

General Equilibrium Theories of the Equity Risk Premium: Estimates and Tests

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Abstract

This paper provides new estimates and tests of a number of leading general equilibrium theories of the price of equity and, to our knowledge, the first estimates of the time-varying equity premia implied by these models. Three general equilibrium theories are examined: the consumption-CAPM with power utility, the Epstein-Zin general equilibrium model with time non-separable preferences and habit-persistence models. We compare these models with the more restrictive unconditional CAPM, and with more general models based on discount factors which encompass all of these models. Rather than use GMM estimation or calibration, we propose a new model, the multi-variate GARCH-covariance-in-mean (MGCM) model that is constrained to satisfy a no-arbitrage condition by including conditional covariance terms in the mean of the asset-pricing equation. This enables us to estimate time-varying risk premia and to investigate the contribution to the risk premium of macroeconomic sources of equity risk. In particular, we examine the role of inflation in pricing the nominal equity risk premium and show that it has importance beyond that implied by the canonical models. Estimates are obtained for monthly data from 1975-2001 for the US and UK stock markets.

1 Introduction

Surprisingly, given the large literature on modelling equity returns using general equilibrium asset-pricing theory and on the equity premium puzzle, to our knowledge, there are no estimates of a time-varying equity risk premium based on these models. Using a new econometric methodology, in this paper we provide estimates both of the parameters of the leading general equilibrium models of equity pricing model and of the implied time-varying equity risk premia. Three general theories are examined: the consumption-based capital asset pricing model (C-CAPM) with power utility, the Epstein-Zin general equilibrium model with time non-separable preferences and various habit-persistence models.

A common feature of these general equilibrium models and the more restrictive CAPM is that they have observable macroeconomic discount factors. Our econometric methodology is based on modelling the joint distribution of equity returns and these observable macroeconomic variables using a new model, the multi-variate GARCH-covariance-in-mean model, in which the asset pricing equations are constrained to satisfy the condition of no-arbitrage. This is achieved by including conditional covariances of the excess equity returns and the macroeconomic factors in the mean of the asset-pricing equations. An immediate implication is that it is necessary for the “in-mean” effects to be conditional covariances rather than conditional variances and, because conditional covariances are required to achieve no-arbitrage, that multi-variate not univariate GARCH models must be used and conditional covariances are required in the mean, not conditional variances. This is in contrast to most of the time-series econometric models of equity in the literature which are univariate and do not include conditional covariances.

Using monthly data for the US and UK from 1975 to 2001 we show that a two-factor model with consumption growth and inflation as the observable factors that does not satisfy the general equilibrium restrictions is found to perform better than both the general equilibrium models and the CAPM. The estimated equity risk premia display considerable time variation, tending to increase sharply during periods of negative excess returns before slowly declining, but no corresponding jump is discernible when excess returns are high. We also confirm previous findings that the estimates of the coefficients of relative risk aversion in the power utility and Epstein-Zin models are implausibly large. It follows that the capital asset pricing model (CAPM) can also be rejected as it ignores significantly priced sources of risk.

Hansen and Singleton (1983) were one of the first to estimate a general equilibrium model of equity returns. Their model was based on power utility and they estimated the Euler equation by GMM. The best-known example of this approach is that of Mehra and Prescott (1985) who used power utility and calibration methods. Their finding that the coefficient of relative risk

aversion must be implausibly large in order to match the observed (ex-post) risk premium is known as the equity premium puzzle. Cochrane (2006) comments: "finance has a lot to say about macroeconomics and it says that something is desperately wrong with most macroeconomic models". The basis of this remark is the failure of the marginal value of wealth to be large enough to "make sense of the basic stock market facts". In other words, the problem is the equity premium puzzle.

The response to the equity premium puzzle has been to try to modify the theory so that it possesses other means of capturing the risk premium, and doesn't have to rely solely on the coefficient of relative risk aversion to do so. One route is to assume that the utility function is time non-separable. This permits the coefficient of relative risk aversion to be different from the elasticity of inter-temporal substitution, and so introduces additional variables into the expression for the equity risk premium. This was also the method employed by Epstein and Zin (1991) who used a constant elasticity version of Kreps and Porteus (1978) time non-separable utility. GMM estimates of the Euler equation were obtained by Epstein and Zin (1991). More ad hoc estimation methods are used by Campbell (1996), and calibration of the risk premium is undertaken by Campbell and Cochrane (1999) and Campbell (2002).

A widely-used alternative time non-separable model to that of Epstein-Zin model are the habit persistence models of Abel (1990) and Constantinides (1990). Their aim is to make the inter-temporal marginal rate of substitution more variable and so be better able to match the observed volatility of equity returns. This approach has been implemented empirically by Abel (1990) using GMM, and by Campbell and Cochrane (1999) using calibration. However, both implementations were based on using unconditional, and not conditional, moments which obscures the fact that these particular models of habitual consumption in effect add nothing to the variability of the risk premium. We show that the success of this approach depends not on the habit-persistence features of these models but on the contemporaneous stochastic properties of the habit process. In our view this undermines the usefulness of habit persistence models in resolving the equity premium puzzle.

Although these general equilibrium models can be written in terms of stochastic discount factors (SDFs), not all asset pricing models based on stochastic factors have a general equilibrium interpretation, see Cochrane (2005a) and Smith and Wickens (2002). General equilibrium models of asset prices are a special case: they involve a particular choice of SDF and strong restrictions on how these factors generate the risk premium. In the more general SDF approach the coefficient on the consumption covariance does not necessarily have the interpretation of the coefficient of relative risk aversion. And other variables, such as output and other rates of return, may be used

to form SDFs. The decision of which variables to employ depends on whether they are significant priced sources of equity risk.

General equilibrium models usually entail that investors are concerned with future consumption, and in particular, consumption next period. The main holders of equity are financial institutions, especially pension funds. They act on behalf of investors' consumption at a much more distant point in the future. In assessing risk, financial institutions focus largely on short-term performance, and on the value of the portfolio. This suggests that the market equity risk premium may be more influenced by short-term price risk than longer-term considerations of consumption associated with the business cycle and inflation. This is examined in Smith, Sorensen and Wickens (2006). Similar considerations led to the finding in Wickens and Smith (2001) that the FOREX risk premium is better explained by short-term exchange rate risk as captured by output and money growth rates, than by consumption.

