sum_{k\in\mathbb{Z}} \chi_A(\omega + k/s).$$

Thus for almost all $\omega \in A$, $\chi_A(\omega + k/s) = 0$ for $k \neq 0$ and the result follows. $\square$

It is implicit in the above that if the sampling theorem holds for one function $f \in S_A$ with Fourier transform supported throughout A, then the null measure translates condition holds and the sampling theorem holds for each $f \in S_A$. This 'global' behaviour of the sampling representation (*i.e.*, if it holds for one function in S_A, it holds for all) can be contrasted with the hybrid Plancherel formula (see Theorem 5 below). If the measure of A is finite, then the representation (8) implies the null translates condition (9). On the other hand the null translates condition (9) alone implies the representation (8).

Theorem 4. *The following are equivalent.*

(4.i) *The set A has finite measure and there exists a function $f \in S_A$ with* $\operatorname{supp} f^\wedge = \bar{A}$ *and with the sampling representation*

$$f(t) = s \sum_{k\in\mathbb{Z}} f(sk)\chi_A^\vee(t - sk),$$

where the series converges absolutely and and uniformly.

(4.ii) *The null intersection condition*

$$\left| A \cap \left(A + \frac{k}{s} \right) \right| = 0.$$

holds for each non-zero $k \in \mathbb{Z}$.

Proof. By taking the Fourier transform of f, it follows that for almost all $\omega \in A$,

$$f^\wedge(\omega) = s \sum_{k\in\mathbb{Z}} f(sk) e^{-2\pi i sk\omega} \chi_A(\omega). \tag{24}$$

Since $|A| < \infty$, $f^\wedge \in L^1 \cap L^2(A)$ and, as shown in Proposition 1, the $1/s$-periodic function $F(\omega) = \sum_{k\in\mathbb{Z}} f^\wedge(\omega + k/s)$ has the Fourier series representation

$$F(\omega) \sim s \sum_{k\in\mathbb{Z}} f(sk) e^{-2\pi i sk\omega} \chi_A(\omega).$$

Therefore, by (24), for almost all $\omega \in A$,

$$f^\wedge(\omega) = \sum_{k\in\mathbb{Z}} f^\wedge(\omega + k/s) \tag{25}$$

and for almost all $\omega + k/s \in A$,

$$f^\wedge(\omega + k/s) = f^\wedge(\omega),$$

while for almost all $\omega + k/s \notin A$, $f^\wedge(\omega + k/s) = 0$. Hence for almost all $\omega \in A$

$$f^\wedge(\omega) = \sum_{\omega + k/s \in A} f^\wedge(\omega) = f^\wedge(\omega) \sum_{k\in\mathbb{Z}} \chi_A(\omega + k/s)$$

and it follows that for almost all $\omega \in A$,

$$1 = \sum_{k\in\mathbb{Z}} \chi_A(\omega + k/s) = 1 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \chi_A(\omega + k/s).$$

Thus for almost all $\omega \in A$, $\sum_{k\in\mathbb{Z}\setminus\{0\}} \chi_A(\omega + k/s) = 0$, *i.e.*

$$|A \cap (A + k/s)| = 0$$

for each non-zero $k \in \mathbb{Z}$ and therefore (4.i) implies (4.ii).

(4.ii) implies (4.i) by Theorem 3. □

Note that (24) is the hybrid form of the Fourier transform

$$f^\wedge(\omega) = \int_{\mathbb{R}} f(t)e^{-2\pi i\omega t}dt.$$

Since $f \in S_A$ implies that the sequence $(f(sk) \colon k \in \mathbb{Z})$ is square integrable, the map $\Phi \colon S_A \to \ell^2(\mathbb{Z})$, given by

$$f \mapsto (f(sk))_{k\in\mathbb{Z}}\,, \tag{26}$$

is well defined and one to one by Theorem 3 ($\Phi(f) = 0$ if and only if $f(sk) = 0$ for each $k \in \mathbb{Z}$, which is equivalent to $f = 0$). The map Φ is not necessarily onto since $\widetilde{\nu}(L^2(A))$ can be strictly contained in $L^2([0, 1/s))$. Now $\ell^2(\mathbb{Z})$ is isomorphic to $L^2([0, 1/s))$, so that when $|A| < 1/s$, there are ℓ^2 sequences which cannot be interpolated by any function in S_A.

