Mathematical Modelling
Lecture 3 – Similitude

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Overview of Course

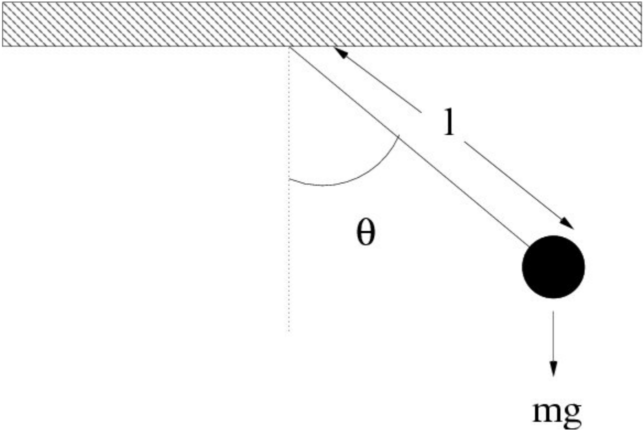
- Model construction ⟶ dimensional analysis
- Experimental input ⟶ fitting
- Finding a 'best' answer ⟶ optimisation
- Tools for constructing and manipulating models ⟶ networks, differential equations, integration
- Tools for constructing and simulating models ⟶ randomness
- Real world difficulties ⟶ chaos and fractals

A First Course in Mathematical Modeling by Giordano, Weir & Fox, pub. Brooks/Cole. Today we're in chapter 8.

Aim

- To exploit dimensional analysis to construct realistic scaled models

You are feeling very sleepy...



Suppose someone wanted to know how the period of a 100m long pendulum varied with the starting angle θ.

A smaller pendulum

100m is a bit big to perform experiments on! Consider a 1m scaled model pendulum:

$$\begin{aligned} \tau &= \sqrt{\frac{l}{g}}h(\theta) \\ l_{actual} &= 100 l_{model} \end{aligned}$$

So provided we start at the same angle θ,

$$\Rightarrow \tau_{actual} = 10\tau_{model}$$

so we can carry out any experiments on a scaled model and then scale up the results.

Beyond pendula

For our 100m pendulum we were able to build a scaled model based on our knowledge of its behaviour.

What do we do when we don't know what the behaviour is?

Dimensional analysis again

Suppose we've done our dimensional analysis and have a set of $(n - m)$ dimensionless products:

$$f(\Pi_1, \Pi_2, ..., \Pi_{n-m}) = 0$$

and we've rearranged it into the form,

$$\Pi_1 = g(\Pi_2, ..., \Pi_{n-m})$$

How do we build a scaled model to perform experiments on?

Scaled modelling

$$\Pi_1 = g\left(\Pi_2, ..., \Pi_{n-m}\right)$$

Remember that the behaviour only depends on the *dimensionless products*. If the dps are the same, the behaviour *must* be the same.

Skyscraper

Suppose we want to know the vibration of a skyscraper. How do we model it?

Identify variables

- height h
- cross-sectional area A
- density ρ
- Young's modulus E
- period τ

Assumptions: ignore density variations, so for example could consider a solid block of steel.

Skyscraper

Variable	Dimension
τ	T
h	L
A	L^2
ρ	ML^{-3}
E	$ML^{-1}T^{-2}$

5 variables, 3 dimensions $\Rightarrow$ 2 dps.

$$\begin{aligned} \Pi &= \tau^\alpha h^\beta A^\gamma \rho^\delta E^\epsilon \\ &= T^{\alpha-2\epsilon} M^{\delta+\epsilon} L^{\beta+2\gamma-3\delta-\epsilon} \end{aligned}$$

Skyscraper

$$\begin{aligned} T &: & \alpha - 2\epsilon &= 0 \\ M &: & \delta + \epsilon &= 0 \\ L &: & \beta + 2\gamma - 3\delta - \epsilon &= 0 \end{aligned}$$

- Dependent variable is τ, so choose $\alpha = 1 \Rightarrow \epsilon = \frac{1}{2} \Rightarrow \delta = -\frac{1}{2}$

$$\Pi_1 = \frac{\tau}{h}\sqrt{\frac{\epsilon}{\rho}}$$

Skyscraper

$$
\begin{aligned}
T &: \quad \alpha - 2\epsilon &= 0 \\
M &: \quad \delta + \epsilon &= 0 \\
L &: \quad \beta + 2\gamma - 3\delta - \epsilon &= 0
\end{aligned}
$$

