Of Bombs and Boats and Mice and Men:

a random tour through some scaling laws

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G I Taylor and the Bomb

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Shock wave radius depends on energy ML^2T^{-2}, air density ML^{-3} and time T

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Dimensional analysis implies radius

$$R \sim \left(\frac{Et^2}{\rho}\right)^{1/5}$$

Scaling Laws

This is a **scale-invariant** process or **scaling law**:

$$y \sim x^a \quad \text{or} \quad y \simeq Cx^a \quad \text{or} \quad \log y \simeq a \log x + \log C$$

Rescaling, multiplying x by λ and y by μ, changes only the constant,

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where $C' = C\lambda^a/\mu$;

the form of the law remains the same

and

there is no preferred scale.

Trees

1'

100'

Yachts

14’

109’

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Speed v depends on length l and acceleration due to gravity, g;

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Speed v depends on length l and acceleration due to gravity, g;

$$v \sim \sqrt{gl}\,.$$

How does yacht price scale with length?

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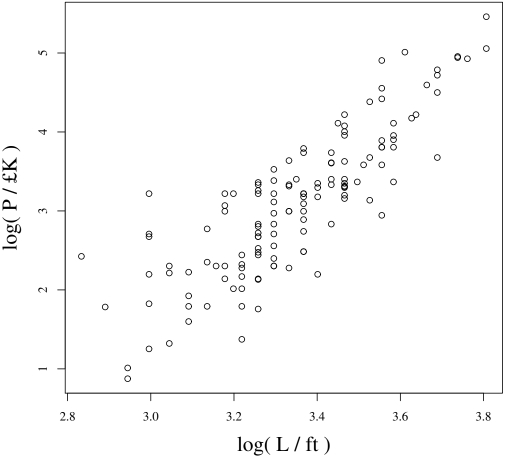

$$P \sim L^{3.8\pm0.2}\,, \qquad \sum R^2 = 0.71$$

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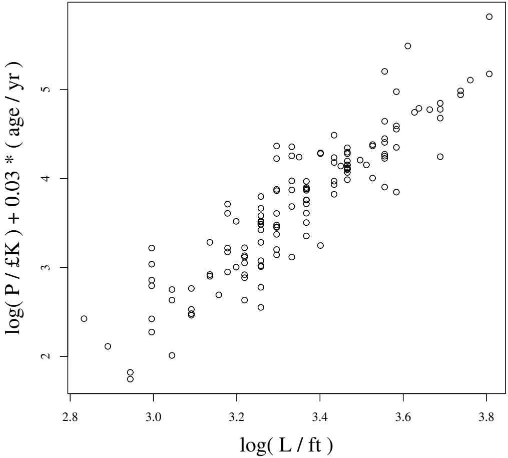

$$P \sim L^{3.5 \pm 0.1} e^{-0.03(\mathrm{age/yr})}, \qquad \sum R^2 = 0.87$$

Roasting times

1 kg

5 kg

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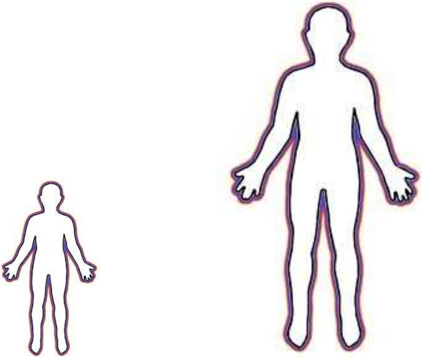
Roasting time t depends on mass $[m] = M$, density $[\rho] = ML^{-3}$, diffusion coefficient $[\kappa] = TL^{-2}$.

So $t \sim \kappa(m/\rho)^{2/3}$. If the partridge takes 1 hr, the turkey takes 3 hrs.