When $|A| < 1/s$, one can consider an auxiliary set $\widetilde{A} \supset A$ which satisfies (9) with $|\widetilde{A}| = 1/s$ and $\nu|_{\widetilde{A}} \colon \widetilde{A} \to [0, 1/s]$ a bijection almost everywhere [12]. Indeed the map $\Phi \colon S_{\widetilde{A}} \to \ell^2(\mathbb{Z})$ given by (26) is an isomorphism with inverse $\Psi \colon \ell^2(\mathbb{Z}) \to S_{\widetilde{A}}$ ($\Psi \circ \Phi = 1$), given by

$$\Psi\left((a_k)_{k\in\mathbb{Z}}\right) = s \lim_{N\to\infty} \sum_{k=-N}^{N} a_k e^{2\pi i\omega ks} \chi_{\widetilde{A}}(\omega). \tag{27}$$

Thus any $\ell^2(\mathbb{Z})$ sequence can be interpolated by a function from $S_{\widetilde{A}}$.

6. When the null translates intersection condition does not hold

A converse to Theorem 4 would be that if A does not satisfy the null translates condition (9), then functions in S_A do not satisfy the hybrid Plancherel condition (7) or allow the reconstruction (8). Indeed if A does not satisfy (9) then, apart from the trivial possibility that $\operatorname{supp} f^\wedge$ is a subset of A satisfying (9), the representation (8) cannot hold. However, the hybrid Plancherel formula (7) can hold for a function in S_A without implying either the null intersection condition (9) or the representation (8). Critically, this also occurs in a non-trivial way, *i.e.*, for a function for which $\operatorname{supp} f^\wedge = \bar{A}$.

Theorem 5. *Suppose that A does not satisfy the null translate intersection condition* (9). *Then there exists $f \in S_A$ satisfying the isometric formula*

$$\int_{\mathbb{R}} |f(t)|^2\, dt = s \sum_{k\in\mathbb{Z}} |f(sk)|^2 = \int_A |f^\wedge(\omega)|^2 d\omega$$

with $\operatorname{supp} f^{\wedge} = \bar{A}$.

Proof. First the existence of a set $E \subset A$ such that $0 < |E| \le 1/s$, $E + k_0/s \subseteq A$ and $|E \cap (E + k_0/s)| = 0$ is established. From the hypothesis that (9) does not hold, there exists $k_0 \in \mathbb{Z}$ such that

$$|A \cap (A + k_0/s)| > 0.$$

Choose $B \subset (A \cap (A + k_0/s)) - k_0/s$ such that $|B| \le 1/s$. Then $B \subset A$ and $B + k_0/s \subset A$. If $|B \cap (B + k_0/s)| = 0$ it suffices to let $E = B$. Otherwise let

$$E = B \setminus (B \cap (B + k_0/s)).$$

Then $E \subset B$ and $E \not\subset (B + k_0/s)$, and there are two cases to consider: either

$$0 < |E| < |B| \le 1/s$$

and E has the required properties, or $|E| = 0$. If $|E| = 0$, then $|B \setminus (B \cap (B + k_0/s))| = 0$, *i.e.*, that B is translation invariant with respect to all multiples of k_0/s. However, it then follows that $\operatorname{ess\,inf} B = -\infty$ and hence that $|B| = \infty$, which contradicts the construction of B which asserts $|B| \le 1/s$. Therefore $|E| \ne 0$.

Let

$$g(t) = (2|E|)^{-1/2} \left(1 - e^{2\pi i k_0 t/s}\right) \chi_E^{\vee}(t)$$

and note that

$$g^{\wedge}(\omega) = (2|E|)^{-1/2} \left(\chi_E(\omega) - \chi_{E+k_0/s}(\omega)\right).$$

Thus $||g|| = 1$ and $g(sm) = 0$ for all $m \in \mathbb{Z}$, whence $\sum_{k\in\mathbb{Z}} |g(k)|^2 = 0$ and the hybrid Plancherel formula does not hold for g. Non-zero functions whose samples are all zero have occurred in sampling theory in connection with the aliasing error. The function g introduced here is based on a family constructed by J. R. Higgins in [15].