- Now choose $\alpha = 0 \Rightarrow \epsilon = 0 \Rightarrow \delta = 0$

$$\Pi_2 = \frac{h^2}{A}$$

Skyscraper

$$\Pi_1 = \frac{\tau}{h}\sqrt{\frac{\epsilon}{\rho}}$$

$$\Pi_2 = \frac{h^2}{A}$$

Now, $f(\Pi_1, \Pi_2) = 0 \Rightarrow \Pi_1 = g(\Pi_2)$

$$\Rightarrow \tau = h\sqrt{\frac{\rho}{\epsilon}}.g\left(\frac{h^2}{A}\right)$$

Design conditions

So we've got our model, but how do we choose a scaled model?

$$\Pi_1 = g\left(\Pi_2, ..., \Pi_{n-m}\right)$$

So provided our scaled model has the same values for $\Pi_2, ..., \Pi_{n-m}$ as the real thing, it will have the same Π_1.

i.e. we need

$$\Pi_2^{model} = \Pi_2^{actual}$$
$$\Pi_3^{model} = \Pi_3^{actual}$$

... and so on. These are called *Design Conditions*.

Skyscraper modelling

In the case of our skyscraper we only have 2dps, so we simply construct a scaled model such that

$$\Pi_2 = \frac{h^2}{A}$$

is the same as the real thing, and then we know that

$$\Pi_1 = \frac{\tau}{h}\sqrt{\frac{\epsilon}{\rho}}$$

will also be the same.

A real skyscraper

Assume $5m$ per floor, 20 storeys:

$$\begin{aligned} h &= 100m \\ A &= 400m^2\,(20m \times 20m) \end{aligned}$$

Let's make our model $1m$ high, and we need:

$$\begin{aligned} \left(\frac{h^2}{A}\right)^{model} &= \left(\frac{h^2}{A}\right)^{actual} = 25 \\ \Rightarrow A^{model} &= \frac{1}{25}m^2 \\ &= 20cm \times 20cm \end{aligned}$$

A real skyscraper

We have

$$\left(\frac{\tau}{h}\sqrt{\frac{E}{\rho}}\right)^{actual} = \left(\frac{\tau}{h}\sqrt{\frac{E}{\rho}}\right)^{model}$$

$$\Rightarrow \tau^{actual} = \tau^{model}\left(\frac{h^{actual}}{h^{model}}\right)\sqrt{\frac{E^{model}\rho^{actual}}{\rho^{model}E^{actual}}}$$

A real skyscraper

Let's assume the real thing is mostly steel:

$$E_{steel} = 209 \times 10^9 Nm^{-2}$$
$$\rho_{steel} = 7.9 \times 10^3 Kgm^{-3}$$

and let's build our model out of perspex:

$$E_{perspex} = 3.0 \times 10^9 Nm^{-2}$$
$$\rho_{perspex} = 1.2 \times 10^3 Kgm^{-3}$$

Scaled model skyscraper

Now we have

$$\tau^{actual} \approx 30.74\tau^{model}$$

So if we measure $\tau^{model} = 0.21s$, then

$$\tau^{actual} \approx 6.5s$$

Fluid dynamics

We often have to use scaled models in fluid dynamics. There are 8 common variables:

Variable	Symbol	Dimension
velocity	v	LT^{-1}
length	r	L
mass density	ρ	ML^{-3}
viscosity	μ	$ML^{-1}T^{-1}$
gravitational acceleration	g	LT^{-2}
speed of sound	c	LT^{-1}
surface tension	σ	MT^{-2}
pressure	p	$ML^{-1}T^{-2}$

Fluid dynamics

8 common variables, 3 dimensional constraints $\Rightarrow$ 5 dimensionless products. The conventional ones are:

Name	Symbol	Definition
Reynolds number	R_e	$\frac{vr\rho}{\mu}$
Froue number	F	$\frac{v^2}{rg}$
Mach number	M	$\frac{v}{c}$
Weber number	W	$\frac{\rho v^2 r}{\sigma}$
Pressure coefficient	P	$\frac{p}{\rho v^2}$

The Reynolds number

Conventional dps simplify the dimensional analysis – just choose the appropriate subset for a given problem, and apply the Buckingham Π-theorem.

The Reynolds number R_e is the only one that contains viscosity, μ. For this reason it is qualitatively different, and its magnitude is often important. In particular it controls the nature of the fluid flow.