Body Mass Index



If mass $M \sim H^3$ and height $H \mapsto 2H$ then $M \mapsto 8M$,

so if BMI$= M/H^2$ then BMI $\mapsto 2$ BMI

Body Mass Index

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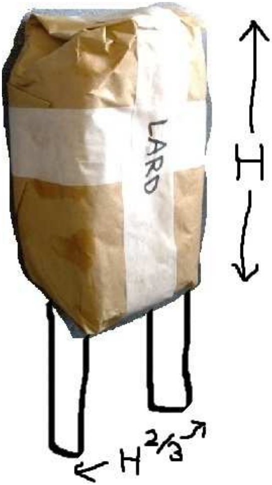

NJM, *Journal of Biomechanics* 2010, arXiv:0910.5834

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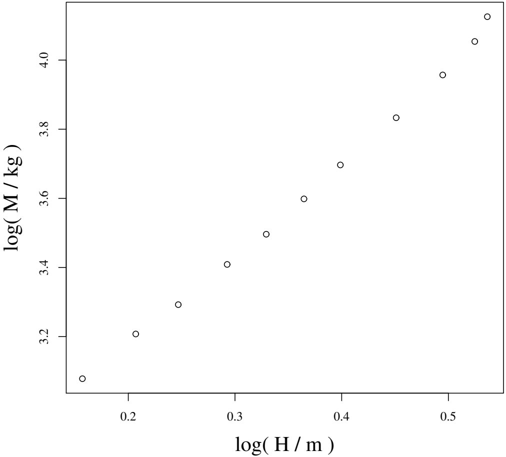
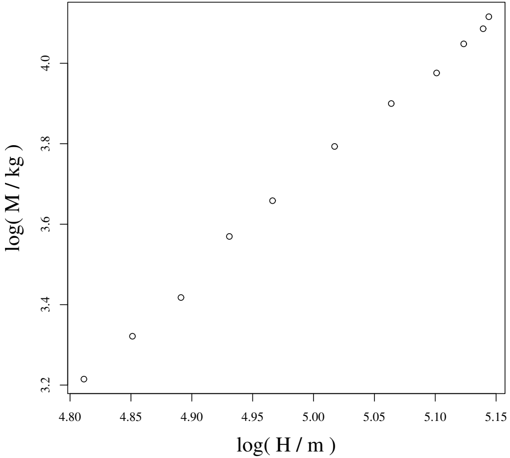


UK: $M \sim H^{2.70\pm0.05}$, $\sum R^2 = 0.996$

Body Mass Index



Hong Kong: $M \sim H^{2.66 \pm 0.05}$, $\sum R^2 = 0.997$

Do mammals obey scaling laws?

20 g

200 kg

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On Being the Right Size, J. B. S. Haldane, 1928 :

'You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away, provided that the ground is fairly soft....'

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$$ma = mg - kAv^2$$

where mass $m \sim L^3$, cross-sectional area $A \sim L^2$, so terminal velocity (at which $a = 0$) $v^2 \sim L^3/L^2 = L$ and $v \sim L^{1/2} \sim m^{1/6}$.

If the mouse's $v = 20$ mph, the bear's $v = 100$ mph.

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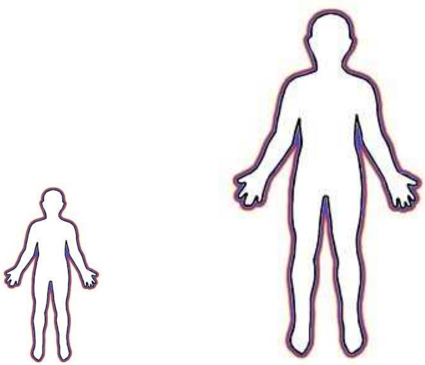
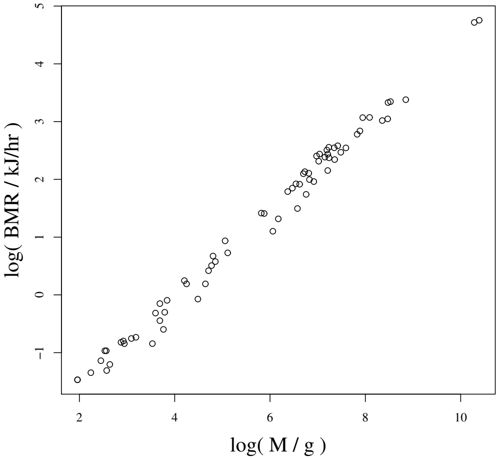
'...A rat is killed, a man is broken, a horse splashes.'