The property that g has zero samples can be exploited as follows. Consider $f = h + \rho g$, $\rho \in \mathbb{C}$. Then

$$s \sum_{k\in\mathbb{Z}} |f(sk)|^2 = s \sum_{k\in\mathbb{Z}} |h(sk) + \rho g(sk)|^2 = s \sum_{k\in\mathbb{Z}} |h(sk)|^2$$

and

$$||f||^2 = ||h + \rho g||^2 = \int_{\mathbb{R}} (h + \rho g)(\bar{h} + \bar{\rho}\bar{g}) = ||h||^2 + \bar{\rho}\langle h, g\rangle + \rho\overline{\langle h, g\rangle} + |\rho|^2,$$

where $\langle h, g\rangle = \int_{\mathbb{R}} h\bar{g}$. Hence, given h, the function f satisfies the hybrid Plancherel formula if ρ can be found such that

$$s \sum_{k\in\mathbb{Z}} |h(sk)|^2 = |\rho|^2 + \bar{\rho}\langle h, g\rangle + \rho\overline{\langle h, g\rangle} + ||h||^2.$$

It is sufficient to find $h \in S_A$, $\rho \in \mathbb{R}$ such that

$$\rho^2 + 2\Re\langle h, g\rangle \rho + ||h||^2 - s \sum_{k\in\mathbb{Z}} |h(sk)|^2 = 0, \tag{28}$$

where $\Re$ is the real part. Such a ρ exists if

$$s\sum_{k\in\mathbb{Z}}|h(sk)|^2-||h||^2>0. \tag{29}$$

Choose $h\in S_A$ such that $h^\wedge(\omega)=e^{-\omega^2/2}\chi_A(\omega)$ and let $H(\omega)=\sum_{k\in\mathbb{Z}}h^\wedge(\omega+k/s)$. Then by arguing as in Proposition 1, it can be verified that

$$\int_0^{1/s}|H(\omega)|^2\,d\omega=s\sum_{k\in\mathbb{Z}}|h(sk)|^2<\infty. \tag{30}$$

Using the absolute convergence of the sum defining H for each $\omega\in(0,1/s)$ to justify changing the order of integration and summation, by direct calculation

$$\begin{aligned}\int_0^{1/s}|H(\omega)|^2\,d\omega&=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}h^\wedge(\omega)\overline{h^\wedge(\omega+k/s)}\,d\omega\\&=\|h\|^2+\sum_{k\in\mathbb{Z}'}\int_{\mathbb{R}}e^{-\omega^2/2}e^{-(\omega+k/s)^2/2}\,d\omega.\end{aligned} \tag{31}$$

Combining (30) and (31)

$$s\sum_{k\in\mathbb{Z}}|h(sk)|^2-||h||^2=\sum_{k\in\mathbb{Z}'}\int_{\mathbb{R}}e^{-\omega^2/2}e^{-(\omega+k/s)^2/2}\,d\omega>0.$$

Therefore, for this particular choice of h, equation (29) is verified and hence a real ρ exists such that $f=h+\rho g$ satisfies equation (7). Finally, note that

$$f^\wedge(\omega)=h^\wedge(\omega)+\rho g^\wedge(\omega)=e^{-\omega^2/2}\chi_A(\omega)+\rho(2|E|)^{-1/2}\left(\chi_E(\omega)-\chi_{E+k_0/s}(\omega)\right)$$

and therefore $f^\wedge$ is non-zero except on a set of measure zero, *i.e.*, that $\operatorname{supp}f^\wedge=\bar{A}$. $\square$

The function f constructed from the particular function h, where $h^\wedge(\omega)=e^{-\omega^2/2}\chi_A(\omega)$, suffices to prove Theorem 5 but other more concrete examples can be calculated. Let $A=(-3/4,-1/4)\cup(1/4,3/4)$ and $s=1$. Then $A\cap(A+1)=(1/4,3/4)$ and A does not satisfy (9) when $k_0=1$. Choose $E=(-3/4,-1/4)$ so that $E+k_0/s=(1/4,3/4)$ and indeed $E\cup(E+k_0/s)=A$. Then it can be calculated that

$$g(t)=\frac{\sin\pi t\sin\pi(t-1/2)/2}{\pi(t-1/2)/2}.$$

Equations (29)–(31) suggest that functions h with $h^\wedge>0$ should be considered. Choose $h\colon\mathbb{R}\to\mathbb{C}$ so that $h^\wedge(\omega)=(1-|\omega|)\chi_A(\omega)$. Then

$$h(t)=\frac{1}{2}\frac{\sin\pi t}{\pi t}\frac{\sin\pi t/2}{\pi t/2}+\frac{3}{8}\left(\frac{\sin\pi t/2}{\pi t/2}-\frac{\sin 3\pi t/2}{3\pi t/2}\right).$$

