Lanchester models and the Battle of Britain

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memorial volume for Rick Rosenthal

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Invented the carburettor, disc brakes, the accelerator pedal;
theory of lift and drag.

Lanchester's equations (1914)

The **aimed-fire** model: $G(t)$ Green units fight $R(t)$ Red units.

$$\frac{dG}{dt} = -rR$$

Green's **instantaneous** loss-rate is proportional to Red numbers

$$\frac{dR}{dt} = -gG$$

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and vice versa. Divide:

$$\frac{dR}{dG} = \frac{gG}{rR} \qquad \text{or} \qquad rR\,dR = gG\,dG$$

and integrate:

$$\frac{1}{2}rR^2 = \frac{1}{2}gG^2 + \text{constant}$$

throughout the battle.

Implications of Lanchester's equations

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suppose we begin with twice as many Reds as Greens, $R_0 = 2G_0$,
but that Greens are three times more effective, $g = 3r$.
Then

$$rR^2 - gG^2 = r(2G_0)^2 - 3rG_0^2 = rG_0^2 > 0,$$

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Concentration is good:
If Red divides its forces, and Green fights each half in turn,
Green wins the first battle, with $\sqrt{2/3} \simeq 80\%$ of G_0 remaining,
Green wins the second battle, with $\sqrt{1/3} \simeq 60\%$ of G_0 remaining.

Some variants of Lanchester's equations

Ancient warfare, along a fixed, narrow battle-line with $N(t)$ fighting on each side:

$$\frac{dG}{dt} = -rN \qquad \frac{dR}{dt} = -gN$$

Modern warfare, but with hidden targets (the **unaimed-fire** model):

$$\frac{dG}{dt} = -rRG \qquad \frac{dR}{dt} = -gGR$$

Either way, dR/dG is now fixed, and the constant quantity is

$$rR - gG\,,$$

which is much more intuitive: fighting strength is just

numbers $\times$ effectiveness.

Asymmetric warfare

Green attacks, Red defends:

$$\frac{dG}{dt} = -rR, \qquad \frac{dR}{dt} = -gG\frac{R}{R_0}$$

and

$$rRR_0 - \frac{1}{2}gG^2$$

is conserved, so that

– Green benefits more from numbers and concentration, but

– needs g to be twice as great, or G_0 to be $\sqrt{2}$ times as great, as in a symmetric aimed-fire battle.

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Red has a defender's advantage

A generalized Lanchester model

fits loss-rates to powers of own and enemy numbers:

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Divide and re-arrange:

$$g G^{g_1 - g_2} \, dG = r R^{r_1 - r_2} \, dR$$

Integrate: the conserved quantity is

$$\frac{r}{\rho} R^{\rho} - \frac{g}{\gamma} G^{\gamma} .$$

where $\rho = 1 + r_1 - r_2$ and $\gamma = 1 + g_1 - g_2$

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where $\rho = 1 + r_1 - r_2$ and $\gamma = 1 + g_1 - g_2$, the **exponents**, capture the conditions of battle:

– Red should concentrate its force if $\rho > 1$, divide if $\rho < 1$.

– if $\rho < \gamma$ then Red has a defender's advantage, by a factor γ/ρ

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(either sets of battles or time-series within battles)

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F W Lanchester
Aircraft in Warfare: the dawn of the fourth arm
(London: Constable & Co., 1916)

How about a (purely) aerial battle?

The Battle of Britain

A battle of attrition and intended annihilation, in which one day's fighting was much like another, the single-seat fighters on each side were well-matched, and all units were seeking engagement.

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Take daily loss-rates for RAF (δR) and Luftwaffe (δG) aircraft and fit to RAF (R) and Luftwaffe (G) daily sortie numbers:

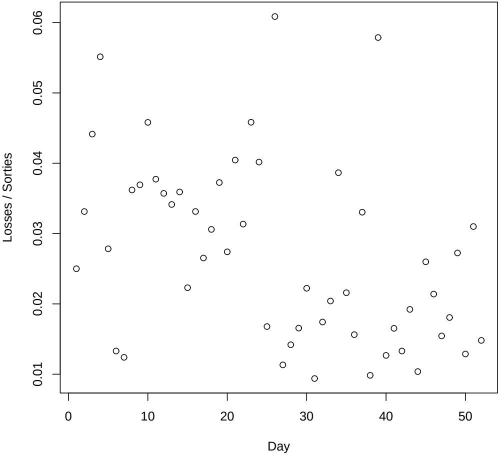
Find the parameters r, r_1, r_2, g, g_1, g_2 for which the data best fit