Metabolic Scaling



height $H \mapsto 2H$, mass $M \mapsto 8M$, power $P \mapsto 4P$, $P \sim M^{2/3}$

Metabolic Scaling



Marsupials: $P \sim M^{0.75 \pm 0.01}$, $\sum R^2 = 0.99$

Kleiber's Law

$$P \sim M^{3/4}$$

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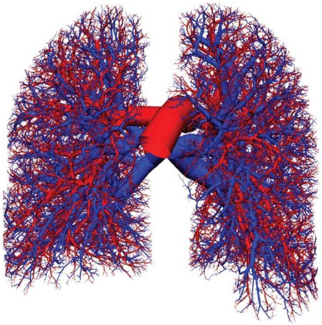

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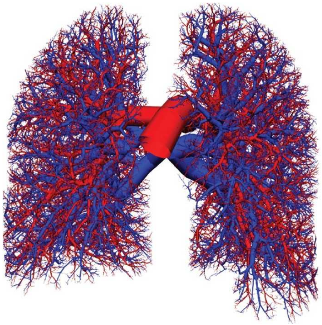
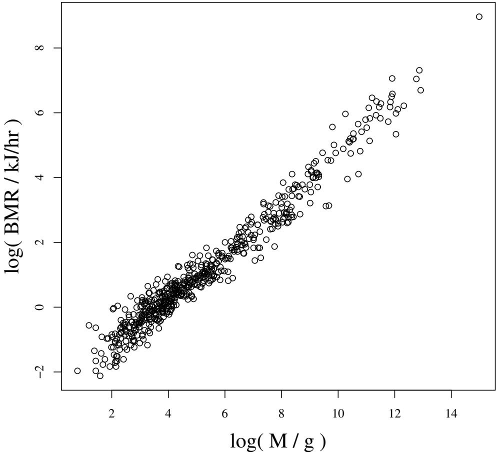


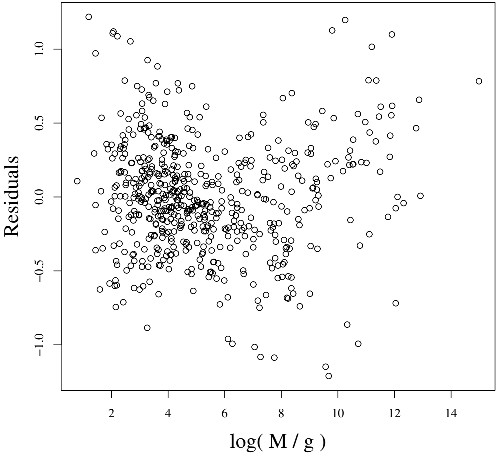
...or $M \sim H^{8/3}$, $P \sim H^2 \;\Rightarrow\; P \sim M^{3/4}$