It can then be determined from (28) that

$$\rho=\sqrt{\left(\frac{(4-\pi)^2}{2\pi^4}+\frac{23}{48}\right)}-\frac{4-\pi}{\sqrt{2}\pi^2}$$

and therefore

$$
\begin{aligned}
f(t) = \frac{1}{2}\frac{\sin \pi t}{\pi t}\frac{\sin \pi t/2}{\pi t/2} + \frac{3}{8}\left(\frac{\sin \pi t/2}{\pi t/2} - \frac{\sin 3\pi t/2}{3\pi t/2}\right) + \\
+ \left(\sqrt{\left(\frac{(4-\pi)^2}{2\pi^4} + \frac{23}{48}\right)} - \frac{4-\pi}{\sqrt{2}\pi^2}\right)\frac{\sin \pi t \sin \pi(t-1/2)/2}{\pi(t-1/2)/2}. \quad (32)
\end{aligned}
$$

Note that $f^\wedge(\omega) = ((1-|\omega|) + \rho e^{i\pi\omega})\,\chi_A(\omega)$ so that $\operatorname{supp} f^\wedge = \bar{A}$. It is readily verified that $f(0) = 1/2$, $f(k) = 0$ for even integers k and $|f(k)| = 1/\pi|k|$ for odd k. Hence

$$
s\sum_{k\in\mathbb{Z}} |f(sk)|^2 = \frac{1}{2} + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{1}{2} + \frac{2}{\pi^2}\frac{\pi^2}{8} = \frac{3}{4}
$$

and it can also be verified, for example by calculating $\|f^\wedge\|$, that $\|f\|_{\mathbb{R}}^2 = 3/4$.

7. Conclusion

Instead of the usual disjoint translates condition in frequency space, a hybrid convolution theorem can be regarded as a basis for the theory of uniform sampling and reconstruction of multi-band signals. The convolution theorem can be derived from a hybrid Plancherel-type isometric formula, allowing sampling theory to be given a logical structure in which the hybrid versions of Plancherel's formula, Parseval's formula and the convolution theorem are at the centre but equivalent to the null translates condition and the representation formula (8), providing the convolution theorem holds for a suitable function, or providing $|A|$ is finite. These conditions cannot be omitted. The function $f \in S_A$ given by (32) satisfies (7) despite the fact that (9) does not hold for A. The reason that functions cannot be reconstructed using (8) from their samples when the null translates condition (9) does not hold is that uniqueness fails — more than one function in the class S_A has the same set of samples $\{f(sk)\colon k \in \mathbb{Z}\}$. This follows from the existence of a non-zero function $g \in S_A$ with zero samples.

Placing the WKS theorem in the abstract harmonic analysis setting reveals more clearly the underlying group structure, the reciprocal character of the discrete sampling subgroup and the discrete group of translates, as well as the relationship between the support of the Fourier transform and the sampling interval [3, 13]. In addition, the confusing nomenclature is avoided. A treatment on these lines will appear in due course [2].