In most existing empirical implementations using the SDF framework, the "factors" are latent variables. Equations for the latent factors must be specified. Since data only exist for the returns, estimates of the parameters of these equations must be obtained from the likelihood function for returns. Estimates of the factors and a time-varying risk premium can then be backed out of the returns data. This is the approach commonly followed for bond pricing and has been used for pricing equity by Bekaert and Grenadier (1999). It is not, however, suitable for the CAPM and C-CAPM because here the factors are observable macroeconomic variables, namely, consumption growth (among others) for the C-CAPM, and the market interest rate in the case of the CAPM. Ang and Piazzesi (2004) combine observed and unobserved factors, but are unable to measure the risk premium. One reason we propose our new econometric methodology is to obtain direct estimates of a time-varying risk premium for the model when the factors are observable. This allows us to examine its properties and, in particular, how it varies over the business cycle.

There is another, rapidly increasing, literature that focusses on the relation between equity returns and their volatility, see for example Campbell (1987), Baillie and DeGennero (1988), Campbell and Hentschel (1992), Glosten, Jagannathan and Runkle (1993), Scruggs (1998) and Lettau and Ludvigson (2003). This is motivated more by statistical than theoretical considerations such as satisfying a no-arbitrage condition. As a result, in principle, these statistical models admit unlimited arbitrage opportunities. One of the main findings is that average equity returns are positively related to their volatility. The usual interpretation, informally offered, is that higher volatility implies greater risk, and larger returns are required to compensate for this. In other words, the volatility is capturing a risk premium. In general, however, this explanation cannot be correct as risk should be expressed in terms of the forecast covariance of returns with other

factors. The volatility of returns can only measure risk if the own return is regarded as a factor.

The paper is set out as follows. In Section 2 we present the asset-pricing models to be estimated. In Section 3 we describe the econometric methodology to be used. The estimates, the test results and plots of the estimated equity risk premium are reported in Section 4. Conclusions are given in Section 5.

2 Theoretical models of the equity risk premium

2.1 Asset pricing of equity returns using stochastic discount factors

P_t , the price of an asset at the beginning of period t , is determined by the expected discounted value of its pay-off at the start of period $t + 1$, namely, X_{t+1} :

$$P_t = E_t[M_{t+1}X_{t+1}] \quad (1)$$

where M_{t+1} is the stochastic discount factor, or pricing kernel (see Cochrane (2005a) for a survey of SDF theory). For equity, the payoff in real terms is $X_{t+1} = P_{t+1} + D_{t+1}$, where D_{t+1} are dividend payments assumed to be made at the start of period $t + 1$. The pricing equation can also be written

$$1 = E_t[M_{t+1} \frac{X_{t+1}}{P_t}] = E_t[M_{t+1}R_{t+1}], \quad (2)$$

where $R_{t+1} = X_{t+1}/P_t = (P_{t+1} + D_{t+1})/P_t$ is the gross real return. If $m_{t+1} = \ln M_{t+1}$, $r_{t+1} = \ln R_{t+1}$ and r_t^f the logarithm of the risk-free rate are jointly normally distributed, then it can be shown that the expected excess real return on equity is given by

$$E_t(r_{t+1} - r_t^f) + \frac{1}{2}V_t(r_{t+1}) = -Cov_t(m_{t+1}, r_{t+1}). \quad (3)$$

The right-hand side is the risk premium and $\frac{1}{2}V_t(r_{t+1})$ is the Jensen effect.

The no-arbitrage condition (3) can also be expressed in terms of nominal returns. If i_{t+1} is the nominal return to equity, i_t^f is the nominal risk-free rate, P_t^c is the consumer price index and inflation is given by $1 + \pi_{t+1} = P_{t+1}^c/P_t^c$, then equation (1) can be re-written as:

$$1 = E_t[M_{t+1} \frac{P_t^c}{P_{t+1}^c} (1 + i_{t+1})]$$

where $M_{t+1} \frac{P_t^c}{P_{t+1}^c}$ is the nominal discount factor. The no-arbitrage condition is now:

$$E_t(i_{t+1} - i_t^f) + \frac{1}{2}V_t(i_{t+1}) = -Cov_t(m_{t+1}, i_{t+1}) + Cov_t(\pi_{t+1}, i_{t+1}). \quad (4)$$

Thus using nominal returns there is an extra term: the conditional covariance of inflation with the nominal (excess) return

A general linear factor model where z_{it} ($i = 1, \dots, n$) are n factors that are jointly log normally distributed with equity returns implies the discount factor

$$m_{t+1} = - \sum_{i=1}^n \beta_i z_{i,t+1}$$

and the no-arbitrage condition

$$\begin{aligned} E_t(r_{t+1} - r_t^f) + \frac{1}{2}V_t(r_{t+1}) &= \sum_i \beta_i Cov_t(z_{i,t+1}, r_{t+1}) \\ &= \sum_i \beta_i f_{it}, \end{aligned}$$

where the f_{it} are known as *common factors*. Such models will not necessarily have a general equilibrium interpretation and the factors may be observed or unobserved processes

It is possible to characterize alternative asset-pricing models according to their stochastic discount factors $z_{i,t+1}$, whether these are defined explicitly or implicitly.

2.2 Consumption-based models

We consider three models: C-CAPM with power utility, the Epstein-Zin time non-separable model and the habit persistence model, which is another time non-separable model.

2.2.1 C-CAPM with power utility

The canonical model of the discount factor is the C-CAPM which implicitly defines M_{t+1} as

$$M_{t+1} = \frac{\beta U'(C_{t+1})}{U'(C_t)}$$

where C_t is consumption and $U(C_t)$ is utility. For the power utility function $U = \frac{C_t^{1-\sigma} - 1}{1-\sigma}$ with constant coefficient of relative risk aversion σ ,

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma}$$

Hence the no-arbitrage condition for real excess returns is

$$E_t(r_{t+1} - r_t^f) + \frac{1}{2}V_t(r_{t+1}) = \sigma Cov_t(\Delta c_{t+1}, r_{t+1})$$

where $c_t = \ln C_t$. In nominal terms we have the nominal discount factor, $\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t^c}{P_{t+1}^c}$ and so the no-arbitrage condition can be written

$$E_t(i_{t+1} - i_t^f) + \frac{1}{2}V_t(i_{t+1}) = \sigma Cov_t(\Delta c_{t+1}, i_{t+1}) + Cov_t(\pi_{t+1}, i_{t+1}) \quad (5)$$

