Yangians and their representations

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ABSTRACT. In this project we give an introduction to Yangians and their representations. In particular we deal with a theorem due to Drinfel'd [**10**] about the adjoint$\oplus$singlet representation of Yangians. Using 'birdtrack' notation we prove this theorem for the series of exceptional Lie algebras and $\mathfrak{su}(n)$.

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Introduction

It was Drinfel'd [**10**] who wrote the defining paper on Yangians in 1985. The motivation was to study the Yang-Baxter Equation (YBE) and the name 'Yangian' was in honour of C. N. Yang who found the first solution of the YBE in terms of a certain formal series [**31**].

Since the 1980's, Yangians have found many other applications in theoretical physics, most notably in its original area of integrable models [**24**]. In the last few years Yangians have received much attention due to the discovery of Yangian symmetries relating to the AdS/CFT correspondence [**6, 9, 4**].

In this project we study the representations of Yangians. More specifically we will prove a theorem of Drinfel'd [**10**] concerning the adjoint$\oplus$singlet representation of Yangians. We complete the proof for the exceptional Lie algebras and $\mathfrak{su}(n)$, however we only manage to give some numerical evidence for $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$. Drinfel'd's paper [**10**] does not contain any proofs, and we believe that a detailed proof has not been published elsewhere[1].

[1]Chari and Pressley claim that they prove it in section 6 of [**7**], but we cannot find the result there.

CHAPTER 1

Yangians and their representations

1.1. Yangians

Throughout this project we will adopt the convention of summing over repeated indicies. Further we assume that we are in a compact space and hence we will not distinguish ‘up’ from ‘down’ indicies. The following definition was given by Drinfel’d in [**10**]. A few years later he gave another (isomorphically equivalent) definition [**11**], which we will not consider here.

Definition 1.1 (Yangian, [**10**]). Let $\mathfrak{g}$ be an N-dimensional simple Lie algebra[1] over $\mathbb{C}$ generated by $\{I_a\}, a = 1, 2, \ldots, N$, with structure constants c_{abc}, such that

$$[I_a, I_b] = c_{abc} I_c \tag{1.1}$$

The *Yangian*, $Y(\mathfrak{g})$, is the enveloping algebra generated by the I’s and a second set of generators $\{J_b\}, b = 1, 2, \ldots, N$, in the adjoint representation of $\mathfrak{g}$, such that

$$[I_a, I_b] = c_{abc} I_c \quad \text{and} \quad [I_a, J_b] = c_{abc} J_c \tag{1.2}$$

$$[J_a, [J_b, I_c]] - [I_a, [J_b, J_c]] = a_{abcdef}\{I_d I_e I_f\} \tag{1.3}$$

$$\begin{aligned}&[[J_a, J_b], [I_r, J_s]] + [[J_r, J_s], [I_a, J_b]] \\ &\qquad = (a_{abcdef} c_{rsc} + a_{rscdef} c_{abc})\{I_d I_e I_f\}\end{aligned} \tag{1.4}$$

where $a_{abcdef} = \frac{1}{24} c_{adi} c_{bej} c_{cfk} c_{ijk}$ and $\{I_d I_e I_f\}$ is the sum of all permutations of $I_d I_e I_f$. Explicitly,

$$\{I_d I_e I_f\} = I_d I_e I_f + I_f I_d I_e + I_e I_f I_d + I_d I_f I_e + I_f I_e I_d + I_e I_d I_f \tag{1.5}$$

Remark 1.2. For $\mathfrak{g} = \mathfrak{su}(2)$ we have that (1.2) imply (1.3), and for $\mathfrak{g} \neq \mathfrak{su}(2)$ condition (1.4) follows from (1.2) and (1.3) [**10**]. Thus for most of our discussion we will ignore (1.4).

The third defining relation (1.3) needs a bit of explanation, since it is far from obvious what it signifies.

[1]with the Lie bracket being the commutator $[A, B] = AB - BA$

1.1.1. Drinfel'd's Third Relation. The Yangian is an example of a Hopf Algebra[2] [**10, 24**] with co-unit $\epsilon : Y(\mathfrak{g}) \to \mathbb{C}$,

$$\epsilon(I_a) = \epsilon(J_a) = 0, \tag{1.6}$$

antipode $s : Y(\mathfrak{g}) \to Y(\mathfrak{g})$,

$$s(I_a) = -I_a \tag{1.7}$$

$$s(J_a) = -J_a + \tfrac{1}{2} c_{abc} I_c I_b, \tag{1.8}$$

and coproduct $\Delta : Y(\mathfrak{g}) \to Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$,

$$\Delta(I_a) = I_a \otimes 1 + 1 \otimes I_a \tag{1.9}$$

$$\Delta(J_a) = J_a \otimes 1 + 1 \otimes J_a + \tfrac{1}{2} c_{abc} I_c \otimes I_b \tag{1.10}$$

Drinfel'd's third relation (1.3) then follows from the requirement for a Hopf algebra that Δ be a homomorphism [**24**]. We show this in Appendix A.

One can also think of the Yangian as a graded algebra where $\deg(I_a) = 0$ and $\deg(J_a) = 1$. Then (1.3) and (1.4) are constraints on how to construct higher-order elements [**24**].

1.2. Relation with the Yang-Baxter Equation

The Yang-Baxter Equation (YBE) was described by McGuire and Yang in connection with the one-dimensional N-body problem [**25, 31**]. It also appeared in areas of statistical mechanics [**3, 2**]. The YBE can be written

$$R_{12}(u-v)R_{13}(v)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{1.11}$$

where the R's are linear operators in the tensor product of two N-dimensional complex spaces (e.g. $\mathbb{C}^N \otimes \mathbb{C}^N$) parametrized by a complex number [**20**].

In the theory of scattering processes the YBE arises as a factorization condition for a multiparticle S-matrix [**20, 24**]. It was therefore much studied by Faddeev and the 'St Petersburg school'[3] in relation with the inverse scattering method, see eg [**21, 29**]. Relationships between the solutions of the YBE and Lie groups soon emerged [**19, 5**] and before Drinfel'd introduced the Yangian in [**10**] similar algebraic structures[4] had been considered by Faddeev and others [**13, 21, 30**]. In his paper [**10**] Drinfel'd showed that the problem of finding rational

[2] see eg [**17**] for a textbook discussion of Hopf algebras

[3] As it was referred to by Molev [**26**]

[4] Now called Yangians for the general linear Lie algebra, $Y(\mathfrak{gl}(n))$ [**26**].

solutions to the Yang-Baxter equation reduced to the problem of finding finite-dimensional irreducible representations of Yangians [**10, 11**]. He proceeded to find such representations (and hence rational solutions to the YBE), thus generalizing the method in [**19**].

Representations of Yangians, and in particular *irreducible* representations, therefore play an important role in the study of the Yang-Baxter equation.

We now proceed with a discussion of some of the representations given by Drinfel'd in [**10**].

1.3. Representations of Yangians

Since $Y(\mathfrak{g})$ contains $\mathfrak{g}$ as a subalgebra, any representation of $Y(\mathfrak{g})$ will also be a representation of $\mathfrak{g}$ [**24**]. Conversely, starting with an irreducible representation ρ of $\mathfrak{g}$ such that $\rho(a_{abcdef}\{I_d I_e I_f\}) = 0$ one can extend it to an irreducible representation of $Y(\mathfrak{g})$ by setting $\rho(J_a) = 0$ [**10**]. The condition $\rho(a_{abcdef}\{I_d I_e I_f\}) = 0$ is neccessary for the representation to be consistent with (1.3) [**24**], and Drinfel'd found such representations for all $\mathfrak{g}$ except e_8 [**10**].

To find finite-dimensional irreducible representations of $Y(\mathfrak{g})$ for *all* $\mathfrak{g}$ Drinfel'd considered the adjoint$\oplus$singlet representation given in Theorem 1.3 below. This representation of $Y(\mathfrak{g})$ is *reducible* as a representation of $\mathfrak{g}$ and, although this is typically the case, it is the only explicitly known such representation [**24**].

THEOREM 1.3 ('Theorem 8' in [**10**]). *Let $\mathfrak{g}$ be a simple, N-dimensional Lie algebra over $\mathbb{C}$ generated by $\{I_a\}, a = 1, \ldots, N$, such that the I's are orthonormal with respect to a fixed, associative[5] inner prodduct $(\cdot, \cdot)$. Let $Y(\mathfrak{g})$ be the Yangian of $\mathfrak{g}$, $v_0 \in \mathbb{C}\backslash\{0\}$ and $x \in \mathfrak{g}$.*

If $\mathfrak{g} = \mathfrak{su}(2)$, then for any $b \in \mathbb{C}$ there is a representation $\rho : Y(\mathfrak{g}) \to$ End$(\mathfrak{g} \oplus \mathbb{C})$ such that

$$\rho(I_a)v_0 = 0 \tag{1.12}$$

$$\rho(I_a)x = [I_a, x] \tag{1.13}$$

$$\rho(J_a)v_0 = bI_a \tag{1.14}$$

$$\rho(J_a)x = (I_a, x)v_0 \tag{1.15}$$

[5]By associative we mean that $(x, [y, z]) = ([x, y], z)$ for all $x, y, z \in \mathfrak{g}$ [**28**].

If $\mathfrak{g} \neq \mathfrak{su}(2)$, then such ρ exists for a unique b. Namely, $b = s^3 c(\mathfrak{g})$, where s^3 is the ratio of the Killing form to the inner product[6] *and*

$$c(\mathfrak{su}(n)) = -\frac{1}{32n^2} \tag{1.16}$$

$$c(\mathfrak{so}(n)) = -\frac{n-4}{16(n-2)^3} \tag{1.17}$$

$$c(\mathfrak{sp}(n)) = -\frac{n+4}{16(n+2)^3} \tag{1.18}$$

and

$$c(\mathfrak{g}) = -\frac{5}{144(N+2)} \tag{1.19}$$

for exceptional $\mathfrak{g}$.

REMARK 1.4. Since $\mathfrak{su}(2) \cong \mathfrak{sp}(2) \cong \mathfrak{so}(3)$, the above expressions (1.16) - (1.18) exclude these cases.

It is the aim of this project to prove this theorem, and in particular to explain where the different values of b come from. The only reason such a ρ would not be a representation is if it is not consistent with (1.3), or (1.4) in the case of $\mathfrak{su}(2)$. By applying ρ to these defining relations of the Yangian we will find the conditions on b as specified in Theorem 1.3.

LEMMA 1.5. *Let ρ be defined as in Theorem 1.3. Then*

$$\rho([J_a, [J_b, I_c]] - [I_a, [J_b, J_c]])v_0 = 0 \tag{1.20}$$

and

$$\rho(a_{abcdef}\{I_d I_e I_f\})v_0 = 0 \tag{1.21}$$

PROOF. That $\rho(a_{abcdef}\{I_d I_e I_f\})v_0 = 0$ follows immediately from $\rho(I_a)v_0 = 0$. Now,

$$\begin{aligned} &\rho([J_a, [J_b, I_c]] - [I_a, [J_b, J_c]]) \\ &\qquad = \rho(J_a[J_b, I_c] - [J_b, I_c]J_a - I_a[J_b, J_c] + [J_b, J_c]I_a) \qquad (1.22) \\ &\qquad = \rho(J_a J_b I_c - J_a I_c J_b - J_b I_c J_a + I_c J_b J_a \\ &\qquad\qquad - I_a J_b J_c + I_a J_c J_b + J_b J_c I_a - J_c J_b I_a) \qquad (1.23) \end{aligned}$$

---

[6]For simple Lie algebras, any associative inner product is a scalar multiple of the Killing form [**12**]

So

$$
\begin{aligned}
\rho([J_a,[J_b,I_c]] - [I_a,[J_b,J_c]])v_0 & \\
= \quad & \rho(J_a)\rho(J_b)\rho(I_c)v_0 - \rho(J_a)\rho(I_c)\rho(J_b)v_0 \\
& - \rho(J_b)\rho(I_c)\rho(J_a)v_0 + \rho(I_c)\rho(J_b)\rho(J_a)v_0 \\
& - \rho(I_a)\rho(J_b)\rho(J_c)v_0 + \rho(I_a)\rho(J_c)\rho(J_b)v_0 \\
& + \rho(J_b)\rho(J_c)\rho(I_a)v_0 - \rho(J_c)\rho(J_b)\rho(I_a)v_0 && (1.24) \\
= & -\rho(J_a)\rho(I_c)bI_b - \rho(J_b)\rho(I_c)bI_a \\
& + \rho(I_c)\rho(J_b)bI_a - \rho(I_a)\rho(J_b)bI_c \\
& + \rho(I_a)\rho(J_c)bI_b && (1.25) \\
= & \, b(-\rho(J_a)[I_c,I_b] - \rho(J_b)[I_c,I_a] \\
& + \rho(I_c)(I_b,I_a)v_0 - \rho(I_a)(I_b,I_c)v_0 \\
& + \rho(I_a)(I_c,I_b)v_0) && (1.26) \\
= & \, b(-\rho(J_a)c_{cbd}I_d - \rho(J_b)c_{cad}I_d) && (1.27) \\
= & -b(c_{cbd}(I_a,I_d) + c_{cad}(I_b,I_d)) && (1.28) \\
= & -b(c_{cbd}\delta_{ad} + c_{cad}\delta_{bd}) && (1.29) \\
= & -b(c_{cbd}\delta_{ad} + c_{cad}\delta_{bd}) && (1.30) \\
= & -b(c_{cba} + c_{cab}) && (1.31) \\
= & -b(c_{cba} - c_{cba}) && (1.32) \\
= & \, 0 && (1.33)
\end{aligned}
$$

where we used the anti-symmetry of the structure constants in the last step. □

We thus conclude that taking $\rho(\cdot)v_0$ is consistent with (1.3) for any value of b. But how about $\rho(\cdot)x$? Since $\{I_a\}$ is a basis of $\mathfrak{g}$ we will only consider the action of $\rho(\cdot)I_x$ on (1.3), where $x = 1,\ldots,N$.

LEMMA 1.6. *Let ρ be defined as in Theorem 1.3. Then*

$$
\begin{aligned}
& \rho([J_a,[J_b,I_c]] - [I_a,[J_b,J_c]])I_x = \\
& \quad b(c_{cxb}\delta_{as} + c_{cbs}\delta_{ax} - c_{abs}\delta_{cx} + c_{acs}\delta_{bx} + c_{axc}\delta_{bs} - c_{axb}\delta_{cs})I_s \qquad (1.34)
\end{aligned}
$$

and

$$\rho(a_{abcdef}\{I_dI_eI_f\})I_x = a_{abcdef}\{c_{fxq}c_{eqr}c_{drs}\}I_s \qquad (1.35)$$

where $\{\cdot\}$ means the sum of all permutations of d, e and f.

PROOF. From (1.23)

$$\begin{aligned}
\rho([J_a,[J_b,I_c]]-[I_a,[J_b,J_c]])I_x = \\
= \quad & \rho(J_a)\rho(J_b)\rho(I_c)I_x-\rho(J_a)\rho(I_c)\rho(J_b)I_x \\
& -\rho(J_b)\rho(I_c)\rho(J_a)I_x+\rho(I_c)\rho(J_b)\rho(J_a)I_x \\
& -\rho(I_a)\rho(J_b)\rho(J_c)I_x+\rho(I_a)\rho(J_c)\rho(J_b)I_x \\
& +\rho(J_b)\rho(J_c)\rho(I_a)I_x-\rho(J_c)\rho(J_b)\rho(I_a)I_x \qquad (1.36)\\
= \quad & \rho(J_a)\rho(J_b)[I_c,I_x]-\rho(J_a)\rho(I_c)(I_b,I_x)v_0 \\
& -\rho(J_b)\rho(I_c)(I_a,I_x)v_0+\rho(I_c)\rho(J_b)(I_a,I_x)v_0 \\
& -\rho(I_a)\rho(J_b)(I_c,I_x)v_0+\rho(I_a)\rho(J_c)(I_b,I_x)v_0 \\
& +\rho(J_b)\rho(J_c)[I_a,I_x]-\rho(J_c)\rho(J_b)[I_a,I_x] \qquad (1.37)\\
= \quad & \rho(J_a)c_{cxs}(I_b,I_s)v_0+\rho(I_c)(I_a,I_x)bI_b \\
& -\rho(I_a)(I_c,I_x)bI_b+\rho(I_a)(I_b,I_x)bI_c \\
& +\rho(J_b)c_{axs}(I_c,I_s)v_0-\rho(J_c)c_{axs}(I_b,I_s)v_0 \qquad (1.38)\\
= {} & b(c_{cxs}(I_b,I_s)I_a+(I_a,I_x)[I_c,I_b] \\
& -(I_c,I_x)[I_a,I_b]+(I_b,I_x)[I_a,I_c] \\
& +c_{axs}(I_c,I_s)I_b-c_{axs}(I_b,I_s)I_c) \qquad (1.39)\\
= {} & b(c_{cxb}I_a+\delta_{ax}c_{cbs}I_s-\delta_{cx}c_{abs}I_s+\delta_{bx}c_{acs}I_s \\
& +c_{axc}I_b-c_{axb}I_c) \qquad (1.40)\\
= {} & b(c_{cxb}\delta_{as}+c_{cbs}\delta_{ax}-c_{abs}\delta_{cx}+c_{acs}\delta_{bx} \\
& +c_{axc}\delta_{bs}-c_{axb}\delta_{cs})I_s \qquad (1.41)
\end{aligned}$$

Now,

$$\begin{aligned}
\rho(a_{abcdef}\{I_dI_eI_f\})I_x &= a_{abcdef}\{\rho(I_d)\rho(I_e)\rho(I_f)I_x\} && (1.42)\\
&= a_{abcdef}\{\rho(I_d)\rho(I_e)[I_f,I_x]\} && (1.43)\\
&= a_{abcdef}\{\rho(I_d)[I_e,[I_f,I_x]]\} && (1.44)\\
&= a_{abcdef}\{[I_d,[I_e,[I_f,I_x]]]\} && (1.45)\\
&= a_{abcdef}\{c_{fxq}[I_d,[I_e,I_q]]\} && (1.46)\\
&= a_{abcdef}\{c_{fxq}c_{eqr}[I_d,I_r]\} && (1.47)\\
&= a_{abcdef}\{c_{fxq}c_{eqr}c_{drs}\}I_s && (1.48)
\end{aligned}$$

□

For the representation ρ to be consistent with (1.3) we thus require that

$$\begin{aligned} b(c_{cxb}\delta_{as} + c_{cbs}\delta_{ax} - c_{abs}\delta_{cx} + c_{acs}\delta_{bx} + c_{axc}\delta_{bs} - c_{axb}\delta_{cs}) \\ = a_{abcdef}\{c_{fxq}c_{eqr}c_{drs}\} \end{aligned} \tag{1.49}$$

for all $a, b, c, x, s = 1, 2, \dots, N$.

This is the condition on b we need to investigate in order to get the values specified in Theorem 1.3.

Note that the RHS of (1.49) when fully expanded has six terms, each with seven structure constants and eight summed and five free indicies. How are we to work with such an expression without getting lost among all the indicies?

CHAPTER 2

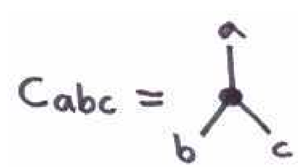
Birdtracks

In order to handle a multitude of structure constants and indicies summed in various ways, we will adopt a diagrammatic notation. This notation, first introduced by Penrose [**27**] and much endorsed and developed by Cvitanović [**8**], is commonly referred to as 'birdtracks' due to its appearance.

In this notation we write a δ as a straight line:

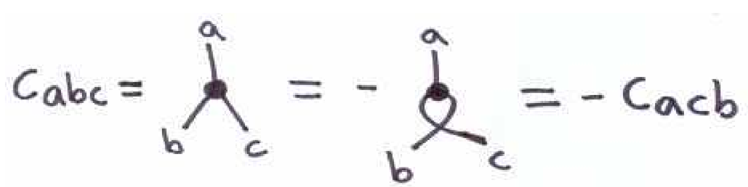
(2.1)

and a structure constant c_{abc} as



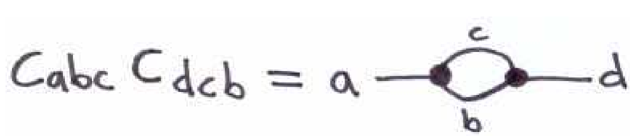
(2.2)

Note that the legs corresponding to the indicies are ordered anticlockwise around the node. The antisymmetry of the structure constants are reflected in the following rule,



(2.3)

Summing over two indicies is represented by joining the legs of those indicies,



(2.4)

However, $c_{abc}c_{dcb} = \kappa(I_a, I_d)$, where $\kappa(\cdot,\cdot)$ is the Killing form. This can be seen by the following argument:

The matrix form of ad I_a with respect to the basis $\{I_a\}$, $a = 1, 2, \ldots, N$, is

$$\text{ad } I_a = \begin{pmatrix} [I_a, I_1] & [I_a, I_2] & \cdots & [I_a, I_N] \\ \Bigg| & \Bigg| & & \Bigg| \end{pmatrix} \tag{2.5}$$

$$= \begin{pmatrix} c_{a11} & c_{a21} & \cdots & c_{aN1} \\ c_{a12} & c_{a22} & \cdots & c_{aN2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{a1N} & c_{a2N} & \cdots & c_{aNN} \end{pmatrix} \tag{2.6}$$

We write this in index notation as $(\text{ad } I_a)_{ij} = c_{aji}$. Hence

$$\kappa(I_a, I_d) = \text{tr}(\text{ad } I_a \circ \text{ad } I_d) \tag{2.7}$$
$$= \text{tr}((\text{ad } I_a)_{ib}(\text{ad } I_d)_{bj}) \tag{2.8}$$
$$= (\text{ad } I_a)_{cb}(\text{ad } I_d)_{bc} \tag{2.9}$$
$$= c_{abc}c_{dcb} \tag{2.10}$$

So $c_{abc}c_{dcb} = \kappa(I_a, I_d)$ and thus $c_{abc}c_{dcb} = s\delta_{ad}$, where s the ratio of the Killing form to the inner product. By renormalization we are free to choose the value of s. Cvitanović [**8**], for example, sets $s = 1$. We will for most part leave the s alone, except for the case of $\mathfrak{su}(n)$ where we will adhere to physics conventions and set $s = -n$.

