# Maximum Sustainable Government Debt in the Perpetual Youth Model*

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## Abstract

The overlapping-generations model of Blanchard, based on a constant probability of death, is used to study the maximum level of government debt consistent with the existence of a steady state equilibrium. In both a small open and a closed economy it is shown that maximum sustainable debt robustly occurs where the consumption of individual households reaches zero, the limit of its feasible range. Taxation absorbs all of the household's labour income here. In a closed economy, at this point the real interest rate also hits a "ceiling" given by a simple combination of preference parameters and the death probability.

## Keywords

maximum sustainable government debt, overlapping generations, uncertain lifetimes

## JEL Classification

E62, H63

## 1. Introduction

Worries about the maximum level of government debt which a country can sustain have come to the fore in the wake of the 2007-9 “credit crunch”. High levels of government debt are the legacy of the cost of bailing out failing banks, of automatic declines in tax revenue due to the recession, and of the deliberate “fiscal stimulus” introduced by many governments to counteract the recession. Although there are many studies of the effects of government debt and deficits on the economy, there are few which have directly addressed the question “What is the maximum level of government debt which a country can sustain?” In practice, this limit must be mainly a political one: some interest groups will be more severely affected by the consequences of high government debt than others, and, depending on their political power, such groups will sooner or later succeed in changing policy in order to halt, or reverse, the increase in debt. Possibly they may even force the government to default. Nevertheless, what drives the political pressures are the economic effects which debt has. It is therefore desirable to study what are the underlying economic limits to the level of government debt. Specifically, we ask whether there is a level of government debt above which no steady-state equilibrium of the economy exists; and, if so, what determines it. Such a limit is inevitably going to be higher than the practical limit which applies once political forces are taken account, and the resource allocation associated with it is certainly going to be more extreme; but since it provides a “backstop” to any practical limit, it is surely useful to know what it is.

In this paper we investigate this question using a well known and very tractable model which allows government debt to have real effects on the economy: namely, the “perpetual youth” model of Blanchard (1985). In Blanchard’s model, which builds on the earlier work of Yaari (1965), agents have a constant probability of death, and thus a finite expected lifetime.[1] This is a way of incorporating overlapping generations (OLGs) which allows us to parameterise the average length of life by varying the probability of death, unlike in the well-

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[1] It is referred to as the “perpetual youth” model by Blanchard and Fischer (1989) because the assumption of a constant probability of death implies that the expected remaining lifetime of an individual is always the same, and so independent of the individual’s age.

known OLG model by Diamond (1965) in which agents live for exactly two periods. It also nests the special case in which agents live forever, which can be captured by setting the probability of death to zero. The latter implies (together with some other assumptions) that "Ricardian Equivalence" holds, and hence provides a useful reference point when assessing the effects of government debt. Blanchard himself used the perpetual youth model to study the effects of changes in government debt. However he did not consider the model's implications for the maximum level of debt which the economy could support, as we do here.

Below we combine a household sector based on a perpetual youth structure with a variety of alternative assumptions about the rest of the economy. As in Diamond (1965) and Blanchard (1985), our focus is on the long run, so that capital accumulation is in general allowed for, prices are flexible and markets are competitive. The economy may be either open or closed. If it is open, then the real interest rate is fixed by the exogenous world interest rate. If it is closed, then the real interest rate is endogenous. The production technology, in which labour and capital are inputs, is allowed to take a fairly general form, and as a special case we also consider pure-exchange economies where there is no production, output being treated as an exogenous endowment.

Our main finding is that in the perpetual youth framework maximum sustainable government debt always occurs at a "degeneracy", defined as where at least one variable (other than debt) reaches the limit of its feasible range. In particular, at such a point the economy hits the limit of its "taxable capacity". Here the taxation necessary to finance the debt interest cannot be increased further without causing agents to be insolvent. Individuals' consumption is driven to zero. If government debt were any higher, individuals' consumption would have to be negative for a steady state equilibrium to exist; but negative consumption is infeasible. We show that the result that maximum sustainable debt occurs at a "degeneracy" is very robust in the perpetual youth model: it holds whether the economy is open or closed; irrespective of the exact form of preferences and production technology; and irrespective of whether there is production at all. The "degeneracy" result is also – at least, for a closed economy – notably different from the result which holds in Diamond's OLG model. In that model, as was shown in Rankin and Roffia (2003), maximum sustainable debt occurs at an "interior maximum", where variables are not at the limit of their feasible ranges. In the perpetual youth model, at the point where the limit to debt is reached the resource allocation is thus more extremely distorted than in the Diamond OLG model. Our second finding is that the value of the maximum sustainable debt: GDP ratio is determined in a very simple way in this model. We show that it equals the wage share in output divided by the relevant interest rate. Although it is not our aim here to carry out serious empirical estimates, rough calculations using this formula suggest that the likely value for the maximum sustainable debt: GDP ratio would be of the order of 10 or more, which is far above any current real-world debt: GDP ratios. Our third finding is that, in a closed economy (in which, unlike in a small open economy, the real interest rate is not exogenous), an alternative way of describing how maximum sustainable debt is reached is to say that it occurs where the interest rate hits a "ceiling". The value of the ceiling is given by a simple combination of the death probability and preference parameters. We show why the ceiling arises by analysing the cross-section composition of household consumption in the steady state.