2.2.2 Time non-separable preferences

Epstein-Zin utility The C-CAPM with power utility uses a von-Neumann-Morgenstern (VNM) type of utility function. It restricts the coefficient of relative risk aversion to be equal to the elasticity of inter-temporal substitution. This restriction can be relaxed using a non-VNM type of utility function which is time non-separable. A general formulation of time non-separable utility proposed by Kreps and Porteus (1987) is

$$\mathcal{U}_t = \mathcal{U}[C_t, E_t(\mathcal{U}_{t+1})]$$

Giovannini and Weil (1989) show that maximising \mathcal{U}_t subject to the budget constraint

$$\begin{aligned} W_{t+1}^R &= R_{t+1}^m (W_t^R - C_t) \\ R_{t+1}^m &= \sum_{j=1}^n \alpha_{kt} R_{j,t+1}, \quad \sum_{j=1}^n \alpha_{kt} = 1 \end{aligned}$$

where R_{t+1}^m is the real return on the market portfolio of all invested real wealth W_t^R , $R_{k,t+1}$ is the return on the k^{th} asset $\{k = 1, \dots, n\}$ and α_{kt} is its portfolio share, gives the following Euler equations for the market and for any two assets k_1 and k_2 :

$$\begin{aligned} E_t[\mathcal{U}_{2,t} \frac{\mathcal{U}_{1,t+1}}{\mathcal{U}_{1,t}} R_{t+1}^m] &= 1 \\ E_t[\mathcal{U}_{2,t} \frac{\mathcal{U}_{1,t+1}}{\mathcal{U}_{1,t}} R_{k_1,t+1}] &= E_t[\mathcal{U}_{2,t} \frac{\mathcal{U}_{1,t+1}}{\mathcal{U}_{1,t}} R_{k_2,t+1}] \end{aligned}$$

where $\mathcal{U}_{1,t} = \frac{\partial \mathcal{U}}{\partial C_t}$ and $\mathcal{U}_{2,t} = \frac{\partial \mathcal{U}}{\partial E_t(\mathcal{U}_{t+1})}$. Thus the real discount factor is

$$M_{t+1} = \mathcal{U}_{2,t} \frac{\mathcal{U}_{1,t+1}}{\mathcal{U}_{1,t}}$$

It follows that

$$E_t[\mathcal{U}_{2,t} \frac{\mathcal{U}_{1,t+1}}{\mathcal{U}_{1,t}} (r_{t+1} - r_t^f)] = 0.$$

Epstein and Zin (1989, 1990, 1991) have implemented a special case of this based on the constant elasticity of substitution (CES) function:

$$\mathcal{U}_t = \left[(1 - \beta) C_t^{1 - \frac{1}{\gamma}} + \beta [E_t(\mathcal{U}_{t+1}^{1 - \sigma})]^{\frac{1 - \frac{1}{\gamma}}{1 - \sigma}} \right]^{\frac{1}{1 - \frac{1}{\gamma}}}$$

where β is the discount rate, σ is the coefficient of relative risk aversion and γ is the elasticity of inter-temporal substitution. In the separable power utility case $\sigma = 1/\gamma$. The Euler equation derived by Epstein and Zin is

$$E_t \left\{ \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\gamma}} R_{t+1}^m \right]^{\frac{1 - \sigma}{1 - \frac{1}{\gamma}}} \right\} = 1$$

implying that the real stochastic discount factor is

$$M_{t+1} = \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\gamma}} \right]^{\frac{1-\sigma}{1-\frac{1}{\gamma}}} (R_{t+1}^m)^{\frac{1-\sigma}{1-\frac{1}{\gamma}} - 1}$$

It follows that,

$$E_t \left\{ \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\gamma}} \right]^{\frac{1-\sigma}{1-\frac{1}{\gamma}}} (R_{t+1}^m)^{\frac{1-\sigma}{1-\frac{1}{\gamma}} - 1} (r_{t+1} - r_t^f) \right\} = 0$$

Assuming log-normality, the no-arbitrage condition is

$$E_t(r_{t+1} - r_t^f) + \frac{1}{2}V_t(r_{t+1}) = -\frac{1-\sigma}{1-\gamma}Cov_t(\Delta c_{t+1}, r_{t+1}) + \frac{1-\gamma\sigma}{1-\gamma}Cov_t(r_{t+1}^m, r_{t+1}),$$

a result first derived by Campbell, Lo and MacKinlay (1997) where $r_{t+1}^m = \ln R_{t+1}^m$.

For nominal returns the Euler equation is

$$E_t \left\{ \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\gamma}} \right]^{\frac{1-\sigma}{1-\frac{1}{\gamma}}} \left(\frac{1+i_{t+1}^m}{1+\pi_{t+1}} \right)^{\frac{1-\sigma}{1-\frac{1}{\gamma}} - 1} \frac{1}{1+\pi_{t+1}} (i_{t+1} - i_t^f) \right\} = 0$$

where i_{t+1}^m is the nominal market return. This gives the no-arbitrage condition

$$E_t(i_{t+1} - i_t^f) + \frac{1}{2}V_t(i_{t+1}) = -\frac{1-\sigma}{1-\gamma}Cov_t(\Delta c_{t+1}, i_{t+1}) - \frac{\gamma(1-\sigma)}{1-\gamma}Cov_t(\pi_{t+1}, i_{t+1}) + \frac{1-\gamma\sigma}{1-\gamma}Cov_t(i_{t+1}^m, i_{t+1})$$

A practical difficulty in implementing this model empirically is obtaining a measure of the market rate of return, especially at monthly frequencies. In principle, the market return should take account of both financial and non-financial assets, including housing and human capital. Due to the problems of obtaining monthly data on non-financial returns, and noting that bond returns are linked to the risk-free rate, we approximate the market return by a fixed weight average of the stock market return and the risk-free rate. Our measure of the market return is therefore:

$$i_{t+1}^m = \theta i_{t+1} + (1-\theta)i_t^f$$

The Epstein-Zin no-arbitrage condition then becomes the fixed parameter model

$$E_t(i_{t+1} - i_t^f) + \left(\frac{1}{2} - \frac{\theta(1-\gamma\sigma)}{1-\gamma} \right) V_t(i_{t+1}) = -\frac{1-\sigma}{1-\gamma}Cov_t(\Delta c_{t+1}, i_{t+1}) - \frac{\gamma(1-\sigma)}{1-\gamma}Cov_t(\pi_{t+1}, i_{t+1}) \quad (6)$$

The interpretation of the coefficients is now changed compared with the no-arbitrage condition for power utility, equation (5). The coefficient on $Cov_t(\Delta c_{t+1}, i_{t+1})$ is no longer the coefficient of relative risk aversion, and the coefficients on $V_t(i_{t+1})$ and $Cov_t(i_{t+1}, \pi_{t+1})$ are no longer restricted.

