When Lanchester met Richardson:
the interaction of warfare with psychology

Niall MacKay

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EPSRC
Engineering and Physical Sciences Research Council

The University of York

Institute of mathematics & its applications

Dynamical systems as an aid to thought

Joshua Epstein, *Why model?* (2008)

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- All the best models are wrong
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- They capture qualitative behaviors of overarching interest
- The issue is whether the model offers a fertile idealization

Dynamical systems as an aid to thought

In this presentation we look at two models:

1. A general Lanchester-Richardson model for insurgencies

MacKay, *When Lanchester met Richardson*, JORS (2014)

2. The Lanchester truel

Kress, Lin & MacKay, *The Attrition Dynamics of Multilateral War*, OR (2018)

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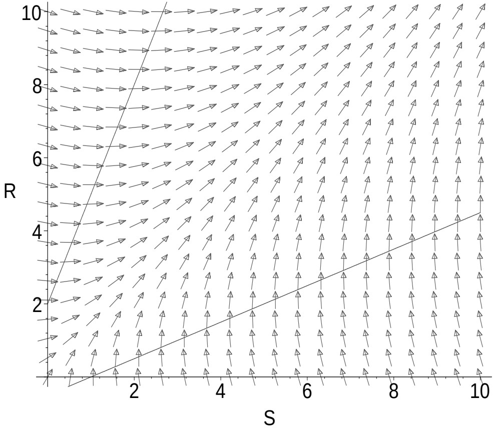
'All that can be proved by mathematics is that certain consequences follow from certain abstract hypotheses.'

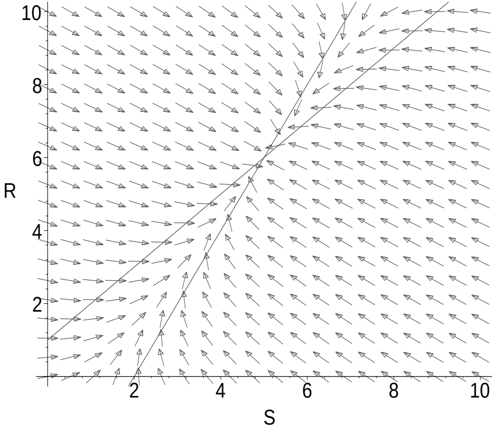
Richardson, *Arms and Insecurity* (1960)

Richardson's arms race



High antagonism: arms race

Richardson's arms race



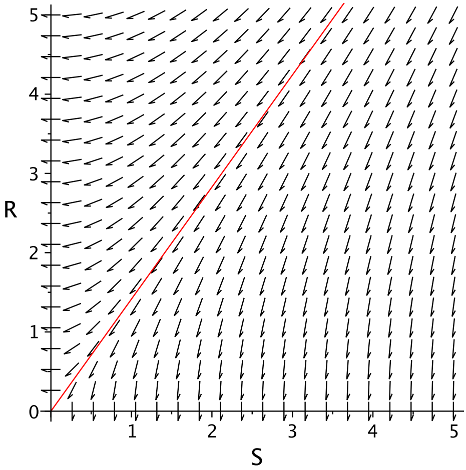
Low antagonism: stalemate

Richardson's arms race

$$\begin{aligned}
\frac{dS}{dt} &= rR - \sigma S + k \\
\frac{dR}{dt} &= sS - \rho R + l
\end{aligned}$$

antagonism quiescence military spend

Lanchester's aimed-fire model



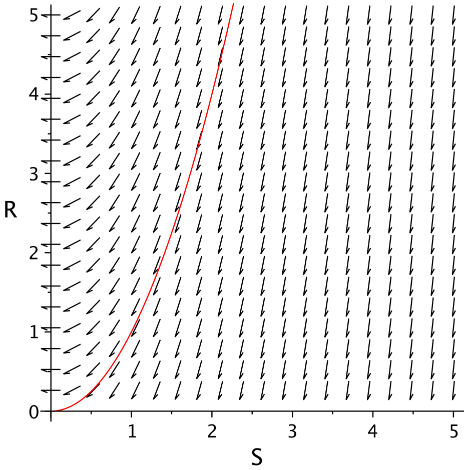
Annihilation

Lanchester's aimed-fire model

$$\frac{dR}{dt} = -dS\,, \qquad \frac{dS}{dt} = -cR$$

and $dS^2 - cR^2$ is constant.