A number of other studies in the literature have examined the "sustainability" of fiscal policy. However, different authors define this term in different ways. Amongst those who define sustainability in terms of the existence of a steady state, the majority have been more interested in the sustainability of deficits rather than debt. Two interesting recent contributions in this spirit, in the context of endogenous growth, are by Brauninger (2005) and Yakita (2008). Yet, even when there is no deficit, a fiscal situation can be unsustainable simply because the inherited stock of debt is too high, which is the problem on which we focus here. Within the Diamond OLG framework, this was examined for a closed economy by Rankin and Roffia, op.cit.; and for a two-country set-up by Farmer and Zotti (2010).

The structure of the the paper is as follows. In Section 2 we present the building blocks of the economy, centred around the perpetual youth framework. In Section 3 we then study maximum sustainable debt in a small open economy version of the model. Section 4 performs a similar analysis for a closed economy. The wider significance of our results is discussed in Section 5.

## 2. The Structure of the Economy

As in Blanchard (1985) we assume time is continuous and that an agent has a constant probability of death, $p$, per unit of time, where $p \geq 0$. The population is constant and its size is normalised to unity. Thus at any point in time, $p$ agents die and $p$ are born. An agent has zero financial wealth at birth and may accumulate or decumulate wealth throughout life. There will hence generally be a distribution of wealth and consumption levels by age across the population. We will index agents by their birthdate, $s$, so that $c(s,t)$ (etc.) denotes the consumption (etc.) at time $t$ of an agent born at time $s$.

The lifetime expected utility maximisation problem of an agent, from the perspective of time $t$, can be written as:

$$\text{maximise} \qquad \int_t^\infty (1-\gamma)^{-1}[c(s,v)]^{1-\gamma} e^{-(\theta+p)(v-t)} dv \tag{1}$$

$$\text{s.t.} \qquad da(s,v)/dv = [r(v)+p]a(s,v) + y^N(v) - c(s,v) \qquad \text{for all } v \geq t.$$

Flow utility takes the "constant elasticity of intertemporal substitution" form, where $1/\gamma \, (\geq 0)$ is the elasticity of intertemporal substitution. It is discounted at the sum of the agent's pure time preference rate, $\theta \, (\geq 0)$ and death probability, $p$ (noting that $e^{-p(v-t)}$ is the probability density of still being alive at date $v$). $a(s,v)$ is the agent's financial wealth at date $v$, which earns real rate of interest $r(v)$. As in Blanchard (1985), perfectly competitive insurance companies provide an annuity scheme whereby an agent who remains alive from one instant to the next earns, in addition, an annuity at the rate $p$ per unit of wealth, while if the agent dies all their wealth is ceded to the insurance company. This yields actuarially fair insurance and in equilibrium the insurance companies make zero profits. $y^N(v)$ is the net-of-tax non-interest income of the agent. In an economy with production this equals the real wage, $\omega(v)$, less a lump-sum tax, $\tau(v)$, since agents have one unit of time which they supply completely inelastically to the labour market. In an economy with no production it equals an exogenous income endowment, $\bar{y}$, less the lump-sum tax.

Solving this optimisation problem yields the Euler equation for optimal intertemporal consumption choice by the individual:

$$[1/c(s,v)]dc(s,v)/dv \;=\; (1/\gamma)[r(v)-\theta]. \tag{2}$$

An individual's optimal consumption at any date *t* can also be expressed as:

$$c(s,t) \;=\; w(s,t)/\Delta(t), \tag{3}$$

where

$$w(s,t) \;\equiv\; a(s,t) \;+\; h(t),$$

$$h(t) \;\equiv\; \int_t^\infty y^N(v)e^{-\int_t^v [r(z)+p]dz}dv$$

$$\Delta(t) \;\equiv\; \int_t^\infty e^{\int_t^v [(1/\gamma-1)r(z)-\theta/\gamma-p]dz}dv.$$

Here, *w*(*s*,*t*) is total lifetime wealth, which is the sum of financial wealth *a*(*s*,*t*) and of "human wealth", *h(t)*. The latter is defined as the present value of current and expected future net-of-tax non-interest income. Since we assume non-interest income and taxation do not depend on the agent's age, *h(t)* is also independent of age, i.e. of the birthdate. Δ(*t*) is the inverse of the propensity to consume out of total wealth. It depends on current and expected future real interest rates, except in the special case where $\gamma = 1$ (which corresponds to the case of a logarithmic utility function), when it reduces to $1/(\theta+p)$.

