Lanchester models of mixed & multilateral fights

Niall MacKay

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EPSRC
Engineering and Physical Sciences Research Council

The University of York

Institute of mathematics & its applications

Dynamical systems as an aid to thought

In this presentation we look at

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Lanchester models for mixed (heterogeneous) forces

MacKay, *Lanchester models for mixed forces...*, JORS 2009

Lin & MacKay, *The optimal policy...*, ORL 2014

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Kress, Lin & MacKay, *The Attrition Dynamics of Multilateral War*, OR 2018

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The point in each is (merely) to illuminate a general connection between assumptions and outcome.

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George Box (1970s)

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‘A model is useful if a better decision is made with the information it adds’

Wayne P. Hughes (1997)

Dynamical systems as an aid to thought

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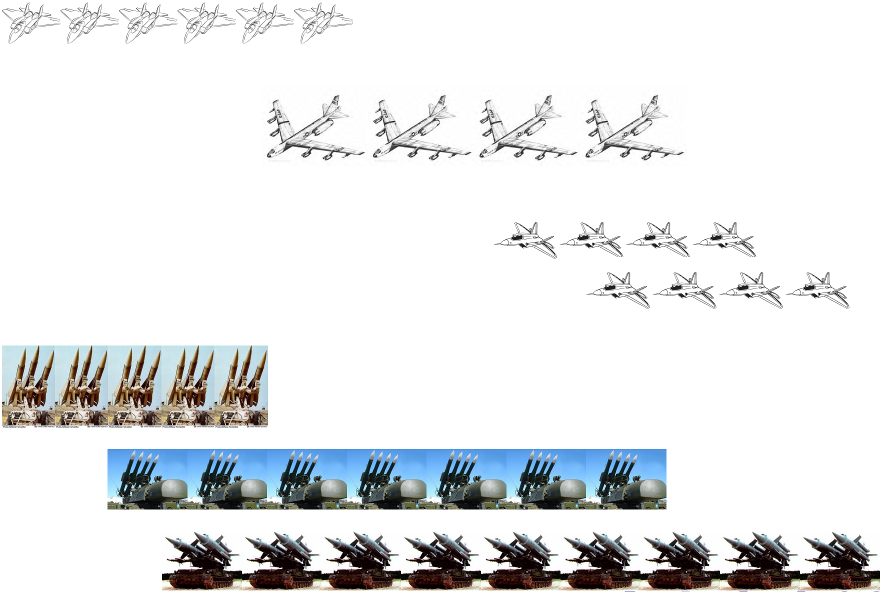
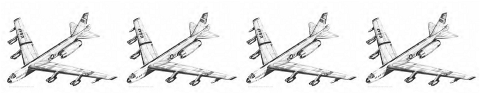
- To illuminate core dynamics
- All the best models are wrong
- But they are fruitfully wrong
- They capture qualitative behaviors of overarching interest
- The issue is whether the model offers a fertile idealization

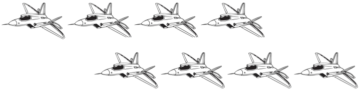
Lanchester models for mixed forces

The Lanchester many-on-many or (m, n) problem



The Lanchester one-on-many or $(1, n)$ problem





Lanchester's equations (1913-14)

$G(t)$ Green units fight $R(t)$ Red units.

Kills are proportional to numbers (a 'target-rich environment'),

$$\frac{dG}{dt} = -rR\,, \qquad \frac{dR}{dt} = -gG\,.$$

Trajectories are hyperbolae

$$rR^2 - gG^2 = \text{constant}.$$

The Lanchester one-on-many problem

$$\frac{dG}{dt} = -\sum_{i=1}^{n} r_i R_i\,, \qquad \frac{dR_i}{dt} = -g_i \gamma_i(t) G\,.$$

Q: What is Green's optimal choice of $\gamma_i(t)$ (with $\gamma_1 + \ldots \gamma_n = 1$) ?

A: The **priority policy**: rank the $g_i r_i$ such that

$$g_1 r_1 < g_2 r_2 < \ldots < g_{n-1} r_{n-1} < g_n r_n,$$

Target only type-n as long as there are any remaining, then type $n-1$, and so on down.