We may write equation (6) as:

$$E_t(i_{t+1} - i_t^f) = \beta_1 V_t(i_{t+1}) + \beta_2 Cov_t(\Delta c_{t+1}, i_{t+1}) + \beta_3 Cov_t(\pi_{t+1}, i_{t+1}) \quad (7)$$

where $\sigma = \beta_2 - \beta_3 + 1$ and $\gamma = \frac{\beta_3}{\beta_2}$.

Habit persistence An alternative example of time non-separable utility is the habit persistence model. This assumes that

$$U_t = U(C_t, X_t)$$

where X_t is the habitual level of consumption. Again the aim is to introduce additional terms in the risk premium. Various specific functional forms for the utility function have been suggested. Abel (1990) proposed the habit persistence utility function

$$U_t = \frac{\left(\frac{C_t}{X_t}\right)^{1-\sigma} - 1}{1-\sigma}$$

where X_t is a function of past consumption, for example $X_t = C_{t-1}^\delta$. The stochastic discount factor becomes $M_{t+1} = \beta \left(\frac{C_{t+1}}{X_{t+1}}\right)^{-\sigma}$. By choosing $x_t = \ln X_t$ suitably, the aim is to produce a discount factor that displays greater volatility and hence a larger risk premium. Assuming log normality, the no-arbitrage condition is

$$E_t(r_{t+1} - r_t^f) + \frac{1}{2}V_t(r_{t+1}) = \sigma Cov_t(\Delta c_{t+1}, r_{t+1}) - \sigma Cov_t(\Delta x_{t+1}, r_{t+1})$$

This equation has been estimated by unconditional moment estimators such as GMM and calibration methods.

Although such estimators allow the extra term to contribute to the explanation of the equity risk premium, by implementing habit persistence in this may they disguise a logical problem, since habit is necessarily a reflection of past consumption it cannot contribute to the risk premium. To see this, consider the *conditional* covariance of returns with the habit term when habits are generated, as here, by $x_{t+1} = \delta c_t$. In this case

$$Cov_t(\Delta x_{t+1}, r_{t+1}) = \delta Cov_t(\Delta c_t, r_{t+1}) = 0$$

and so the habit persistence model would be of no help in providing additional terms in the risk premium. Thus, if risk arises from conditional and not unconditional covariation then the risk premium is unchanged in this habit persistence model.

Constantinides (1990) proposed a different utility function with habits of form

$$U_t = \frac{(C_t - \lambda X_t)^{1-\sigma} - 1}{1-\sigma}$$

The stochastic discount factor implied by this is

$$M_{t+1} = \beta \left(\frac{C_{t+1} - \lambda X_{t+1}}{C_t - \lambda X_t}\right)^{-\sigma}$$

Campbell and Cochrane (1999) implemented this with the restriction that $\lambda = 1$. They introduced the concept of surplus consumption, defined as $S_t = \frac{C_t - X_t}{C_t}$. The discount factor can then be written

$$M_{t+1} = \beta \left(\frac{C_{t+1} S_{t+1}}{C_t S_t} \right)^{-\sigma}$$

Assuming log normality, the no-arbitrage condition is

$$E_t(r_{t+1} - r_t^f) + \frac{1}{2}V_t(r_{t+1}) = \sigma Cov_t(\Delta c_{t+1}, r_{t+1}) + \sigma Cov_t(s_{t+1}, r_{t+1})$$

where $s_{t+1} = \ln S_{t+1}$. Once again, the attraction of this approach is the inclusion of the extra term to account for the risk premium; the problem lies in the way it has been implemented.

Campbell and Cochrane assume that s_{t+1} is generated by an AR(1) process with a disturbance term whose variance depends on s_t . They do not estimate the resulting model but, like Constantinides, use calibration methods based on unconditional moments. By calibrating the *unconditional* variance of the error term of the AR process suitably, it is possible to force the covariance between r_{t+1} and s_{t+1} to be of the necessary size.

Consider once more the conditional moment version of the model on the assumption that s_{t+1} is the AR(1) process

$$s_{t+1} = \alpha s_t + \xi_{t+1}$$

It follows that

$$Cov_t(s_{t+1}, r_{t+1}) = Cov_t(\xi_{t+1}, r_{t+1})$$

Hence, if s_{t+1} is generated in this way, then it is necessary for the innovation in surplus consumption to have a non-zero conditional covariance with returns. We note that if $X_t = C_{t-1}$ then $S_t \simeq \Delta c_{t+1}$. The additional habit term would then add nothing to the explanation of the risk premium as a conditional covariance with Δc_{t+1} is already present. Only if ξ_{t+1} is an independent source of risk is the risk premium affected.

2.3 CAPM

The CAPM relates the expected excess return on equity to the excess return of the market portfolio through

$$E_t(r_{t+1} - r_t^f) = \beta_t E_t(r_{t+1}^m - r_t^f)$$

where

$$\beta_t = \frac{Cov_t(r_{t+1}^m, r_{t+1})}{V_t(r_{t+1}^m)}$$

and

$$E_t(r_{t+1}^m - r_t^f) = \sigma_t V_t(r_{t+1}^m).$$

Thus beta is in general time-varying. It follows that we can re-write CAPM as

$$E_t(r_{t+1} - r_t^f) = \sigma_t \text{Cov}_t(r_{t+1}^m, r_{t+1})$$

We note that $(1 + r_{t+1}^m) = \frac{W_{t+1}^R}{W_t^R}$, where W_t^R is real wealth, and if consumption is proportional to wealth, as in the life cycle model, we obtain

$$E_t(r_{t+1} - r_t^f) = \sigma_t \text{Cov}_t(\Delta c_{t+1}, r_{t+1})$$

Thus we can interpret the CAPM as a single factor model defined by $\sigma_t(1 + r_{t+1}^m) = \sigma_t \frac{\Delta W_{t+1}^R}{W_t^R}$.

There is no Jensen effect because the assumption of log-normality is not made.

The corresponding expression for nominal returns is

$$E_t(i_{t+1} - i_t^f) = \sigma_t \text{Cov}_t(i_{t+1}^m, i_{t+1})$$

If $\sigma_t = \sigma$, as in the unconditional CAPM, and the market return is again approximated by a weighted average of the return on equity and the risk-free rate so that $i_{t+1}^m = \theta i_{t+1} + (1 - \theta) i_t^f$ then this would become

$$E_t(i_{t+1} - i_t^f) = \theta \sigma V_t(i_{t+1})$$

All of these models for CAPM are therefore special cases of the general model of discount factors, equation (9) below.