We next turn from individual to aggregate consumption. We will generally denote aggregate variables in the upper case and individual variables in the lower case. Aggregating by birthdate across all individuals alive at time *t* gives:

$$C(t) \;\equiv\; \int_{-\infty}^t pe^{-p(t-s)}c(s,t)ds. \tag{4}$$

Notice that the individual consumption function, (3), is linear in wealth and its form does not depend on the agent's birthdate. Hence an identical relationship applies between the corresponding aggregate variables:

$$C(t) \;=\; W(t)/\Delta(t). \tag{5}$$

In the case of the Euler equation we cannot simply replace individual by aggregate variables. Nevertheless we can still derive an "aggregate" counterpart of (2), which is as follows:

$$dC(t)/dt = (1/\gamma)[r(t)-\theta]C(t) - pA(t)/\Delta(t). \tag{6}$$

Compared with (2), the difference is that, for $p \neq 0$, there is an additional, negative, effect in the equation arising from aggregate financial wealth, $A(t)$. A similar equation is derived by Blanchard (1985).

When there is production, we assume that labour and capital are combined to produce output through a production function with constant returns to scale. Since aggregate labour input has been normalised to unity, we can represent technology simply by:

$$Y = F(K), \qquad (F' > 0, F'' < 0)$$

where the variable for labour input has been suppressed. Profit maximisation and competitive markets yield the usual factor-price conditions:

$$r = F'(K), \quad \omega = F(K) - KF'(K). \tag{7}$$

Lastly, consider the government's budget constraint. We shall abstract from government spending in the sense of purchases of output. Taxation, as noted, takes the form of an equal lump-sum tax, $\tau$, on each agent. Since the population is of size one, aggregate and individual taxation are equal: $T = \tau$. The government will normally balance its budget, so that, letting $D$ denote the outstanding stock of government debt, we have:

$$T = rD. \tag{8}$$

This is appropriate since we shall focus on steady states of the model and, without a source of permanent growth, the stock of government debt must be held constant over time in order to have a steady state. However, we shall permit occasional one-off increases or decreases in the stock of debt, implemented through an equal tax cut or tax increase on each agent, with a corresponding discontinuous jump in $D$. At such points the government will run a deficit or surplus for an instant and (8) will momentarily be violated.

## 3. A Small Open Economy

A country which can trade freely in world goods and capital markets and is of negligible size relative to the rest of the world (hence "small"), faces an exogenous world real interest rate, $r^*$. Together with the factor-price conditions (7), this determines the country's capital stock, real wage and output levels. We can denote these values as:

$$\bar{K} \equiv F'^{-1}(r^*), \quad \bar{\omega} \equiv F(\bar{K}) - \bar{K}F'(\bar{K}), \quad \bar{Y} \equiv F(\bar{K}). \tag{9}$$

The given $r^*$ thus effectively makes all the variables on the production side of the economy exogenous. Clearly, then, they are not affected by government debt.

In the open economy, the aggregate financial wealth of domestic households, $A$, has three components:

$$A = \bar{K} + F + D. \tag{10}$$

These are the domestic capital stock ($\bar{K}$), domestic government debt ($D$), and net foreign assets held by households ($F$). The latter are claims on foreigners which are accumulated or decumulated as a result of surpluses or deficits in the current account of the balance of payments. The balance-of-payments equation states this:

$$\dot{F} = r^*F + F(\bar{K}) - C. \tag{11}$$

In this equation, $\dot{F} \equiv dF/dt$, $F(\bar{K}) - C$ is the trade surplus (excess of domestic production over absorption), and $r^*F$ are net interest receipts from abroad. The right-hand side is hence the current account surplus. Note that the constancy of $K$ implies that investment is zero.[2]

To study the behaviour of aggregate consumption, we shall use the "aggregate Euler equation", (6). Since $r(t) = r^*$ for all $t$, $\Delta(t)$, the inverse propensity to consume out of wealth, becomes exogenous: the relevant formula for it is given below. In the case of the small open economy, (6) can thus be expressed as:

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[2] This is true except in situations in which $r^*$ or $F(.)$ change, which we shall not be concerned with here. We also abstract from depreciation of capital.

$$\dot{C} \;=\; (1/\gamma)[r^{*}-\theta]C \;-\; (p/\Delta)[\bar{K}+D+F], \tag{12}$$

$$\text{where} \quad 1/\Delta \;=\; p+(1/\gamma)\theta+(1-1/\gamma)r^{*}.$$

The small open economy macroeconomic model thus consists of the pair of differential equations (11) and (12), which jointly govern the evolution of *F* and *C*. This is essentially the same model as in Blanchard (1985). Although *K*, and thus *Y*, are made exogenous by the small open economy assumption, the country's aggregate consumption, and also its GNP (≡ $\bar{Y}+r^{*}F$) are able to vary, because the country's citizens are able to run up or run down their stock of net foreign assets.

Our interest here is in steady states. In a steady state of this economy, the current account must be in balance, so that $\dot{F}=0$. Imposing this on (11) reminds us that *C* = $\bar{Y}+r^{*}F$ in a steady state, so that steady state consumption is higher or lower as net foreign assets are higher or lower (respectively). This is because higher net foreign assets mean higher interest receipts, which enable home residents to permanently consume more. Government debt has a negative effect on steady state consumption because, although in a small open economy such debt does not crowd out capital, it does crowd out net foreign assets. We can see this by setting $\dot{F}=\dot{C}=0$ in (11) and (12) and solving for *C*:

$$C \;=\; \frac{p[\theta+\gamma p+(\gamma-1)r^{*}]}{(p+r^{*})(\theta+\gamma p-r^{*})}\{F(\bar{K})-r^{*}[\bar{K}+D]\}. \tag{13}$$

(13) shows that the long-run effect of an increase in *D* on *C* is negative, since the expression multiplying {.} is positive.[3] Intuitively, government debt provides domestic households with an alternative form of financial wealth to net foreign assets, so that they now need to hold less of the latter in order to have a sufficient quantity of a "store of value" which will enable them to shift consumption from earlier to later in their lives.[4] These are standard results which were first shown for this model by Blanchard (1985).