We thus have the following diagrammatic rule,

(2.11)

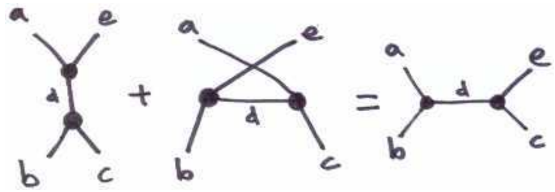
The elements of a Lie algebra satisfy the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \text{ for all } x, y, z \in \mathfrak{g} \tag{2.12}$$

In terms of structure constants this reads,

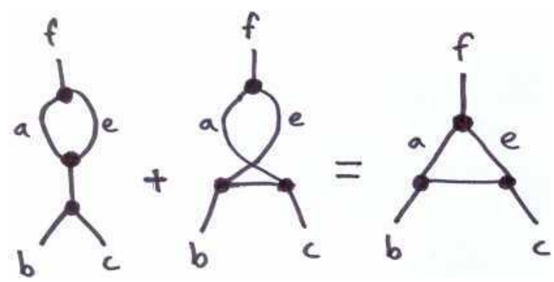
$$[I_a, [I_b, I_c]] + [I_b, [I_c, I_a]] + [I_c, [I_a, I_b]] = 0 \tag{2.13}$$
$$\Leftrightarrow c_{ade}c_{bcd} + c_{bde}c_{cad} + c_{cde}c_{abd} = 0 \tag{2.14}$$
$$\Leftrightarrow c_{ade}c_{bcd} + c_{bde}c_{cad} = c_{ced}c_{abd} \tag{2.15}$$

The birddtrack version of the Jacobi identity is then [**8**],



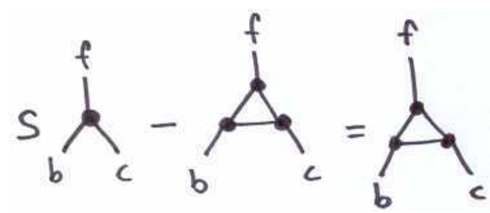
(2.16)

If we introduce a third structure constant, c_{fae}, on both sides of (2.16) and sum over repeated indicies we have,



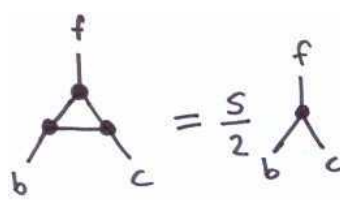
(2.17)

Using (2.11) and (2.3) this becomes



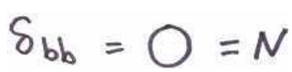
(2.18)

By rearranging we obtain a rule for how to reduce a '3-loop',

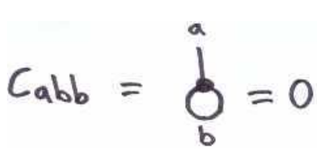


(2.19)

Finally we note that $\delta_{bb} = N$, and by antisymmetry $c_{abb} = 0$. In birdtrack form these statements are:



(2.20)



(2.21)

We will often drop the letters labelling the different indicies when they are implicit.

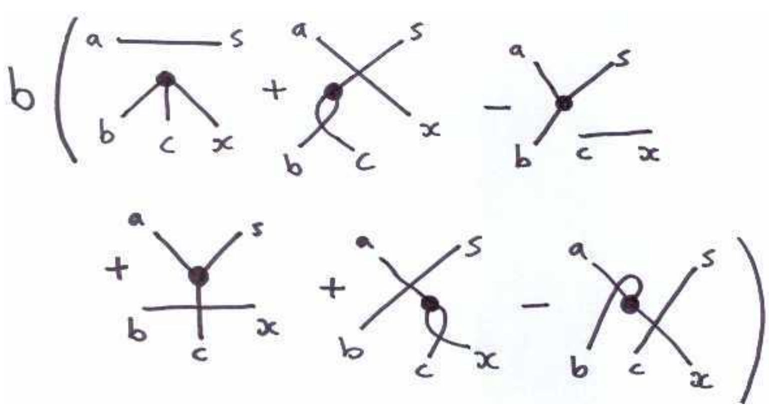
CHAPTER 3

Proving Theorem 1.3

How does our condition on b, (1.49), look in birdtrack notation? Recall that (1.49) is

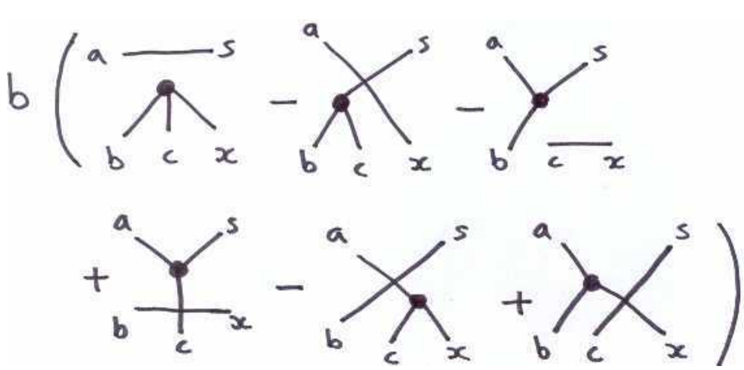
$$
\begin{aligned}
b(c_{cxb}\delta_{as} + c_{cbs}\delta_{ax} - c_{abs}\delta_{cx} + c_{acs}\delta_{bx} + c_{axc}\delta_{bs} - c_{axb}\delta_{cs})& \\
= a_{abcdef}\{c_{fxq}c_{eqr}c_{drs}\}&
\end{aligned}
$$

The LHS of this is,



(3.1)

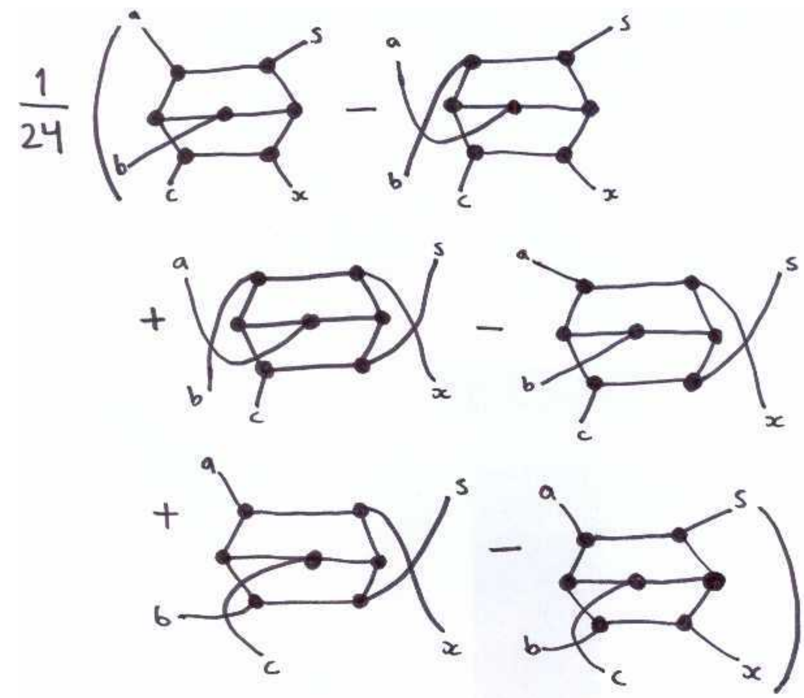
or



(3.2)

where we used (2.3).

The RHS of (1.49) is (see Appendix B),

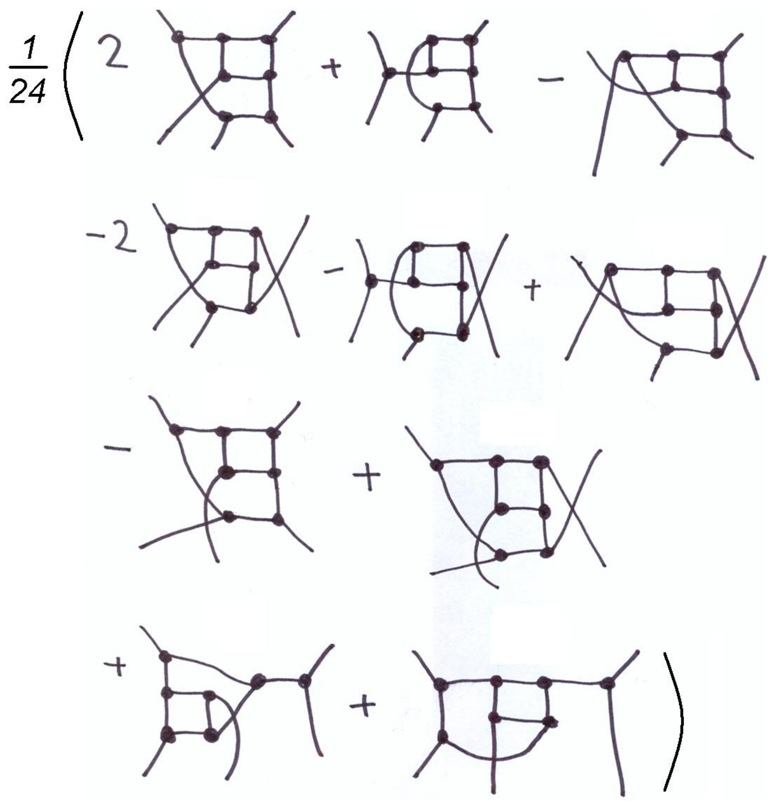


(3.3)

We want to show that the above six 'bipentagons' reduce[1] to a multiple of (3.2). Thereby we can find the value b must take for the representation in Theorem 1.3 to exist. The full details of how we do this can be found in Appendix B. In this section we will give a brief summary.

By using identities derived from the Jacobi identity we show that (3.3) equals the following:

[1]By 'reduce' we mean removing loops in the sense of (2.19) and (2.11).

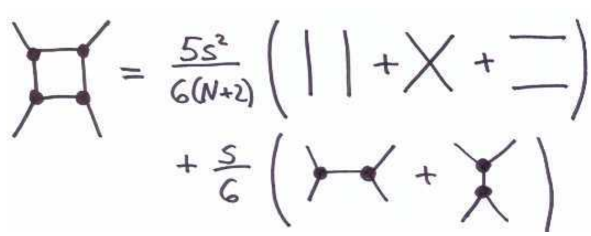


(3.4)

As seen in the last equation, we need a way to reduce a '4-loop' in order to proceed with the calculation. Unlike (2.11) and (2.19), the way in which we can reduce a 4-loop will be different for different Lie algebras.

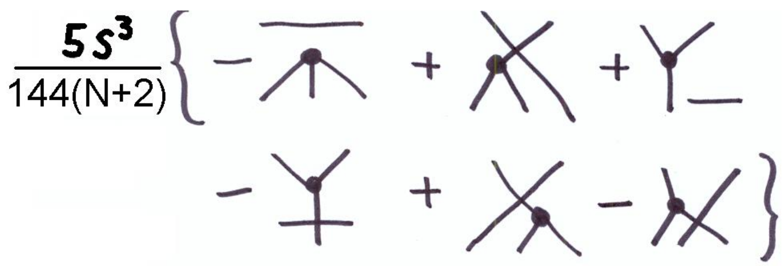
3.1. The exceptional Lie algebras

For the series of exceptional Lie algebras ($a_2 = \mathfrak{su}(3), g_2, d_4 = \mathfrak{so}(8), f_4, e_6, e_7, e_8$) we have the following identity [**8**]:



(3.5)

Using this, we show that (3.4) reduces to



(3.6)

Comparing this to (3.2) we find that b has to equal $\frac{-5s^3}{144(N+2)}$ for (1.49) to hold. Hence the representation ρ in Theorem 1.3 is consistent with (1.3) if and only if

$$b = \frac{-5s^3}{144(N+2)} \tag{3.7}$$

for the Lie algebras of the exceptional series

$$a_2 = \mathfrak{su}(3), g_2, d_4 = \mathfrak{so}(8), f_4, e_6, e_7, e_8$$

including its classical elements. We can verify that the different expressions for b in Theorem 1.3 co-incide for the classical algebras in the exceptional series. For $\mathfrak{su}(3)$ (1.16) is

$$\frac{-1}{32n^2} = \frac{-1}{32 \times 3^2} = \frac{-1}{288}$$

which equals (1.19):

$$\frac{-5}{144(N+2)} = \frac{-5}{144(8+2)} = \frac{-1}{288}$$

For $\mathfrak{so}(8)$ (1.17) is

$$-\frac{n-4}{16(n-2)^3} = -\frac{8-4}{16(8-2)^3} = \frac{-1}{864}$$

and (1.19) is

$$\frac{-5}{144(N+2)} = \frac{-5}{144(28+2)} = \frac{-1}{864}$$

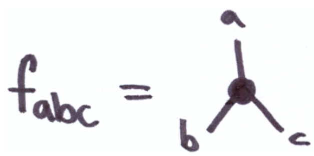
in perfect agreement.

3.2. The Lie algebras $\mathfrak{su}(n)$

For $\mathfrak{g} = \mathfrak{su}(n)$ we will adopt the notation commonly used in physics litterature and make indirect use of Gell-Mann matrices λ_j [**14**]. These are traceless, hermitian $n \times n$ matrices satisfying the multiplication law

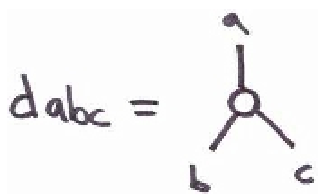
$$\lambda_j \lambda_k = \frac{2}{n}\delta_{jk} + (d_{jkl} + i f_{jkl})\lambda_l \tag{3.8}$$

where the d-tensors are completely symmetric and the f-tensors are the structure constants of $\mathfrak{su}(n)$ (which are anti-symmetric) [**22, 23**]. In birdtrack notation we will write the structure constants f as usual:



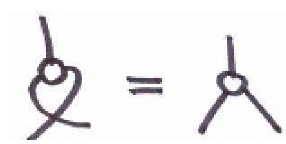
(3.9)

while for the d-tensors we will use a white node:



(3.10)

Since the d-tensors are symmetric we have the following rule:



(3.11)

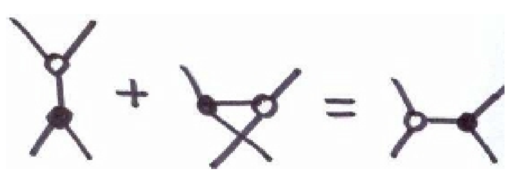
Throughout this section we will follow [**23, 22, 1**] and use the normalization $s = -n$:

(3.12)

From [**22**] we have the following Jacobi-type identity:

$$f_{ilm}d_{mjk} + f_{jlm}d_{imk} + f_{klm}d_{ijm} = 0 \tag{3.13}$$

whose birdtrack equivalent can be written as

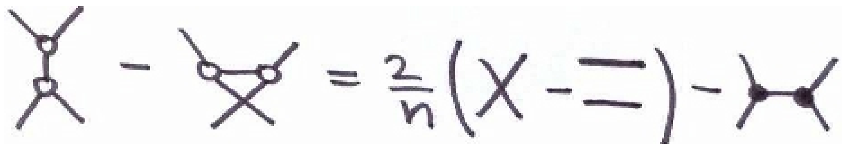


(3.14)

Further we have the result[2][**22**]

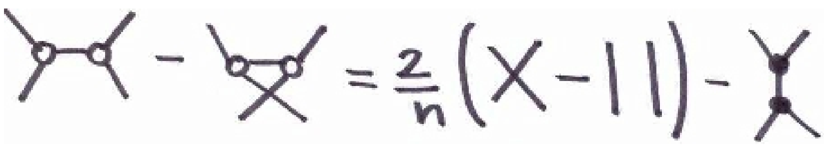
$$f_{ijm}f_{klm} = \tfrac{2}{n}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (d_{ikm}d_{jlm} - d_{jkm}d_{ilm}) \tag{3.15}$$

which we can write as



(3.16)

Rotating the above by 90 degrees we get



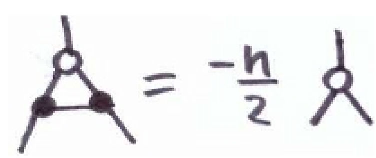
(3.17)

and by combining the last two equations we have

(3.18)

In [**22**] we also find identities for how to reduce '3-loops', of which we shall only need to use the following:

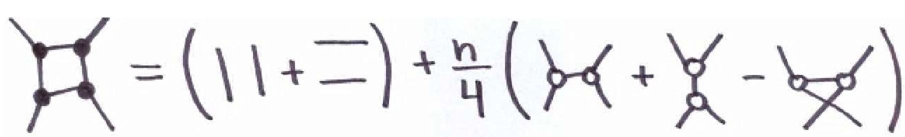
(3.19)



(3.20)

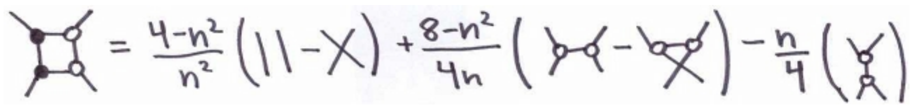
Note that the first of these is our familiar result (2.19) with the normalization $s = -n$.

From [**1**] we get the following identities for reducing '4-loops'



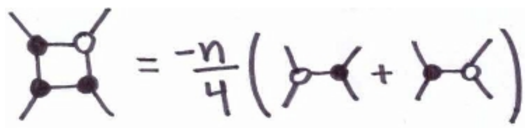
(3.21)

[2]This is a generalization of the familiar $\mathfrak{su}(2)$ result $\epsilon_{ijm}\epsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$



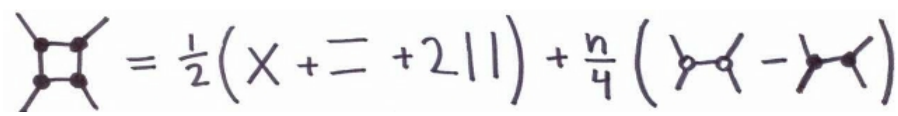
(3.22)

and

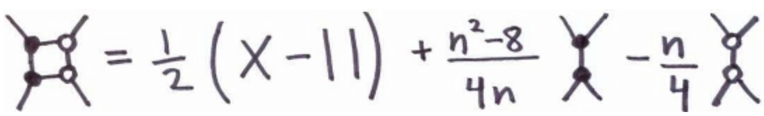


(3.23)

We will sometimes find it convenient to use (3.16) - (3.18) and rewrite the first two '4-loop' identities as:



(3.24)



(3.25)

These are all the identities we need to tackle equation (3.4). We do so in Appendix C and only state the result here.

We find that (3.4) equals

(3.26)

Comparing this with (3.2) we conclude that for $\mathfrak{su}(n)$, $n \geq 3$, the representation ρ of Theorem 1.3 is consistent with (1.3) if and only if

$$b = \frac{3n}{4 \times 24} = \frac{-(-n)^3}{32n^2} = \frac{-s^3}{32n^2} \tag{3.27}$$

Recall that we used the normalisation $s = -n$ in our calculations.

Since $\mathfrak{su}(2)$ only has three generators, at least two of the free indicies in (3.26) and (3.2) have to co-incide. It is a straight-forward calculation to verify that these expressions vanish when joining any two legs. So for $\mathfrak{su}(2)$ both (3.26) and (3.2) are zero and hence does not restrict the values b can take.

3.3. The Lie algebras $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$

Unfortunately we do not know how to reduce a '4-loop' for the algebras $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$. Cvitanović [**8**] only gives a formula for turning a '4-loop' in the adjoint representation into an expression in the defining representation - an expression we do not know how to evaluate. It is possible that the results we need exist in litterature but we have not been able to find them, yet.

To get some results we do numerical calculations for $\mathfrak{so}(5)$ and hence, by isomorphism, $\mathfrak{sp}(4)$.

The details of this calculation can be found in Appendix E.

We find that b has to take the value specified in Theorem 1.3 for (1.49) to hold. We only show that this is a *neccessary* condition, whereas Theorem 1.3 states that it is a *neccessary and sufficient* condition. Still it gives some justification where the values of b come from.

Since $\mathfrak{sp}(4) \cong \mathfrak{so}(5)$ we have also shown that (1.17) is a neccessary condition for the result to hold when $\mathfrak{g} = \mathfrak{sp}(4)$. We note that

$$c(\mathfrak{sp}(4)) = -\frac{4+4}{16(4+2)^3} = -\frac{1}{16 \times 27} \tag{3.28}$$

and

$$c(\mathfrak{so}(5)) = -\frac{5-4}{16(5-2)^3} = -\frac{1}{16 \times 27} \tag{3.29}$$

So $c(\mathfrak{sp}(4)) = c(\mathfrak{so}(5))$ as it should.

Further, $\mathfrak{so}(8)$ is in the exceptional series, and we have shown previously that Theorem 1.3 holds in this case.

There is one more isomorphism between the simple Lie algebras: $\mathfrak{su}(4) \cong \mathfrak{so}(6)$. It is easily verified that $c(\mathfrak{su}(4)) = c(\mathfrak{so}(6))$. We have thus already proved Theorem 1.3 for $\mathfrak{g} = \mathfrak{so}(6)$ in the last section.

To summarise, we have proved Theorem 1.3 for $\mathfrak{so}(8)$ and $\mathfrak{so}(6)$, and showed that b has to take the specified values for $\mathfrak{so}(5)$ and $\mathfrak{sp}(4)$ *if* such a ρ exists.

3.4. The Lie algebra $\mathfrak{su}(2)$

When $\mathfrak{g} = \mathfrak{su}(2)$ we need to look at how the representation ρ in Theorem 1.3 acts on (1.4) instead of (1.3). This since (1.3) follows from (1.2) for $\mathfrak{su}(2)$ [**10**]. The claim of Theorem 1.3 is that the given representation is consistent with (1.4) for *all* values of b. This requires that both sides of (1.4) is zero when we apply $\rho(\cdot)v_0$ and $\rho(\cdot)I_x$.

LEMMA 3.1. *Let ρ be defined as in Theorem 1.3. Then,*

$$\rho([[J_a, J_b], [I_r, J_s]] + [[J_r, J_s], [I_a, J_b]])v_0 = 0 \tag{3.30}$$

and

$$\rho((a_{abcdef}c_{rsc} + a_{rscdef}c_{abc})\{I_d, I_e, I_f\})v_0 = 0 \tag{3.31}$$

Proof. That $\rho((a_{abcdef}c_{rsc} + a_{rscdef}c_{abc})\{I_d, I_e, I_f\})v_0 = 0$ follows directly from $\rho(I_a)v_0 = 0$. Now,

$$\begin{aligned}
&[[J_a, J_b], [I_r, J_s]] + [[J_r, J_s], [I_a, J_b]] \\
&\quad = [J_a, J_b][I_r, J_s] - [I_r, J_s][J_a, J_b] + (a, b) \leftrightarrow (r, s) &(3.32)\\
&\quad = (J_aJ_b - J_bJ_a)(I_rJ_s - J_sI_r) \\
&\quad - (I_rJ_s - J_sI_r)(J_aJ_b - J_bJ_a) + (a, b) \leftrightarrow (r, s) &(3.33)\\
&\quad = J_aJ_bI_rJ_s - J_aJ_bJ_sI_r - J_bJ_aI_rJ_s + J_bJ_aJ_sI_r - I_rJ_sJ_aJ_b \\
&\qquad + I_rJ_sJ_bJ_a - J_sI_rJ_aJ_b + J_sI_rJ_bJ_a + (a, b) \leftrightarrow (r, s) &(3.34)
\end{aligned}$$

When we apply $\rho(\cdot)v_0$ to the above the terms of the form $XXXI$ and $XIJJ$ will be zero. This since $\rho(I_a)v_0 = 0$ and $\rho(J_a)$ takes an element of $\mathfrak{g}$ to $\mathbb{C}$ and vice versa.