The Lanchester one-on-many problem

For any **semi-dynamical** policy, such that $\gamma_i(t) = \frac{\nu_i R_i}{\sum \nu_i R_i}$,

$$Q := G^2(t) - \sum_{i,j} \frac{r_i \nu_j + r_j \nu_i}{g_i \nu_i + g_j \nu_j} R_i(t) R_j(t)$$

is constant. The optimal policy is that which maximizes Q and thereby G when all Red units are destroyed, and is $\nu_i/\nu_{i+1} = 0$, $i = 1, \ldots n-1$.

MacKay, Lanchester models for mixed forces with semi-dynamical target allocation, *Journal of the Operational Research Society* **60** (2009) 1421-1427

The Lanchester one-on-many problem

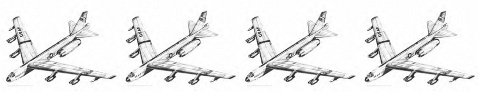
Under **fully dynamical** policies, in which $\gamma_i(t)$ and $\nu_i(t)$ may vary arbitrarily, Q is no longer conserved: rather

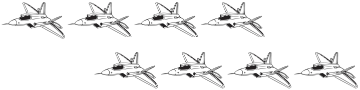
$$\frac{dQ(t)}{dt} = 2\sum_{i<j} \frac{g_j r_j - g_i r_i}{g_i} \frac{d}{dt}\left(\frac{1}{g_i \nu_i/\nu_j + g_j}\right) R_i(t) R_j(t)\,.$$

Any departure from $\nu_i/\nu_j = 0$ causes an irreversible reduction of Q, so the priority policy is still optimal.

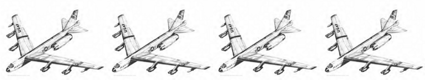
Lin & MacKay, The optimal policy for the one-against-many heterogeneous Lanchester model, *Operations Research Letters* **42** (2014) 473-477.

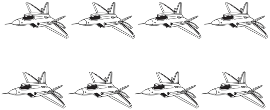
The Lanchester one-on-many or $(1, n)$ problem





The Lanchester $(2, 2)$ problem





The Lanchester $(2,2)$ problem

$$
\begin{aligned}
\frac{dG_i}{dt} &= -\sum_{j=1}^{2} r_{ij}\rho_{ij}(t)R_j, \qquad \sum_{i=1}^{2}\rho_{ij} = 1 \\
\frac{dR_i}{dt} &= -\sum_{j=1}^{2} g_{ij}\gamma_{ij}(t)G_j, \qquad \sum_{i=1}^{2}\gamma_{ij} = 1
\end{aligned}
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Q: What are the optimal choices of $\rho_{ij}(t)$ and $\gamma_{ij}(t)$?

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What is 'optimal' depends on what your opponent does

The Lanchester $(2,2)$ problem

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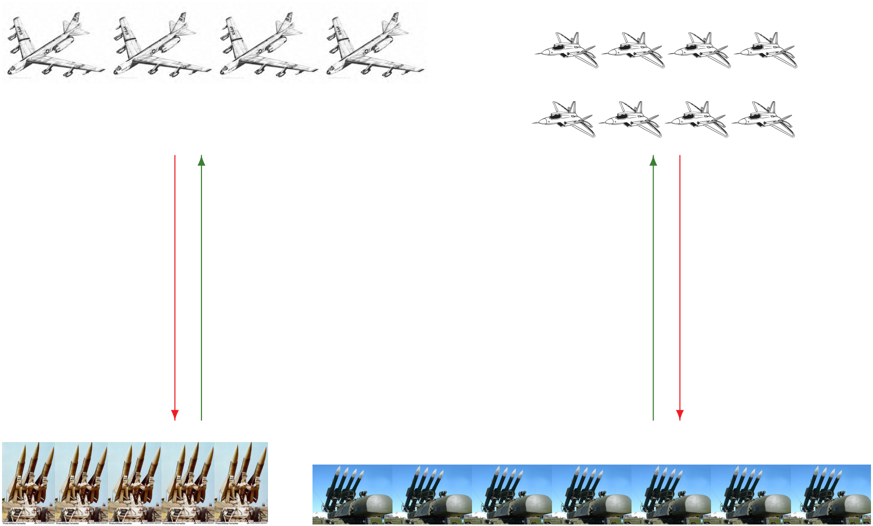
For general r_{ij} and g_{ij}, a type-by-type priority policy is **not** optimal