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[3] Here we assume that the exogenous parameters ($\theta,\gamma,p,r^{*}$) are such that $\theta+\gamma p-r^{*}$ > 0. In fact this condition must be satisfied for a valid steady state equilibrium of the model to exist at all: we explain why in Section 4.

[4] Indeed, the steady state equilibrium level of *F* could be negative, i.e. the country could be a net debtor. In this case an increase in domestic government debt just increases domestic residents' indebtedness to foreigners.

Let us now consider what happens to steady state values as the stock of government debt becomes very large. (13) shows that *C* is a decreasing linear function of *D*. It is therefore obvious that there exists a critical value of *D* which drives *C* to zero. A negative value of *C* makes no economic sense, so the level of *D* at which *C* becomes zero is the "maximum sustainable" level of government debt (as we defined it in the Introduction) for this economy. If *D* is greater than this critical value, no steady state equilibrium of the economy exists. What we see in this economy, then, is that the point at which a steady state equilibrium ceases to exist is a "degeneracy", by which we mean that, at such a point, at least one economic variable (*C*, in this case) has reached the limit of its feasible range, so that the steady state equilibrium is "degenerate". The economy has, in effect, collapsed in the steady state corresponding to this value of *D*. An alternative way in which the maximum sustainable level of debt could in principle be reached, if the economy were different, is at an "interior maximum". This has the property that no variable has reached the limit of its feasible range. We will discuss this second type of limit further when we look at the closed economy. The conclusion that, in a small open economy, the limit to debt is reached at a "degeneracy" also holds when the Diamond OLG structure is used, as Rankin and Roffia (2003) note.

To understand better what happens when *D* reaches its maximum sustainable value in this economy, let us set *C* to zero in (13). The value of *D* at which this occurs clearly satisfies:

$$\begin{aligned} r^*D &= F(\bar{K}) - r^*\bar{K} \\ &= \bar{\omega}, \end{aligned} \tag{14}$$

where the second line follows from (7). In other words, interest payments on government debt – and thus, from the government's budget constraint, (8), the tax, $\tau$, faced by a household – equal the real wage, at the point where the limit to debt is reached. Suppose government debt is increased in discontinuous small steps, where between the increases the economy is allowed to converge to the corresponding new steady state, where one exists.[5] As this happens, the tax burden on households steadily increases, since higher taxes are

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[5] Blanchard (1985) discusses the dynamic convergence process. For brevity, we omit analysis of this here.

necessary to finance the government's higher interest payments. Eventually there comes a point at which all of a household's labour income is taken in taxation. In terms of the notation in (1), the household's net-of-tax non-interest income, $y^N = \bar{\omega} - \tau$, hits zero. In such a steady state, a household has zero "human wealth", since the expected present value of $y^N$ is (recalling the definition in (3)):

$$y^N / (r^* + p), \tag{15}$$

which is hence also zero. This means that a newborn household, which by assumption has no financial wealth, has zero total lifetime wealth, and is therefore on the brink of insolvency. The household can therefore afford no consumption. Any further small increase in government debt would cause $\tau$ to exceed $\bar{\omega}$, and thus would make newborn households insolvent. Thus another way of describing how the maximum sustainable level of debt is reached, in this "degeneracy" case, is to say that the economy has reached the limit of its "taxable capacity" at this point. The government is extracting as much tax revenue as it can from households.

This perspective also helps to explain a simple formula which we can derive for maximum sustainable debt. From (14), the limit to government debt can be expressed as:

$$D = \bar{\omega} / r^*. \tag{16}$$

Recalling that the population, and thus aggregate labour supply, is of size one, this says that maximum sustainable debt equals total labour income divided by the world interest rate. (Equivalently, the maximum sustainable debt: GDP ratio equals the wage share in GDP divided by the world interest rate.) Intuitively, this is because, using the government budget constraint, (8), government debt in a steady state can be written as $D = T/r^*$, which says that debt equals the present value of tax revenue; while in turn the maximum feasible tax revenue, as just seen, equals total labour income.

## 4. A Closed Economy

In a closed economy, the real interest rate becomes endogenous. Consequently the capital stock and output are also endogenous. On the transition path between steady states, investment will now be non-zero.