2.3.1 The relation between equity returns and volatility

Much of the current view of the relation between the expected excess return and its volatility dates from Campbell (1987) who finds that the relation is generally insignificantly positive when the conditional variance term is modelled as a GARCH-in-mean term. This result on the monthly CRSP market-wide index has been further reinforced by Baillie and DeGennero (1990) using a similar methodology. Glosten et al (1993) have subsequently found evidence of a significant negative relationship between mean and conditional variance. They found that including seasonal effects in the GARCH-M model appears to be of importance. Campbell and Hentschel (1992) and Scruggs (1998), using additional factors, find more support for a positive (partial) relation between mean and variance.

A closely related approach is that of Lettau and Ludvigson (2003) who examine the behaviour of the Sharpe ratio using an ad hoc model that relates the Sharpe ratio to the conditional standard deviation of returns. It is not necessary to use an ad hoc model, however, as it is possible to infer the behaviour of the Sharpe ratio from the no-arbitrage condition by re-writing it as

$$\frac{E_t(r_{t+1} - r_t^f)}{SD_t(r_{t+1})} = -\frac{1}{2} SD_t(r_{t+1}) - \rho_t(m_{t+1}, r_{t+1}) SD_t(m_{t+1}) \quad (8)$$

where $\rho_t(m_{t+1}, r_{t+1})$ is the conditional correlation and SD_t denotes a conditional standard deviation. These models can easily be tested within the SDF framework since they imply that conditional covariances can be excluded from the conditional mean and the coefficient on the own conditional variance should not be restricted to satisfy the Jensen effect.

2.4 General discount factor models

Instead of basing the choice of discount factor model on general equilibrium considerations or on the CAPM, it is possible to allow the factors to have unrestricted coefficients. We examine such general discount factor models with consumption, inflation, output and the market rate of return as factors. A possible justification for this approach is that the main holders of equity are financial institutions, especially pension funds. They act on behalf of investors' consumption at a much more distant point in the future. In assessing risk, financial institutions focus largely on short-term performance, and on the current value of the portfolio. This suggests that the market equity risk premium may be more influenced by short-term price risk than longer-term considerations of consumption. The sort of factors that are likely to affect the price of equity in the short term are associated with the business cycle and inflation. We therefore examine whether output is an additional source of equity risk to consumption and inflation without seeking to give this model a general equilibrium interpretation.

2.5 A taxonomy of models

The following is a summary of the models above:

1. C-CAPM

- (a) based on time-separable power utility model
 - (i) with real returns
 - (ii) with nominal returns
- (b) based on the Epstein-Zin time non-separable utility model
 - (i) with nominal returns and an explicit market return
 - (ii) with nominal returns but no explicit market return

2. General Discount Factor Models

- (a) based on two macroeconomic factors: consumption and inflation
- (b) based on three macroeconomic factors: consumption, inflation and industrial production
- (c) based on four factors: three macroeconomic factors and the market return

These can all be represented as restricted versions of the SDF model

$$E_t(i_{t+1} - i_t^f) = \beta_1 V_t(i_{t+1}) + \beta_2 Cov_t(\Delta c_{t+1}, i_{t+1}) + \beta_3 Cov_t(\pi_{t+1}, i_{t+1}) \quad (9) \\ + \beta_4 Cov_t(\Delta q_{t+1}, i_{t+1}) + \beta_5 Cov_t(i_{t+1}^m, i_{t+1})$$

where q_t is the logarithm of a measure of output.

The various models are summarised in Table 1 below. All of the models except the first assume that returns are in nominal terms. The first model is the C-CAPM with *ex-post* real returns. The different models can be distinguished by the restrictions they impose on β_i . All of these models restrict the coefficient on the conditional variance of equity returns to be $\beta_1 = -1/2$ (the Jensen effect). There are two exceptions; these are where the model includes the market return as a factor. As the market portfolio includes equity, we replace the covariance involving the market return with the variance of equity and remove the restriction on β_1 . This gives an alternative way of expressing the Epstein-Zin (model 4), and the four variable model which includes the market return as a factor (model 7). In our estimation models we have only the equity market return to measure the market return and therefore models 3 and 4 are equivalent as are models 7 and 9. Similarly, the CAPM which involves only the covariance of the asset with the market, can be re-expressed in terms of the conditional variance of the return to equity. The statistical approach relating the conditional mean to its conditional variance also implies that β_1 is unrestricted and all other $\beta_i = 0$ $\{i = 2, \dots\}$.¹

¹ The general unrestricted model (9) in Table 1 does not satisfy the no-arbitrage condition as it does not restrict $\beta_1 = -1/2$; it can therefore be interpreted as another example of the statistical approach.

<i>Model</i>	β_1	β_2	β_3	β_4	β_5
C-CAPM					
1. PU real	$-\frac{1}{2}$	σ	0	0	0
2. PU nominal	$-\frac{1}{2}$	σ	1	0	0
3. EZ: nominal with market return	$-\frac{1}{2}$	$-\frac{1-\sigma}{1-\gamma}$	$-\frac{\gamma(1-\sigma)}{1-\gamma}$	0	$\frac{1-\gamma\sigma}{1-\gamma}$
4. EZ: nominal without market return	$-\frac{1}{2} + \frac{\theta(1-\gamma\sigma)}{1-\gamma}$	$-\frac{1-\sigma}{1-\gamma}$	$-\frac{\gamma(1-\sigma)}{1-\gamma}$	0	0
SDF					
5. Two factors	$-\frac{1}{2}$	β_2	β_3	0	0
6. Three factors	$-\frac{1}{2}$	β_2	β_3	β_4	0
7. Four factors	β_1	β_2	β_3	β_4	0
Other models					
8. CAPM	$\theta\sigma$	0	0	0	0
9. General unrestricted	β_1	β_2	β_3	β_4	β_5

Table 1: Restrictions on the no-arbitrage condition

3 The Econometric Framework

We need to model the distribution of the excess return on equity jointly with the macroeconomic factors in such a way that the mean of the conditional distribution of the excess return in period $t + 1$, given information available at time t , satisfies the no-arbitrage condition. The conditional mean of the excess return involves selected time-varying second moments of the joint distribution. We therefore require a specification of the joint distribution that admits a time-varying variance-covariance matrix. We propose a new model, the multi-variate GARCH-covariance-in-mean (MGCM) model.