The Lanchester $(2, 2)$ problem



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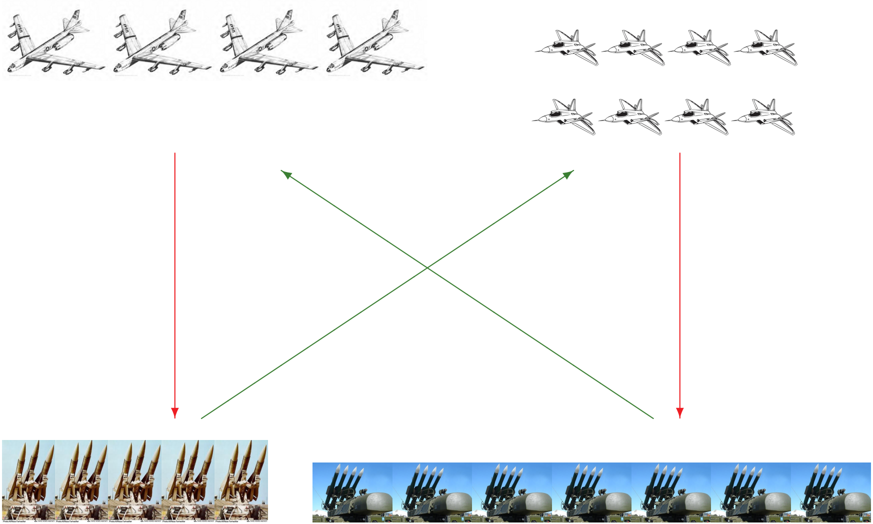

Lanchester models for multilateral fights

Tri- and multilateral war

Tri- and multilateral war

'If each of three pairs of nations is separately unstable then the triplet is necessarily unstable' [but] if each of the three pairs [is] stable [then] the triplet of nations may [nevertheless] be unstable'

Richardson, *Arms and Insecurity* (1960)

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'the world will for most of the time be content with just enough stability'

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'the triadic situation often favors the weak over the strong'

Caplow, *Coalitions in the Triad* (1956)

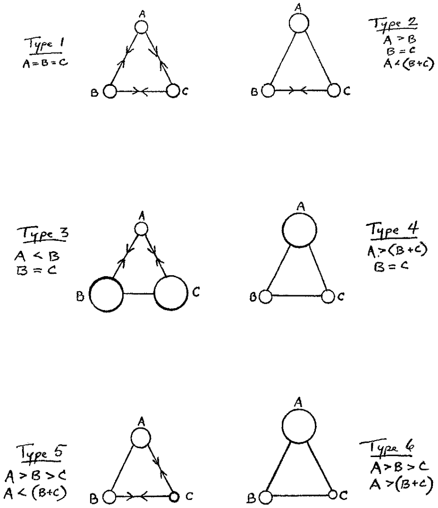
Type 1
A = B = C
A
B
C
Type 2
A > B
B = C
A < (B+C)
A
B
C
Type 3
A < B
B = C
A
B
C
Type 4
A > (B+C)
B = C
A
B
C
Type 5
A > B > C
A < (B+C)
A
B
C
Type 6
A > B > C
A > (B+C)
A
B
C

FIGURE 1

The Truel

The Truel

The sequential random truel:

The Truel

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Players A, B, C

The Truel

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take turns to shoot, the shooter on each turn being chosen at random and aiming at their most accurate opponent,

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take turns to shoot, the shooter on each turn being chosen at random and aiming at their most accurate opponent,

with hitting probabilities a, b, c such that $a > b > c$.

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Typically $P(C \text{ win}) > P(B) > P(A)$.

For example, let $a = \frac{4}{5}$, $b = \frac{3}{5}$, $c = \frac{2}{5}$.

Then $P(A) = \frac{8}{27}$, $P(B) = \frac{9}{27}$, $P(C) = \frac{10}{27}$.

The Truel

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Better marksmanship can hurt!

Brams and Kilgour, *The Truel* (1997)

The Truel

Variants may be simultaneous, have limited ammunition, allow formation of coalitions, assume perfect anticipation.

The Truel

Variants may be simultaneous, have limited ammunition, allow formation of coalitions, assume perfect anticipation.

Some conclusions are robust:
the weakness of being the best marksman, the fragility of pacts.

Often these conclusions are counterintuitive or paradoxical.