The equations which describe the macroeconomic equilibrium are as follows. First, the economy's income-expenditure identity tells us that investment equals output minus consumption:

$$\dot{K} \;=\; F(K) - C\,. \tag{17}$$

Aggregate financial assets consist just of capital and government debt, since there are no net foreign assets: $A = K + D$. Hence the "aggregate Euler equation", (6), can be written:

$$\dot{C} \;=\; (1/\gamma)[F'(K) - \theta]C \;-\; p(K + D)/\Delta\,, \tag{18}$$

(in which we have also used (7)). $\Delta$, the inverse propensity to consume out of wealth, is time-varying in the closed economy since the interest rate is time-varying. By differentiating the definition of $\Delta$ with respect to $t$ we can derive an equation for its rate of change:

$$\dot{\Delta} \;=\; [p + \theta/\gamma + (1 - 1/\gamma)F'(K)]\Delta \;-\; 1\,. \tag{19}$$

The closed-economy macroeconomic model thus consists of the three simultaneous differential equations (17)-(19), in ($K$,$C$,$\Delta$). This is as in Blanchard (1985).

As before, we will focus on steady states. Setting $\dot{K} = \dot{C} = \dot{\Delta} = 0$ in (17)-(19), and rearranging the resulting equations to eliminate the steady state values ($C$,$\Delta$), we can reduce the system to an equation in the steady state value of $K$ alone:

$$(1/\gamma)[F'(K) - \theta]F(K) \;=\; p[p + \theta/\gamma + (1 - 1/\gamma)F'(K)][K + D]\,. \tag{20}$$

This determines $K$ as an implicit function of $D$. Since $K$ has a unique negative relationship with $r$ via $r = F'(K)$, we can equivalently re-express (20) as an equation which implicitly determines $r$:

$$(1/\gamma)[r-\theta]F(F'^{-1}(r)) \;=\; p[p+\theta/\gamma+(1-1/\gamma)r][F'^{-1}(r)+D], \qquad (21)$$

where $F'^{-1}(r)$ (=$K$) denotes the inverse of the function $F'(K)$.

Let us now study the relationship (21) in more detail. We would expect it to imply that an increase in $D$ causes an increase in $r$, and thus a decrease in $K$. This is the standard result, found also in Blanchard (1985) (and indeed in Diamond (1965)) that, in a closed economy, higher government debt raises the steady state real interest rate and crowds out the private capital stock. We will confirm this below. However, here we are not just interested in a small change in debt; we also wish to study potentially very large changes. To do this we need to understand not only the "local" properties of the relationship (21), but also its "global" properties, for a wide range of values of $D$.

We cannot obtain a closed-form solution for $r$ as a function of $D$ from (21), but we can obtain a closed-form solution for $D$ as a function of $r$:

$$D \;=\; \frac{(1/\gamma)[r-\theta]}{p[p+\theta/\gamma+(1-1/\gamma)r]}F(F'^{-1}(r)) \;-\; F'^{-1}(r). \qquad (22)$$

Notice that the first right-hand term corresponds to $A$ and the second to $K$. Hence (22) can be described as saying that, for any given steady state value of $r$, the level of government debt which would support this value equals the difference between households' aggregate demand for financial assets at that value of $r$, and firms' desired stock of physical capital at that value of $r$.[6]

Before analysing the function (22) further, we digress to highlight a crucial point. This is, namely, that in the perpetual youth model, for a steady state equilibrium to exist the real interest rate cannot exceed a certain critical "ceiling" value. To see why, consider the problem of aggregating individual households' consumption levels in a steady state. The general relationship of aggregate to individual consumption is given by (4). Now, in a steady state, integrating the Euler equation for an individual household, (2), we can readily obtain:

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[6] Since households' "demand for financial assets" could also be termed their "supply of financial capital", while $F'^{-1}(r)$ is clearly firms' demand for physical capital, (22) equivalently says that the government debt which supports a given steady state $r$ equals the private sector's "excess supply" of capital at that value of $r$.

$$c(s,t) \;=\; c(s,s)e^{(1/\gamma)(r-\theta)(t-s)}. \tag{23}$$

This says that, for a household born at date $s$, consumption at date $t$ will have grown relative to consumption at birth, $c(s,s)$, by a factor which depends on the age of the household, $t$-$s$, and on the growth rate of individual consumption, $(1/\gamma)(r\text{-}\theta)$. Note moreover that, in a steady state, the cross-section distribution of consumption by age must be constant over time. Hence consumption of the newborn does not depend on the date at which they are born, i.e. $c(s,s)$ is independent of $s$. Let us call this value $\overline{c(s,s)}$. Then by substituting (23) into (4) we can calculate aggregate consumption as:

$$C(t) \;=\; \overline{c(s,s)}p\int_{-\infty}^{t} e^{[(1/\gamma)(r-\theta)-p](t-s)}ds. \tag{24}$$

Here we have factored out $\overline{c(s,s)}$, which is valid because of its independence from $s$. It is now apparent that $C(t)$ can only take a finite value if the value of the integral in (24) is finite, and that the integral in (24) will only be finite if $(1/\gamma)(r\text{-}\theta) - p$ is negative. In other words, the condition for aggregate consumption in a steady state to be finite is that:

$$r \;<\; \theta + \gamma p. \tag{25}$$

This therefore places a ceiling on the value of the interest rate. Intuitively, if the interest rate is large, consumption across the population increases rapidly with age. In calculating aggregate consumption, the increase which occurs with age is in general offset by the decline in the size of any given age cohort with age, and this is enough to keep aggregate consumption finite However, if $r$ is too large, the death rate, $p$, is insufficient to stop aggregate consumption from "exploding" as we sum across progressively older age cohorts.[7]