Let $\mathbf{x}_{t+1} = (r_{t+1} - r_t^f, z_{1,t+1}, z_{2,t+1}, \dots)'$, where $z_{1,t+1}, z_{2,t+1}, \dots$ include the macroeconomic

variables that give rise to the factors in the SDF through their conditional covariances with the excess return. In principle, they may also include additional variables that are jointly distributed with these macroeconomic variables as this may improve the estimate of the joint distribution. The MGCM model can then be written

$$\mathbf{x}_{t+1} = \boldsymbol{\alpha} + \boldsymbol{\Gamma}\mathbf{x}_t + \mathbf{B}\mathbf{g}_t + \varepsilon_{t+1}$$

where

$$\varepsilon_{t+1} \mid I_t \sim D[0, \mathbf{H}_{t+1}]$$

$$\mathbf{g}_t = \text{vech}\{\mathbf{H}_{t+1}\}$$

The *vech* operator converts the lower triangle of a symmetric matrix into a vector. The distribution D can be any multivariate distribution. The first equation of the model is restricted to satisfy the no-arbitrage condition. Thus, in general, the first row of $\boldsymbol{\Gamma}$ is zero and the first row of \mathbf{B} is $(-\frac{1}{2}, -\beta_{11}, -\beta_{12}, -\beta_{13}, \dots)$.

It will be noted that the theory requires that the macroeconomic variables display conditional heteroskedasticity. This is something not traditionally assumed in macro-econometrics, but is present in our data. Ideally, we would like to use high frequency data for asset returns, but very little macroeconomic data are published for frequencies higher than one month, and then only a few variables are available. Although more macroeconomic variables are published at lower frequencies, they tend not to display conditional heteroskedasticity at these frequencies.

Whilst the MGCM model is convenient, it is not ideal. First, it is heavily parameterised which can create numerical problems in finding the maximum of the likelihood function due to the likelihood being relatively flat, and hence uninformative. Second, asset returns tend to be excessively volatile. Assuming a non-normal distribution such as a t -distribution can sometimes help in this regard by allowing thicker tails. We deal with the major outlier in the sample in October 1987 by experimenting with a simple dummy variable. In principle, a stochastic volatility model, which includes an extra random term in the variance, could capture these extreme values. Unfortunately, as far as we are aware, no multivariate stochastic model with covariance-in-mean effects in the conditional covariances has been proposed in the literature.

In view of the need to restrict the number of coefficients being estimated, a commonly used specification of \mathbf{H}_{t+1} is the Constant Conditional Correlation model of Ding and Engle (1994) where the dynamics of the conditional covariances are driven by individual GARCH processes for the variances of each variable. Given that the SDF approach focusses on the importance of the contribution of covariances, restricting their dynamics in this way, and not allowing the

correlations to be time-varying, seems too restrictive.²

As a result, we specify \mathbf{H}_{t+1} using the BEKK model originally proposed by Engle and Kroner (1995). This takes the form:

$$vech(\mathbf{H}_{t+1}) = \mathbf{\Lambda} + \sum_{i=0}^{p-1} \mathbf{\Phi}_i vech(\mathbf{H}_{t-i}) + \sum_{j=0}^{q-1} \mathbf{\Theta}_j vech(\varepsilon_{t-j} \varepsilon'_{t-j})$$

where $\mathbf{\Lambda}$, $\mathbf{\Phi}$ and $\mathbf{\Theta}$ may be unrestricted. With $n - 1$ factors z_{it} then $\mathbf{\Phi}$ and $\mathbf{\Theta}$ are both square matrices of size $n(n + 1)/2$ and $\mathbf{\Lambda}$ is a size $n(n + 1)/2$ vector. A formulation of this model that economises on parameters yet retains flexibility is the error-correction formulation or VECM BEKK:

$$\mathbf{H}_{t+1} = \mathbf{V}'\mathbf{V} + \mathbf{A}'(\mathbf{H}_t - \mathbf{V}'\mathbf{V})\mathbf{A} + \mathbf{B}'(\varepsilon_t \varepsilon'_t - \mathbf{V}'\mathbf{V})\mathbf{B}.$$

where the first term on the RHS is the long-run or unconditional covariance matrix. This can be initialised with starting values from sample averages or estimated directly. The remaining terms capture short-run deviations from this long run. A restricted version of this formulation is to specify \mathbf{V} to be lower triangular and \mathbf{A} and \mathbf{B} to be symmetric matrices which further reduces parameter numbers.

A comparison of the number of parameters required for $n = 3$ and $p = q = 1$ is: BEKK = $n(n + 1)/2 + (p + q)n^2(n + 1)^2/4 = 78$; VECM BEKK = $3n^2 = 27$; Restricted VECM BEKK = $3n(n + 1)/2 = 18$; Constant Correlation = $3n + n(n - 1)/2 = 12$. We require that the covariance function is stationary. This is satisfied if the absolute value of the eigenvalues of $(\mathbf{A} \otimes \mathbf{A}) + (\mathbf{B} \otimes \mathbf{B})$ lie inside the unit circle where \otimes is the Kronecker product.

The structure of the VECM BEKK model that we employ is common to all of the models that we estimate. In each case we condition on the same set of variables in the macroeconomic environment even though the terms in the no-arbitrage condition for the excess return differ between models. Thus the vector \mathbf{x}_{t+1} for the models of the nominal equity return is $\mathbf{x}_{t+1} = (i_{t+1} - i_t^f, \pi_{t+1}, \Delta c_{t+1}, \Delta q_{t+1})'$ whilst that for the real return is: $\mathbf{x}_{t+1} = (r_{t+1} - r_t^f, \pi_{t+1}, \Delta c_{t+1}, \Delta q_{t+1})'$. A first order vector autoregression for the macroeconomic variables is found to be sufficient to capture the serial dependence in their means; a VECM BEKK(1,1) model is found to be adequate for the multivariate variance-covariance process. For greater generality we employ a multivariate t -distribution.

² The attraction of the reduction in parameterisation offered by the CCC model has led to an extension to the dynamics in the DCC proposed by Engle (2002). A feature of this approach is 2 step estimation. This is, however, not feasible in our context as elements of the estimated conditional covariance matrix appear in the mean asset pricing equations.

4 Results and Estimated Risk Premia

4.1 The Data

The data are monthly for the US (1975.6-2001.12) and UK (1975.6-2001.12). The US data consists of the excess return on equity of Fama and French, real non-durable growth consumption from FRED (Federal Reserve Economic Data provided by the Federal Reserve Bank of St Louis), CPI inflation from Datastream and the industrial production volume index from Datastream. The UK data are the MSCI total equity return index from Datastream, total real non-durable consumption growth specially provided by the NIESR, RPI inflation and the volume index of industrial production both from Datastream. All data are expressed in equivalent annual percentages.