Deitchman's 'guerrilla' model



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$$\frac{dR}{dt} = -dRS\,, \qquad \frac{dS}{dt} = -cR$$

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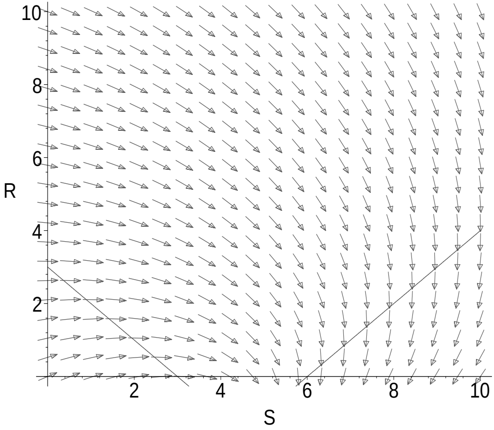
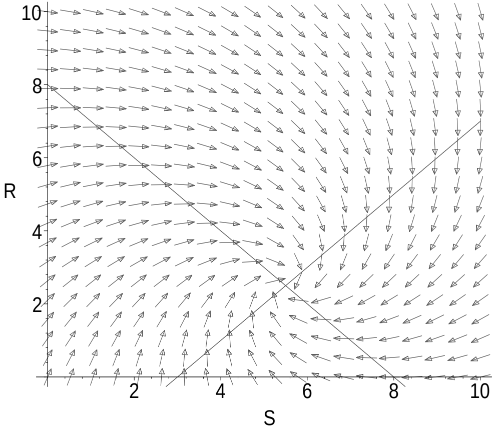


Low antagonism: S wins

Richardson + Lanchester



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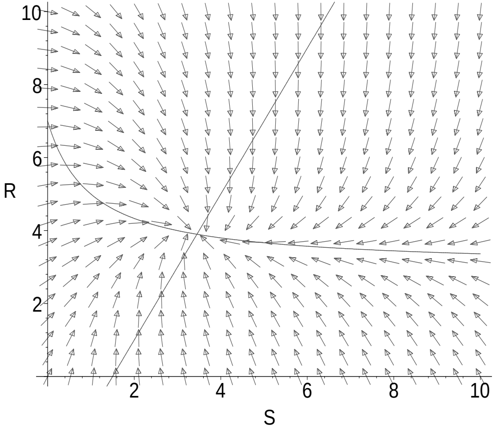

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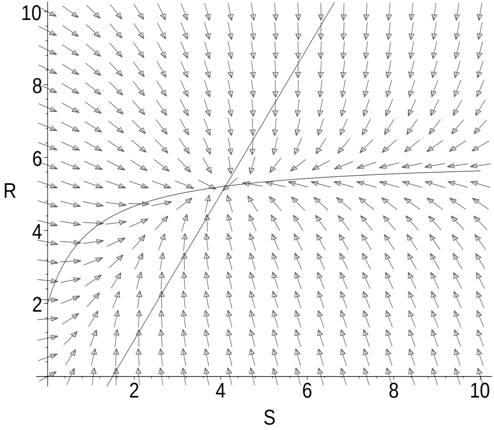

High antagonism: stalemate

When Lanchester met Richardson: conclusion

At the same level of argument as

Richardson: antagonism plus quiescence leads to arms race or stalemate

Lanchester: aimed fire leads to disproportionate value for numbers and concentration

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Richardson: antagonism plus quiescence leads to arms race or stalemate

Lanchester: aimed fire leads to disproportionate value for numbers and concentration

we have that

asymmetric attrition generalizing Deitchman's insurgency
i.e. insurgent losses scale faster with overall numbers than state losses

$+$ antagonism characteristic of Richardson

$\Rightarrow$ **stalemate**

MacKay, *When Lanchester met Richardson*, JORS (2014)

Tri- and multilateral war

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On N nations:
'the world will for most of the time be content with just enough stability'

'the triadic situation often favors the weak over the strong'

Caplow, *Coalitions in the Triad* (1956)

The Truel

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The sequential random truel:

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Players A, B, C

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Typically $P(C \text{ win}) > P(B) > P(A)$.

For example, let $a = \frac{4}{5}$, $b = \frac{3}{5}$, $c = \frac{2}{5}$.

Then $P(A) = \frac{8}{27}$, $P(B) = \frac{9}{27}$, $P(C) = \frac{10}{27}$.

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Better marksmanship can hurt!

Brams and Kilgour, *The Truel* (1997)

The Truel

Variants may be simultaneous, have limited ammunition, allow formation of coalitions, assume perfect anticipation.

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Variants may be simultaneous, have limited ammunition, allow formation of coalitions, assume perfect anticipation.

Some conclusions are robust:
the weakness of being the best marksman, the fragility of pacts.

Often these conclusions are counterintuitive or paradoxical.

The Lanchester Truel

$$\begin{aligned}
\dot{A} &= \qquad\qquad\qquad\quad -b(1-\beta)B \quad -c\gamma C \\
\dot{B} &= \qquad -a\alpha A \qquad\qquad\qquad -c(1-\gamma)C \\
\dot{C} &= -a(1-\alpha)A \qquad -b\beta B
\end{aligned}$$

Let $a > b > c > 0$ and begin with $A = A_0$, $B = B_0$, $C = C_0$.
The truel finishes when at most one player remains.

