Mathematical Modelling
Lecture 2 – Dimensional Analysis

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Overview of Course

- Model construction $\rightarrow$ dimensional analysis
- Experimental input $\rightarrow$ fitting
- Finding a ‘best’ answer $\rightarrow$ optimisation
- Tools for constructing and manipulating models $\rightarrow$ networks, differential equations, integration
- Tools for constructing and simulating models $\rightarrow$ randomness
- Real world difficulties $\rightarrow$ chaos and fractals

Aim

To identify the relevant parameters and relationships for real-world problems and hence guide experimental design
Functions of several variables

Suppose we have a function of $x$, $y$ and $z$ and we know that it is linear in $x$, $y$ and $z$ – i.e. if we fix $y$ and $z$, and plot $f$ against $x$ we get a straight line; and the same if we fix $x$ and $z$ and vary $y$ etc. How many different terms are there?

$$f(x) = mx + c$$

A straight line has 2 parameters (slope and intercept), and we have 3 variables ($x,y,z$). Does this mean we have $2 \times 3 = 6$ parameters in total?
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Functions of several variables

\[ f(x, y, z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 xy + a_6 xz + a_7 yz + a_8 xyz \]

We have \( 2^3 = 8 \).

This gets worse very quickly if it isn’t linear, but quadratic (\( 3^3 = 27 \)), cubic (\( 4^3 = 64 \)) or even higher order.

To get these parameters from experiment, we need at least one experimental measurement per parameter.
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You are feeling very sleepy...

What affects the period $\tau$ of the pendulum?
You are feeling very sleepy...

Perhaps $\tau = f(l, m, g, \theta)$. What is $f$?
Pendula

\[ \tau = f(l, m, g, \theta) \]

Could use experiments to determine \( f \). How many measurements do we need?

- Quadratic \( \Rightarrow \) 3 parameters
- 4 independent variables
- \( \Rightarrow 3^4 = 81 \) total parameters
- \( \Rightarrow 81 \) expt measurements!
$P$ parameters per variable, $V$ variables $\Rightarrow P^V$ total parameters.

But what if we can reduce the number of variables that need to be studied? Then we have a big saving!

*Dimensional analysis* does this by considering *dimensionless products*.

NB Dimensions are not the same as units!
Dimensional analysis

**Dimensions**  M, L, T (and K, C, etc)

**Product**  (includes quotients) e.g. [area] = $L^2$
[energy] = $ML^2T^{-2}$, etc.

**Dimensionless product**  = combination s.t. dimensions are $M^0L^0T^0$
Dimensional compatibility

When adding terms in an equation they must all have the same dimension.

\[ s = ut + \frac{1}{2}at^2 \]

- \( u \) is a velocity, \( LT^{-1} \), \( t \) is a time \( T \)
  \( \Rightarrow \) \( ut \) is \( LT^{-1} \cdot T = L \), i.e. a length

- \( a \) is acceleration, \( LT^{-2} \)
  \( \Rightarrow \) \( at^2 \) is also a length

You cannot add apples and oranges! Terms must be **dimensionally compatible**.
Dimensional homogeneity

The equation should be true regardless of units. This is achieved if the left- and right-hand sides have the same dimensions.

- \( s = \frac{1}{2} gt^2 \)
  
  \( g \) is an acceleration \( LT^{-2} \), so \( \frac{1}{2} gt^2 \) is \( L \)
  
  \( \rightarrow \) dimensionally homogeneous

- \( s = 4.6 t^2 \)
  
  Dimensionally inhomogeneous \( \rightarrow \) different answer if measure time in seconds, minutes, hours...
Dimensionless products

Products of variables which are dimensionless are always dimensionally homogeneous.

If a real world problem can be modelled by a dimensionally homogeneous equation (and no logarithms) then we can find the form of that equation using dimensional analysis.
What are the dimensions of a general product $m^\alpha g^\beta \tau^\gamma l^\delta \theta^\epsilon$?

$M^{\alpha} L^{\beta+\delta} T^{\gamma-2\beta}$
Pendulum analysis

\[ M^\alpha L^{\beta+\delta} T^{\gamma-2\beta} \] is dimensionless iff

\[
\begin{align*}
M : & \quad \alpha = 0 \\
L : & \quad \beta + \delta = 0 \\
T : & \quad \gamma - 2\beta = 0
\end{align*}
\]

which gives an infinite set of solutions – not enough equations!