Table 2 reports various unconditional moment descriptive statistics for the data, including skewness, kurtosis, autocorrelations of levels and squares and correlations between the variables.

	$i_{t+1}^{us} - i_t^{us}$	$i_{t+1}^{uk} - i_t^{uk}$	π_{t+1}^{us}	π_{t+1}^{uk}	Δc_{t+1}^{us}	Δc_{t+1}^{uk}	Δq_{t+1}^{us}	Δq_{t+1}^{uk}
Mean	6.2508	5.3242	4.5212	6.0657	2.6784	2.9605	2.7923	1.3879
Std. Dev	54.2440	60.4400	3.6252	7.3884	7.7575	9.5286	8.2967	15.644
Skewness	-1.0174	-0.9398	0.8340	1.8082	-0.0308	-1.4379	0.0055	-0.1004
Excess Kurtosis	3.9780	4.5802	1.0578	6.5753	0.4453	14.4820	1.7726	7.0462
Normality	50.4180	65.9750	37.0090	117.0600	3.6367	273.2900	32.6320	232.6600
$\rho(x_t, x_{t-1})$	0.0339	0.0199	0.7233	0.3903	-0.3686	-0.1959	0.2884	-0.2978
$\rho(x_t, x_{t-2})$	-0.0874	-0.1147	0.5993	0.2793	-0.0178	0.1464	0.2483	0.0082
$\rho(x_t, x_{t-3})$	-0.0556	-0.0948	0.5921	0.2177	0.1587	-0.0759	0.2116	0.1176
$\rho(x_t^2, x_{t-1}^2)$	0.0225	0.0861	0.7879	0.1669	0.1011	0.2516	0.1529	0.4476
$\rho(x_t^2, x_{t-2}^2)$	0.0288	0.0731	0.7108	0.0972	0.0164	0.0301	0.0940	0.0284
$\rho(x_t^2, x_{t-3}^2)$	0.0150	-0.0189	0.6399	0.2107	0.0358	0.1624	0.0512	0.0125
$\rho(x_t, i_{t+1}^{us} - i_t^{us})$	1	0.621	-0.125	-0.037	0.157	0.035	-0.079	-0.029
$\rho(x_t, i_{t+1}^{uk} - i_t^{uk})$	0.621	1	-0.113	-0.004	0.154	0.070	-0.025	0.003
$\rho(x_t, \pi_{t+1}^{us})$	-0.125	-0.113	1	0.456	-0.166	-0.098	0.011	-0.059
$\rho(x_t, \pi_{t+1}^{uk})$	-0.037	-0.004	0.456	1	-0.125	-0.259	-0.002	0.010
$\rho(x_t, \Delta c_{t+1}^{us})$	0.157	0.154	-0.166	-0.125	1	0.120	0.141	0.101
$\rho(x_t, \Delta c_{t+1}^{uk})$	0.035	0.070	-0.098	-0.259	0.120	1	-0.003	0.155
$\rho(x_t, \Delta q_{t+1}^{us})$	-0.079	-0.025	0.011	-0.002	0.141	-0.003	1	0.237
$\rho(x_t, \Delta q_{t+1}^{uk})$	-0.029	0.003	-0.059	0.010	0.101	0.155	0.237	1

Table 2. Descriptive statistics for the raw data: US and UK

Thus the excess returns and most of the macroeconomic variables display excess kurtosis and non-normality, and the inflation rates show volatility persistence. The excess returns are positively correlated with consumption growth and negatively with inflation. Consumption growth is negatively correlated with inflation in both countries. All variables are positively correlated with their counterparts in the other country, not an issue that we follow up in this paper.

4.2 Estimates

4.2.1 A complete set of estimates for C-CAPM

To illustrate, a full set of model estimates with their restrictions is reported for the variables $x(t+1)' = \{i_{t+1} - i_t^f, \pi_{t+1}, \Delta c_{t+1}, \Delta q_{t+1}\}$ for the US. The model is C-CAPM with power utility and nominal (ex-post real) returns. An additional variable, output growth, is included in the joint distribution but not in the no-arbitrage condition. Output growth is included in the conditioning model in order to improve our ability to model the conditional covariances. A dummy variable for October 1987 is also included on the grounds that the excess return for this particular observation is drawn from a different distribution. This is to prevent an extreme observation from contaminating the estimates of the GARCH parameters for the whole sample.³

$\mathbf{g}_t = (V_t(i_{t+1}), Cov_t(\pi_{t+1}^{us}, i_{t+1}^{us}), Cov_t(\Delta c_{t+1}^{us}, i_{t+1}^{us}), Cov_t(\Delta q_{t+1}^{us}, i_{t+1}^{us}))'$ and \mathbf{A} and \mathbf{B} are symmetric matrices.

The main points to note at this stage are that the multivariate GARCH process is well determined with the macroeconomic variables showing significant conditional heteroskedasticity. The conditional covariance of returns with consumption is highly significant, but the size of the coefficient, namely 328.7, implies an implausibly large coefficient of relative risk aversion. This is, of course, a common finding in consumption-based models estimated with consumption data, see for example Campbell (2002) and Yogo (2006).

³ The method of analysis we use allows for risk quantities based on covariances with excess returns to vary over time. Thus we estimate substantial time variation in risk premia. Whilst breaks in the parameters of this model over time are possible, we have examined the residuals for evidence of changes in fit and find no evidence for such breaks. We have found one substantial outlier and allow for this in our estimation.

Estimates of C-CAPM with power utility and real returns for the US 1975.1-2001.11

$$\begin{aligned}
 \mathbf{x}_{t+1} &= \boldsymbol{\alpha} + \boldsymbol{\Gamma}\mathbf{x}_t + \mathbf{B}\mathbf{g}_t + \delta d_t + \varepsilon_{t+1} \\
 \varepsilon_{t+1} &| I_t \sim t[0, \mathbf{H}_{t+1}] \\
 \mathbf{H}_{t+1} &= \mathbf{V}'\mathbf{V} + \mathbf{A}'(\mathbf{H}_t - \mathbf{V}'\mathbf{V})\mathbf{A} + \mathbf{B}'(\varepsilon_t \varepsilon_t' - \mathbf{V}'\mathbf{V})\mathbf{B} \\
 \mathbf{g}_t &= \text{vech}\{\mathbf{H}_{t+1}\} \\
 \mathbf{x}_{t+1} &= \{r_{t+1} - r_t^f, \pi_{t+1}, \Delta c_{t+1}, \Delta q_{t+1}\}'
 \end{aligned}$$