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References

1. C. Bardaro and G. Vinti, *Uniform convergence and rate of approximation for a nonlinear version of the general sampling operator*, Result. Math. **34** (1998), 224–240.
2. M. G. Beaty and M. M. Dodson, *Shannon's sampling theorem and harmonic analysis*, Preprint.
3. ———, *Abstract harmonic analysis and the sampling theorem*, Sampling theory in Fourier and signal analysis: Advanced topics (J. R. Higgins and R. L. Stens, eds.), Oxford University Press, 1999, pp. 233–265.
4. M. G. Beaty, M. M. Dodson, and J. R. Higgins, *Approximating Paley-Wiener functions by smoothed step functions*, J. Approx. Th. **78** (1994), 433–445.
5. J. L. Brown, Jr., *Sampling expansions for multiband signals*, IEEE Trans. Acoust. Speech Signal Process. **ASSP-33** (1985), 312–315.
6. P. Butzer and A. Gessinger, *The approximate sampling theorem, Poisson's sum formula, a decomposition theorem for Parseval's equation and their interconnections. The heritage of P. L. Chebyshev: a Festschrift in honor of the 70th birthday of R. J. Rivlin*, Ann. Numer. Math. **4** (1997), 143–160.
7. P. L. Butzer, *A survey of the Whittaker-Shannon sampling theorem and some of its extensions*, J. Math. Res. Exposition **3** (1983), 185–212.
8. P. L. Butzer, M. Hauss, and R. L. Stens, *The Sampling Theorem and its unique role in various branches of Mathematics*, Mathematical Sciences, Past and Present, 300 years of Mathematische Gesellschaft in Hamburg, Mitteilungen Math. Ges. Hamburg, Hamburg, 1990.
9. P. L. Butzer, G. Schmeisser, and R. L. Stens, *Sampling in Theory and Practice*, ch. 2, An Introduction to Sampling Analysis, Plenum/Kluwer, 2000.
10. P. L. Butzer, W. Splettstößer, and R. L. Stens, *The sampling theorem and linear prediction in signal analysis*, Jber. d. Dt. Math.-Verein. **3** (1988), 1–70.
11. P. L. Butzer and R. L. Stens, *Sampling theory for not necessarily band-limited functions: a historical overview*, SIAM Review **34** (1992), 40–53.
12. M. M. Dodson and A. M. Silva, *Fourier analysis and the sampling theorem*, Proc. Royal Irish Acad. **85A** (1985), 81–108.
13. M. M. Dodson, A. M. Silva, and V. Souček, *A note on Whittaker's cardinal series in harmonic analysis*, Proc. Edinb. Math. Soc. **29** (1986), 349–357.
14. J. R. Higgins, *Five short stories about the cardinal series*, Bull. Amer. Math. Soc. **12** (1985), 45–89.
15. ———, *Another look at aliasing in Paley-Wiener spaces*, Proc. 13'th IMACS World Congress on Computation and Applied Mathematics, 1991, pp. 256–257.
16. ———, *Sampling theory in Fourier and signal analysis: Foundations*, Clarendon Press, Oxford, 1996.
17. J. R. Higgins, G. Schmeisser, and J. J. Voss, *The sampling theorem and several equivalent results in analysis*, J. Comput. Anal. Appl. **2** (2000), 333–371.
18. J. R. Higgins and R. L. Stens, *Sampling theory in Fourier and signal analysis: Advanced topics*, Oxford University Press, Oxford, 1999.
19. A. J. Jerri, *The Shannon sampling theorem - its various extensions and applications: a tutorial review*, Proc. IEEE **65** (1977), 1565–1596.
20. I. Kluvánek, *Sampling theorem in abstract harmonic analysis*, Mat.-Fyz. Casopis Sloven. Akad. Vied. **15** (1965), 43–48.
21. A. Kohlenberg, *Exact interpolation of bandlimited functions*, J. Appl. Phys. **24** (1953), 1432–1436.
22. K. S. Krishnan, *A simple result in quadrature*, Nature **162** (1948), 215.
23. H. J. Landau, *Necessary density conditions for sampling and interpolation of certain entire functions*, Acta Mathematica **117** (1967), 37–52.
24. ———, *Sampling, data transmission and the Nyquist rate*, Proc. IEEE **55** (1967), 1701–1706.
25. S. P. Lloyd, *A sampling theorem for stationary (wide sense) stochastic processes*, Proc. Amer. Math. Soc. **92** (1959), 1–12.

26. R. J. Marks II, *Introduction to Shannon sampling and interpolation theory*, Springer-Verlag, New York, 1991.
27. R. J. Marks II (ed.), *Advanced topics in Shannon sampling and interpolation theory*, Springer-Verlag, New York, 1993.
28. Q. I. Rahman and G. Schmeisser, *The summation formulae of Poisson, Plana, Euler-Maclaurin and their relationship*, J. Math. Sci. (Part I) **28** (1994), 151–171.
29. W. Rudin, *Real and complex analysis*, McGraw Hill, New York, 1986.
30. C. E. Shannon, *A mathematical theory of communication*, Bell Sys. Tech. J. **27** (1948), 379–423, 623–656.
31. ______, *Communication in the presence of noise*, Proc. IRE **37** (1949), 10–21.
32. E. T. Whittaker, *On the functions which are represented by the expansions of the interpolation theory*, Proc. Roy. Soc. Edinburgh **35** (1915), 181–194.
33. J. M. Whittaker, *Interpolatory function theory*, Cambridge Tracts in Mathematics and Mathematical Physics, Cambridge University Press, Cambridge, 1935.
34. A. Zygmund, *Trigonometric series*, Cambridge University Press, Cambridge, 1959.

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