- \( \alpha = 0 \implies m \) cannot appear in the model
- \( \theta \) has no units \( \implies \) value is arbitrary
Dimensionless products

There are three basic rules when forming dimensionless products:

1. Choose the dependent variable to appear once
2. Choose any variable that always appears in each dimensional equation
3. Choose any variable that always has zero exponent (e.g. $\theta$)
Our dependent variable is $\tau$, so choose $\gamma = 1$

But $\gamma - 2\beta \implies \beta = \frac{1}{2}$

$\delta = -\beta = -\frac{1}{2}$

$\epsilon$ is arbitrary, so choose $\epsilon = 0$

Thus our first dimensionless product is

$$\Pi_1 = m^0 g^{\frac{1}{2}} \tau^{\frac{1}{2}} l^{-\frac{1}{2}} \theta^0 = \tau \sqrt{\frac{g}{l}}$$
Pendulum – dimensionless product 2

- Already have $\tau$ in first product
- For second product choose $\gamma = 0$
- $\implies \beta = 0$
- $\implies \delta = 0$
- $\epsilon$ arbitrary – choose $\epsilon = 1$ (already used $\epsilon = 0$)

Thus our second dimensionless product is

$$\Pi_2 = m^0 g^0 \tau^0 l^0 \theta^1 = \theta$$
Pendulum analysis

Our dimensionless pendulum equation will relate the dps in some way

\[ \Pi_1 = f(\Pi_2) \]

\[ \Rightarrow \tau \sqrt{\frac{g}{l}} = f(\theta) \]

\[ \tau = \sqrt{\frac{l}{g} f(\theta)} \]

Quick check:

- LHS has dimension \( T \)
- RHS has dimension \( \sqrt{\frac{L}{LT^{-2}}} = T \)
- Equation is \textit{dimensionally homogeneous}
Pendulum analysis

\[ \tau = \sqrt{\frac{l}{g}} f(\theta) \]

What have we learned?

- \( \tau \) does not depend on \( m \) \( \Rightarrow \) ‘only’ \( 5^3 = 125 \) experiments
- changing units of time cannot change the actual period \( \tau \) – there’s a corresponding change in ‘\( g \)’ \( \Rightarrow \) equation is dimensionally homogeneous.

We shall make this a bit more rigorous in a moment.

But for now...
Pendulum analysis

\[ \tau = \sqrt{\frac{l}{g}} \cdot h(\theta) \]

If we keep \( \theta = \theta_0 = \text{constant} \) and vary \( l \) then

\[ \frac{\tau_1}{\tau_2} = \sqrt{\frac{l_1}{l_2}} \]

i.e.

- \( \tau \propto \sqrt{l} \) \textit{regardless of} \( h \).
- \( \Rightarrow \) graph of \( \tau \) against \( \sqrt{l} \) should be linear
- \( \Rightarrow \) simple test requiring only 5 points!

If test fails, we go back and check our assumptions...
Pendulum analysis

What about $h(\theta)$? Fix $l = l_0$ and vary $\theta$:

$$\frac{\tau_1}{\tau_2} = \frac{h(\theta_1)}{h(\theta_2)}$$

⇒ plot a graph of $\tau$ vs. $\theta$ (or better, $\tau \sqrt{\frac{g}{l}}$ vs. $\theta$)

- Can get $h(\theta)$ directly! Another insight gained ...

NB Whilst can do SHO analytically, cannot do the general case for arbitrary $\theta$ as it is non-linear ...
A pendulum

t\left(\frac{g}{l}\right)^{1/2}

\begin{align*}
\text{SHO} \\
\theta
\end{align*}
Original problem had 1 dependent and 4 independent variables

We had 3 dimensional constraints

Hence need $5 - 3 = 2$ dimensionless products

Result was an equation determined up to an unknown function of 1 dimensionless product
A problem with \( n \) variables and \( m \) independent dimensional constraints can be written in dimensionally homogeneous form using \((n - m)\) dimensionless products (dps) as

\[
f (\Pi_1, \Pi_2, \ldots \Pi_{n-m}) = 0
\]
Example 1

\[ n = 4, \ m = 3 \]

- ⇒ need 1 product including the dependent variable
- ⇒ \( f (\Pi_1) = 0 \) so can solve to get \( \Pi_1 = \text{constant} \)
- ⇒ can get dependent variable = (unknown constant) \( \times \) (other variables)
Example 2

\[ n = 5, \ m = 3 \]

- ⇒ need 2 products
- ⇒ \( f (\Pi_1, \Pi_2) = 0 \)
- can solve to get \( \Pi_1 = h(\Pi_2) \)
- ⇒ can get equation up to an unknown function of a single dimensionless product, as with the pendulum
Example 3

\[ n = 6, \ m = 3 \]

- need 3 products
- \( f (\Pi_1, \Pi_2, \Pi_3) = 0 \)
- so can solve to get \( \Pi_1 = h (\Pi_2, \Pi_3) \)
- can get equation up to an unknown function of two dimensionless products, etc.