Flow

$$R_e = \frac{vr\rho}{\mu}$$

- $R_e < 2000 \rightarrow$ laminar flow
- $R_e > 3000 \rightarrow$ turbulent flow

The critical value at which flow becomes turbulent is difficult to predict.

A submarine

Let's model the drag force D on a submarine.

$$D = p \times A$$

What does D depend on?

- Fluid velocity v
- Fluid density ρ
- Viscosity μ
- Velocity of sound c
- Size and shape

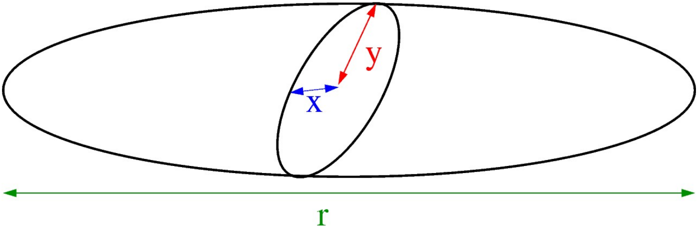
Example – a submarine

How can we model the size and shape of a submarine?

What shape does a submarine have?

A submarine

Let's model with an ellipsoid



Use length r and two shape factors $\frac{x}{r}$ and $\frac{y}{r}$ (dimensionless!).

A submarine

Now have the drag force D is

$$D = f\left(v, \rho, \mu, c, r, \frac{x}{r}, \frac{y}{r}\right)$$

We have 6 dimensioned variables, and 3 dimensional constraints $\Rightarrow$ need 3 dimensionless constants.

Choose

$$\begin{aligned} R_e &= \frac{vr\rho}{\mu} \\ M &= \frac{v}{c} \\ P &= \frac{p}{\rho v^2} \end{aligned}$$

A submarine

$$R_e = \frac{vr\rho}{\mu}$$
$$M = \frac{v}{c}$$
$$P = \frac{p}{\rho v^2}$$

... but where's D?! Recall:

$$\begin{aligned} D &= p \times A \\ &= \kappa p r^2 \\ \Rightarrow P &= \frac{p}{\rho v^2} \\ &= \frac{D}{\kappa \rho v^2 r^2} \end{aligned}$$

A submarine

Now we have

$$\begin{aligned} f\left(R_e, M, P, \frac{x}{r}, \frac{y}{r}\right) &= 0 \\ \Rightarrow P &= g\left(R_e, M, \frac{x}{r}, \frac{y}{r}\right) \\ \Rightarrow D &= \kappa\rho v^2 r^2 g\left(R_e, M, \frac{x}{r}, \frac{y}{r}\right) \end{aligned}$$

So we need:

- same $\frac{x}{r}$ and $\frac{y}{r}$ (i.e. geometrically similar model)
- $R_e^{model} = R_e^{actual}$ (same flow conditions)
- $M^{model} = M^{actual}$ (same compressibility, shock-waves etc.)

i.e. 4 design conditions.

A submarine

If we use the same fluid for the model:

$$
\begin{aligned}
R_e^{model} &= R_e^{actual} \\
\Rightarrow (vr)^{model} &= (vr)^{actual}
\end{aligned}
$$

But:

$$
\begin{aligned}
M^{model} &= M^{actual} \\
\Rightarrow v^{model} &= v^{actual} \\
\Rightarrow r^{model} &= r^{actual}
\end{aligned}
$$

Our model has to be the same length as the real thing! Oh dear...

A submarine

In order to satisfy all 4 design conditions, we will need to find a different fluid. This can be tricky to get right!

If we do manage it, then

$$\begin{aligned} P^{model} &= P^{actual} \\ \Rightarrow \left(\frac{D}{\rho v^2 a^2}\right)^{model} &= \left(\frac{D}{\rho v^2 a^2}\right)^{actual} \\ \Rightarrow \frac{D^{act}}{D^{mod}} &= \left(\frac{\rho^{act}}{\rho^{mod}}\right)\left(\frac{v^{act}}{v^{mod}}\right)^2\left(\frac{a^{act}}{a^{mod}}\right)^2 \end{aligned}$$

Then all we have to do is measure D^{model} and we can compute D^{actual}.

Summary

We've seen how the dimensionless products lead naturally to design conditions.

- Buckingham: $\Pi_1 = g(\Pi_2, \Pi_3, ...)$
- For an accurate scaled model, $\Pi_2^{model} = \Pi_2^{real}$ etc. These are the Design Conditions for the model.
- Π_1 allows us to convert between the model behaviour and the behaviour of the real system.