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1. Introduction

In his seminal work on the mathematics of warfare, Lanchester's purpose was to make a crucial distinction between what he called 'ancient' warfare, in which combat is essentially a set of duels, and 'modern' war, in which, with the advent of long-range aimed projectile weapons, combatants with the advantage of numbers could concentrate their fire, many-on-one.^[?] When these assumptions are built into the simplest possible dynamical system, ancient war produces a **linear law**, in which fighting strength is given by units' individual effectiveness multiplied by their numbers, while modern war results in a **square law**, in which strength is individual effectiveness times numbers *squared*. Which of these holds for air combat? Lanchester thought the answer was clear: air combat is modern, square-law war, and he included his original articles in a book on *Aircraft in Warfare*.^[?] The same answer has seemed obvious to modern commentators, too.^[?]

In the last couple of decades the standard way to perform a Lanchestrian campaign analysis has been to fit the two sides' loss rates to (possibly different) monomial scalings—one writes each side's loss rate as a power of its own numbers multiplied by a power of its opponents'.^[?] Unsurprisingly, such loss rates are usually roughly in proportion to overall numbers, as both common sense and the statistical effects of aggregation would lead us to expect (see below). However, the answers to all tactical and operational questions follow rather from the *ratio* of these loss rates, the casualty exchange ratio (CER), and indeed it is the expression for this that leads to the dynamical systems' conserved quantities which result in the linear and square laws and their generalizations. Essentially, by taking the quotient of the loss rates we have eliminated the explicit time dependence: the point is that a battle's outcome depends on the ratio of losses sustained, not on the rate of its progress. Yet even Morse and Kimball, who number among the fathers operations research, treat the CER separately from the Lanchester equations.^[?]

To see how this works, let B and R be Blue and Red sortie rates, and $-dB/dt$ and $-dR/dt$ be the corresponding loss rates. Lanchester's aimed-fire equations are

$$\frac{dB}{dt} = -rR, \quad \frac{dR}{dt} = -bB, \quad (1)$$

where r (resp. b) are the losses caused by Red (resp. Blue) per sortie. Upon dividing, we see that the CER

$$\frac{dR}{dB} = \frac{bB}{rR}, \quad (2)$$

which we separate to give $bB dB = rR dR$ and integrate to give the square law $\frac{1}{2}rR^2 = \frac{1}{2}bB^2 + \text{constant}$. Thus the signature of the square law is that the CER is proportional to the force ratio (FR) B/R , while that of the linear law $rR = bB + \text{constant}$ is that the CER is constant.

One might be tempted to test between linear and square laws by performing a simple linear regression fit of the CER to the FR. However, this would be mistaken, for the conclusions are not invariant under $B \leftrightarrow R$, because the significance of β in fitting $\frac{dB}{dR} = \alpha + \beta \frac{R}{B}$ is different from that in fitting $\frac{dR}{dB} = \alpha + \beta \frac{B}{R}$. Rather one should seek the scaling of $\frac{dR}{dB}$ with $\frac{B}{R}$ by fitting

$$\frac{dR}{dB} = \alpha \left(\frac{B}{R} \right)^\beta, \quad (3)$$

equivalent to performing a linear regression of the logarithm of the CER onto that of the FR. From a statistical perspective, the point is that residuals with respect to a given FR are given the same weight as those with respect to its reciprocal. Evidence for the square law rather than the linear law would then be for $\beta = 1$ to be preferred to $\beta = 0$, while any evidence for $\beta \neq 0$ points to a non-trivial dependence of CER on FR.

2. Two campaigns of World War Two

Here we look at two campaigns for which we have finely-resolved sortie and loss data, the 1940 Battle of Britain and the 1941-45 carrier-based Pacific air campaign. For the former we have daily data, while for the latter the data are at the level of a carrier operation, ranging from a single raid to several weeks' action. Of course, ideal data would be for individual, indivisible engagements. When one sums a number of smaller, independent engagements, the resulting loss rates are pushed towards linear dependence on sortie numbers (owing, at root, to what mathematicians call 'Jensen's inequality'). To the extent to which larger loss rates are due to their being the sums of more rather than larger engagements, aggregation effects can obscure evidence for the square law.