The Lanchester Truel

$$
\begin{aligned}
\frac{dA}{dt} &= -b(1-\beta)B - c\gamma C \\
\frac{dB}{dt} &= -a\alpha A - c(1-\gamma)C \\
\frac{dC}{dt} &= -a(1-\alpha)A - b\beta B
\end{aligned}
$$

$a > b > c > 0, \quad 0 \leq \alpha, \beta, \gamma \leq 1.$

The Lanchester Truel

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$a > b > c > 0, \quad 0 \le \alpha, \beta, \gamma \le 1.$

Begin with $A = A_0, B = B_0, C = C_0$. The truel finishes when at most one player remains.

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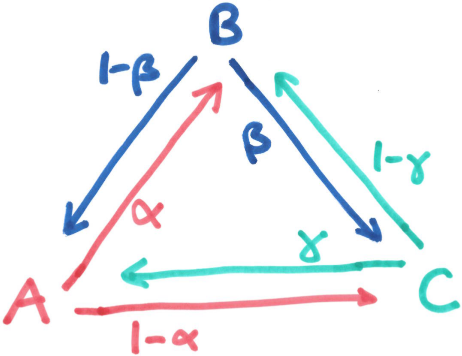
What happens next?


B
1-β
β
1-γ
α
γ
A
C
1-α

The Lanchester Truel

In the paper we generalize to

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- N players

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- N players
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- linear-law rather than square-law fire

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Here we keep it simple.

The Lanchester Truel

$$
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\dot{A} &= && -b(1-\beta)B && -c\gamma C \\
\dot{B} &= -a\alpha A && && -c(1-\gamma)C \\
\dot{C} &= -a(1-\alpha)A && -b\beta B &&
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Let $0 \leq \alpha, \beta, \gamma \leq 1$.
We now have a dynamical game in which the decision parameters are α (for A), β for B, γ for C.

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\end{aligned}$$

Let $a > b > c > 0$ and begin with $A = A_0, B = B_0, C = C_0$.
The truel finishes when at most one player remains.

Let $0 \leq \alpha, \beta, \gamma \leq 1$.
We now have a dynamical game in which the decision parameters are α (for A), β for B, γ for C.

There is (in general) no quadratic conserved quantity, no 'Square Law', and thus no preferred objective function.

Theorem

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If the **objective function** for each player is
its numbers minus others' numbers, *e.g.* (for A) $A_\infty - B_\infty - C_\infty$,

then

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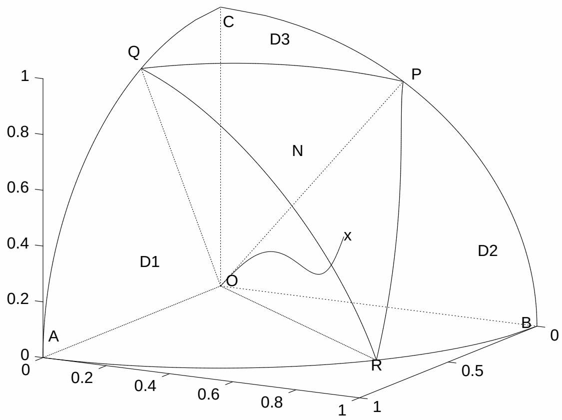
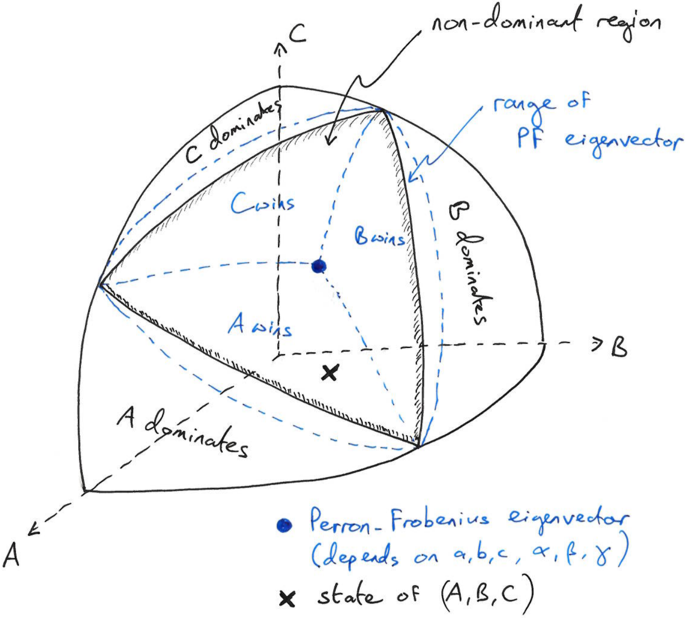
then

either one force can beat the other two together,

or the outcome is mutual annihilation

The Lanchester Truel




non-dominant region
range of PF eigenvector
C
B
A
C dominates
B dominates
A dominates
C wins
B wins
A wins
Perron-Frobenius eigenvector (depends on a, b, c, α, β, γ)
× state of (A, B, C)