Metabolic Scaling



Eutheria: $P \sim M^{0.72 \pm 0.01}$, $\sum R^2 = 0.96$

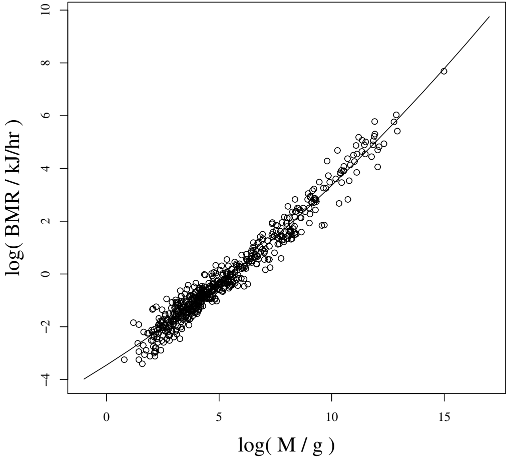
Metabolic Scaling



Eutheria residuals

Metabolic Scaling

Kolokotrones et al., *Curvature in metabolic scaling*, Nature 2010



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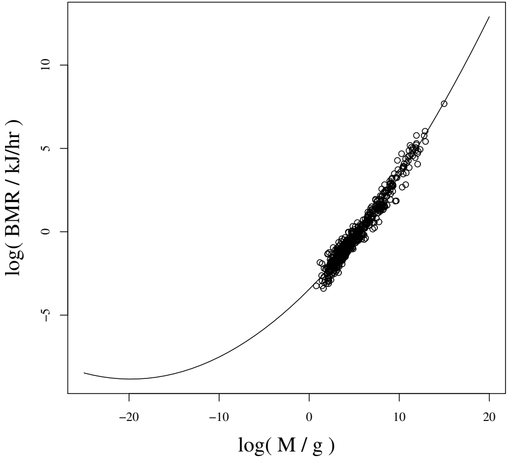

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Metabolic Scaling

A scaling law

$$P = CM^{\beta_1}$$

may be written

$$\log P = \beta_0 + \beta_1 \log M \quad \text{where} \quad C = e^{\beta_0}.$$

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Multiplying M by λ, or (equivalently) adding $\log \lambda$ to $\log M$, just changes the constant C to $C' = Ce^{\lambda}$:

the power β_1 is unchanged, while $\beta_0 \mapsto \beta_0 + \log \lambda$.

Metabolic Scaling

$$\log(P) = \beta_0 + \beta_1 \log(M) + \beta_2 \log^2(M)$$

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This is no longer scale-invariant:

Multiplying M by λ, or (equivalently) adding $\log \lambda$ to $\log M$, changes the linear term and thus the form of the law.

Metabolic Scaling

$$\log\left(\frac{P}{P_0}\right) = \beta_0 + \beta_1 \log\left(\frac{M}{M_0}\right) + \beta_2 \log^2\left(\frac{M}{M_0}\right).$$

Metabolic Scaling

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Suppose now that we choose a different scale M_0'.

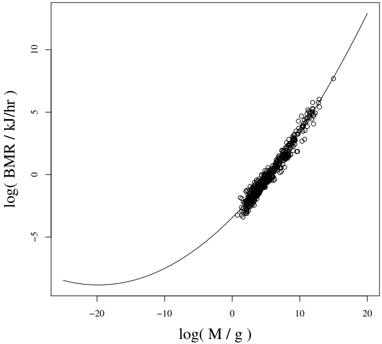
Let $\mu = \log(M_0'/M_0)$.

Then

$$\log\left(\frac{P}{P_0}\right) = \beta_0+\beta_1\mu+\beta_2\mu^2 + (\beta_1+2\beta_2\mu)\log\left(\frac{M}{M_0'}\right)+\beta_2 \log^2\left(\frac{M}{M_0'}\right).$$

Metabolic Scaling

NJM, Comment on Kolokotrones et al., *Journal of Theoretical Biology* 2011



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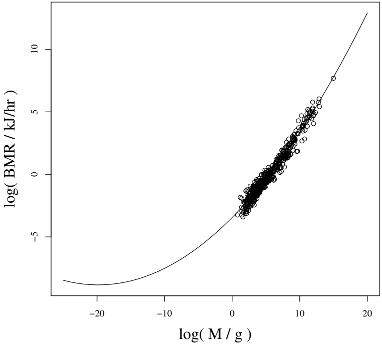

The quadratic curve is fitted with no preferred choice of origin, so the values of the linear and constant terms are meaningless.

Metabolic Scaling

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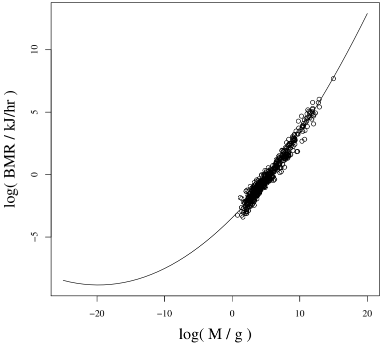

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A quadratic of any fixed curvature can approximate a line arbitrarily well over a finite interval if that interval is sufficiently far from the turning point.

...or $M \sim H^{8/3}$, $P \sim aH^{5/3} + bH^2 \;\Rightarrow\; P \sim aM^{5/8} + bM^{3/4}$

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The End