The Battle of Britain, 1940

Let B be Royal Air Force (RAF) and R (estimated) Luftwaffe daily sortie numbers, and $\delta B := -dB/dt$ and $\delta R := -dR/dt$ be RAF and Luftwaffe daily loss rates, discretized to $dt = 1$ day.^[?] The logarithm of the CER = $\frac{\delta R}{\delta B}$ is plotted against that of the FR = $\frac{R}{B}$ in Figure 1(a). If we fit

$$\frac{\delta R}{\delta B} = \alpha \left(\frac{B}{R} \right)^\beta \quad (4)$$

then we find $\beta = 0.26 \pm 0.17$ with a t -test significance of $p = 0.13$. In fact the battle naturally splits into two phases: a first, more intense phase up to 15th September 1940, for which $\beta = -0.007 \pm 0.217$ ($p = 0.98$); and a second phase after this, for which $\beta = 0.45 \pm 0.30$ ($p = 0.14$).

All fits have a proportion of variance explained (which, at the risk of ambiguity, we write conventionally as $\sum R^2$) less than 0.05. Thus, in the Battle of Britain, it seems that there is no strong evidence for the square law, or for any dependence of the CER on the FR.

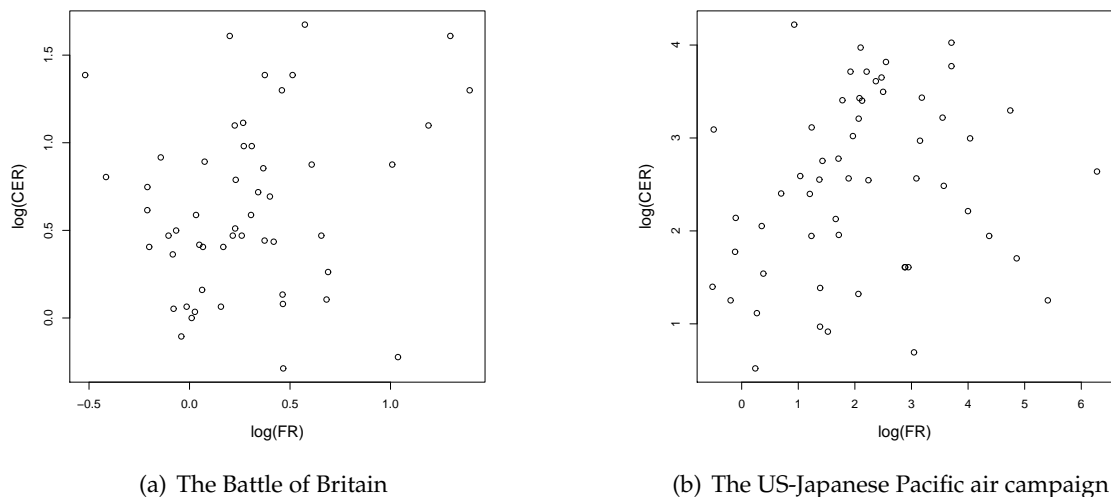


Figure 1: Log-log plot of Casualty Exchange Ratio against Force Ratio

The Pacific Air War, 1941-1945

Here we let B be US Navy (USN) and R estimated Imperial Japanese Navy (IJN) sortie numbers, and δB and δR be USN and IJN losses in the corresponding operation.^[?,?] The log-log plot of CER against FR is Figure 1(b). We find $\beta = 0.13 \pm 0.09$ with $p = 0.13$. This recedes further from significance if we again split the campaign into its two very different phases, either side of the lull in the summer of 1943. (The USN fared vastly better in the second phase, with the introduction of Hellcats and the relative improvement in its pilots' experience levels.) For the first phase we find $\beta = -0.006 \pm 0.16$ ($p = 0.97$); while for the second $\beta = -0.005 \pm 0.092$ ($p = 0.96$). Again all $\sum R^2 < 0.05$. Thus, once again, we see no evidence that air combat is other than a set of duels.

As we noted, aggregation effects could conceivably be masking evidence for the square law, especially in the case of the Pacific war. Nevertheless the absence of evidence for dependence of the CER on the FR is striking.

3. The Air Campaign and evidence for the square law

In his seminal book *The Air Campaign*, John Warden argues consistently for the importance of numbers and concentration in air combat: for example, 'a primary goal of the operational commander ought to be to make sure that his forces outnumber the enemy every time they meet'.^[?] This is not a new position. In considering the Battle of Britain, for example, it is strik-

ing how closely Warden’s arguments parallel those used between the world wars in lectures at the RAF Staff College.^[?] Yet the tactic which won the Battle of Britain and the (air) Battle of Malta, due to Keith Park, was very different: forward interception by minimal numbers. In contrast, the emphasis of Leigh-Mallory and his proteges on concentration (exemplified in the ‘Big Wing’) proved disastrous, both in the 1941 fighter sweeps over France and in the early stages of the Battle of Malta.^[?,?]