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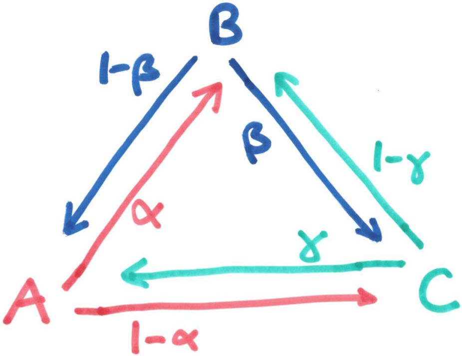
There is (in general) no quadratic conserved quantity, no 'Square Law', and thus no preferred objective function.



B
1-β
β
1-γ
α
γ
A
C
1-α

Theorem

The Lanchester Truel

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If the **objective function** for each player is
its numbers minus others' numbers, *e.g.* (for A) $A_\infty - B_\infty - C_\infty$,

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either one force can beat the other two together,

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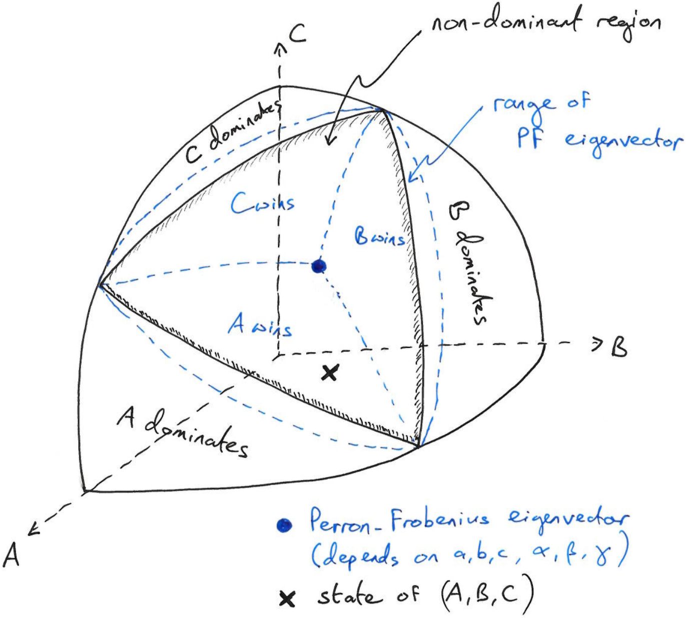

non-dominant region
range of
PF eigenvector
C
B
A
C dominates
B dominates
A dominates
C wins
B wins
A wins
Perron-Frobenius eigenvector
(depends on a, b, c, α, β, γ)
state of (A, B, C)

Lemma 1: The range of • encloses the non-dominant region, with equality when $a = b = c$.
(Blue dashed triangle encloses hachured black triangle.)

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• *simply chases the state* $\times$.

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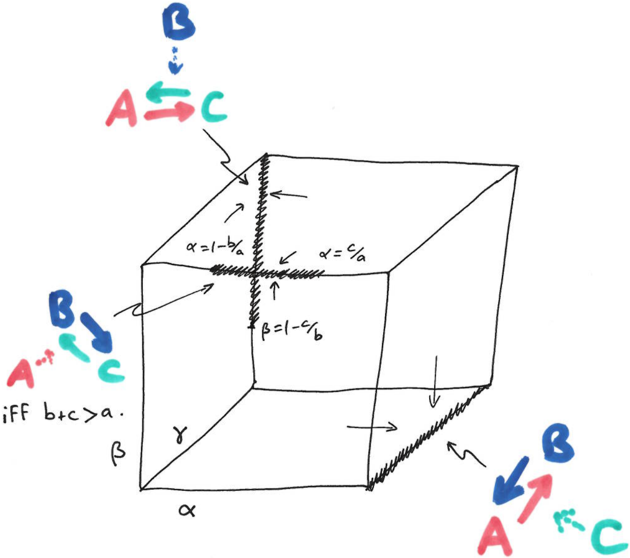
Suppose that the only thing a force values is reducing its own casualty rate:

A wants to maximize $\ddot{A}$, likewise for B and C.

Then the equilibria are...


B
A
C
$\alpha = 1 - b/a$
$\alpha = c/a$
$\beta = 1 - c/b$
B
A
C
iff $b+c>a$.
β
γ
α
B
A
C

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- long-term victory, to maximize $X_\infty - Y_\infty - Z_\infty$,
  then either one player can beat the others put together, or the outcome is total annihilation
- short-term reduction of loss rate $-\dot{X}$,
  then fire distributions approach stable states in which two players target only each other, and the weakest player has an advantage because they are least capable of hurting the others.

Kress, Lin & MacKay, *The Attrition Dynamics of Multilateral War*, OR (2018)

Thank you for listening