We always need to construct \((n-m)\) independent dps, with the dependent variable only appearing once, e.g. in \( \Pi_1 \).

NB Good to put the most sensitive variables into the dp with the independent variable (e.g. \( \Pi_1 \)) to minimise the amount of unknown behaviour and simplify experiments.
Buckingham’s Π-Theorem Example

Predict the period of 2 masses \((m_1 & m_2)\) orbiting each other at a distance \(R\) apart, in vacuum

<table>
<thead>
<tr>
<th>Variable</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tau)</td>
<td>(T)</td>
</tr>
<tr>
<td>(m_1)</td>
<td>(M)</td>
</tr>
<tr>
<td>(m_2)</td>
<td>(M)</td>
</tr>
<tr>
<td>(R)</td>
<td>(L)</td>
</tr>
<tr>
<td>(G)</td>
<td>(L^3M^{-1}T^{-2})</td>
</tr>
</tbody>
</table>

So we have 5 variables and 3 dimensions \(\Rightarrow\) need 2 dps
Buckingham’s $\Pi$-Theorem Example

General form of dp:

$$\Pi = \tau^\alpha m_1^\beta m_2^\gamma R^\delta G^\epsilon$$
$$= T^\alpha M^\beta M^\gamma L^\delta L^{3\epsilon} M^{-\epsilon} T^{-2\epsilon}$$
$$= M^{\beta + \gamma - \epsilon} L^{3\epsilon + \delta} T^{\alpha - 2\epsilon}$$

i.e. coefficients:

- $T: \alpha - 2\epsilon = 0$
- $M: \beta + \gamma - \epsilon = 0$
- $L: \delta + 3\epsilon = 0$
Buckingham’s Π-Theorem Example

\[ T : \quad \alpha - 2\epsilon = 0 \]
\[ M : \quad \beta + \gamma - \epsilon = 0 \]
\[ L : \quad \delta + 3\epsilon = 0 \]

\( \Pi_1 : \) include \( \tau \) once \( \Rightarrow \alpha = 1 \Rightarrow \epsilon = \frac{1}{2} \Rightarrow \delta = -\frac{3}{2} \Rightarrow \beta + \gamma = \frac{1}{2} \)

so free choice, e.g. \( \beta = 1/2 \) and \( \gamma = 0 \)

\( \Rightarrow \Pi_1 = \tau m_1^{1/2} R^{-3/2} G^{1/2} = \tau \sqrt{\frac{m_1 G}{R^3}} \)
Buckingham’s Π-Theorem Example

\[ T : \quad \alpha - 2\epsilon = 0 \]
\[ M : \quad \beta + \gamma - \epsilon = 0 \]
\[ L : \quad \delta + 3\epsilon = 0 \]

\( \Pi_2 : \) set \( \alpha = 0 \) \( \Rightarrow \epsilon = 0 \) \( \Rightarrow \delta = 0 \) \( \Rightarrow \beta + \gamma = 0 \) so free choice except must not choose same as before, e.g. \( \beta = 1 \) and \( \gamma = -1 \)

\[ \Rightarrow \Pi_2 = \frac{m_1}{m_2} \]
Buckingham’s Π-Theorem Example

Hence \( f (\Pi_1, \Pi_2) = 0 \)

\[ \Rightarrow \tau = \sqrt{\frac{R^3}{m_1 G}} \cdot h \left( \frac{m_1}{m_2} \right) \]

Exact analytic answer: \( \tau = 2\pi \sqrt{\frac{R^3}{G(m_1+m_2)}} \)

\[ \Rightarrow h \left( \frac{m_1}{m_2} \right) = \frac{1}{\sqrt{1 + \frac{m_1}{m_2}}} \]
Summary of Methodology

1. Decide your \( n \) variables, hence \( m \) dimensional constraints.
2. Form complete set of \((n - m)\) dimensionless products (dps):
   - dependent variable only appears once (e.g. in \( \Pi_1 \))
   - put most sensitive variables into same dp
   - check each dp found has no dimensions!
3. Apply Buckingham’s \( \Pi \)-Theorem and hence solve for dependent variable.
4. Test assumptions made (e.g. \( \tau \propto \sqrt{l} \) for pendulum).
5. Conduct further experiments necessary to find any unknown functions, or further computations based upon the dps found.