Mathematical Modelling

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Overview of Course

- Model construction $\longrightarrow$ dimensional analysis
- Experimental input $\longrightarrow$ fitting
- Finding a ‘best’ answer $\longrightarrow$ optimisation
- Tools for constructing and manipulating models $\longrightarrow$ networks, differential equations, integration
- Tools for constructing and simulating models $\longrightarrow$ randomness
- Real world difficulties $\longrightarrow$ chaos and fractals

Along the way we shall deal with:

- problem identification
- model construction & selection
- identification of key variables and collecting data
- validating the model
- calculation of solutions & simulation

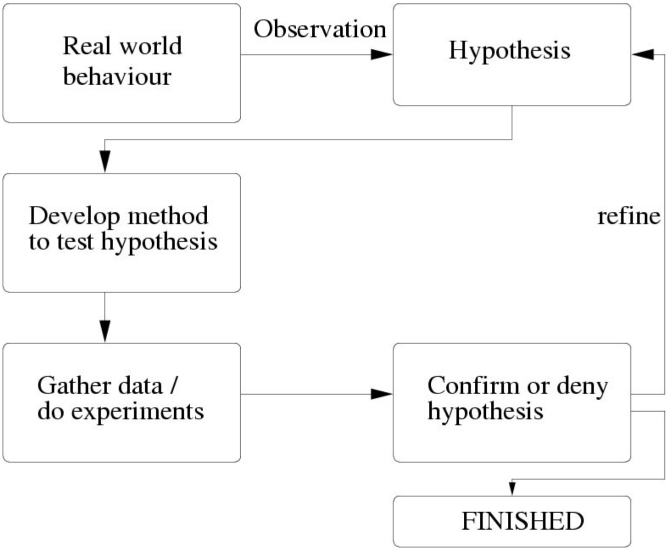
A First Course in Mathematical Modeling by Giordano, Weir & Fox, pub. Brooks/Cole.

Introduction (GWF chapter 2)

Q. What is a model?

A. "A mathematical construct designed to study a particular real-world system or phenomenon"

Simple view of science



Science vs Modelling

- Science works by testing hypotheses
- Scientists ask ‘is this correct?’
- Modelling works with simpler assumptions
- Modellers ask ‘is this reasonable?’

The Modelling Process

1. Identify the problem
   What is *'the question'*?
2. Make assumptions
   Simplify, simplify – Henry David Thoreau
3. Solve/Interpret the model
   What does our model say?
4. Validate/Verify the model
   - Is our model reasonable?
   - Does it answer *'the question'*?

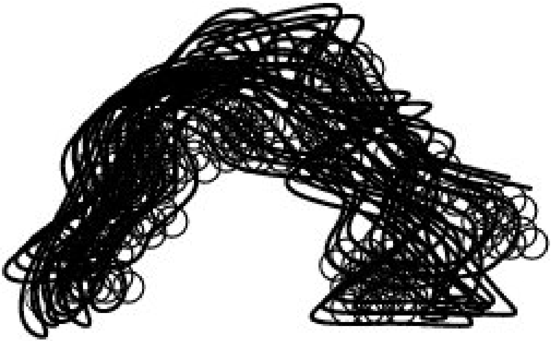
is that it? No!

The Modelling Process

- Identify the problem
- Make assumptions
  Simplify, simplify – Henry David Thoreau
- Solve/Interpret the model
- Validate/Verify the model
- Implement the model
- Maintain the model

The best model?

Tradeoff between



simplicity

refinement

The best model?

There might be several different models that can be made for a given question on a given phenomenon ...

Q. How do we know which one is best?

A. Choose *robust* over *fragile*

Robust vs Fragile

robust has OK conclusions even if not all the underlying assumptions are satisfied exactly.

fragile has invalid conclusions if not all assumptions are satisfied exactly.

and

- check the *sensitivity* of conclusions to small changes in initial conditions
- a less sensitive model is more robust $\Rightarrow$ better

Population Growth

- 1798: Thomas Malthus wrote “An Essay on the Principle of Population”
- Based on USA Census data
- The first model of population growth

The Malthus model

$$N(t) = N_0 e^{(\alpha-\beta)t}$$

The Malthus model

Year	USA population
1790	3.9 million
1800	5.3 million

So,

$$\begin{array}{lllllll} 1790 & \longrightarrow & t=0 & \longrightarrow & N_0 = 3.9 & & \\ 1800 & \longrightarrow & t=1 & \longrightarrow & N_1 = 5.3 & = & N_0 e^{(\alpha-\beta)1} \\ 1810 & \longrightarrow & t=2 & \longrightarrow & N_2 = 3.9 \exp\left(\ln\left[\frac{5.3}{3.9}\right] \times 2\right) & = & 7.2 \end{array}$$

i.e. model predicted population in 1810 to be 7.2 million.