Let us now return to the function (22). A special case of (22) which it is helpful to consider first is where the economy contains no production. Thus, imagine that output is just an exogenous endowment, $\overline{Y}$. In this pure exchange economy, government debt can still affect the distribution of consumption between agents of different ages and thus the real interest rate; but it cannot affect aggregate consumption. In (22), output is in general given by

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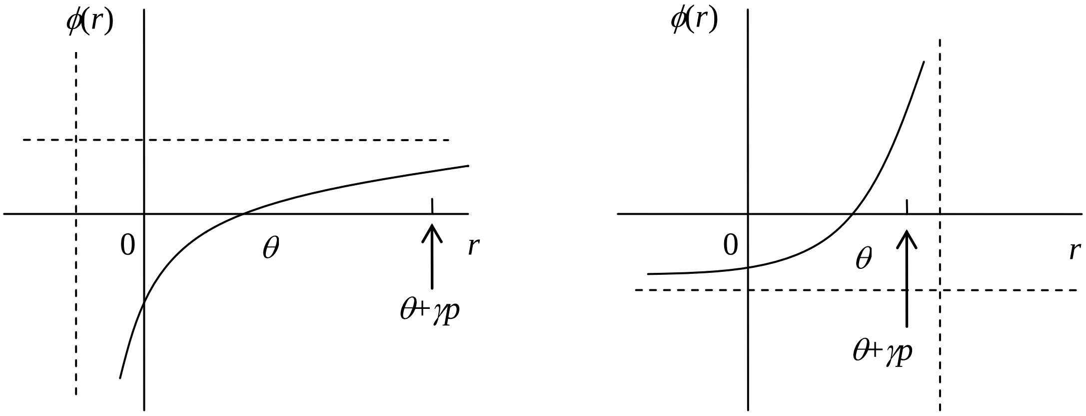
[7] Blanchard (1985) also acknowledges this ceiling on the interest rate at the end of section II of his paper. However he merely records it in passing. It does not play a significant role in his main results.

$F(F'^{-1}(r))$, so in this special case this term is replaced by $\bar{Y}$. Moreover, in the absence of physical capital, the second right-hand term vanishes. (22) then reduces to:

$$\begin{aligned} D &= \frac{(1/\gamma)[r-\theta]}{p[p+\theta/\gamma+(1-1/\gamma)r]}\bar{Y} \\ &\equiv \phi(r)\bar{Y}\,. \end{aligned} \tag{26}$$

When $\gamma = 1$ (i.e. utility of consumption is logarithmic), the function $\phi(r)$ is clearly linear and increasing in $r$. For $\gamma \neq 1$, on the other hand, the graph of $\phi(r)$ is a rectangular hyperbola. Its exact shape differs depending on whether $\gamma > 1$ or $\gamma < 1$. We sketch it in Figure 1: panel (a) for $\gamma > 1$ and panel (b) for $\gamma < 1$. By closer inspection of (26), it is straightforward to see that in all cases it must be upward-sloping and cut the horizontal axis at $r = \theta$. When $\gamma > 1$, it is continuous for all $r \geq \theta$ and tends to a horizontal asymptote at $1/p(\gamma\text{-}1)$. When $\gamma < 1$, over the range $r \geq \theta$ it is discontinuous at, and tends to a vertical asymptote at, $r = (\theta+\gamma p)/(1-\gamma)$.



(a) $\gamma > 1$ (b) $\gamma < 1$

Figure 1

To consider the maximum sustainable government debt in this pure exchange economy, we now ask: what is the value of $r$ that maximises $D$, subject to the constraint that a steady state equilibrium exists? The latter constraint means that the interest-rate ceiling (25) must be respected. The answer is immediately apparent from Figure 1: regardless of whether $\gamma > 1$, $\gamma$

= 1 or $\gamma < 1$, $\phi(r)$, and thus $D$, is maximised at the "ceiling" value of $r$, i.e. where $r = \theta + \gamma p$. (In the case $\gamma < 1$, notice that the ceiling value lies "to the left of" the vertical asymptote.)[8]

There is another way of seeing what happens as the economy approaches its maximum sustainable debt level. So long as $r < \theta + \gamma p$ holds, (24) can be evaluated as:

$$C(t) = \frac{\gamma p}{\theta + \gamma p - r} \overline{c(s,s)}. \tag{27}$$

We would of course expect the coefficient of proportionality between "newborn" consumption and aggregate consumption to be positive, and this is clearly true when $r < \theta + \gamma p$. We can also see that for $r > \theta + \gamma p$, it becomes negative. A negative value does not make sense, since neither $C(t)$ nor $\overline{c(s,s)}$ can be negative. The explanation is that, when $r > \theta + \gamma p$, (27) is incorrect as the evaluation of the integral in (24), this integral being infinite in such a case. Now consider what happens as $r$ approaches $\theta + \gamma p$ from below. The coefficient in (27) tends to infinity. However, in general equilibrium, $C(t)$ equals the exogenous $\bar{Y}$. Hence newborn consumption is driven to zero as $r$ approaches $\theta + \gamma p$. Intuitively, this must occur because, in a cross-section of the population, consumption rises very quickly with age when $r$ is large. To keep aggregate consumption equal to the given output, consumption of the newborn must shrink, since at given $r$ this shrinks consumption at all ages (see again (23)). As $r$ approaches the critical "ceiling" value, this argument requires newborn consumption actually to shrink to zero.