So,

$$\begin{aligned}
&\rho([[J_a, J_b], [I_r, J_s]] + [[J_r, J_s], [I_a, J_b]])v_0 \\
&\quad = \rho(J_aJ_bI_rJ_s)v_0 - \rho(J_bJ_aI_rJ_s)v_0 - \rho(I_rJ_sJ_aJ_b)v_0 \\
&\qquad + \rho(I_rJ_sJ_bJ_a)v_0 + (a, b) \leftrightarrow (r, s) &(3.35)\\
&\quad = \rho(J_aJ_bI_r)bI_s - \rho(J_bJ_aI_r)bI_s - \rho(I_rJ_sJ_a)bI_b \\
&\qquad + \rho(I_rJ_sJ_b)bI_a + (a, b) \leftrightarrow (r, s) &(3.36)\\
&\quad = \rho(J_aJ_b)b[I_r, I_s] - \rho(J_bJ_a)b[I_r, I_s] - \rho(I_rJ_s)b(I_a, I_b)v_0 \\
&\qquad + \rho(I_rJ_s)b(I_b, I_a) + (a, b) \leftrightarrow (r, s) &(3.37)\\
&\quad = \rho(J_aJ_b)bc_{rst}I_t - \rho(J_bJ_a)bc_{rst}I_t - \rho(I_rJ_s)b\delta_{ab}v_0 \\
&\qquad + \rho(I_rJ_s)b\delta_{ab}v_0 + (a, b) \leftrightarrow (r, s) &(3.38)\\
&\quad = \rho(J_a)bc_{rst}(I_b, I_t)v_0 - \rho(J_b)bc_{rst}(I_a, I_t)v_0 + (a, b) \leftrightarrow (r, s) &(3.39)\\
&\quad = b^2c_{rst}\delta_{bt}I_a - b^2c_{rst}\delta_{at}I_b + (a, b) \leftrightarrow (r, s) &(3.40)\\
&\quad = b^2(c_{rst}\delta_{bt}I_a - c_{rst}\delta_{at}I_b + c_{abt}\delta_{st}I_r - c_{abt}\delta_{rt}I_s) &(3.41)\\
&\quad = b^2(c_{rsb}I_a - c_{rsa}I_b + c_{abs}I_r - c_{abr}I_s) &(3.42)
\end{aligned}$$

Since there are only three generators in $\mathfrak{su}(2)$ we have $a, b, r, s \in \{1, 2, 3\}$. Thus for $c_{rsb} \neq 0$ we need $\{r, s, b\} = \{1, 2, 3\}$. This means that either $a = b$, $a = r$, or $a = s$.

If $a = b$ then,

$$b^2(c_{rsb}I_a - c_{rsa}I_b + c_{abs}I_r - c_{abr}I_s)$$
$$= b^2(c_{rsb}I_b - c_{rsb}I_b + c_{bbs}I_r - c_{bbr}I_s) \tag{3.43}$$
$$= b^2(c_{rsb}I_b - c_{rsb}I_b) \tag{3.44}$$
$$= 0 \tag{3.45}$$

If $a = r$ then,

$$b^2(c_{rsb}I_a - c_{rsa}I_b + c_{abs}I_r - c_{abr}I_s)$$
$$= b^2(c_{rsb}I_r - c_{rsr}I_b + c_{rbs}I_r - c_{rbr}I_s) \tag{3.46}$$
$$= b^2(c_{rsb}I_r + c_{rbs}I_r) \tag{3.47}$$
$$= b^2(c_{rsb}I_r - c_{rsb}I_r) \tag{3.48}$$
$$= 0 \tag{3.49}$$

If $a = s$ then,

$$b^2(c_{rsb}I_a - c_{rsa}I_b + c_{abs}I_r - c_{abr}I_s)$$
$$= b^2(c_{rsb}I_s - c_{rss}I_b + c_{sbs}I_r - c_{sbr}I_s) \tag{3.50}$$
$$= b^2(c_{rsb}I_s - c_{sbr}I_s) \tag{3.51}$$
$$= b^2(c_{rsb}I_s - c_{rsb}I_s) \tag{3.52}$$
$$= 0 \tag{3.53}$$

So if $c_{rsb} \neq 0$, then $b^2(c_{rsb}I_a - c_{rsa}I_b + c_{abs}I_r - c_{abr}I_s) = 0$. Similarly, assuming $c_{rsa} \neq 0$, $c_{abs} \neq 0$, and $c_{abr} \neq 0$ we also get that

$$b^2(c_{rsb}I_a - c_{rsa}I_b + c_{abs}I_r - c_{abr}I_s) = 0$$

Hence $\rho([[J_a, J_b], [I_r, J_s]] + [[J_r, J_s], [I_a, J_b]])v_0 = 0$.

□

We have thus showed that taking $\rho(\cdot)v_0$ is consistent with (1.4), for any value of b.

Lemma 3.2. *Let ρ be defined as in Theorem 1.3. Then,*

$$\rho([[J_a, J_b], [I_r, J_s]] + [[J_r, J_s], [I_a, J_b]])I_x = 0 \tag{3.54}$$

and

$$\rho((a_{abcdef}c_{rsc} + a_{rscdef}c_{abc})\{I_d, I_e, I_f\})I_x = 0 \tag{3.55}$$

PROOF. From (3.34)

$$\begin{aligned}
&[[J_a, J_b], [I_r, J_s]] + [[J_r, J_s], [I_a, J_b]] \\
&\qquad = J_aJ_bI_rJ_s - J_aJ_bJ_sI_r - J_bJ_aI_rJ_s + J_bJ_aJ_sI_r - I_rJ_sJ_aJ_b \\
&\qquad\quad + I_rJ_sJ_bJ_a - J_sI_rJ_aJ_b + J_sI_rJ_bJ_a + (a,b) \leftrightarrow (r,s) \qquad (3.56)
\end{aligned}$$

When we apply $\rho(\cdot)I_x$ to the above, the terms of the form $XXIJ$ and $IJJJ$ will be zero. So,

$$\begin{aligned}
&\rho([[J_a, J_b], [I_r, J_s]] + [[J_r, J_s], [I_a, J_b]])I_x \\
&\qquad = -\rho(J_aJ_bJ_sI_r)I_x + \rho(J_bJ_aJ_sI_r)I_x \\
&\qquad\quad - \rho(J_sI_rJ_aJ_b)I_x + \rho(J_sI_rJ_bJ_a)I_x + (a,b) \leftrightarrow (r,s) \qquad (3.57)\\
&\qquad = -\rho(J_aJ_bJ_s)[I_r, I_x] + \rho(J_bJ_aJ_s)[I_r, I_x] - \rho(J_sI_rJ_a)(I_b, I_x)v_0 \\
&\qquad\quad + \rho(J_sI_rJ_b)(I_a, I_x)v_0 + (a,b) \leftrightarrow (r,s) \qquad (3.58)\\
&\qquad = -\rho(J_aJ_bJ_s)c_{rxt}I_t + \rho(J_bJ_aJ_s)c_{rxt}I_t - \rho(J_sI_rJ_a)\delta_{bx}v_0 \\
&\qquad\quad + \rho(J_sI_rJ_b)\delta_{ax}v_0 + (a,b) \leftrightarrow (r,s) \qquad (3.59)\\
&\qquad = -\rho(J_aJ_b)c_{rxt}\delta_{st}v_0 + \rho(J_bJ_a)c_{rxt}\delta_{st}v_0 - \rho(J_sI_r)\delta_{bx}bI_a \\
&\qquad\quad + \rho(J_sI_r)\delta_{ax}bI_b + (a,b) \leftrightarrow (r,s) \qquad (3.60)\\
&\qquad = b(-\rho(J_a)c_{rxt}\delta_{st}I_b + \rho(J_b)c_{rxt}\delta_{st}I_a - \rho(J_s)\delta_{bx}c_{rat}I_t \\
&\qquad\quad + \rho(J_s)\delta_{ax}c_{rbt}I_t) + (a,b) \leftrightarrow (r,s) \qquad (3.61)\\
&\qquad = b(-c_{rxt}\delta_{st}\delta_{ab}v_0 + c_{rxt}\delta_{st}\delta_{ba}v_0 - \delta_{bx}c_{rat}\delta_{st}v_0 \\
&\qquad\quad + \delta_{ax}c_{rbs}\delta_{st}) + (a,b) \leftrightarrow (r,s) \qquad (3.62)\\
&\qquad = bv_0(\delta_{ax}c_{rbs} - \delta_{bx}c_{ras} + \delta_{rx}c_{asb} - \delta_{sx}c_{arb}) \qquad (3.63)
\end{aligned}$$

By an argument very similar to that in the proof of Lemma 3.1, it can be shown that

$$\delta_{ax}c_{rbs} - \delta_{bx}c_{ras} + \delta_{rx}c_{asb} - \delta_{sx}c_{arb} = 0$$

Hence $\rho([[J_a, J_b], [I_r, J_s]] + [[J_r, J_s], [I_a, J_b]])I_x = 0$.

Now,

$$\begin{aligned}
&\rho((a_{abcdef}c_{rsc} + a_{rscdef}c_{abc})\{I_d, I_e, I_f\})I_x \\
&\qquad = (a_{abcdef}c_{rsc} + a_{rscdef}c_{abc})\{\rho(I_dI_e)[I_x, I_f]\} \qquad (3.64)\\
&\qquad = (a_{abcdef}c_{rsc} + a_{rscdef}c_{abc})\{\rho(I_d)[[I_x, I_f], I_e]\} \qquad (3.65)\\
&\qquad = (a_{abcdef}c_{rsc} + a_{rscdef}c_{abc})\{[[[I_x, I_f], I_e], I_d]\} \qquad (3.66)
\end{aligned}$$

where $\{\cdot\}$ means the sum of all permutations of the indicies d, e, f.

This expression was evaluated numerically using the computer software Matlab, see Appendix D. The result was that

$$\rho((a_{abcdef}c_{rsc} + a_{rscdef}c_{abc})\{I_d, I_e, I_f\})I_x = 0 \qquad (3.67)$$

$\square$

Hence, taking $\rho(\cdot)I_x$ is consistent with (1.4), for any value of b.

So indeed, the constant b in the representation of Theorem 1.3 can take any value, when $\mathfrak{g} = \mathfrak{su}(2)$.

Conclusion

In this project we have studied Yangians and their representations. We have given some historical background of the topic and briefly discussed its connection with the Yang-Baxter equation. We proved a theorem from Drinfel'd's paper [**10**] about the adjoint$\oplus$singlet representation of $Y(\mathfrak{g})$ for the Lie algebras of the exceptional series and $\mathfrak{su}(n)$. For the algebras $\mathfrak{so}(5)$ and thus $\mathfrak{sp}(4)$ we only managed to give some numerical verifications of the theorem. The reason we could not complete the proof for these algebras was that we did not find an identity for reducing a '4-loop' in these cases. It is possible that such identities already exists in litterature, and further searches could thus be fruitful. Another option would be to learn more about Cvitanović's birdtrack methods and thereby (hopefully) be able to derive the result. It is possible that Drinfel'd had a better method than ours for proving Theorem 1.3 and it would be interesting to know how he arrived at the result, in particular if he did so without a long explicit calculation.

Finally we would like to point out that Yangians have found many other applications than the Yang-Baxter equation, for example on both sides of the AdS/CFT correspondence [**6, 9, 15**] and thus they continue to be of importance to modern theoretical physics.

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APPENDIX A

Drinfel'd's Third Relation

It is far from obvious what the relation (1.3) signifies in Definition 1.1 of the Yangian. Recall that (1.3) is

$$[J_a,[J_b,I_c]]-[I_a,[J_b,J_c]]=a_{abcdef}\{I_d,I_e,I_f\}$$

We saw that one can give a Hopf algebra structure to the Yangian by defining a co-unit, antipode and coproduct. The coproduct $\Delta : Y(\mathfrak{g}) \to Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$ is defined as follows:

$$\Delta(I_a)=I_a\otimes 1+1\otimes I_a \tag{A.1}$$

$$\Delta(J_a)=J_a\otimes 1+1\otimes J_a+\tfrac{1}{2}c_{abc}I_c\otimes I_b \tag{A.2}$$

We will show that the relation (1.3) follows from the requirement that Δ be a homomorphism. To do so we follow the outline of the argument in [**24**] which originates from personal correspondence with Drinfel'd.

Let $u_{ab}\in\mathbb{C}$ be such that $u_{ab}=-u_{ba}$ and

$$u_{ab}[I_a,I_b]=0 \tag{A.3}$$

If we require Δ to be a homomorphism we have the following result [**24**] [1]

LEMMA A.1.

$$\begin{aligned}&u_{ab}(\Delta([J_a,J_b])-1\otimes[J_a,J_b]-[J_a,J_b]\otimes 1)\\&\qquad=\tfrac{1}{4}u_{ab}c_{ade}c_{bfg}c_{dfh}(I_eI_g\otimes I_h+I_h\otimes I_eI_g)\end{aligned} \tag{A.4}$$

[1]In [**24**] the result differs from mine by a factor of 2. We assume that this is a typo.

PROOF.

$$
\begin{aligned}
& u_{ab}(\Delta([J_a, J_b]) - 1 \otimes [J_a, J_b] - [J_a, J_b] \otimes 1) \\
& = u_{ab}([\Delta(J_a), \Delta(J_b)]) - 1 \otimes [J_a, J_b] - [J_a, J_b] \otimes 1) && \text{(A.5)} \\
& = u_{ab}([J_a \otimes 1 + 1 \otimes J_a + \tfrac{1}{2} c_{ade} I_e \otimes I_d, J_b \otimes 1 + 1 \otimes J_b + \tfrac{1}{2} c_{bfg} I_g \otimes I_f] \\
& \qquad - 1 \otimes [J_a, J_b] - [J_a, J_b] \otimes 1) && \text{(A.6)} \\
& = u_{ab}(\tfrac{1}{4} c_{ade} c_{bfg} [I_e \otimes I_d, I_g \otimes I_f] + [J_a \otimes 1 + 1 \otimes J_a, \tfrac{1}{2} c_{bfg} I_g \otimes I_f] \\
& \qquad + [\tfrac{1}{2} c_{ade} I_e \otimes I_d, J_b \otimes 1 + 1 \otimes J_b] && \text{(A.7)} \\
& = u_{ab}(\tfrac{1}{4} c_{ade} c_{bfg} (I_e I_g \otimes I_d I_f - I_g I_e \otimes I_f I_d) + \tfrac{1}{2} c_{bfg} I_g \otimes [J_a, I_f] \\
& \qquad + \tfrac{1}{2} c_{bfg} [J_a, I_g] \otimes I_f + \tfrac{1}{2} c_{ade} I_e \otimes [I_d, J_b] + \tfrac{1}{2} c_{ade} [I_e, J_b] \otimes I_d) && \text{(A.8)} \\
& = \tfrac{1}{4} u_{ab} c_{ade} c_{bfg} (I_e I_g \otimes I_d I_f + ([I_e, I_g] - I_e I_g) \otimes I_f I_d) \\
& \qquad + \tfrac{1}{2} u_{ab} c_{bfg} c_{afk} I_g \otimes J_k + \tfrac{1}{2} \underbrace{u_{ab} c_{bfg} c_{agk} J_k \otimes I_f}_{f \leftrightarrow g} \\
& \qquad + \tfrac{1}{2} \underbrace{u_{ab} c_{ade} c_{dbk} I_e \otimes J_k}_{e \to g, d \to f} + \tfrac{1}{2} \underbrace{u_{ab} c_{ade} c_{ebk} J_k \otimes I_d)}_{d \to g, e \to f} && \text{(A.9)} \\
& = \tfrac{1}{4} u_{ab} c_{ade} c_{bfg} (I_e I_g \otimes [I_d, I_f] + [I_e, I_g] \otimes I_f I_d) \\
& + \tfrac{1}{2} u_{ab} (c_{bfg} c_{afk} + c_{afg} c_{fbk}) I_g \otimes J_k - \tfrac{1}{2} u_{ab} (c_{bfg} c_{afk} + c_{afg} c_{fbk}) J_k \otimes I_g && \text{(A.10)}
\end{aligned}
$$

where we have renamed bound indicies in the last step. By the Jacobi identity $c_{bfg} c_{afk} = -c_{afg} c_{kfb} - c_{kfg} c_{bfa} = -c_{afg} c_{fbk} - c_{kfg} c_{abf}$. Hence (A.10) is

$$
\begin{aligned}
& \tfrac{1}{4} u_{ab} c_{ade} c_{bfg} (I_e I_g \otimes [I_d, I_f] + [I_e, I_g] \otimes I_f I_d) \\
& - \tfrac{1}{2} u_{ab} c_{abf} c_{kfg} (I_g \otimes J_k - J_k \otimes I_g) && \text{(A.11)}
\end{aligned}
$$

Since $u_{ab}[I_a, I_b] = 0$, we have that $u_{ab}c_{abf} = 0$. So the last term in (A.11) is zero. Thus,

$$\begin{aligned}
&u_{ab}(\Delta([J_a, J_b]) - 1 \otimes [J_a, J_b] - [J_a, J_b] \otimes 1) \\
&= \tfrac{1}{4} u_{ab} c_{ade} c_{bfg} (I_e I_g \otimes [I_d, I_f] + [I_e, I_g] \otimes I_f I_d) && \text{(A.12)} \\
&= \tfrac{1}{4} u_{ab} c_{ade} c_{bfg} c_{dfh} I_e I_g \otimes I_h + \tfrac{1}{4} \underbrace{u_{ab} c_{ade} c_{bfg} c_{egh} I_h \otimes I_f I_d}_{e \leftrightarrow f, d \leftrightarrow g} && \text{(A.13)} \\
&= \tfrac{1}{4} u_{ab} c_{ade} c_{bfg} c_{dfh} I_e I_g \otimes I_h + \tfrac{1}{4} u_{ab} c_{agf} c_{bed} c_{fdh} I_h \otimes I_e I_g && \text{(A.14)} \\
&= \tfrac{1}{4} u_{ab} c_{ade} c_{bfg} c_{dfh} I_e I_g \otimes I_h - \tfrac{1}{4} \underbrace{u_{ab} c_{afg} c_{bde} c_{dfh} I_h \otimes I_e I_g}_{a \leftrightarrow b} && \text{(A.15)} \\
&= \tfrac{1}{4} u_{ab} c_{ade} c_{bfg} c_{dfh} I_e I_g \otimes I_h - \tfrac{1}{4} u_{ba} c_{bfg} c_{ade} c_{dfh} I_h \otimes I_e I_g && \text{(A.16)} \\
&= \tfrac{1}{4} u_{ab} c_{ade} c_{bfg} c_{dfh} (I_e I_g \otimes I_h + I_h \otimes I_e I_g) && \text{(A.17)}
\end{aligned}$$

where we used $u_{ab} = -u_{ba}$.

□

Since $u_{ab}c_{abc} = 0$ we can write

$$u_{ab} = v_{lma} c_{lmb} - v_{lmb} c_{lma} \tag{A.18}$$

for some anti-symmetric tensor v_{lma}. This is a non-trivial result, equivalent to the second homology $H_2(\mathfrak{g})$ being zero [**24**].

So,

$$\begin{aligned}
&u_{ab}(\Delta([J_a, J_b]) - 1 \otimes [J_a, J_b] - [J_a, J_b] \otimes 1) \\
&= \tfrac{1}{4} (v_{lma} c_{lmb} - v_{lmb} c_{lma}) c_{ade} c_{bfg} c_{dfh} (I_e I_g \otimes I_h + I_h \otimes I_e I_g) && \text{(A.19)}
\end{aligned}$$

Now we make repeated use of the Jacobi identity to obtain the following result

Lemma A.2. $(v_{lma} c_{lmb} - v_{lmb} c_{lma}) c_{ade} c_{bfg} c_{dfh} = 2 v_{lma} c_{mgb} c_{aed} c_{lhf} c_{bdf}$

Proof. First we will show that

$$v_{lma} c_{lmb} c_{ade} c_{bfg} c_{dfh} = 2 v_{lma} (c_{mgb} c_{ade} c_{bdf} c_{lfh} + c_{mgb} c_{ade} c_{dlf} c_{bfh}) \tag{A.20}$$

We do this by using the Jacobi identity twice.

$$
\begin{aligned}
v_{lma}c_{lmb}c_{ade}c_{bfg}c_{dfh} &= v_{lma}\underbrace{c_{lmb}c_{gbf}}_{\text{use Jacobi}}c_{ade}c_{dfh} && \text{(A.21)}\\
&= -v_{lma}\big(c_{mgb}c_{lbf}+c_{glb}c_{mbf}\big)c_{ade}c_{dfh} && \text{(A.22)}\\
&= -v_{lma}\big(c_{mgb}c_{ade}\underbrace{c_{lbf}c_{dfh}}_{\text{use Jacobi}}+c_{glb}c_{ade}\underbrace{c_{mbf}c_{dfh}}_{\text{use Jacobi}}\big) && \text{(A.23)}\\
&= v_{lma}\big(c_{mgb}c_{ade}\big(c_{bdf}c_{lfh}+c_{dlf}c_{bfh}\big)\\
&\quad+c_{glb}c_{ade}\big(c_{bdf}c_{mfh}+c_{dmf}c_{bfh}\big)\big) && \text{(A.24)}\\
&= v_{lma}c_{mgb}c_{ade}c_{bdf}c_{lfh}+v_{lma}c_{mgb}c_{ade}c_{dlf}c_{bfh}\\
&\quad+\underbrace{v_{lma}c_{glb}c_{ade}c_{bdf}c_{mfh}}_{l\leftrightarrow m}+\underbrace{v_{lma}c_{glb}c_{ade}c_{dmf}c_{bfh}}_{l\leftrightarrow m} && \text{(A.25)}\\
&= v_{lma}c_{mgb}c_{ade}c_{bdf}c_{lfh}+v_{lma}c_{mgb}c_{ade}c_{dlf}c_{bfh}\\
&\quad+v_{mla}c_{gmb}c_{ade}c_{bdf}c_{lfh}+v_{mla}c_{gmb}c_{ade}c_{dlf}c_{bfh} && \text{(A.26)}\\
&= 2v_{lma}\big(c_{mgb}c_{ade}c_{bdf}c_{lfh}+c_{mgb}c_{ade}c_{dlf}c_{bfh}\big) && \text{(A.27)}
\end{aligned}
$$

where we used the anti-symmetry of v_{lma}.