$$\delta R = g\,G^{g_1} R^{r_2}\,, \qquad \delta G = r\,R^{r_1} G^{g_2}$$

by linear regression onto

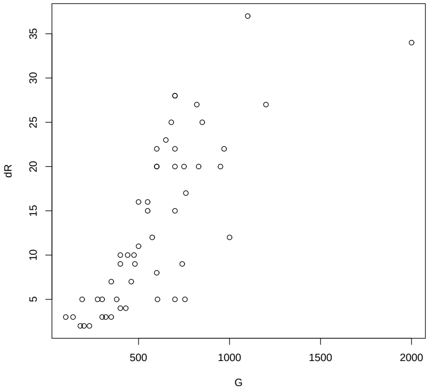
$$\log \delta R = \log g + g_1 \log G + r_2 \log R\,, \quad \log \delta G = \log r + r_1 \log R + g_2 \log G$$

Are the days independent?

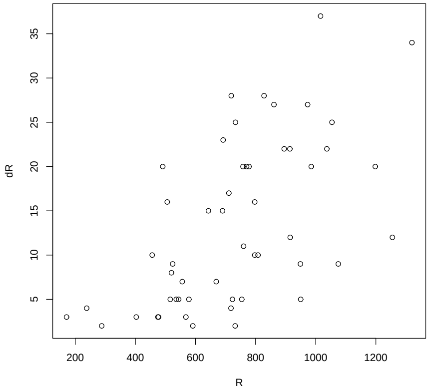


Losses per sortie $(\delta G + \delta R)/(G + R)$ *vs* day

RAF losses

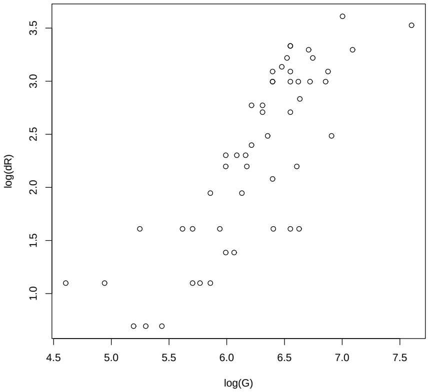


δR vs G

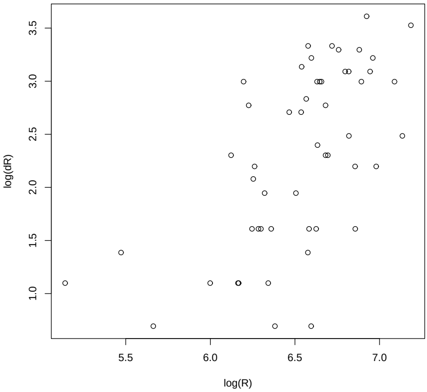


δR vs R

RAF losses



$\log \delta R$ *vs* $\log G$



$\log \delta R$ *vs* $\log R$

RAF losses

$$\frac{dR}{dt} = -g\,G^{1.12\pm0.17}\,R^{0.18\pm0.25}$$

$$= -g\,G^{1.2} \qquad (\Sigma R^2 = 0.66)$$

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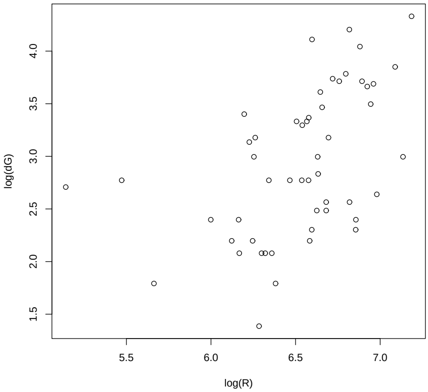
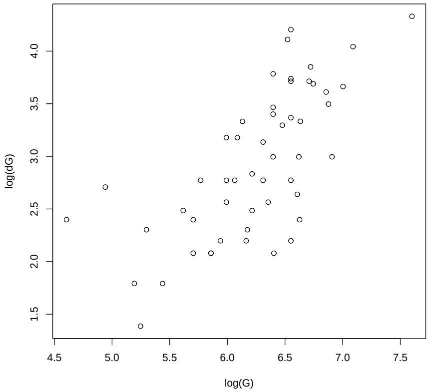
Hooray for Lanchester!

Luftwaffe losses



$\log \delta G$ *vs* $\log R$



$\log \delta G$ *vs* $\log G$

Luftwaffe losses

$$\frac{dG}{dt} = -rR^{0.00\pm0.25}\,G^{0.86\pm0.18}$$

$$= -gG^{0.9} \qquad (\Sigma R^2 = 0.49)$$

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Not so good.