If newborn consumption is zero, then human wealth must also equal zero, which in turn means that taxation must completely exhaust the household's non-interest income. This is the same situation as was encountered in the small open economy. The reasoning, as in Section 3, is that newborn households have no financial wealth and so their consumption is given by:

$$c(s,s) = h(s)/\Delta \tag{28}$$

(which has been obtained from the individual's consumption function, (3), with $\Delta$ here taking its steady state value of $[p + (1/\gamma)\theta + (1 - 1/\gamma)r]^{-1}$). Moreover, human wealth in the steady

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[8] Since the constraint (25) defines the permissible values of $r$ as an "open" interval, we should strictly speaking refer to the "supremum", rather than the "maximum", of $D$. However the use of more ordinary language does not cause confusion here.

state of the pure exchange economy is $h(s) = (\bar{y} - \tau)/(r + p)$ (cf. (15)), where $\bar{y}$ (= $\bar{Y}$) is the individual's endowment income. Thus, as *D* is increased and *r* approaches its ceiling value, this is associated with a steady state in which $\tau$ (= *T* = *rD*) approaches $\bar{y}$. Hence we see that in this version of the closed economy, just as in the small open economy, at the maximum sustainable government debt, the economy reaches the limit of its taxable capacity. As in the small open economy, the maximum occurs at a "degenerate" steady state, where newborn consumption is driven to its lowest feasible value, the zero value. A difference from the small open economy, however, is that *aggregate* consumption does not go to zero.

Having analysed the special case of a pure exchange economy, we return to the economy with production. Using (7) and the definition of $\phi(r)$ in (26), we can rearrange the function (22) as:

$$D = \phi(r)\{\omega(r) - [1/\phi(r) - r]F'^{-1}(r)\}, \tag{29}$$

where $\omega(r)$ $[\equiv F(F'^{-1}(r)) - rF'^{-1}(r)]$ gives the real wage as a function of *r*. Compared to its counterpart (26) which applied in the pure exchange economy, (29) replaces the exogenous $\bar{Y}$ term by {.}, which is another function of *r*. We already know that $\phi(r)$ is unambiguously increasing in *r* for all *r* in the interval $[\theta, \theta + \gamma p]$. Consider now how the term {.} varies with *r* over this interval. $F'^{-1}(r)$ (firms' demand for capital) is decreasing in *r*, while $\omega(r)$ is easily shown to be increasing.[9] It is also clear that the term $[1/\phi(r) - r]$ is decreasing in *r*. Given that $[1/\phi(r) - r]$ is moreover always positive[10], we can then see that {.} is unambiguously increasing in *r*. (29) therefore consists of the product of two positively-valued functions, both of which are increasing in *r*, for *r* in the range of interest. It follows that, in the economy with production, it is still the case that *D* is unambiguously increasing in *r*, for *r* in the relevant range. Therefore, just as in the pure exchange economy, government debt in the economy with production reaches its maximum sustainable value at the interest rate ceiling.[11]

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[9] At this point let us further assume that $F(.)$ satisfies the "Inada conditions" that $\lim_{K \to 0} F'(K) = \infty$, $\lim_{K \to \infty} F'(K) = 0$. This will ensure that $F'(K) = r$ has a solution for all $r \in [\theta, \theta + \gamma p]$.

[10] $1/\phi(r) - r = (r + p)(\theta + \gamma p - r)/(r - \theta)$, which is positive for all $r \in (\theta, \theta + \gamma p)$.

[11] See also footnote 8.

In the economy with production, output declines as government debt increases because the economy's capital stock declines. This is obvious from the fact that $K = F'^{-1}(r)$. Note, however, that $K$ and $Y$ do not in general tend to zero as the maximum sustainable debt is approached. Rather, they tend to $K = F'^{-1}(\theta + \gamma p)$ and $Y = F(F'^{-1}(\theta + \gamma p))$, which will generally be positive.[12] Nevertheless, as in the pure exchange economy, as the interest rate ceiling is approached, the consumption of newborn agents is driven to zero and all of a household's non-interest income (which, here, means wage income) is taken in taxation. The arguments that we used above to show this for the pure exchange economy clearly still apply here. Maximum sustainable debt therefore once more occurs at a "degeneracy".