By a similar argument one can show that

$$
v_{lmb}c_{lma}c_{ade}c_{bfg}c_{dfh} = 2v_{lma}\big(c_{meb}c_{afg}c_{hld}c_{bdf}+c_{meb}c_{afg}c_{bhd}c_{ldf}\big)
$$

Thus

$$
\begin{aligned}
&\big(v_{lma}c_{lmb}-v_{lmb}c_{lma}\big)c_{ade}c_{bfg}c_{dfh}\\
&\qquad= 2v_{lma}\big(c_{mgb}c_{ade}c_{bdf}c_{lfh}-c_{meb}c_{afg}c_{hld}c_{bdf}\\
&\qquad\quad+c_{mgb}c_{ade}c_{dlf}c_{bfh}-c_{meb}c_{afg}c_{bhd}c_{ldf}\big) && \text{(A.28)}
\end{aligned}
$$

Now,

$$
\begin{aligned}
&v_{lma}\big(c_{mgb}c_{ade}c_{bdf}c_{lfh}-c_{meb}c_{afg}c_{hld}c_{bdf}\big)\\
&= v_{lma}c_{mgb}c_{aed}c_{lhf}c_{bdf}-\underbrace{v_{lma}c_{meb}c_{agf}c_{lhd}c_{bdf}}_{m\leftrightarrow a,b\to d\to f\to b} && \text{(A.29)}\\
&= v_{lma}c_{mgb}c_{aed}c_{lhf}c_{bdf}-v_{lam}c_{aed}c_{mgb}c_{lhf}c_{dfb} && \text{(A.30)}\\
&= v_{lma}c_{mgb}c_{aed}c_{lhf}c_{bdf}+v_{lma}c_{aed}c_{mgb}c_{lhf}c_{bdf} && \text{(A.31)}\\
&= 2v_{lma}c_{mgb}c_{aed}c_{lhf}c_{bdf} && \text{(A.32)}
\end{aligned}
$$

And

$$\begin{aligned}
&v_{lma}(c_{mgb}c_{ade}c_{dlf}c_{bfh} - c_{meb}c_{afg}c_{bhd}c_{ldf}) \\
&= v_{lma}c_{mgb}c_{aed}c_{bhf}c_{lfd} - \underbrace{v_{lma}c_{meb}c_{agf}c_{bhd}c_{lfd}}_{m\leftrightarrow a, b\to d\to f\to b} &\text{(A.33)}\\
&= v_{lma}c_{mgb}c_{aed}c_{bhf}c_{lfd} - v_{lam}c_{aed}c_{mgb}c_{dhf}c_{lbf} &\text{(A.34)}\\
&= v_{lma}c_{mgb}c_{aed}(\underbrace{c_{bhf}c_{lfd}}_{\text{use Jacobi}} + c_{dhf}c_{lbf}) &\text{(A.35)}\\
&= v_{lma}c_{mgb}c_{aed}(-c_{hlf}c_{bfd} - c_{lbf}c_{hfd} + c_{dhf}c_{lbf}) &\text{(A.36)}\\
&= -v_{lma}c_{mgb}c_{aed}c_{lhf}c_{bdf} &\text{(A.37)}
\end{aligned}$$

So, from (A.28), we have

$$(v_{lma}c_{lmb} - v_{lmb}c_{lma})c_{ade}c_{bfg}c_{dfh} = 2v_{lma}c_{mgb}c_{aed}c_{lhf}c_{bdf}$$

□

Using this result in equation (A.19) we get,

$$\begin{aligned}
&u_{ab}(\Delta([J_a, J_b]) - 1\otimes[J_a, J_b] - [J_a, J_b]\otimes 1) \\
&= \tfrac{1}{2}v_{lma}c_{mgb}c_{aed}c_{lhf}c_{bdf}(I_eI_g\otimes I_h + I_h\otimes I_eI_g) &\text{(A.38)}
\end{aligned}$$

From our expression (A.18) we have

$$\begin{aligned}
&u_{ab}(\Delta([J_a, J_b]) - 1\otimes[J_a, J_b] - [J_a, J_b]\otimes 1) \\
&= (v_{lma}c_{lmb} - v_{lmb}c_{lma})(\Delta([J_a, J_b]) - 1\otimes[J_a, J_b] - [J_a, J_b]\otimes 1) &\text{(A.39)}\\
&= v_{lma}c_{lmb}(\Delta([J_a, J_b]) - 1\otimes[J_a, J_b] - [J_a, J_b]\otimes 1) \\
&\quad - \underbrace{v_{lmb}c_{lma}(\Delta([J_a, J_b]) - 1\otimes[J_a, J_b] - [J_a, J_b]\otimes 1)}_{a\leftrightarrow b} &\text{(A.40)}\\
&= v_{lma}c_{lmb}(\Delta([J_a, J_b]) - 1\otimes[J_a, J_b] - [J_a, J_b]\otimes 1) \\
&\quad - v_{lma}c_{lmb}(\Delta([J_b, J_a]) - 1\otimes[J_b, J_a] - [J_b, J_a]\otimes 1) &\text{(A.41)}\\
&= 2v_{lma}c_{lmb}(\Delta([J_a, J_b]) - 1\otimes[J_a, J_b] - [J_a, J_b]\otimes 1) &\text{(A.42)}\\
&= 2v_{lma}c_{blm}(\Delta([J_a, J_b]) - 1\otimes[J_a, J_b] - [J_a, J_b]\otimes 1) &\text{(A.43)}
\end{aligned}$$

Equation (A.38) thus reads

$$\begin{aligned}
&2v_{lma}c_{blm}(\Delta([J_a, J_b]) - 1\otimes[J_a, J_b] - [J_a, J_b]\otimes 1) \\
&= \tfrac{1}{2}v_{lma}c_{mgb}c_{aed}c_{lhf}c_{bdf}(I_eI_g\otimes I_h + I_h\otimes I_eI_g) &\text{(A.44)}\\
&= 12v_{lma}a_{lmahge}(I_eI_g\otimes I_h + I_h\otimes I_eI_g) &\text{(A.45)}
\end{aligned}$$

where we used the expression $a_{abcdef} = \frac{1}{24}c_{adi}c_{bej}c_{cfk}c_{ijk}$, as in Definition 1.1.

Now we take the anti-symmetric sum of all permutations of the indicies l, m, a

$$\begin{aligned}
&2v_{[lma]}c_{b[lm}(\Delta([J_{a]}, J_b]) - 1 \otimes [J_{a]}, J_b] - [J_{a]}, J_b] \otimes 1) \\
&= 12 v_{[lma]} a_{[lma]hge}(I_e I_g \otimes I_h + I_h \otimes I_e I_g)
\end{aligned} \tag{A.46}$$

where $[lma] = lma + mal + alm - lam - aml - mla.$

Lemma A.3. *We can express the LHS of* (1.3) *as*

$$[J_l, [J_m, I_a]] - [I_l, [J_m, J_a]] = \tfrac{1}{2} c_{b[lm}[J_{a]}, J_b] \tag{A.47}$$

Proof. By using the Jacobi identity[2],

$$[J_l, [J_m, I_a]] - [I_l, [J_m, J_a]] = [J_l, [J_m, I_a]] + [J_m, [J_a, I_l]] + [J_a, [I_l, J_m]] \tag{A.48}$$

$$= c_{mab}[J_l, J_b] + c_{alb}[J_m, J_b] + c_{lmb}[J_a, J_b] \tag{A.49}$$

$$= \tfrac{1}{2} c_{b[lm}[J_{m]}, J_b] \tag{A.50}$$

where we used the anti-symmetry of the structure constants. $\square$

Equation (A.46) is then

$$\begin{aligned}
4v_{[lma]}(&\Delta([J_l, [J_m, I_a]] - [I_l, [J_m, J_a]]) \\
&- 1 \otimes ([J_l, [J_m, I_a]] - [I_l, [J_m, J_a]]) \\
&- ([J_l, [J_m, I_a]] - [I_l, [J_m, J_a]]) \otimes 1)) \\
= 12 v_{[lma]} &a_{[lma]hge}(I_e I_g \otimes I_h + I_h \otimes I_e I_g)
\end{aligned} \tag{A.51}$$

To see what the RHS of this expression is we use the following Lemma from [**18**]:

Lemma A.4.

$$a_{[lma]hge}(I_e I_g \otimes I_h + I_h \otimes I_e I_g) = a_{lmahge}\{I_e I_g \otimes I_h + I_h \otimes I_e I_g\} \tag{A.52}$$

where $\{\cdot\}$ *means the sum of all permutations of the indicies* h, e *and* g.

Proof.

$$a_{lmahge}\{I_e I_g \otimes I_h + I_h \otimes I_e I_g\} = a_{lma\{hge\}}(I_e I_g \otimes I_h + I_h \otimes I_e I_g) \tag{A.53}$$

[2]We can use the Jacobi identity for the J's as well since our Lie bracket $[\cdot, \cdot]$ is just a commutator

and

$$\begin{aligned}
24 \times a_{lma\{hge\}} &= c_{lhi}c_{mgj}c_{aek}c_{ijk} + \underbrace{c_{lgi}c_{mej}c_{ahk}c_{ijk}}_{i\to j\to k\to i} + \underbrace{c_{lei}c_{mhj}c_{agk}c_{ijk}}_{i\to k\to j\to i} \\
&+ \underbrace{c_{lhi}c_{mej}c_{agk}c_{ijk}}_{j\leftrightarrow k} + \underbrace{c_{lgi}c_{mhj}c_{aek}c_{ijk}}_{i\leftrightarrow j} + \underbrace{c_{lei}c_{mgj}c_{ahk}c_{ijk}}_{i\leftrightarrow k} \qquad \text{(A.54)} \\
&= c_{lhi}c_{mgj}c_{aek}c_{ijk} + c_{lgj}c_{mek}c_{ahi}c_{jki} + c_{lek}c_{mhi}c_{agj}c_{kij} \\
&+ c_{lhi}c_{mek}c_{agj}c_{ikj} + c_{lgj}c_{mhi}c_{aek}c_{jik} + c_{lek}c_{mgj}c_{ahi}c_{kji} \qquad \text{(A.55)} \\
&= c_{lhi}c_{mgj}c_{aek}c_{ijk} + c_{ahi}c_{lgj}c_{mek}c_{ijk} + c_{mhi}c_{agj}c_{lek}c_{ijk} \\
&- c_{lhi}c_{agj}c_{mek}c_{ijk} - c_{mhi}c_{lgj}c_{aek}c_{ijk} - c_{ahi}c_{mgj}c_{lek}c_{ijk} \qquad \text{(A.56)} \\
&= 24 \times a_{[lma]hge} \qquad \text{(A.57)}
\end{aligned}$$

So

$$a_{lmahge}\{I_eI_g \otimes I_h + I_h \otimes I_eI_g\} = a_{[lma]hge}(I_eI_g \otimes I_h + I_h \otimes I_eI_g) \qquad \text{(A.58)}$$

$\square$

Equation (A.51) is then

$$\begin{aligned}
&v_{[lma]}(\Delta([J_l,[J_m,I_a]] - [I_l,[J_m,J_a]]) \\
&\qquad - 1 \otimes ([J_l,[J_m,I_a]] - [I_l,[J_m,J_a]]) \\
&\qquad - ([J_l,[J_m,I_a]] - [I_l,[J_m,J_a]]) \otimes 1)) \\
&= 3v_{[lma]}a_{lmahge}\{I_eI_g \otimes I_h + I_h \otimes I_eI_g\} \qquad \text{(A.59)}
\end{aligned}$$

Applying Δ to the RHS of (1.3) and requiring it to be a homomorphism we get the following outcome.

Lemma A.5.

$$\begin{aligned}
&\Delta(a_{lmahge}\{I_hI_gI_e\}) \\
&\qquad = a_{lmahge}(1 \otimes \{I_hI_eI_g\} + \{I_hI_eI_g\} \otimes 1 \\
&\qquad\qquad + 3\{I_eI_g \otimes I_h + I_h \otimes I_eI_g\}) \qquad \text{(A.60)}
\end{aligned}$$

PROOF. Requiring Δ to be a homomorphism we have,

$$\Delta\{I_h I_g I_e\} = \{\Delta(I_h)\Delta(I_g)\Delta(I_e)\} \tag{A.61}$$
$$= \{(1 \otimes I_h + I_h \otimes 1)(1 \otimes I_g + I_g \otimes 1)(1 \otimes I_e + I_e \otimes 1)\} \tag{A.62}$$
$$= \{I_h I_g I_e \otimes 1 + I_h I_g \otimes I_e + I_h I_e \otimes I_g + I_g I_e \otimes I_h + 1 \otimes I_h I_g I_e + I_g \otimes I_h I_e + I_e \otimes I_h I_g + I_h \otimes I_g I_e\} \tag{A.63}$$
$$= 1 \otimes \{I_h I_g I_e\} + \{I_h I_g I_e\} \otimes 1 + 3\{I_g I_e \otimes I_h + I_h \otimes I_g I_e\} \tag{A.64}$$
$$= 1 \otimes \{I_h I_e I_g\} + \{I_h I_e I_g\} \otimes 1 + 3\{I_e I_g \otimes I_h + I_h \otimes I_e I_g\} \tag{A.65}$$

So,

$$\Delta(a_{lmahge}\{I_h I_g I_e\}) = a_{lmahge}(1 \otimes \{I_h I_e I_g\} + \{I_h I_e I_g\} \otimes 1 + 3\{I_e I_g \otimes I_h + I_h \otimes I_e I_g\}) \tag{A.66}$$

$\square$

Using this result in (A.59) we get

$$v_{[lma]}(\Delta([J_l,[J_m,I_a]] - [I_l,[J_m,J_a]]) - 1 \otimes ([J_l,[J_m,I_a]] - [I_l,[J_m,J_a]]) - ([J_l,[J_m,I_a]] - [I_l,[J_m,J_a]]) \otimes 1)) = v_{[lma]}\Delta(a_{lmahge}\{I_h I_g I_e\}) - v_{[lma]}a_{lmahge}(1 \otimes \{I_h I_e I_g\} + \{I_h I_e I_g\} \otimes 1) \tag{A.67}$$

Requiring (A.67) to hold for all v, we obtain

$$[J_l,[J_m,I_a]] - [I_l,[J_m,J_a]] = a_{lmahge}\{I_h I_g I_e\} \tag{A.68}$$

which is the third defining relation of the Yangian, (1.3).

APPENDIX B

Full calculation for the exceptional algebras

In this section we will show in full detail that (1.49) holds if and only if $b = \frac{-5s^3}{144(N+2)}$. Recall that (1.49) is

$$
\begin{aligned}
b(c_{cxb}\delta_{as} + c_{cbs}\delta_{ax} - c_{abs}\delta_{cx} + c_{acs}\delta_{bx} + c_{axc}\delta_{bs} - c_{axb}\delta_{cs}) \\
= a_{abcdef}\{c_{fxq}c_{eqr}c_{drs}\}
\end{aligned}
$$

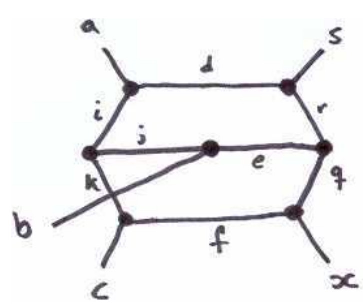
or

$$
\begin{aligned}
24b(c_{cxb}\delta_{as} + c_{cbs}\delta_{ax} - c_{abs}\delta_{cx} + c_{acs}\delta_{bx} + c_{axc}\delta_{bs} - c_{axb}\delta_{cs}) \\
= 24a_{abcdef}\{c_{fxq}c_{eqr}c_{drs}\}
\end{aligned} \quad \text{(B.1)}
$$

Writing out the RHS of this in full we get,

$$
\begin{aligned}
24a_{abcdef}\{c_{fxq}c_{eqr}c_{drs}\} = (c_{adi}c_{bej}c_{cfk}c_{ijk}(&c_{fxq}c_{eqr}c_{drs} + c_{exq}c_{dqr}c_{frs} \\
&+ c_{dxq}c_{fqr}c_{ers} + c_{fxq}c_{dqr}c_{ers} + c_{exq}c_{fqr}c_{drs} \\
&+ c_{dxq}c_{eqr}c_{frs}))
\end{aligned} \quad \text{(B.2)}
$$

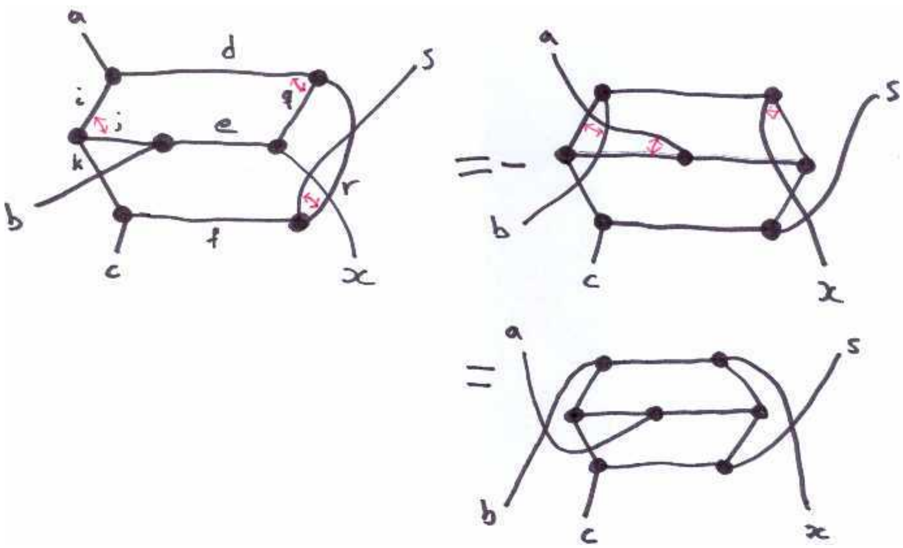
Now we proceed to write this in diagrammatic notation. Starting with $c_{adi}c_{bej}c_{cfk}c_{ijk}c_{fxq}c_{eqr}c_{drs} = c_{aid}c_{bej}c_{cfk}c_{ikj}c_{fxq}c_{eqr}c_{drs}$, we write this in birdtrack notation as:



(B.3)

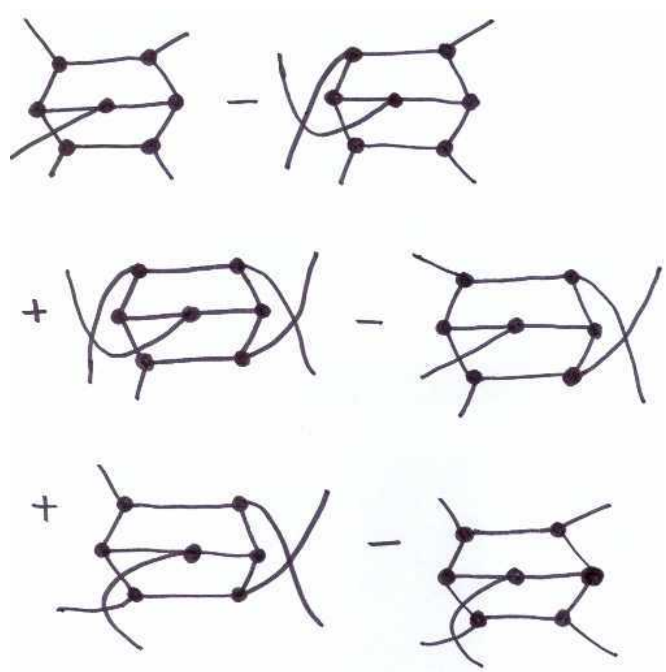
We will call this type of figure a '*bipentagon*'.

The next term in the RHS of (B.2) is $c_{aid}c_{bej}c_{cfk}c_{ikj}c_{exq}c_{dqr}c_{frs}$:



(B.4)

which is also a bipentagon, but with some of the free indicies swapped. By similiar methods we find that the birdtrack version of the RHS of (B.2) is is



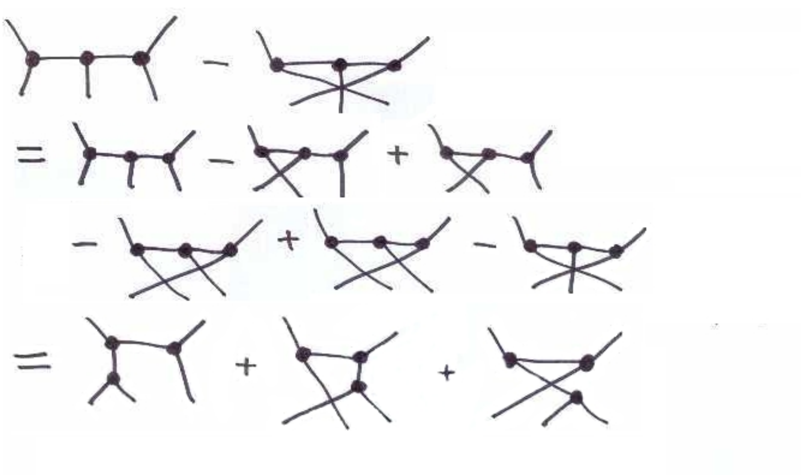
(B.5)

We have dropped the explicit labelling of indicies and will continue to do so for the rest of this section.

The Jacobi identity (2.16) can be written

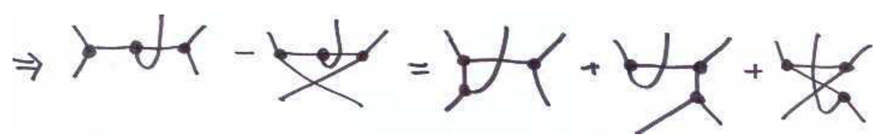
(B.6)

Now,



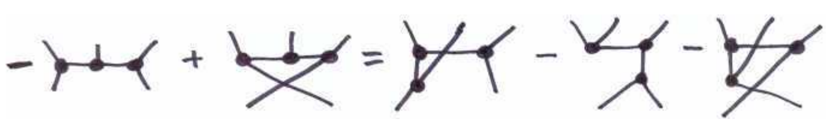
(B.7)

where we used the Jacobi identity. Moving one of the free indicies 'up' we get



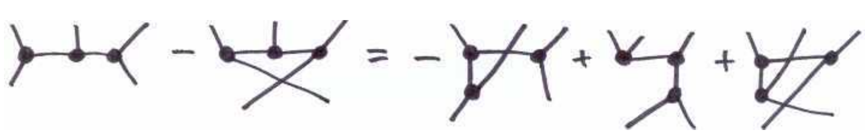
(B.8)

which implies that



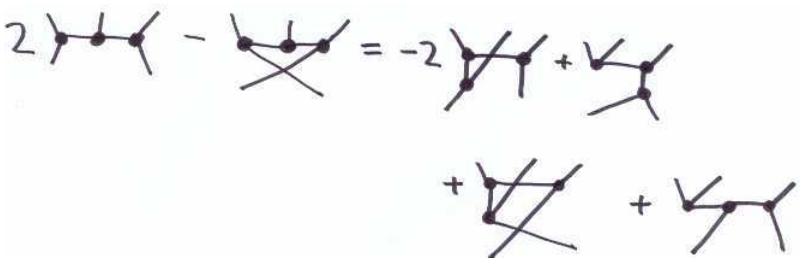
(B.9)

We have thus derived the rule



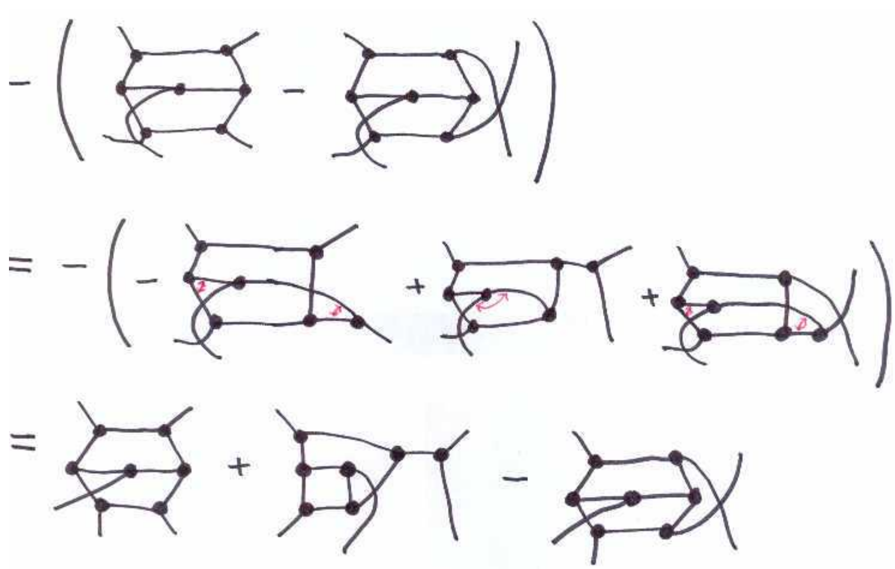
(B.10)

By a similiar argument (adding and subtracting the same terms and using the Jacobi identity) we can derive the following result:



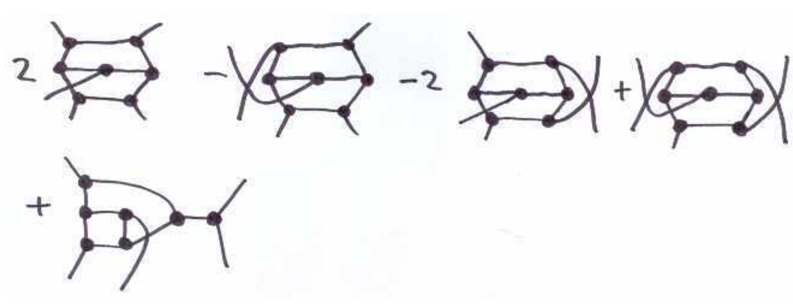
(B.11)

Now we use (B.10) on the last two terms of (B.5).