Year	USA population	Malthus Model
1790	3.9 million	3.9 million
1800	5.3 million	5.3 million
1810	7.2 million	7.2 million

A perfect model?

So, is that it? A perfect model of population?

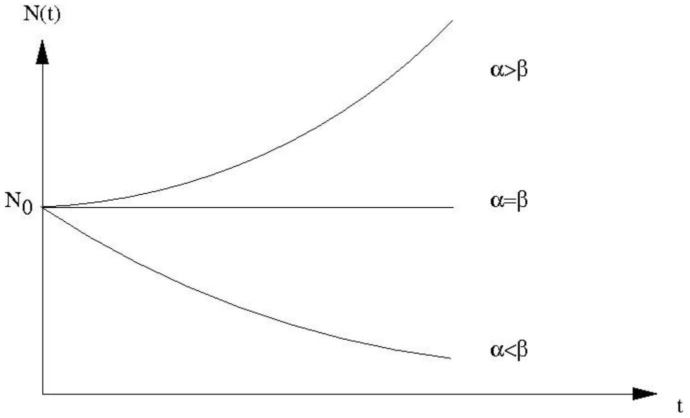
NO! Repeat steps 5 & 6 ... gather more (later) data...

Data vs Malthus!

Year	USA Population (millions)	Malthus Model (millions)
1820	9.6	9.8
1830	12.9	13.3
1840	17.1	18.1
1850	23.2	24.6
1860	31.4	33.4
1870	38.6	45.4
1880	50.2	61.7
1890	62.9	83.8
1900	76.0	113.9
1910	92.0	154.7
1920	106.5	210.3
1930	123.2	285.8

Problems with Malthus' model

Malthus' model is too simple – always results in exponential growth/decay

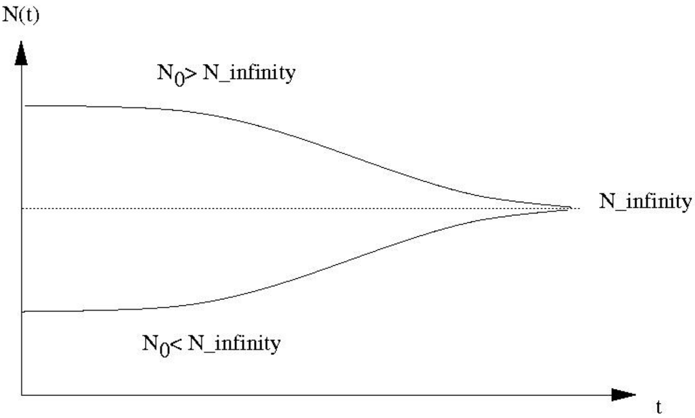


Verhulst model

- 1837: Verhulst proposed ‘constraints’ due to finite resources
- Hence there is an *optimum* (carrying) *capacity* N_∞

$$\frac{dN}{dt} = (\alpha - \beta)\left(1 - \frac{N}{N_\infty}\right) N \qquad (1)$$

Verhulst model



Verhulst model

The Verhulst equation is very similar to the *logistic equation*

$$\frac{dN}{dt} = \alpha \left(1 - \frac{N}{N_\infty}\right) N - \beta N \tag{2}$$

- Restricted birth
- Unrestricted death

We will meet this again in later lectures...

USA Census Data

Year	USA Population (millions)	Malthus Model (millions)	Verhulst Model (millions)
1820	9.6	9.8	9.7
1830	12.9	13.3	13.0
1840	17.1	18.1	17.4
1850	23.2	24.6	23.0
1860	31.4	33.4	30.2
1870	38.6	45.4	38.1
1880	50.2	61.7	49.9
1890	62.9	83.8	62.4
1900	76.0	113.9	76.5
1910	92.0	154.7	91.6
1920	106.5	210.3	107.0
1930	123.2	285.8	122.0

More detailed models

But what about...

- competition for resources?
- emigration & immigration?
- changing birth & death rates?
- changes in ‘demographics’ (e.g. population age profile)?

World population

A recent study in the journal Nature (2000) considered *total planet population*.

- Used sub-models – 13 regions with different statistical distributions for birth/death/fertility vs age/immigration etc.
- Thousands of simulations $\longrightarrow$ 'most likely' scenario
- The result? Most robust future is for population saturation in 2080 at around 15 billion people, followed by gentle decline. Phew!