These results for the closed economy contrast with those obtained using Diamond's (1965) OLG framework. As Rankin and Roffia (2003) showed, in Diamond's framework maximum sustainable debt is reached at an "interior maximum", not a "degeneracy". In the case of an interior maximum, the counterpart of the function (22) is non-monotonic in *r* over the feasible range of *r*, so that the value of *r* at which *D* reaches a maximum is not at the limit of the feasible range of *r*. Nor is it at the limit of any other variable associated with the steady state, which also means that the economy is not at the limit of its taxable capacity. It should be highlighted that, neither in the closed-economy perpetual youth model nor in the closed-economy Diamond model, is the nature of the maximum sensitive to the exact specification of households' preferences or firms' technology. In this paper we have used the fairly general "constant elasticity of intertemporal substitution" utility function, (1), and we have seen that the value of this elasticity, $1/\gamma$, is not relevant for the result. We have also seen that the shape of the production function, *F*(.), is not critical for the result, and indeed the nature of the maximum is unaffected by whether there is production at all. The crucial aspect must hence be the different "demographic" assumptions of the two models.

As in Section 3, we can derive a simple expression for the maximum sustainable debt level. Since the latter occurs where wage income is entirely absorbed by taxation, the formula (16) still applies, except that the "world" interest rate used in (16) is now replaced by the

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[12] See again footnote 9.

"ceiling" interest rate. Denoting $\theta + \gamma p$ as $\hat{r}$, and the associated steady state values of other variables as $\hat{Y}$, etc., we have:

$$D/Y = \hat{\omega}/\hat{r}\hat{Y}. \tag{30}$$

This says that the maximum sustainable debt: GDP ratio equals the wage share in GDP evaluated at the maximum sustainable debt level, $\hat{\omega}/\hat{Y}$, divided by the ceiling interest rate, $\hat{r}$. A very simple case is where the wage share in GDP is independent of the debt level, as arises with a Cobb-Douglas production function such as $Y = K^{\alpha}$ ($0 < \alpha < 1$). In this case (30) reduces to:

$$D/Y = (1-\alpha)/(\theta + \gamma p). \tag{31}$$

From this it is clear that any change in an exogenous parameter which raises the ceiling interest rate reduces the maximum sustainable debt: GDP ratio. Hence the maximum sustainable debt: GDP ratio is reduced by greater "impatience" by households (higher $\theta$), by lower flexibility regarding the timing of consumption (higher $\gamma$), and by lower life expectancy (higher $p$).[13]

## 5. Discussion

As made clear in the Introduction, our aim in this paper has been to explore the limits to government debt in a well-known and tractable model of overlapping generations. The situations studied here are hypothetical extremes. We have acknowledged that, in practice, political constraints on how much debt can be issued will become binding well before the constraints of pure economic feasibility. To keep the analysis simple, we have also abstracted from many "realistic" features of actual economies. For all these reasons, the model as it stands does not permit a serious quantitative assessment of maximum sustainable debt. Nevertheless, a quick view of the kinds of numbers implied may contribute something of

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[13] Note that we should resist any temptation to apply the formula when $p = 0$. In that case any level of government debt is sustainable, since it has no real effects.

interest. In the formula (31), a widely accepted value for the wage share in GDP, $1-\alpha$, is 2/3; while a consensual value for the pure time preference rate, $\theta$, might be 0.02 (measuring time in years). A commonly used number for the elasticity of intertemporal substitution in consumption ($1/\gamma$) is 1. Less consensus exists over what is a reasonable value for the death probability, $p$. The expected lifetime of an individual in the model is $1/p$. This might suggest setting $1/p$ at about 70 years, and thus $p$ at, say, 0.014. Adopting these parameter values, (31) yields a maximum sustainable debt: GDP ratio of 19.7. This is clearly a value far above any actual debt: GDP ratio that is currently observed.[14] On the other hand, a case for less favourable choices of parameter values could be made. Some empirical studies of consumption suggest an elasticity of intertemporal substitution of about 0.5, i.e. of $\gamma = 2$. It could also be argued that $1/p$ should be equated to the expected remaining lifetime, not of the newborn, but of an average member of the population, which would reduce it to about 35 (so $p = 0.028$, say). With these alternative values, (31) yields a ceiling on $D/Y$ of 8.82. This is still well above current actual debt: GDP ratios.

We have noted that the perpetual youth framework implies that maximum sustainable debt is reached in a different way from the way it is reached in the Diamond OLG framework (at least, in a closed economy): namely, at a “degeneracy” not an “interior maximum”. It is not possible to say unambiguously which of these analyses is “correct”. As a broad proposition, the “interior maximum” which occurs in the Diamond closed-economy framework seems likely to result from the real interest rate being more strongly affected by debt in that framework than is the case in the perpetual youth framework. This proposition is suggested by the facts, first, that when the real interest rate is pegged to the world interest rate, as happens in a small open economy, then even in the Diamond framework maximum sustainable debt occurs at a “degeneracy”; and, second, that in the closed-economy perpetual youth framework, where it occurs at a degeneracy, there are distinct limits on the range within which the interest rate can move. Further, it is notable that empirical work which has attempted to identify a relationship between government debt and the real interest rate has

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[14] For example, the UK debt: GDP ratio reached a historical peak of 2.5 in 1945.

found that, while such a relationship exists, it is not particularly strong.[15] Evidence of this kind, therefore, would tend to favour the “perpetual youth” framework over the “Diamond” one.

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[15] See, for example, Bernheim (1987).

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