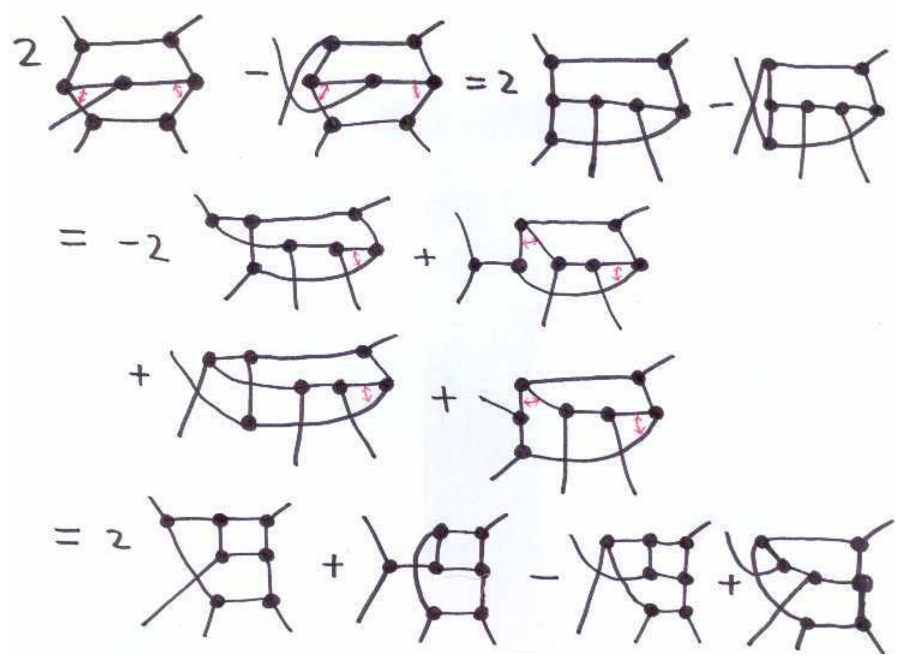
(B.12)

Thus (B.5) reads



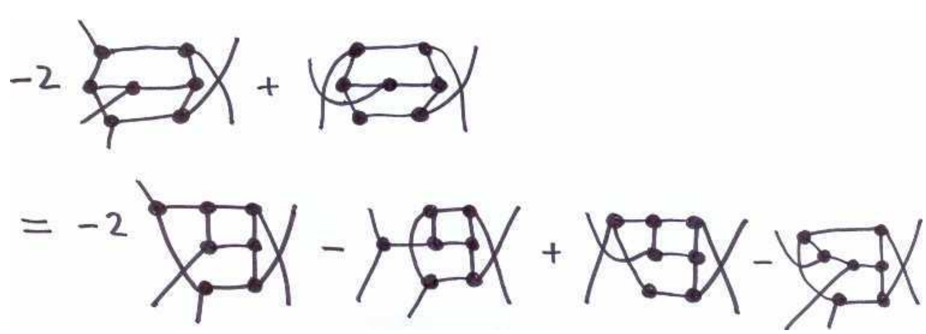
(B.13)

Now we apply (B.11) to the terms of the above expression



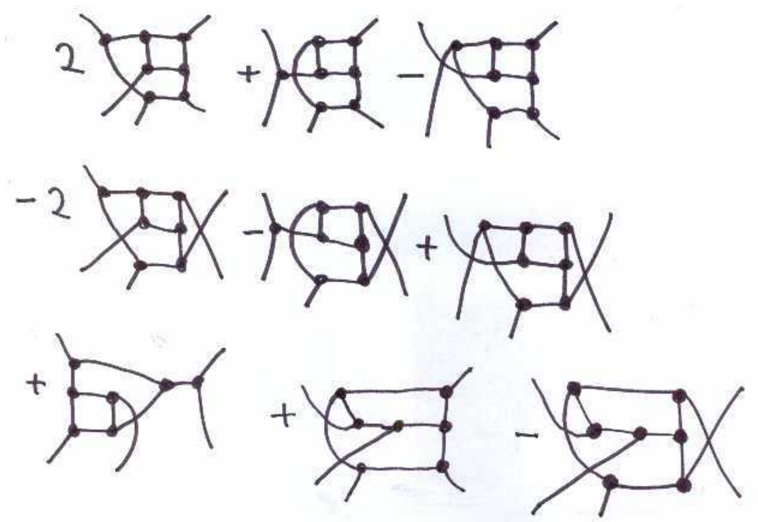
(B.14)

and similarly,



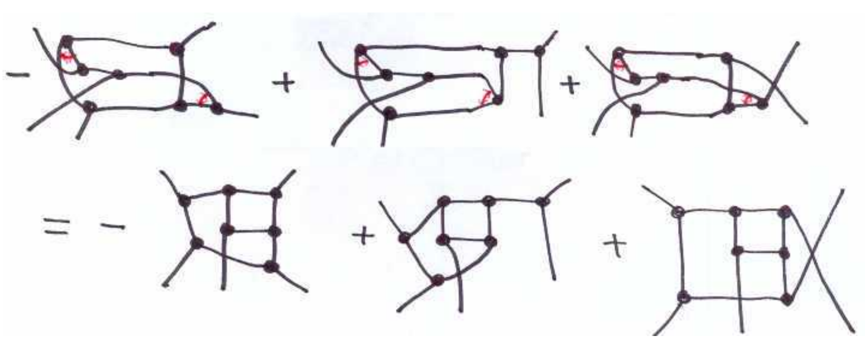
(B.15)

Equation (B.13) is then



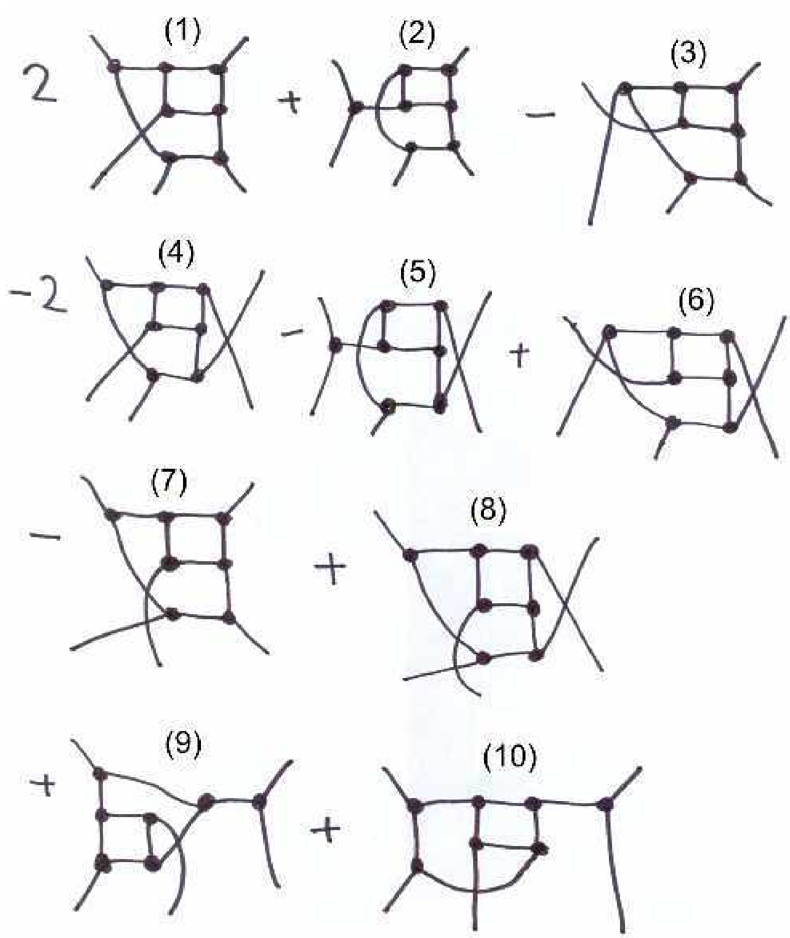
(B.16)

We use (B.10) on the last two terms:



(B.17)

Finally, from (B.16), we end up with the following expression for the RHS of (B.1) :

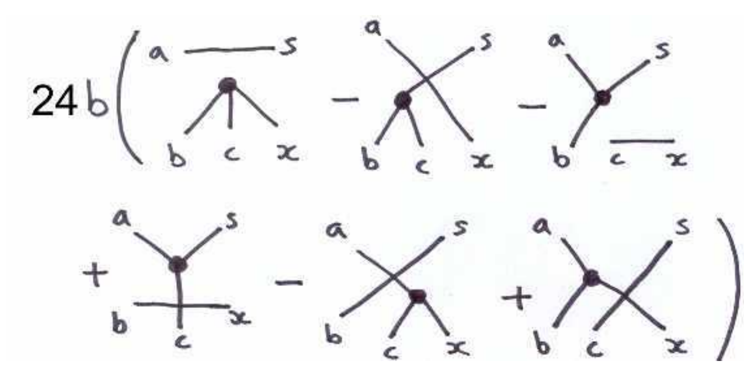


(B.18)

Our aim is to show that this equals

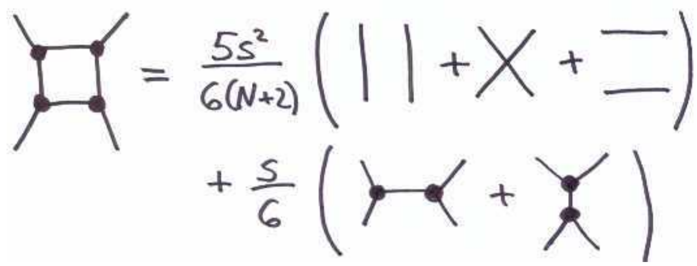
$$24b(c_{cxb}\delta_{as} + c_{cbs}\delta_{ax} - c_{abs}\delta_{cx} + c_{acs}\delta_{bx} + c_{axc}\delta_{bs} - c_{axb}\delta_{cs})$$

if and only if b takes a specific value, namely $b = \frac{-5s^3}{144(N+2)}$. The above expression in birdtrack form is (see (3.2)):



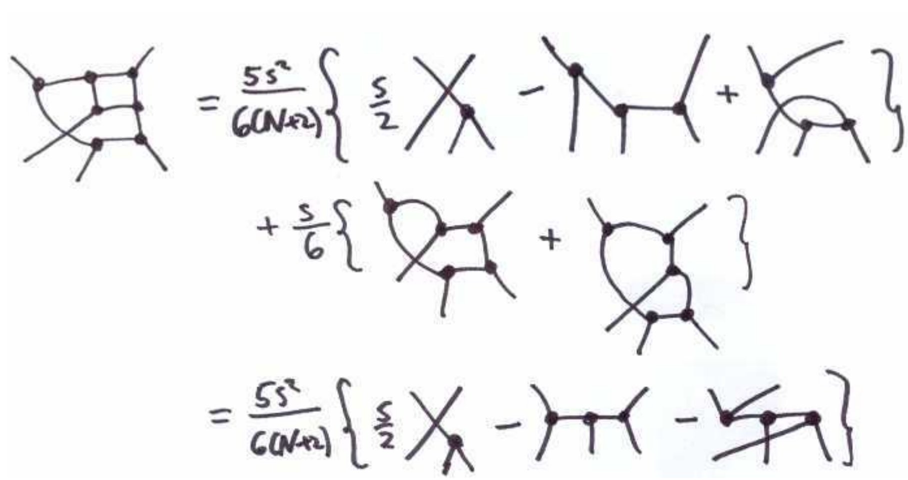
(B.19)

For the exceptional series of Lie algebras Cvitanović [**8**] gives the following rule for reducing a '4-loop':



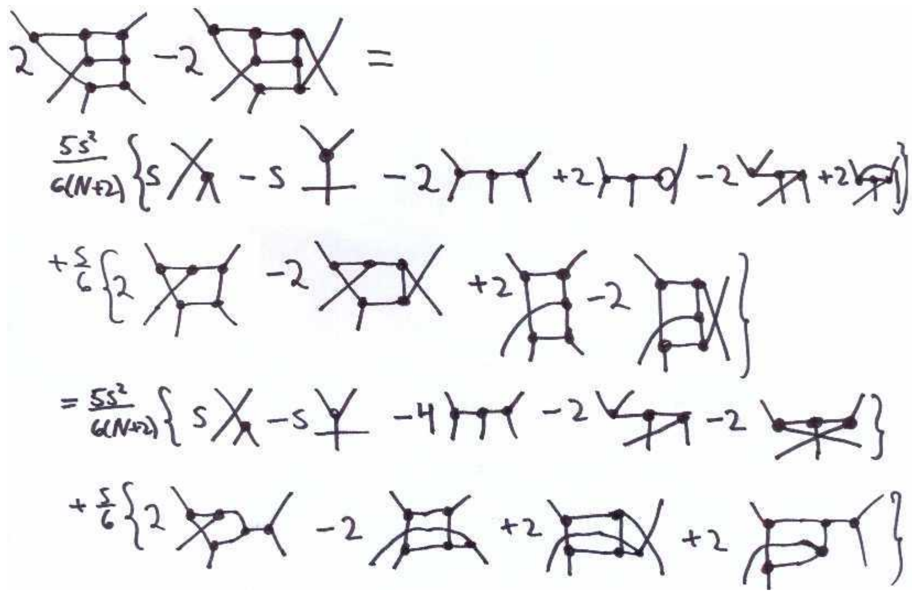
(B.20)

We use this rule on the terms in (B.18), starting with the figure labelled (1):



(B.21)

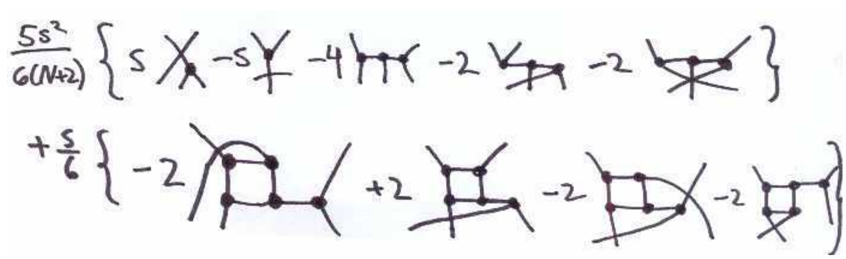
Noting that the figure labelled (4) in (B.18) is the same as figure (1) but with the two rightmost indicies swapped, we have that (1)+(4) is



(B.22)

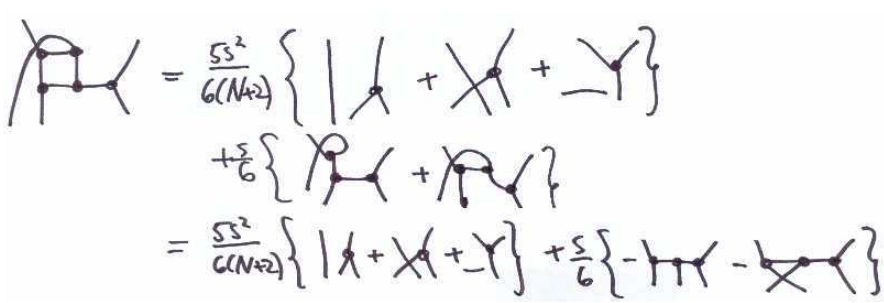
where we used the Jacobi identity in the last step.

Further, this equals

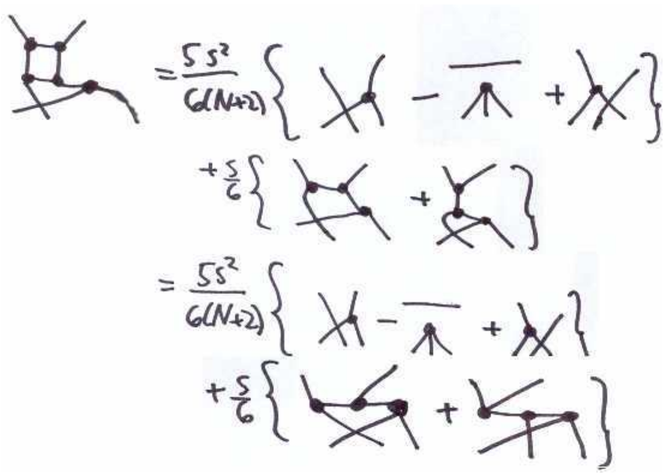


(B.23)

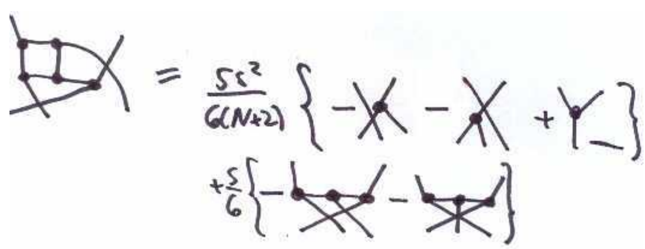
Now, the '4-loops' in (B.23) reduce as follows:



(B.24)

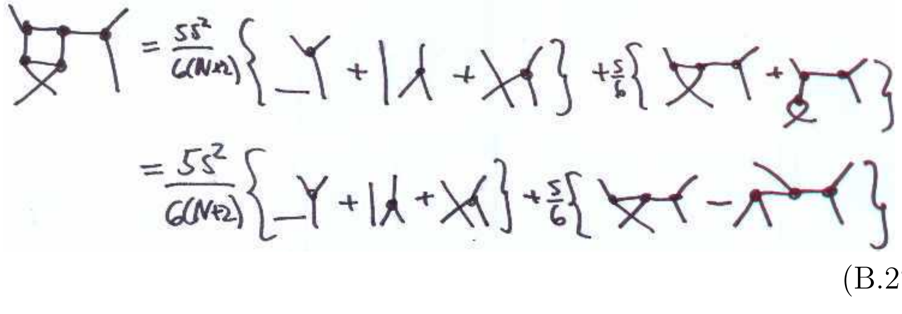


(B.25)



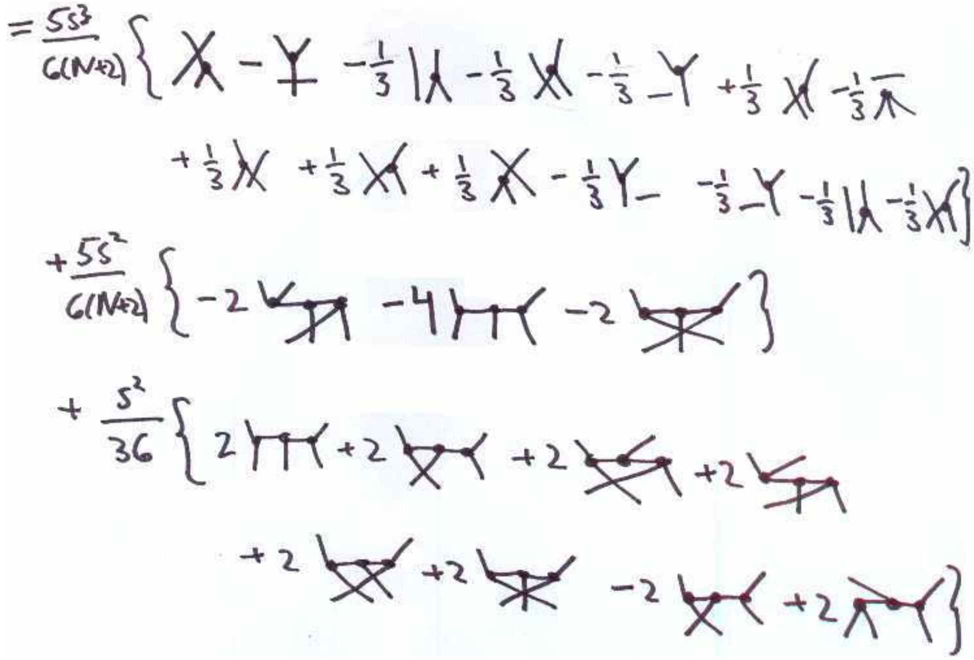
(B.26)

and

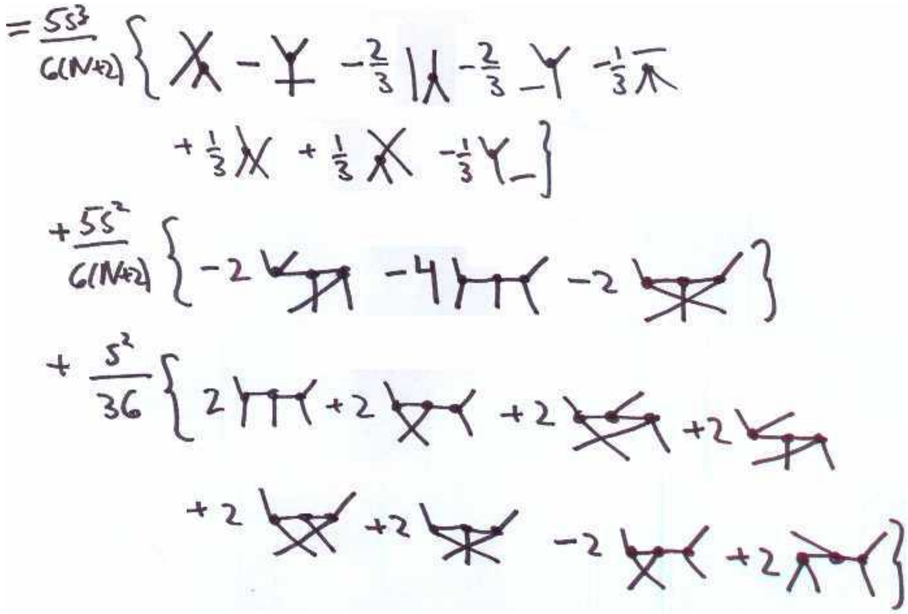


(B.27)

Hence (1)+(4) in (B.18) equals

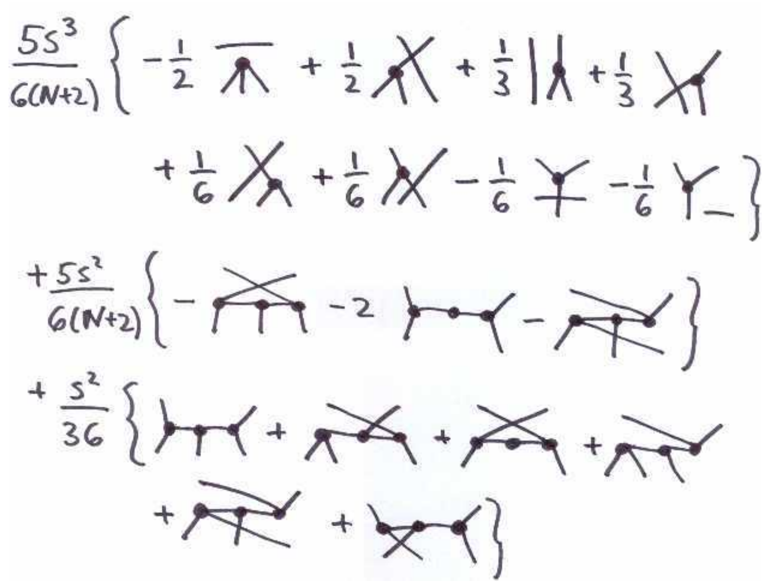


(B.28)



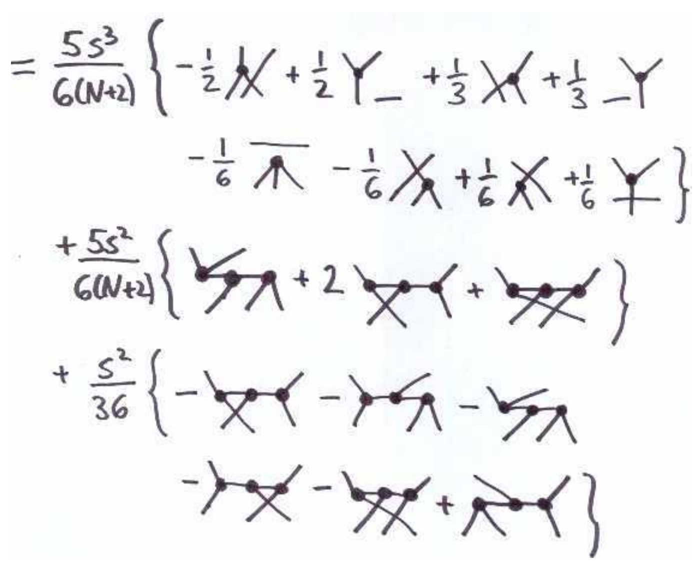
(B.29)