In Warden’s view, the special feature of the Battle of Britain as a *defensive* campaign was Dowding’s use of a strategic reserve.^[?] But, if a scaling law for loss rates indeed underlies the Battle of Britain data, it certainly does not result in Lanchester’s square law, but rather in the ‘defender’s advantage’ of cover and concealment, and not of concentration, for which the prototype variant of Lanchester’s equations is that of guerrilla warfare.^[?,?] Even though *The Air Campaign* nowhere mentions Lanchester’s equations, it is effectively advocating the square law, and its mathematical argument rests on the CER: ‘Loss rates vary disproportionately with the ratio of forces involved’.^[?] This in turn rests on a rather problematic 1970 study.^[?]

The analysis in this study utilizes two sets of data, each consisting of sortie and loss numbers of the form considered above. One is of the 26 months of the air campaign over Korea (Figure 2(a)). The time periods have the merit of being identical, but 30 times longer than in the daily rates used in the Battle of Britain, so that aggregation problems will be consistently much more severe. Further, points with high CER $\frac{\delta R}{\delta B}$ are very sensitive to changes in monthly Blue losses, which were often as low as one or two. For these data, $\beta = 0.23 \pm 0.24$, $p = 0.35$, and $\sum R^2 < 0.04$, so that there is again no evidence of sensitivity of the CER to the FR.

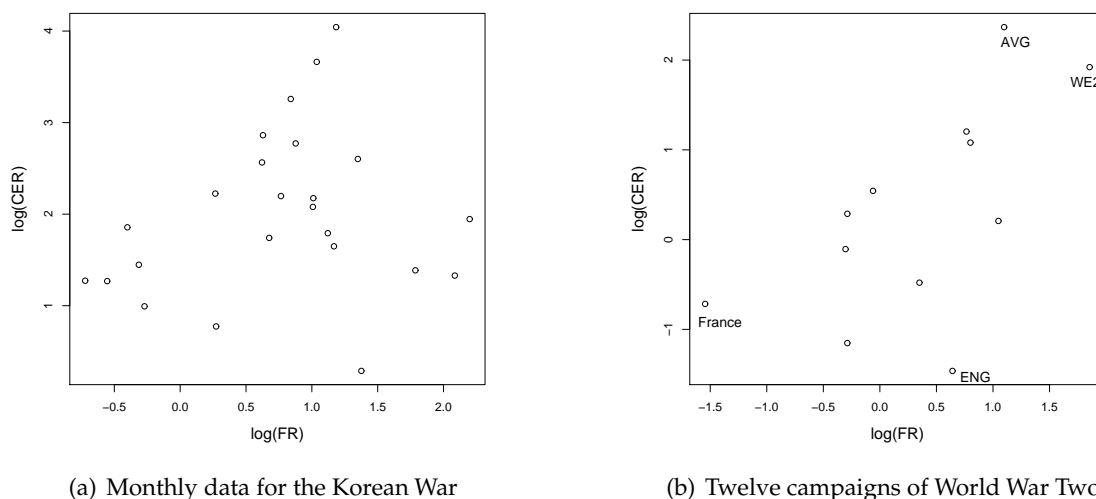


Figure 2: Ratio plots of data used in the 1970 study.^[?]

The other set is of twelve World War Two (WWII) campaigns, and is the only one of the four sets which appears to support proportionality of the CER to the FR. Unfortunately it is also

by far the most problematic of the data sets. Each of its points is a campaign, and these vary hugely in duration and magnitude, ranging from Midway (3 days, and about 200 sorties) to the second half of the 1944 battle over western Europe (3 months and over 200,000 sorties). Independent of aggregation issues, whereas in the other sets one might expect some degree of homogeneity among the data points, in this set we have different campaigns, fought at different stages of the war, between different antagonists and using different airplanes. Thus, in particular, and unlike for the other three sets, we might expect each data point to vary not only in B, R but also in b, r : the study is guilty of what Helmbold calls ‘the Constant Fallacy’.^[?]

The plot of these points is shown in Figure 2(b), in which we have identified some of the outlying points: the second half of the 1944 battle over western Europe (WE2), the 1941-2 campaign of the American Volunteer Group in China (AVG), the 1940 Battle for France (France), and the 1942 Eastern New Guinea campaign (ENG). This data set has $\beta = 0.82 \pm 0.33$ at $p = 0.03$ and $\sum R^2 = 0.38$, and so (provided one believes that b/r is the same for all twelve campaigns) provides some evidence for the square law.