The Lanchester Truel

Lemma 1: The range of • encloses the non-dominant region, with equality when $a = b = c$.
(Blue dashed triangle encloses hachured black triangle.)

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(Players' optimal strategy is to shift $\bullet$ onto the state $\times$, which then remains static, resulting in collective annihilation.)

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Then • simply chases the state $\times$.

So what happened to the ubiquitous truel idea, that the weakest is surprisingly strong?

The Lanchester Truel

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It's all in the choice of the objective function.

The Lanchester Truel

So what happened to the ubiquitous truel idea, that the weakest is surprisingly strong?

It's all in the choice of the objective function.

Suppose that the only thing a force values is reducing its own casualty rate:

A wants to maximize $\ddot{A}$, likewise for B and C.

The Lanchester Truel

Impose a rapid dynamical game on α, β, γ:

$$\frac{1}{\tau}\frac{d\alpha}{dt} = b(1-\beta) - c\gamma$$

The Lanchester Truel

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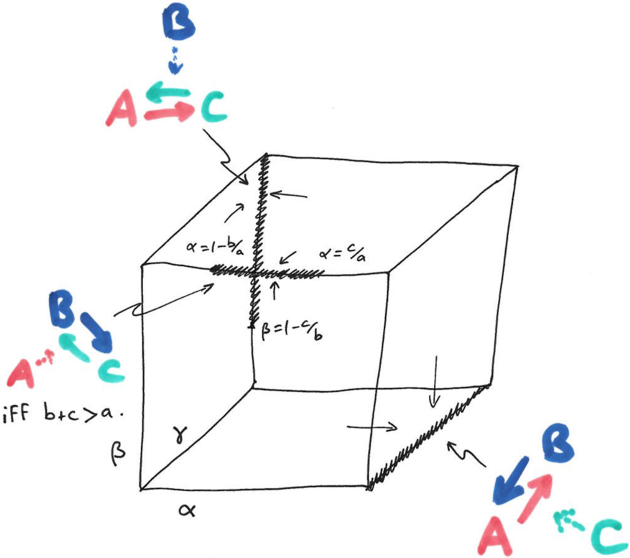
$$\frac{1}{\tau}\frac{d\alpha}{dt} \quad = \quad b(1-\beta) - c\gamma \quad \left(\propto \frac{d\ddot{A}}{d\alpha}\right)$$

The Lanchester Truel

Impose a rapid dynamical game on α, β, γ:

$$\begin{aligned}
\frac{1}{\tau}\frac{d\alpha}{dt} &= b(1-\beta) - c\gamma \quad \left(\propto \frac{d\ddot{A}}{d\alpha}\right) \\
\frac{1}{\tau}\frac{d\beta}{dt} &= c(1-\gamma) - a\alpha \\
\frac{1}{\tau}\frac{d\gamma}{dt} &= a(1-\alpha) - b\beta.
\end{aligned}$$


B
A
C
$\alpha = 1 - b/a$
$\alpha = c/a$
$\beta = 1 - c/b$
B
A
C
iff b+c>a.
γ
β
α
B
A
C

The Lanchester Truel: conclusion

If X's objective is

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  then either one player can beat the others put together, or the outcome is total annihilation

The Lanchester Truel: conclusion

If X's objective is

- long-term victory, to maximize $X_\infty - Y_\infty - Z_\infty$,
  then either one player can beat the others put together, or the outcome is total annihilation
- short-term reduction of loss rate $-\dot{X}$,
  then fire distributions approach stable states in which two players target only each other, and the weakest player has an advantage because they are least capable of hurting the others.

Kress, Lin & MacKay, *The Attrition Dynamics of Multilateral War*, OR (2018)

Lanchester models of multilateral war

In multilateral war

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- which can be averted only if an external player intervenes
    - to make one player dominant
    - to enforce coalitions or political settlements

Thank you for listening