By noting that figure (3) in (B.18) is the same as (1) but with the left top and bottom indicies (or the 'a' and 'b' indicies) swapped, and similiarly for (6) and (4), we have that '(3)+(6)=-$\frac{1}{2}$((1)+(4))' with these indicies swapped. So (3)+(6) in (B.18) is:



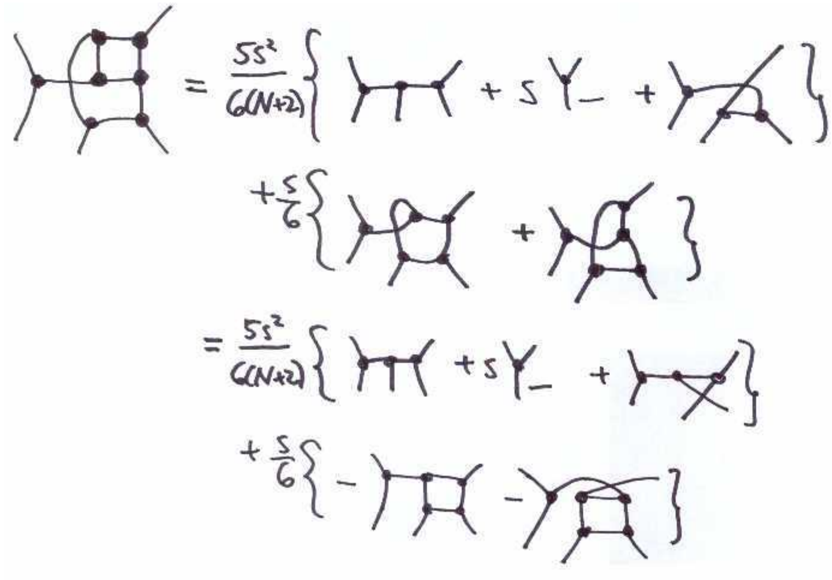
(B.30)

Next we note that figure (7) in (B.18) is the same as (1) but with the bottom left and bottom middle (or the 'b' and 'c' indicies) swapped, and similiarly for (8) and (4), we thus have that '(7)+(8)=-$\frac{1}{2}$((1)+(4))' with those indicies swapped. So (7)+(8) in (B.18) is:



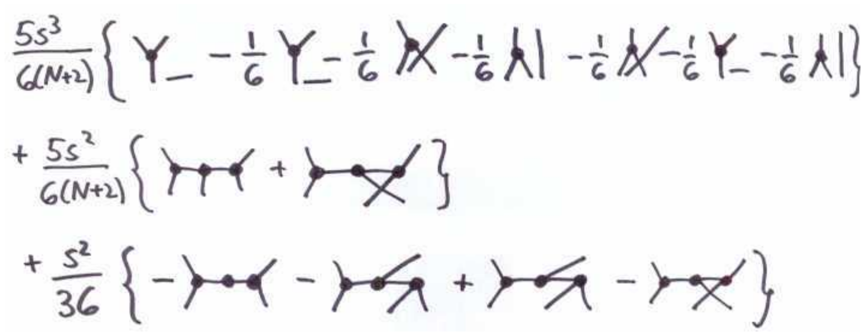
(B.31)

Figure (2) in (B.18) is



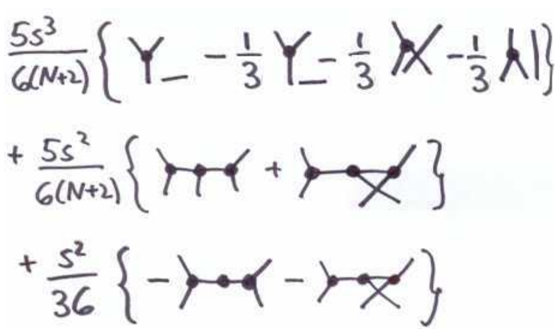
(B.32)

Using (B.20) this is



(B.33)

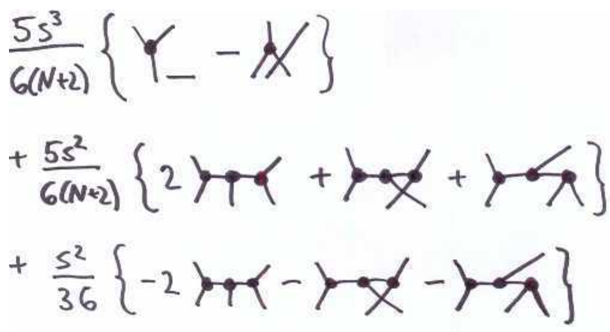
which equals



(B.34)

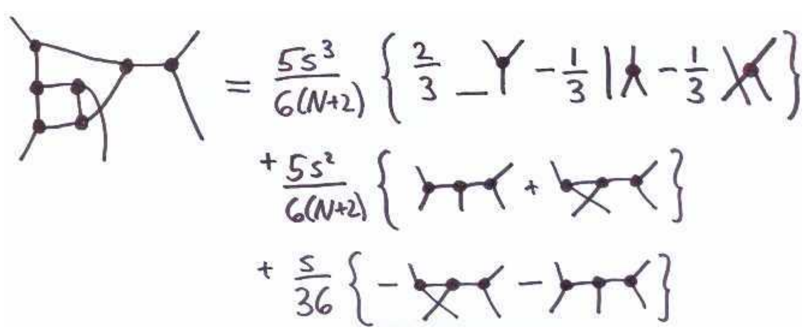
Noting that figure (5) is the same as figure (2), but with opposite sign and the top and bottom right indicies swapped, we have that

(2)+(5) is



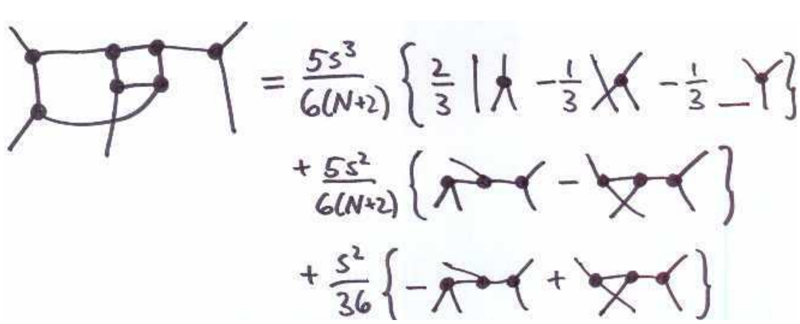
(B.35)

Using (B.20) we can show that figure (9) in (B.18) is:



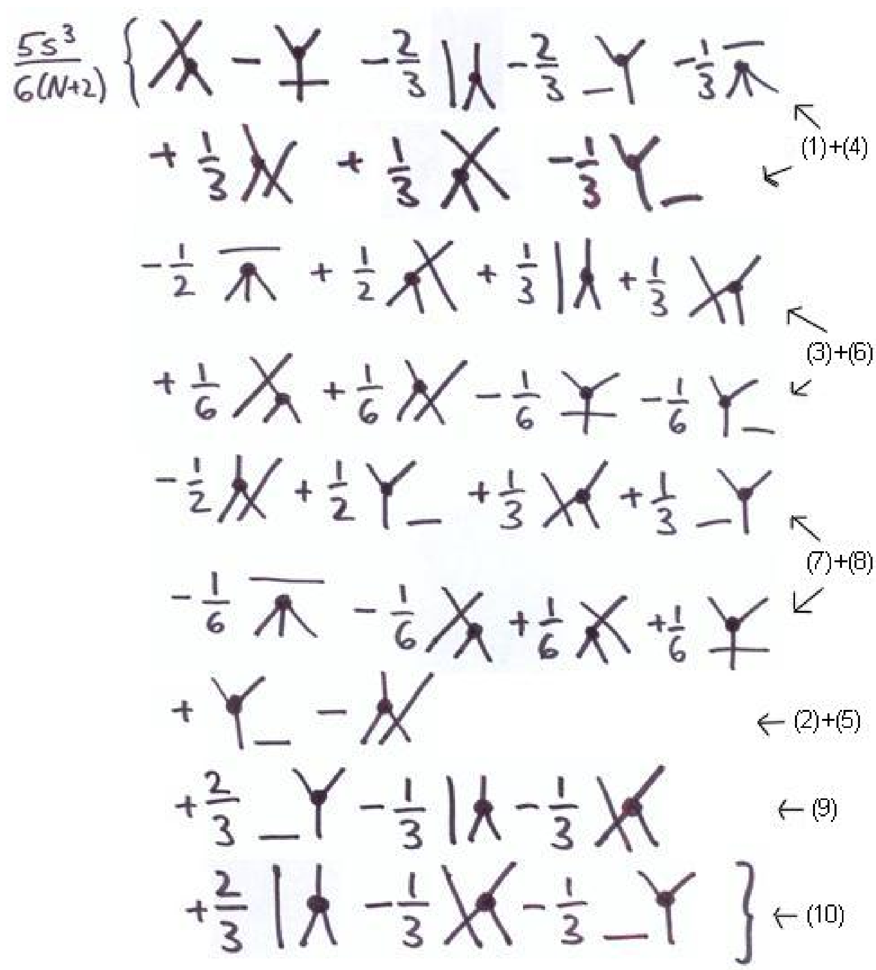
(B.36)

and (10) is:

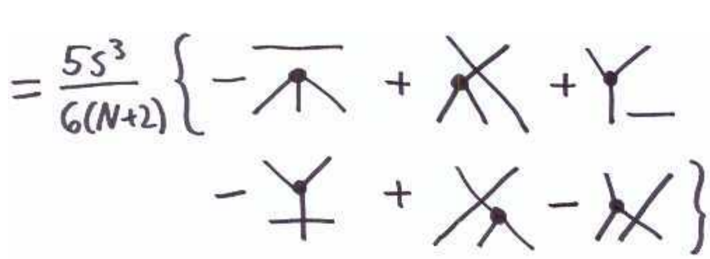


(B.37)

Now we add it all together. Starting with the terms of order s^3:

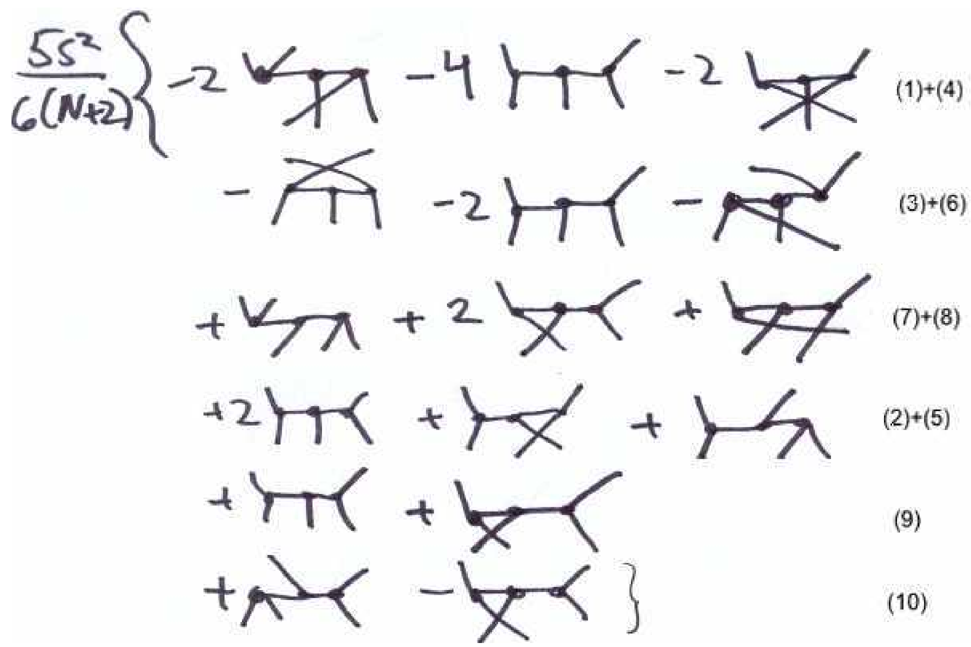


(B.38)



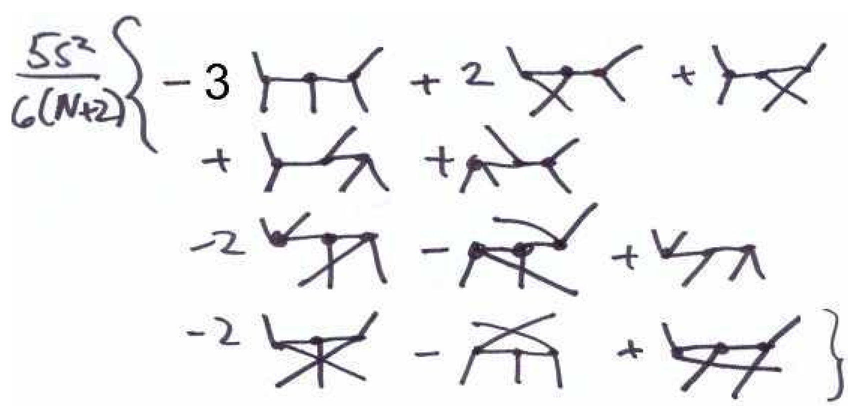
(B.39)

The terms with a factor of $\frac{5s^2}{6(N+2)}$ are



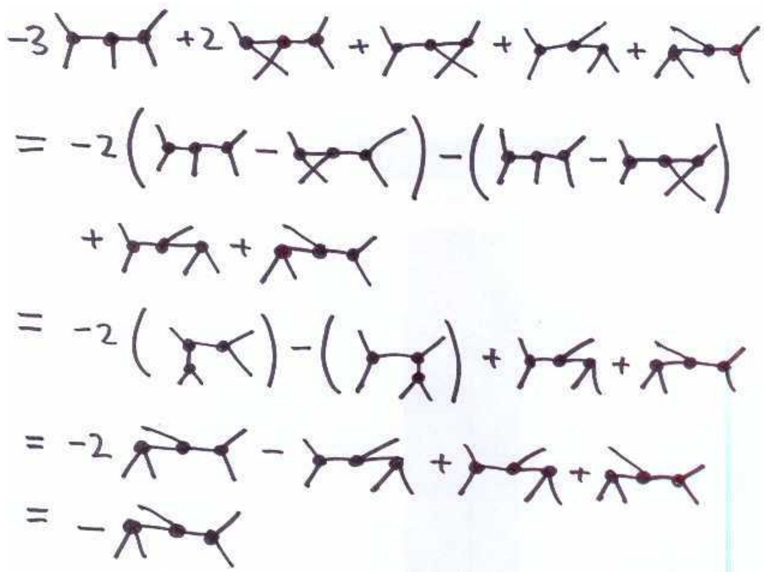
(B.40)

which is equal to



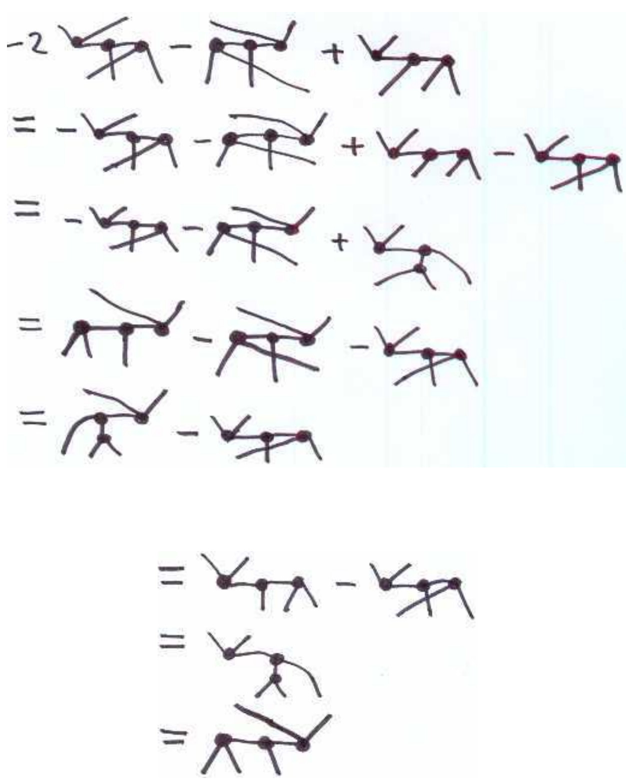
(B.41)

Now, using the Jacobi identity



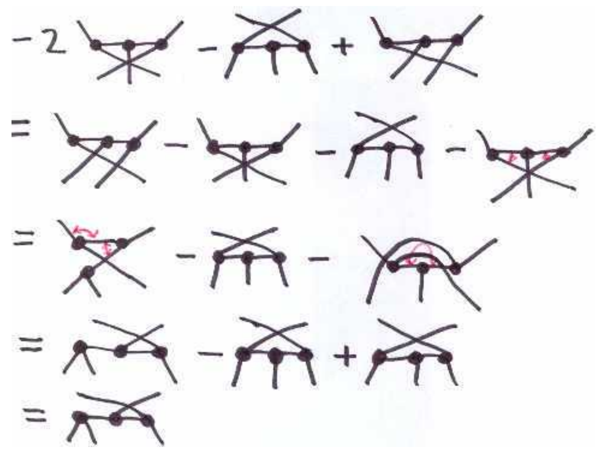
(B.42)

and



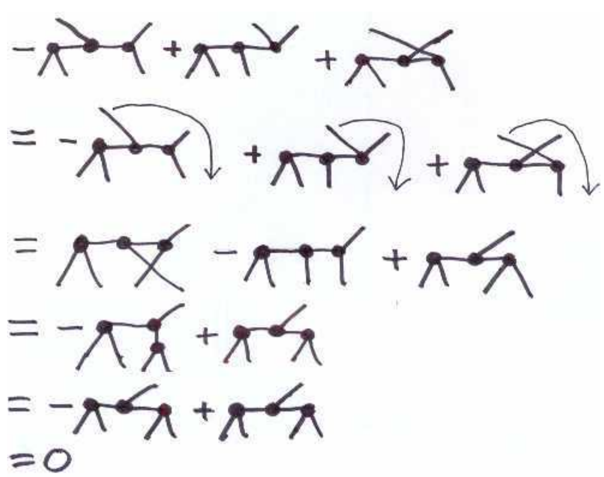
(B.43)

Further



(B.44)

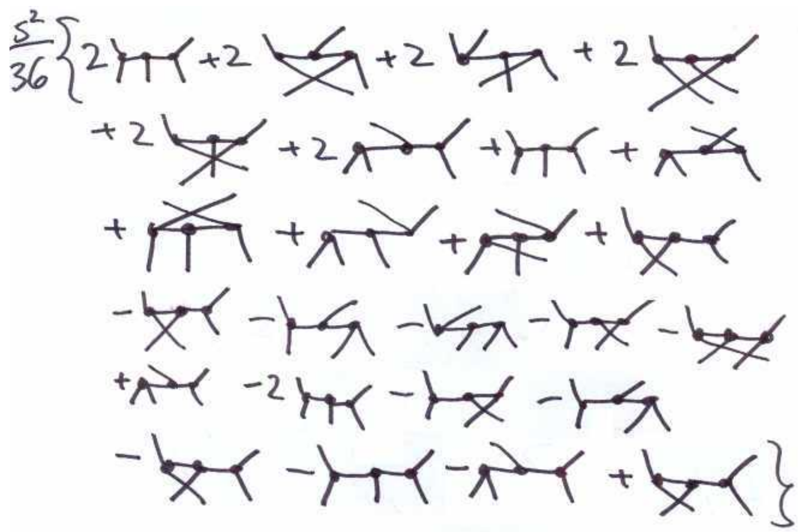
So (B.41) is equal to



(B.45)

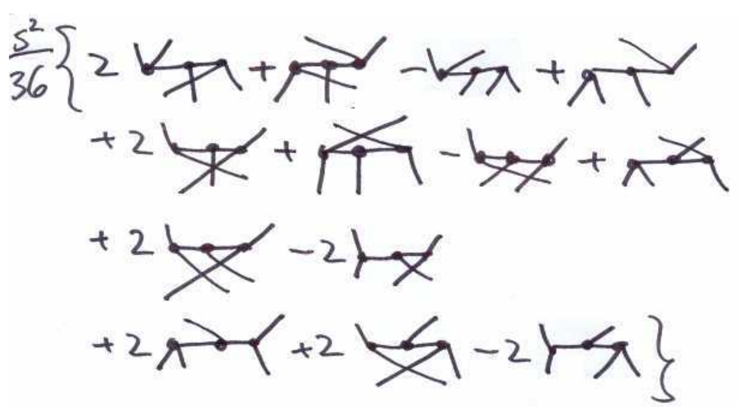
where we moved one of the free indicies to make the use of the Jacobi identity more clear. Hence the terms with a factor of $\frac{5s^2}{6(N+2)}$ cancel each other.