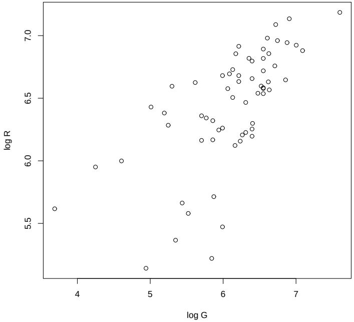
In fact, fitting to R alone,

$$\frac{dG}{dt} \propto R^{0.87\pm0.22}$$

explains only $\Sigma R^2 = 0.24$.

Subtleties I

G and R are highly correlated (0.74):



log R *vs* log G

and so the overall powers in the loss-rates, $g_1 + r_2$ and $r_1 + g_2$, are better-determined than their constituents: variation is less significant *along* the lines of constant $g_1 + r_2$ and $r_1 + g_2$ than *orthogonal* to them.

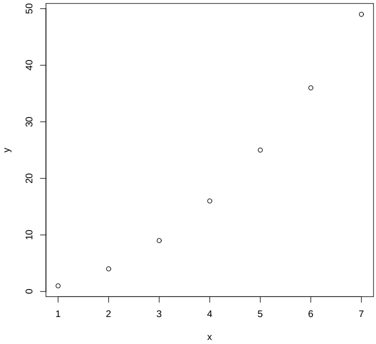
Subtleties II

When $g_1 + r_2 \neq 1$ or $r_1 + g_2 \neq 1$, autonomous battles ('raids') should not be aggregated into daily data.

If they are, the effect is to push the overall powers $g_1 + r_2$ and $r_1 + g_2$ away from their true values and towards one, and to reduce the quality of the fit.

Subtleties II

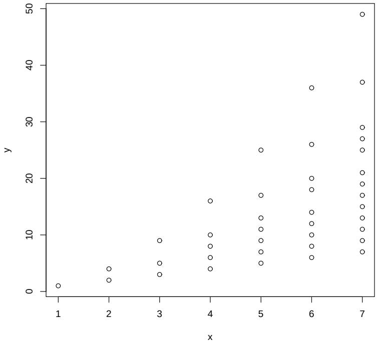
Example: $y = x^2$



has $\log y = 2 \log x$, of course.

Subtleties II

Example: $y = x^2$ and sums of these: *e.g.* not only $(3, 9)$ but also $(1 + 2, 1 + 4) = (3, 5)$ and $(1 + 1 + 1, 1 + 1 + 1) = (3, 3)$.



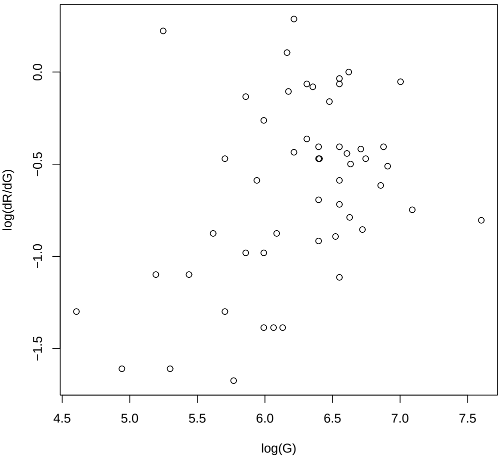
and the best fit is now $\log y = 1.5 \log x$, with $\Sigma R^2 = 0.6$.

Subtleties III

The good correlations we saw earlier are really not so surprising, since the natural null hypothesis is of an overall linear dependence of loss rates on sortie numbers.

All **tactical** implications follow from the constant quantities, which in turn follow from the dependence of the loss **ratio** $\delta R/\delta G$ on R and G, which is poorly modelled.

Subtleties III



$\log(\delta R/\delta G)$ *vs* $\log G$

Overall

$$\frac{dR}{dt} = -g\,G^{1.12\pm0.17}\,R^{0.18\pm0.25}\,, \qquad \frac{dG}{dt} = -r\,R^{0.00\pm0.25}\,G^{0.86\pm0.18}$$

suggests $\gamma = 1 + g_1 - g_2 \simeq 1.3$, $\rho = 1 + r_1 - r_2 \simeq 0.8$, but these are poorly modelled.

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The results of which we can be most sure are differences of $g_1 + r_2$ or $r_1 + g_2$ from one: and we found

$$g_1 + r_2 = 1.30, \qquad r_1 + g_2 = 0.86,$$

and thus

$$\gamma - \rho = g_1 + r_2 - r_1 - g_2 = 0.44.$$

We can conclude with fair confidence that $\gamma > 1$ and that $\rho < 1$, and with much more confidence that $\gamma > \rho$.

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The achievement of Keith Park (Commander, 11 Group, RAF Fighter Command) lay in creating and exploiting this advantage:

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How to win a battle:
'Get there first with the most men' (Nathan Bedford Forrest)
Given the choice, in this case, first is better.

What, finally, of Lanchester's laws?

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Perhaps it's a good thing his laws were not acted upon.