The (anonymous) author of the 1970 study claims to have found a good fit to (what are effectively, although not identified as) Lanchester’s aimed-fire equations, but in fact the argument is circular: the paper posits relationships of the form $\delta B/B \propto (B/R)^{-1}$ and $\delta R/R \propto (R/B)^{-1}$ and then finds the best-fitting constants of proportionality (without regard to goodness of fit).^[?] This amounts to finding the best-fitting values of b and r in (??), and thereby of b/r in a *posited* proportionality (??) of the CER to the FR. It is not a test of the truth of such a relationship, or of the applicability of the square law.

4. Conclusion

Does air combat obey the square law? It would be highly tendentious to cite the last data set, of disparate WWII campaigns, as evidence that it does. When we probe further, using data resolved down to months, individual carrier operations or days, which we would expect to reduce aggregation problems and thereby improve the evidence for the square law, in fact the evidence melts away. If one wishes to believe that air combat is Lanchestrian—that *mere* concentration of numbers is advantageous in air combat, all other things being equal—a more convincing argument is needed.

Acknowledgments

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References

- [1] F. W. Lanchester, *Engineering* **98** (1914) 422-423 and 452-453.
- [2] F. W. Lanchester, *Aircraft in Warfare: the Dawn of the Fourth Arm* (London: Constable & Co., 1916). The chapter which deals with the mathematical combat models is reprinted in J. Newman ed., *The World of Mathematics* vol.4, 2138-2157 (New York: Simon and Schuster, 1956; also Mineola, NY: Dover, 2000). A later letter (F. W. Lanchester, Letter to the Editor, *Engineering* **154** (1942) 174) concerning the activities of the American Volunteer Group in China makes clear his view that air combat obeys the square law.
- [3] J. A. Warden III, *The Air Campaign: planning for combat* (Washington: Brassey's, 1989)
- [4] See, for example, J. Bracken, *Lanchester models of the Ardennes campaign*, *Nav. Res. Log.* **42** (1995) 559-577; J. M. Epstein, *Nonlinear Dynamics, Mathematical Biology, and Social Science*, Santa Fé Institute Studies in the Sciences of Complexity (New York: Perseus, 1999); T. W. Lucas and T. Turkes, *Fitting Lanchester models to the battles of Kursk and Ardennes*, *Nav. Res. Log.* **51** (2004) 95-116
- [5] P. M. Morse and G. E. Kimball, *Methods of Operations Research* (Cambridge: MIT Press and New York: Wiley, 1951), section 3.2.1
- [6] I. R. Johnson and N. J. MacKay, Lanchester models and the Battle of Britain, *Naval Research Logistics* **58** (2011) 210-222
- [7] The data is from *Naval Aviation Combat Statistics—World War II*, OPNAV-P-23V No. A129 (Washington, DC: Air Branch, Office of Naval Intelligence, Office of the Chief of Naval Operations, June 1946), Table 14: Air Operations and Results for Individual Carrier Operations and Phases Thereof, p40-43. A study analogous to that of the Battle of Britain was carried out in B. Matthews, *Fitting generalized Lanchester models to the Pacific air war*, University of York BSc dissertation, 2008.
- [8] Warden, *op. cit.*, p142. See also p60-61.
- [9] See, for example, A. B. Ellwood, RAF Staff College 17th course, *Air warfare III: fighter operations*, March 1939, held at the RAF museum London in AIR 69/223, and discussed in Niall MacKay and Christopher Price, *Safety in Numbers: Ideas of concentration in RAF fighter defence from Lanchester to the Battle of Britain*, *History* **96** (2011) 304-325.
- [10] Stephen Bungay, *The Most Dangerous Enemy: a History of the Battle of Britain* (London, 2000).
- [11] Keith Park, 1968 letter to Dowding, RAF Museum London AC95/124/12.
- [12] Warden, *op. cit.*, p102-104.
- [13] S. J. Deitchman, *A Lanchester model of guerrilla warfare*, *Operations Research* **10** (1962) 818-827
- [14] Warden, *op. cit.*, p61. Indeed Warden goes further, saying 'dependence of the casualty exchange ratio on the force ratio "is not linear; it is exponential" '. It is not clear what this means, although it may be a misinterpretation of the final formulae in the 1970 study.
- [15] *The relationship between sortie ratios and loss rates for air-to-air battle engagements during World War II and Korea*, Saber Measures (Charlie) (Washington, DC: Headquarters, USAF, Assistant Chief of Staff, Studies and Analysis, 1970), document call no. M-U 42210-75
- [16] R. L. Helmbold, *The Constant Fallacy*, *Eur. J. Op. Res.* **75** (1994) 647-658