The terms with a factor of $\frac{s^2}{36}$ are



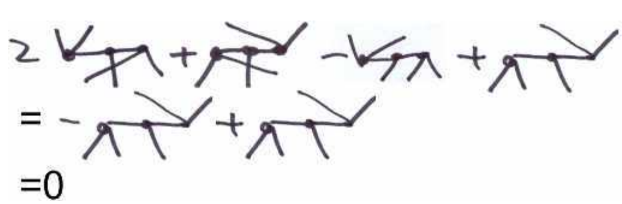
(B.46)

which equals



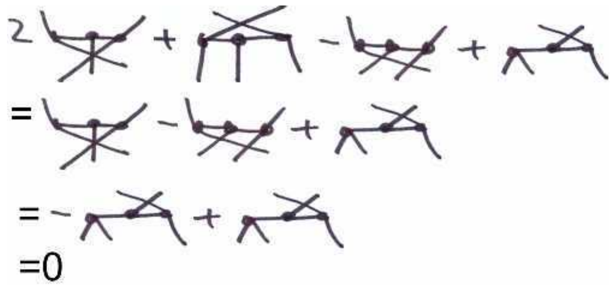
(B.47)

Now, similarly to (B.43)



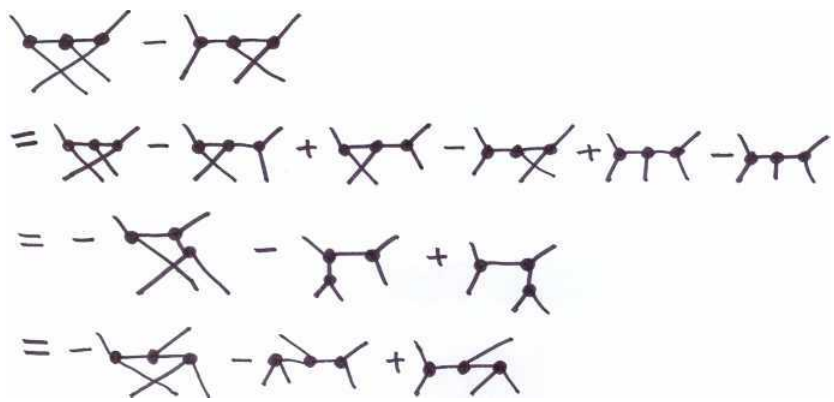
(B.48)

and similarly to (B.44)



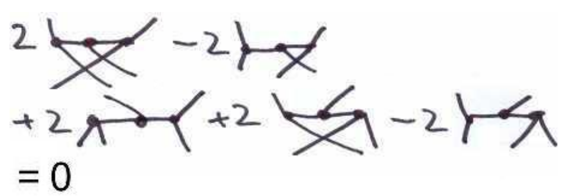
(B.49)

Now,



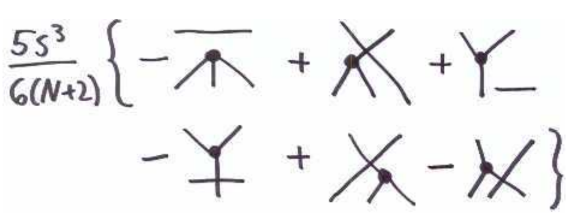
(B.50)

So,



(B.51)

Hence the terms with a factor of $\frac{s^2}{36}$ cancel as well.
We thus conclude that (B.18) equals



(B.52)

In classical notation, this means that (see (B.19))

$$\begin{aligned}24a_{abcdef}\{c_{fxq}c_{eqr}c_{drs}\} &= \\ &\frac{-5}{6(N+2)}(c_{cxb}\delta_{as} + c_{cbs}\delta_{ax} - c_{abs}\delta_{cx} \\ &\qquad + c_{acs}\delta_{bx} + c_{axc}\delta_{bs} - c_{axb}\delta_{cs})\end{aligned} \tag{B.53}$$

Thus

$$\begin{aligned}b(c_{cxb}\delta_{as} + c_{cbs}\delta_{ax} - c_{abs}\delta_{cx} + c_{acs}\delta_{bx} + c_{axc}\delta_{bs} - c_{axb}\delta_{cs}) \\ = a_{abcdef}\{c_{fxq}c_{eqr}c_{drs}\}\end{aligned}$$

if and only if

$$\begin{aligned}b &= \frac{-5}{24 \times 6(N+2)} \\ &= \frac{-5}{144(N+2)}\end{aligned} \tag{B.54}$$

We have thus shown that the representation ρ in Theorem 1.3 is consistent with (1.3) if and only if

$$b = \frac{-5}{144(N+2)} \tag{B.55}$$

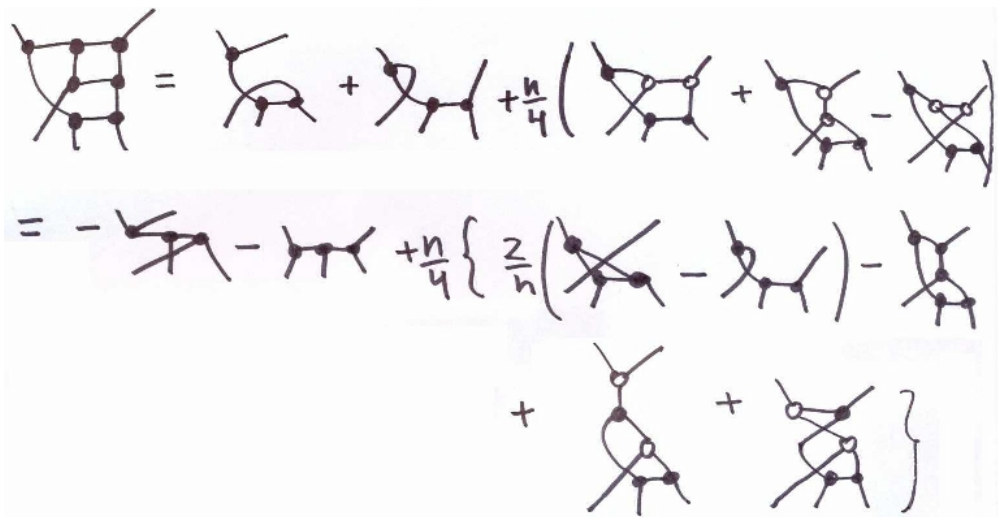
for the exceptional Lie algebras.

APPENDIX C

Full calculation for $\mathfrak{su}(n)$

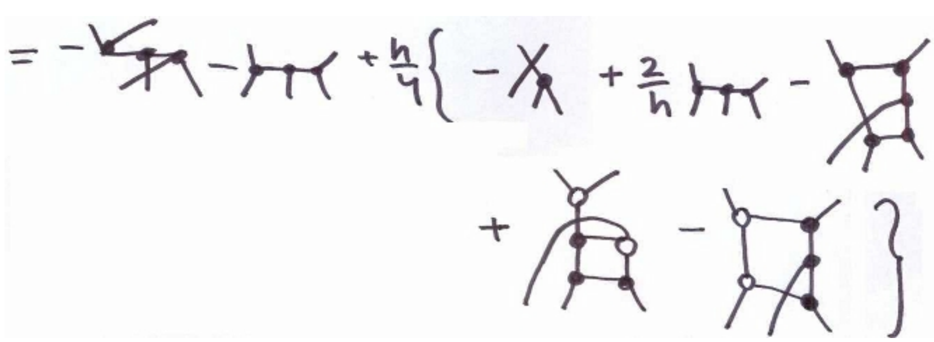
Here we give the details of how we calculate the expression (3.4) for $\mathfrak{g} = \mathfrak{su}(n)$. We make use of the identities (3.13) - (3.25).

Using (3.21) on figure (1) in (B.18) we have



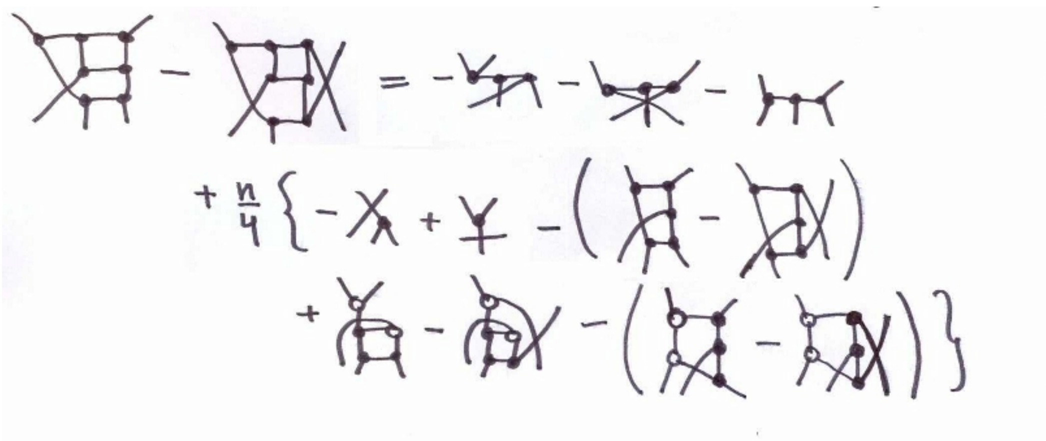
(C.1)

using (3.21) and (3.16). This simplifies to



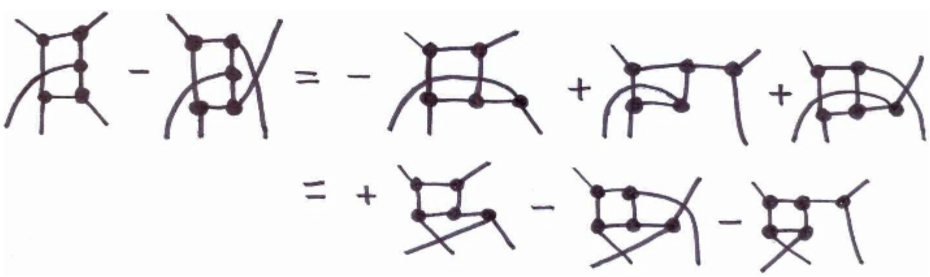
(C.2)

Thus



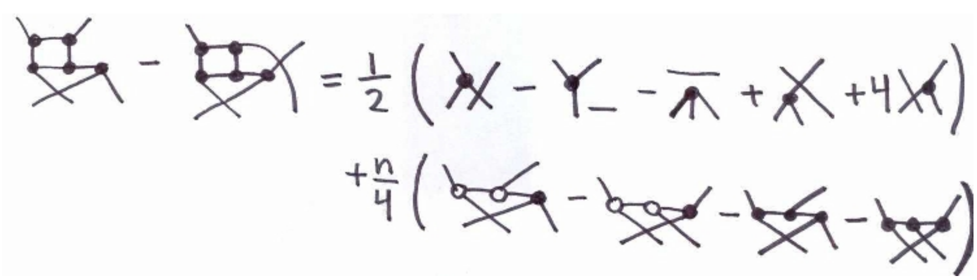
(C.3)

Using (B.10) we have



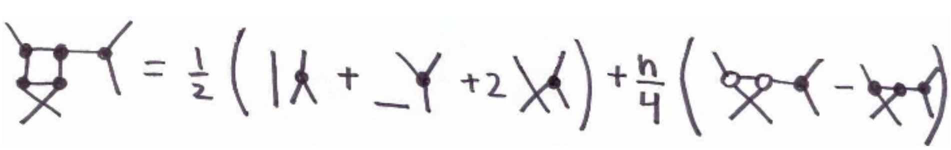
(C.4)

where



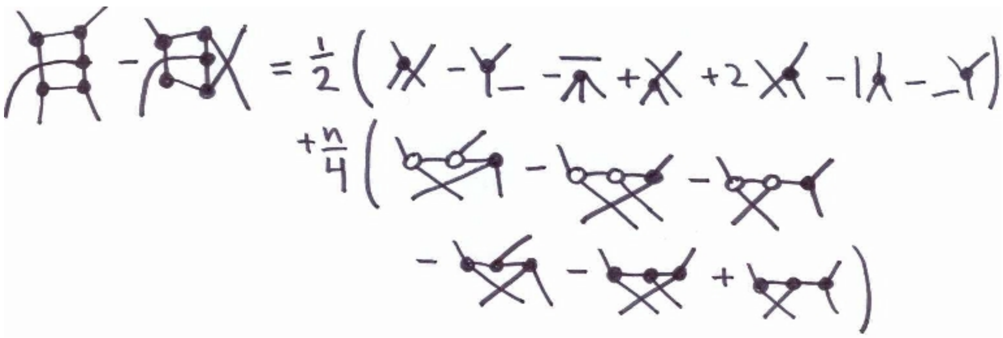
(C.5)

and



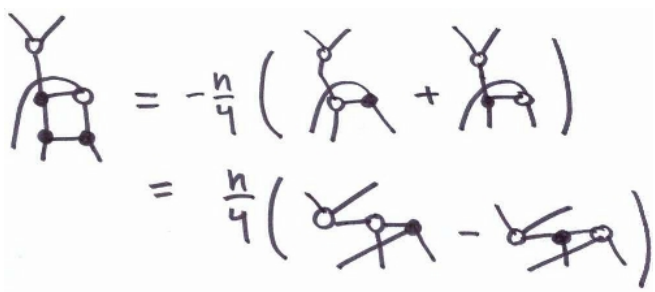
(C.6)

so



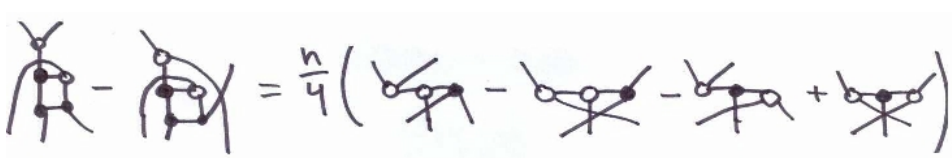
(C.7)

Further



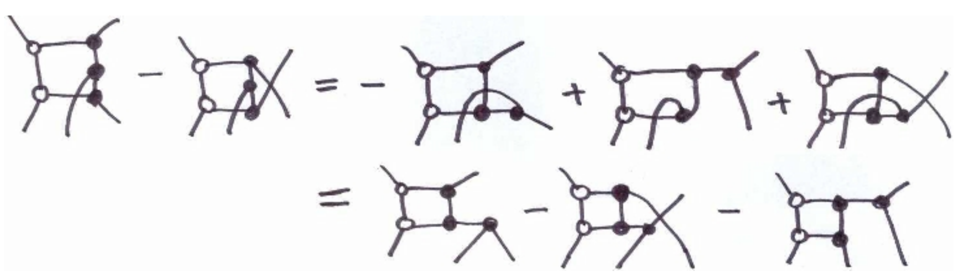
(C.8)

so

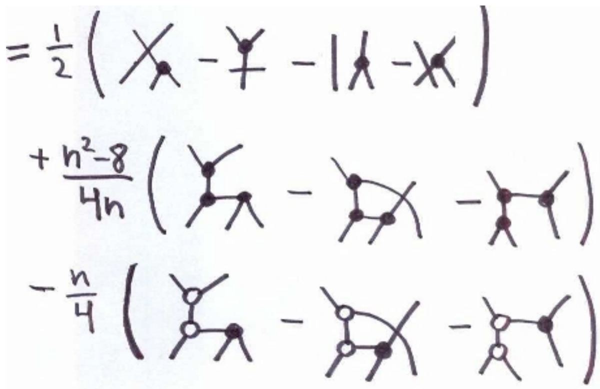


(C.9)

Using (B.10) we have

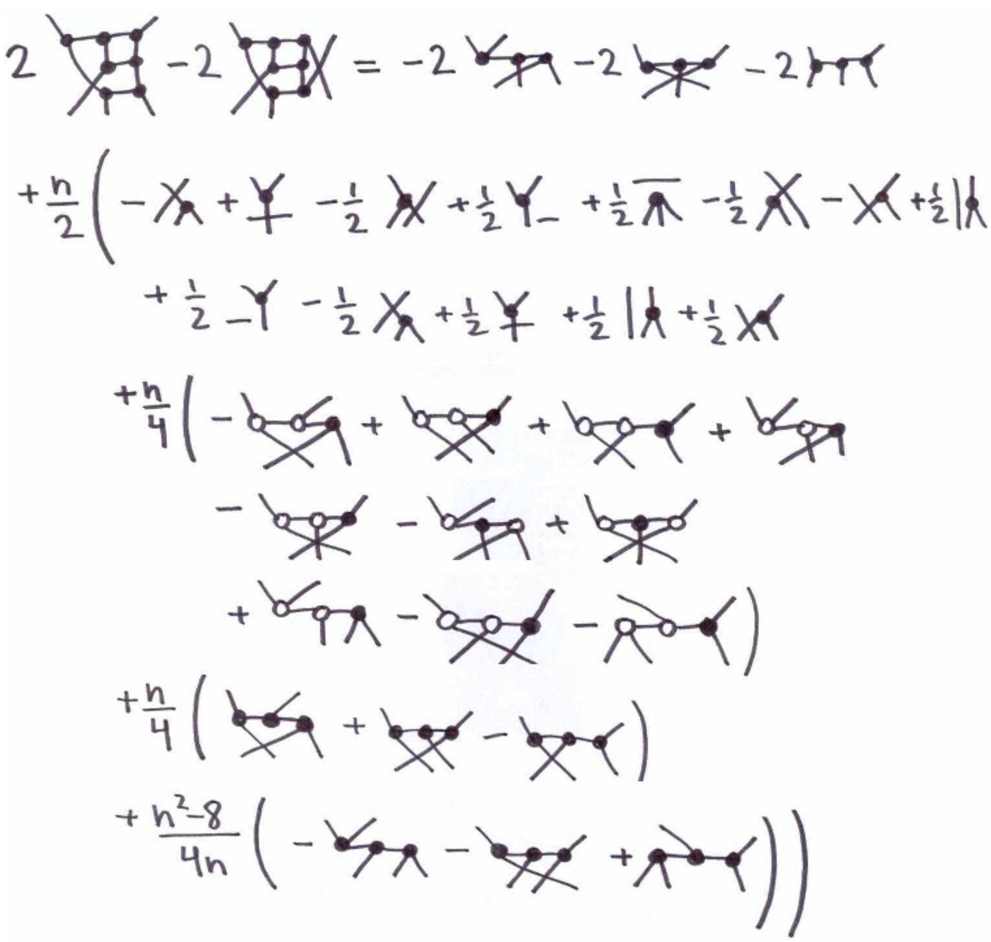


(C.10)



(C.11)

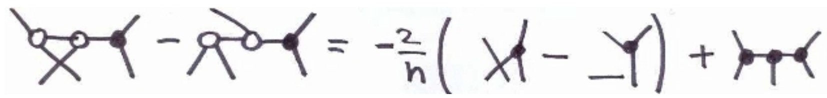
Using all this we get that



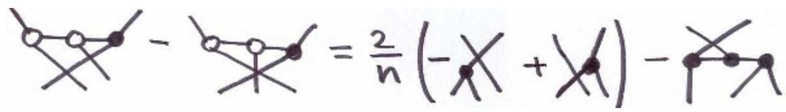
(C.12)

Now, using the Jacobi identity and (3.16) - (3.18) we have

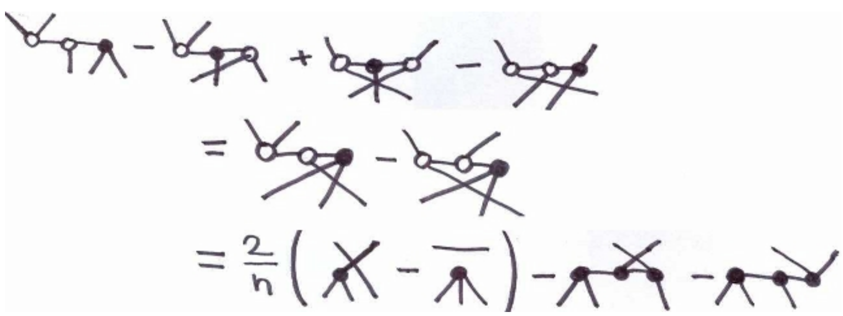
(C.13)



(C.14)



(C.15)

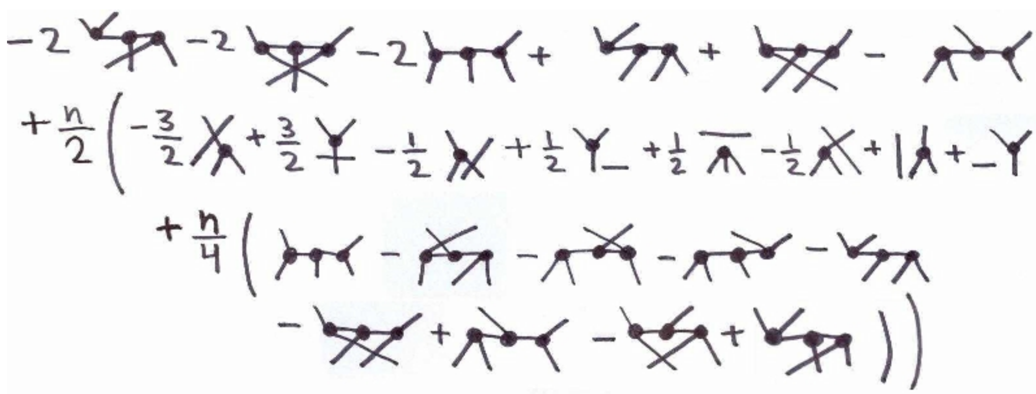


(C.16)

and

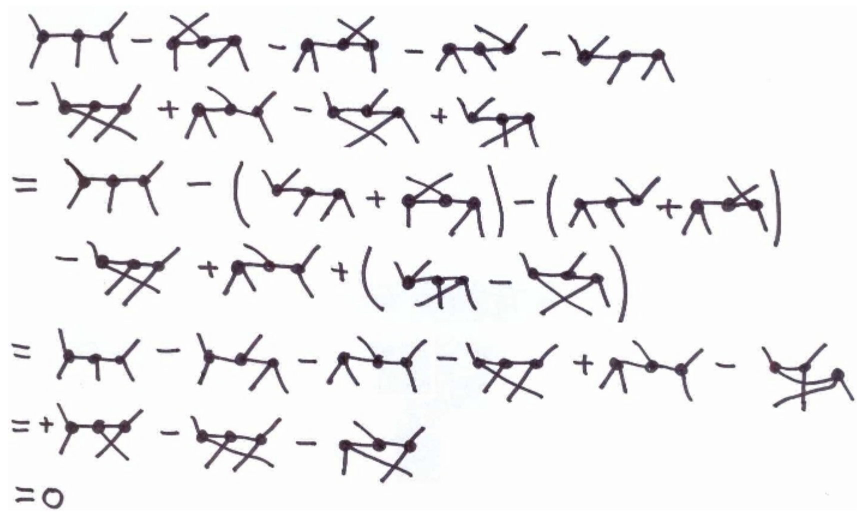
(C.17)

So, (C.12) is



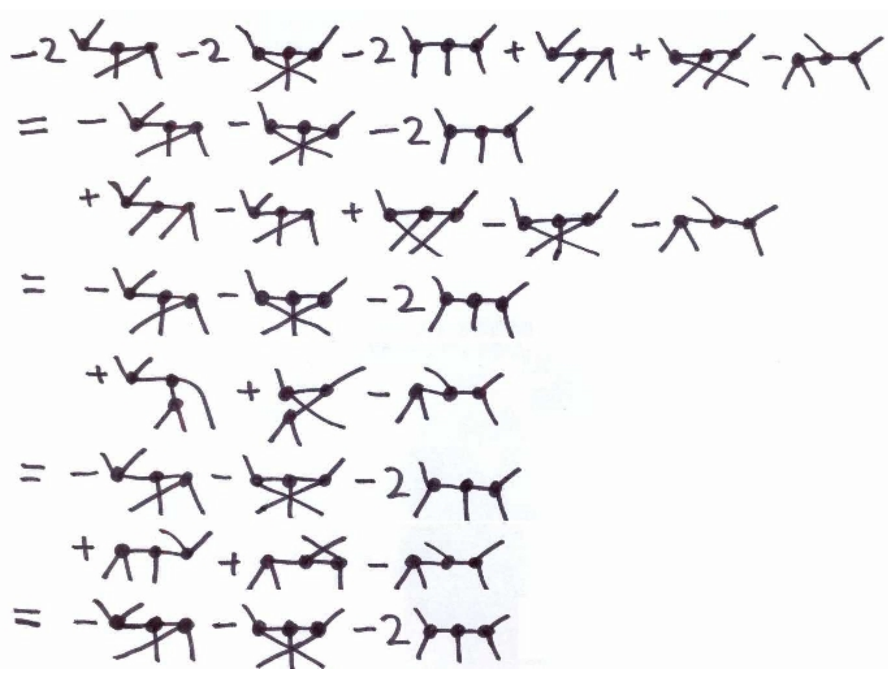
(C.18)

We can simplify this using the Jacobi identity. Firstly,



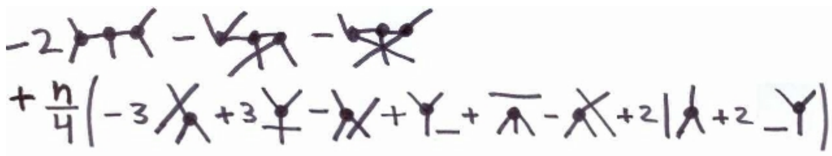
(C.19)

Secondly,



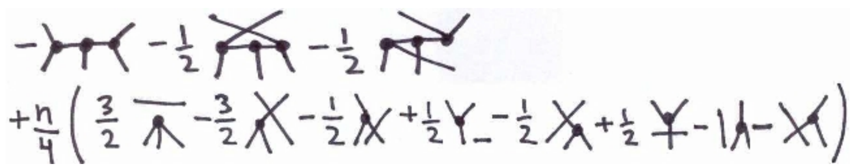
(C.20)

Thus (1)+(4) of (3.4) is:



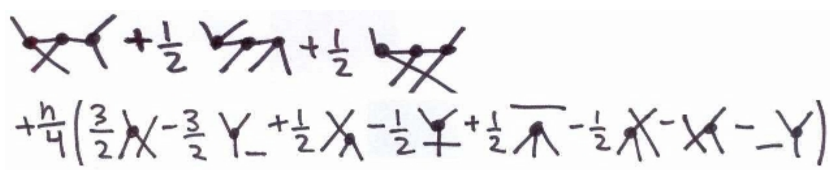
(C.21)

Since '(3)+(6)=-$\frac{1}{2}$((1)+(4))' with the left top and bottom indicies (or the 'a' and 'b' indicies) swapped, we have that (3)+(6) in (B.18) is:



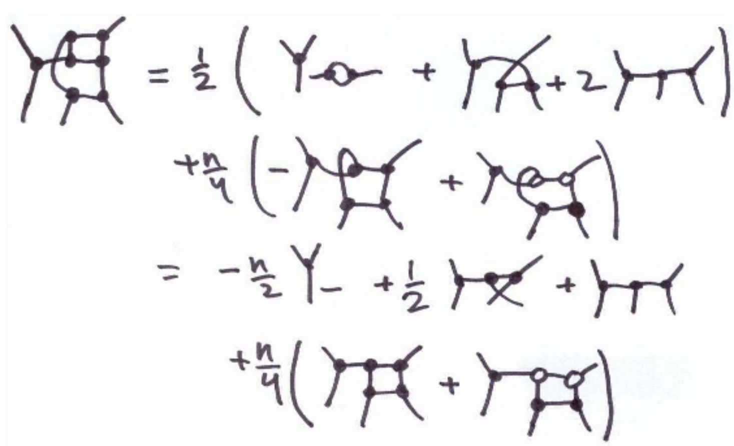
(C.22)

Next we note that '(7)+(8)=-$\frac{1}{2}$((1)+(4))' with the bottom left and bottom middle (or the 'b' and 'c' indicies) swapped, so (7)+(8) in (B.18) is:



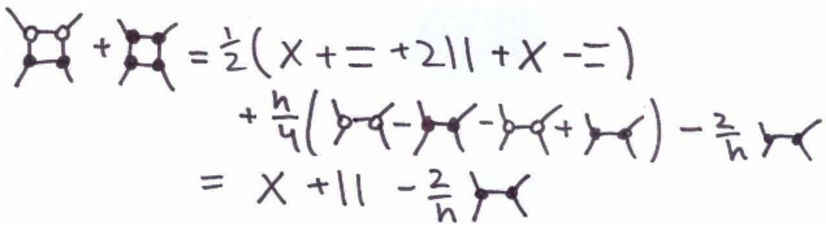
(C.23)

Figure (2) in (B.18) is:



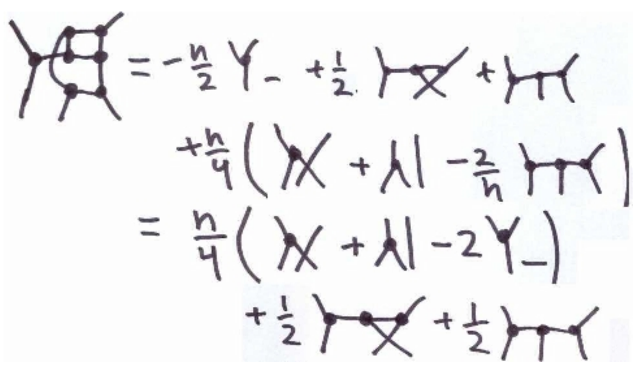
(C.24)

Now, using (3.24) and (3.25), we have



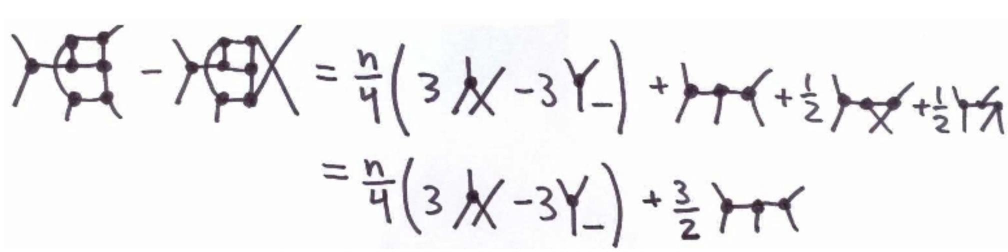
(C.25)

So



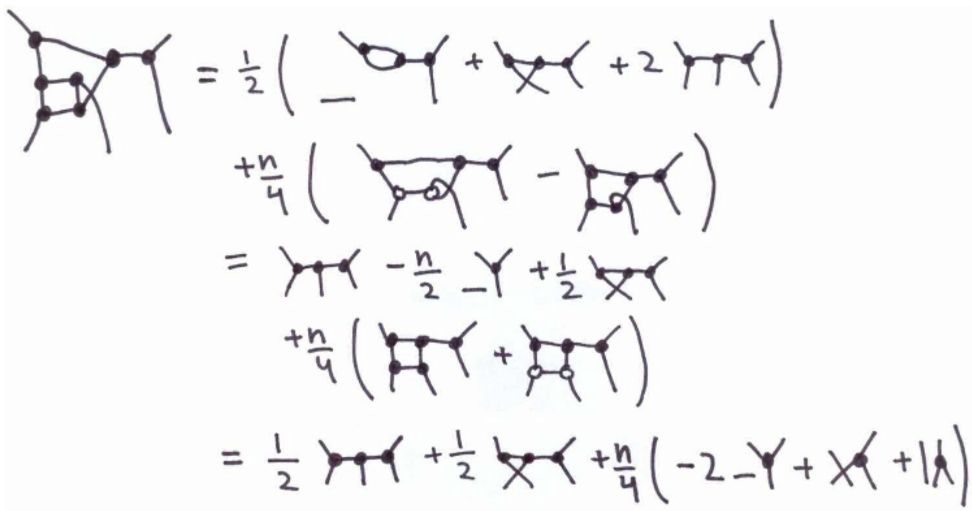
(C.26)

and hence (2)+(5) in (B.18) equals:



(C.27)

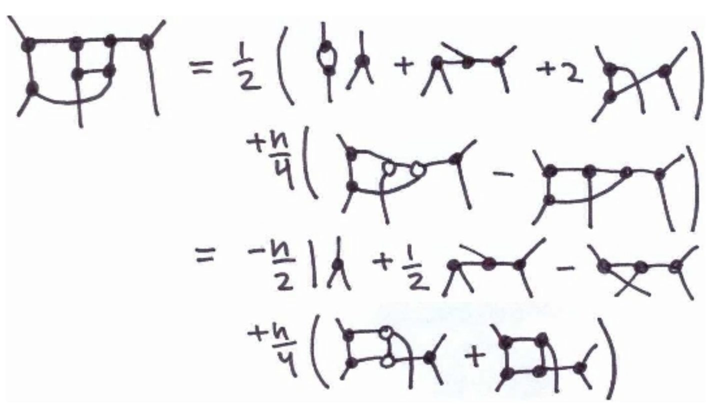
Figure (9) in (B.18) is:



(C.28)

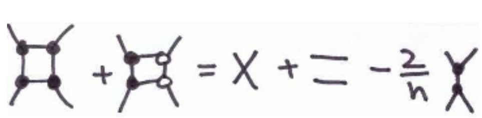
where we made use of (C.25).

Figure (10) in (B.18) is:



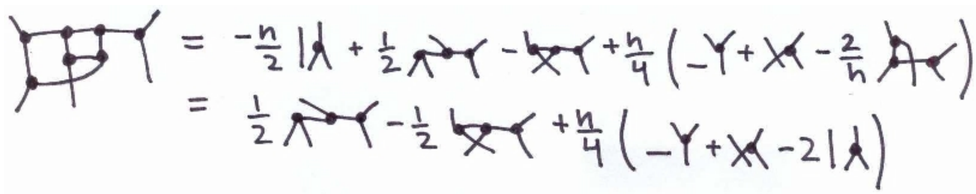
(C.29)

Rotating (C.25) we get



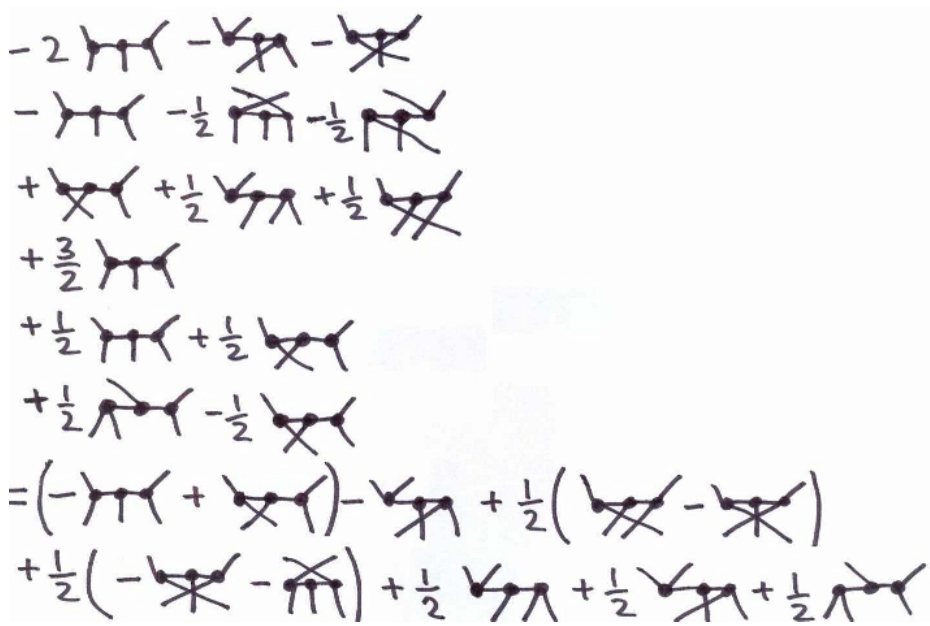
(C.30)

So figure (10) is



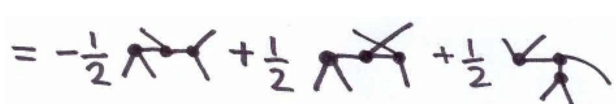
(C.31)

Now we add together the terms in (B.18). First we see that the terms with three structure constants cancel:



(C.32)

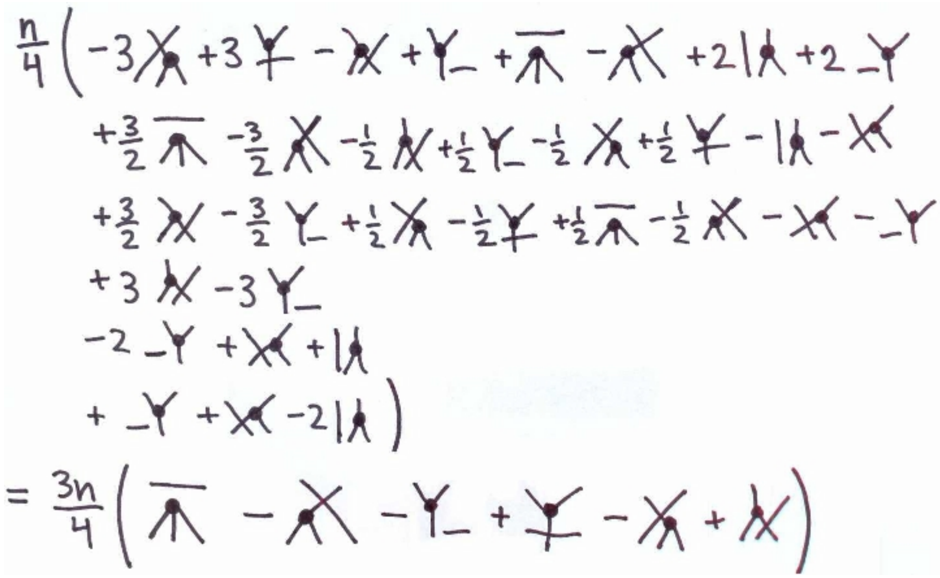
(C.33)



(C.34)

(C.35)

The terms with one structure constant are:



(C.36)

So (3.4) equals

(C.37)

Comparing this with (3.2) we conclude that the representation ρ of Theorem 1.3 is consistent with (1.3) if and only if

$$b = \frac{3n}{4 \times 24} = \frac{-(-n)^3}{32n^2} = \frac{-s^3}{32n^2} \tag{C.38}$$

since we have used the normalisation $s = -n$ in our calculations.

Thus we have proved the result for $\mathfrak{su}(n)$.

APPENDIX D

Numerical calculation for $\mathfrak{su}(2)$

In this appendix we prove the statement (3.67) in the proof of Lemma 3.2. That is

$$\rho((a_{abcdef}c_{rsc} + a_{rscdef}c_{abc})\{I_d, I_e, I_f\})I_x = 0$$

By (3.66),

$$\begin{aligned}\rho((a_{abcdef}c_{rsc} + a_{rscdef}c_{abc})\{I_d, I_e, I_f\})I_x \\ = (a_{abcdef}c_{rsc} + a_{rscdef}c_{abc})\{[\ [\ [I_x, I_f], I_e], I_d]\}\end{aligned} \tag{D.1}$$

In order to evaluate the RHS of this expression we use the computer software Matlab. First we choose three anti-hermitian matrices generating $\mathfrak{su}(2)$,

$$I_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{D.2}$$

$$I_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{D.3}$$

$$I_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{D.4}$$

These matrices are orthonormal with respect to the inner product

$$(A, B) = \tfrac{1}{2}\mathrm{tr}(AB^\dagger) \tag{D.5}$$

This inner product is associative because of the cyclicity of the trace and the anti-hermiticity of the chosen basis,

$$\begin{aligned}
(I_a, [I_b, I_c]) &= \tfrac{1}{2}tr\{I_a(I_bI_c)^\dagger - I_a(I_cI_b)^\dagger\} && \text{(D.6)}\\
&= \tfrac{1}{2}tr\{I_aI_c^\dagger I_b^\dagger - I_aI_b^\dagger I_c^\dagger\} && \text{(D.7)}\\
&= \tfrac{1}{2}tr\{-I_aI_c^\dagger I_b + I_aI_bI_c^\dagger\} && \text{(D.8)}\\
&= \tfrac{1}{2}tr\{-I_bI_aI_c^\dagger + I_aI_bI_c^\dagger\} && \text{(D.9)}\\
&= \tfrac{1}{2}tr\{[I_a, I_b]I_c^\dagger && \text{(D.10)}\\
&= ([I_a, I_b], I_c) && \text{(D.11)}
\end{aligned}$$

Hence this inner product satisfies the requirements of Theorem 1.3.

We then use the code below to evaluate the RHS of (D.1). The functions below should be saved as M-files with file names '`function.m`'. The other segments of code should be executed in the command window, in order of appearance.

```
global n
global dim

n=2;
dim=n*n-1; %=3

%---------------I.m-------------------------------
function [ Ik ] = I( k )
%Takes input k, returns matrix Ik, for k=1,2,3


global n

Ik=zeros(n);

if k==1
    Ik=[1i 0; 0 -1i];
end

if k==2
    Ik=[0 1i ; 1i 0];
end

if k==3
    Ik=[0 1; -1 0];
end

end
%------------------inner.m------------------------

function [ in ] = inner( A,B )
%INNER calculates the inner product of A and B

global n

in=trace(A*B')/2;

%B' means B hermitian transpose
end
%-----------------com.m----------------------------
function [ com ] = com( X, Y )
%COM returns [X,Y]=X*Y-Y*X

 com=X*Y-Y*X;

end
%-----------------------------------------------------
```

```
%creates the structure constants and places them in a 3x3x3 array
%the values of the constants are then obtained by writing e.g. c(1,2,3)
global c
global dim
global n


c= cat( 3, zeros(1,dim), zeros(1,dim), zeros(1,dim));

for i=1:dim
    for j=1:dim

M=com(I(i),I(j));

       for k=1:dim

           c(i,j,k)=inner(M,I(k));

       end
    end
end
%----------------a.m--------------------------------

function [ A ] = a( la, mu, nu, alf, bet, gam )
%Calculates the a's as defined by Drinfel'd

global c
global n
global dim

A=0;
for i=1:dim
    for j=1:dim
        for k=1:dim
            A=A+c(la,alf,i)*c(mu,bet,j)*c(nu,gam,k)*c(i,j,k);
        end
    end
end

A=A/24;

end
%------------------rhs.m----------------------------

function [ Sr ] = rhs( la, mu, r, s, x )
%RHS evaluates RHS for the given values of the free indicies
%

global n
global dim
global c

S=zeros(n,n);
```

```
for alf=1:dim
    for bet=1:dim
        for gam=1:dim
            for nu=1:dim


S=S+(a(la,mu,nu, alf, bet, gam)*c(r,s,nu) + a(r,s,nu, alf, bet, gam)
   *c(la,mu,nu) )*(com(I(alf), com(I(bet), com(I(gam),I(x))))
   + com(I(bet), com(I(gam), com(I(alf),I(x))))
   + com(I(gam), com(I(alf), com(I(bet),I(x))))
   + com(I(alf), com(I(gam), com(I(bet),I(x))))
   + com(I(gam), com(I(bet), com(I(alf),I(x))))
   + com(I(bet), com(I(alf), com(I(gam),I(x)))));

   %The above expression "S=S+ ..." should be written on ONE line

            end
        end
    end
end

Sr=S;
%-----------------------------------------------------

%tests if rhs=0 for all i,j,k,l,m
%The running time for this is about 2 minutes
global n
global dim

err=[]; %to store error values

for i=1:dim
    for j=1:dim
        for k=1:dim
            for l=1:dim
                for m=1:dim

                if isequal(rhs(i,j,k,l,m),zeros(n,n))

                else
                    err= [err , i, j,  k, l, m,0];
                    %if rhs is not zero, store the values of i,j,k,l,m
                end
                end
            end
         end
    end
end

err %display errors. If empty, rhs=0 for all i,j,k,l,m

%-----------------------------------------------------
```

We get the output

```
err = []
```

which means that the RHS of (D.1) is zero, for all $a, b, r, s, x = 1, 2, 3$.

Hence

$$\rho((a_{abcdef}c_{rsc} + a_{rscdef}c_{abc})\{I_d, I_e, I_f\})I_x = 0.$$

APPENDIX E

Numerical calculation for $\mathfrak{so}(5)$

Using Matlab we evaluate both sides of the equation:

$$\begin{aligned} b(c_{cxb}\delta_{as} + c_{cbs}\delta_{ax} - c_{abs}\delta_{cx} + c_{acs}\delta_{bx} + c_{axc}\delta_{bs} - c_{axb}\delta_{cs}) \\ = a_{abcdef}\{c_{fxq}c_{eqr}c_{drs}\} \end{aligned} \tag{E.1}$$

which gives the same condition on b as (1.49).
First we chose an anti-hermitian basis of $\mathfrak{so}(5)$:

$$I_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{E.2}$$

$$I_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{E.3}$$

$$\vdots \tag{E.4}$$

$$I_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{E.5}$$

These matrices are orthonormal with respect to the associative[1] inner product

$$(A, B) = \tfrac{1}{2}\mathrm{tr}(AB^\dagger) \tag{E.6}$$

We now use the code below. The functions below should be saved as M-files with file names '`function.m`'. The other segments of code should be executed in the command window, in order of appearance.

[1]See Appendix D for a proof of associativity

```
%--------------I.m----------------------------
function [ I ] = I( n )
%returns basis element of so(5)

I=zeros(5,5);

if n >= 1 && n <=4
    I(1,n+1)=1;
end

if n>=5 && n<=7
    I(2,n-2)=1;
end

if n==8
    I(3,4)=1;
end

if n==9
    I(3,5)=1;
end

if n==10
    I(4,5)=1;
end

I=I-I';
%I' means the transpose of I
end

%---------------inner.m------------------

function [ in ] = inner( A, B )
%calculates the inner product of A and B

% so5
%
% A=I(a);
% B=I(b);
in=trace(A*B')/2;

end

%-----------------com.m------------------------

function [ com ] = com( X, Y )
%Calculates the commutator of X and Y

com=X*Y-Y*X;

end
```

```
%---------------------------------------------

%creates the structure constants

global c

c= cat( 3, zeros(1,10), zeros(1,10), zeros(1,10));

for i=1:10
    for j=1:10

M=com(I(i),I(j));

       for k=1:10

           c(i,j,k)=inner(M,I(k));

       end

    end
end

%-------------------a.m-------------------------

function [ A ] = a( lam, mu, nu, alf, bet, gam )
%calculates the a's in D3

global c

A=0;
for i=1:10
    for j=1:10
        for k=1:10
            A=A+c(lam,alf,i)*c(mu,bet,j)*c(nu,gam,k)*c(i,j,k);
        end
    end
end

A=A/24;

end

%------------kill.m-----------------------

function [ kill ] = kill( a,b )
%calculates the killing form of I(a) and I(b)

global c

kill=0;
for i=1:10
    for j=1:10
```

```
    kill=kill+ c(a,j,i)*c(b,i,j);
    end
end

end

%-----------testkill.m-----------------------

%calculates the Killing form between all the basis elements and
%places the result in a matrix

K=zeros(10,10);

for a=1:10
    for b=1:10
        K(a,b)=kill(a,b);
    end
end

K

%--------------------lhs.m-------------------

function [ Sl ] = lhs( la, mu, nu, x)
%evlauates LHS of D3

global c

S=zeros(5,5);

for si=1:10

    S= S + (c(nu, x, mu)*delta(la,si) + c(nu,mu,si)*delta(la,x) -
     c(la,mu,si)*delta(nu,x) + c(la,nu,si)*delta(mu,x)
     + c(la,x,nu)*delta(mu,si) - c(la,x,mu)*delta(nu,si))*I(si);

            %NOTE: the above expression 'S=S+...' should be written on
            the same line

end

Sl=S;

end

%---------rhs.m-----------------------------------

function [ Sr ] = rhs( la, mu, nu, x )
%Evaluates rhs of D3

S=zeros(5,5);

for alf=1:10
```

```
    for bet=1:10
        for gam=1:10

S=S+a(la,mu,nu, alf, bet, gam)*(com(I(alf), com(I(bet), com(I(gam),I(x))))
 + com(I(bet), com(I(gam), com(I(alf),I(x)))) + com(I(gam), com(I(alf),
 com(I(bet),I(x)))) + com(I(alf), com(I(gam), com(I(bet),I(x))))+
 com(I(gam), com(I(bet), com(I(alf),I(x))))+ com(I(bet), com(I(alf),
 com(I(gam),I(x)))));

            %NOTE: the above expression 'S=S+...' should be written on
            the same line

        end
    end
end

Sr=S;

end

%-------------------------------------------
```

We can now write for example

```
lhs(1,2,3,4)
rhs(1,2,3,4)
```

to find that both the LHS and RHS of (E.1) is zero when $a = 1$, $b = 2$, $c = 3$ and $x = 4$.

If we instead calculate

```
lhs(1,2,3,5)
rhs(1,2,3,5)
```

we get the outputs

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{E.7}$$

and

$$\begin{pmatrix} 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{E.8}$$

respectively. By (E.1) we thus have the neccessary condition $b = \frac{1}{2}$.

The value of b in Theorem 1.3 is

$$b = -s^3 \frac{n-4}{16(n-2)^3} = \frac{-s^3}{16 \times 3^3}$$

By executing

```
testkill
```

we see that the ratio of the Killing form to the inner product is $s = -6$.

Hence the value of b as given by Theorem 1.3 is

$$b = \frac{-(-6)^3}{16 \times 3^3} = \frac{1}{2}$$

in our case.

Our numerical results thus verifies Theorem 1.3. However it should be noted that our calculation only gives a *neccessary* condition on b. For a sufficient result we would need to do calculations for *all* $a, b, c, x = 1, 2, \ldots, 10$.

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