$$\boldsymbol{\alpha} = \begin{pmatrix} 0.0000 \\ (-) \\ 0.0009 \\ (4.63) \\ 0.0040 \\ (7.02) \\ 0.0029 \\ (4.99) \end{pmatrix}, \boldsymbol{\Gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.002 & 0.645 & 0.040 & 0.006 \\ (0.90) & (15.02) & (2.14) & (0.32) \\ 0.015 & -0.307 & -0.402 & -0.043 \\ (1.83) & (2.57) & (7.98) & (0.80) \\ 0.007 & -0.224 & -0.080 & 0.297 \\ (0.95) & (1.89) & (141) & (5.08) \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -\frac{1}{2} & 1 & 328.7 & 0 \\ (3.10) & & & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 0.913 & - & - & - \\ (24.34) & & & \\ 0.003 & 0.909 & - & - \\ (0.27) & (25.73) & & \\ -0.002 & 0.039 & 0.901 & - \\ (0.07) & (1.69) & (16.07) & \\ 0.207 & 0.035 & -0.219 & -0.478 \\ (2.38) & (0.47) & (1.44) & (3.47) \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0.110 & - & - & - \\ (3.46) & & & \\ -0.000 & 0.304 & - & - \\ (0.07) & (4.90) & & \\ 0.032 & 0.036 & -0.043 & - \\ (3.71) & (1.60) & (0.77) & \\ 0.034 & 0.031 & 0.478 & 0.571 \\ (3.01) & (1.07) & (0.90) & (6.36) \end{pmatrix}$$

$$\mathbf{V}'\mathbf{V} = \begin{pmatrix} 2415 & - & - & - \\ -0.8 & 8.3 & - & - \\ 43.8 & -0.9 & 46.4 & - \\ -22.0 & 2.5 & 9.6 & 63.6 \end{pmatrix}, \text{correlation matrix} = \begin{pmatrix} 1.00 & - & - & - \\ -0.01 & 1.00 & - & - \\ 0.13 & -0.05 & 1.00 & - \\ -0.06 & 0.11 & 0.18 & 1.00 \end{pmatrix}$$

The estimate of the 1987.10 dummy d_t is $\delta = -0.250$ (1.80). The three largest eigenvalues for the GARCH process are 0.961, 0.949, 0.929 and the smallest is -0.544 . The annualised mean log excess return is 6.25% and mean risk premium is 11.31%. The percentage of the variance of the excess return explained by variations in the risk premium is 2.2%. Absolute t -statistics are in parentheses.

4.2.2 Estimates of the no-arbitrage equation

US Estimates of all of the models in Table 1 are presented in Table 3 for US data including those for the C-CAPM in real terms presented in full above. In addition to the parameter estimates in the mean equation, we present a number of summary statistics. The estimates of the dummy variable for October 1987 are all generally significant. The estimates of the degrees of freedom refer to the t -distribution. Absolute t -statistics are in parentheses.

US	PU real	PU	EZ	SDF2	SDF3	SDF4	CAPM
Model	1	2	3&4	5	6	7&9	8
$V_t(i_{t+1}^{us})$	-0.5	-0.5	-3.615 (0.75)	-0.5	-0.5	-3.544 (0.72)	4.727 (3.60)
$C_t(\Delta c_{t+1}^{us}, i_{t+1}^{us})$	328.7 (3.10)	328.5 (3.10)	358.79 (2.38)	296.56 (2.94)	294.33 (2.89)	356.66 (2.34)	—
$C_t(\pi_{t+1}^{us}, i_{t+1}^{us})$	—	-1	-423.29 (1.65)	-421.47 (1.69)	-460.65 (1.68)	-434.52 (1.58)	—
$C_t(\Delta q_{t+1}^{us}, i_{t+1}^{us})$	—	—	—	—	28.979 (0.29)	8.540 (0.08)	—
Dummy, d_t	-0.250 (1.80)	-0.250 (1.80)	-0.246 (1.95)	-0.247 (1.90)	-0.246 (2.02)	-0.246 (1.98)	-0.254 (1.47)
Deg of Freedom	13.53 (3.25)	13.53 (3.25)	13.79 (3.13)	13.74 (3.21)	13.88 (3.15)	13.83 (3.10)	12.61 (3.07)
Log-likelihood	4439.37	4439.37	4441.27	4440.84	4440.89	4441.27	4436.43
Mean risk premium	12.01%	12.00%	8.85%	9.99%	9.75%	8.83%	10.69%
$ \lambda_{\max} $	0.961	0.961	0.966	0.966	0.966	0.966	0.982
$\bar{\varepsilon}_{t+1}$	-2.853	-2.854	-1.828	-2.812	-2.753	-1.843	-2.109
$V(\phi_{t+1})$	58.882	58.881	94.076	76.868	77.885	93.920	0.0174
$\frac{V(\phi_{t+1})}{V(i_{t+1}^{us} + \frac{1}{2}V_t(i_{t+1}^{us}) - \hat{\alpha}D)}$	0.0225	0.0225	0.0359	0.0293	0.0297	0.0358	0.000066
Key: $ \lambda_{\max} $ =maximum eigenvalue, $\bar{\varepsilon}_{t+1}$ =mean error, $V(\phi_{t+1})$ =risk premium unconditional variance							
$\frac{V(\phi_{t+1})}{V(i_{t+1}^{us} + \frac{1}{2}V_t(i_{t+1}^{us}) - \hat{\alpha}D)}$ =proportion of variation of adjusted excess return explained by the risk premium							

Table 3. Estimates of no-arbitrage equations for the US

Clearly there is little difference between the estimates of C-CAPM with power utility for real and nominal returns (models 1 and 2) and the conclusions reached earlier with the respect to nominal returns apply again. Estimates of the Epstein-Zin model in which the market return is assumed to consist of equity (models 3 and 4) are not significantly different from those for power utility. A likelihood ratio test of no difference between the two models has a $\chi^2(2)$ test statistic of 3.80, which is marginally significant at the 0.15 level. Even though the coefficient on the covariance term with inflation is bordering on significance, we conclude that in terms of

accounting for the equity risk premium the Epstein-Zin model offers only a minor improvement over power utility. To set against this, the implied estimate of the CRRA is now even larger than for power utility as $\sigma = \beta_2 - \beta_3 + 1 = 783$; the estimate of the elasticity of inter-temporal substitution is $\gamma = \frac{\beta_3}{\beta_2} = -1.18$.

Turning to the models with stochastic discount factors that satisfy the no-arbitrage condition (due to the constraint on the coefficient of the conditional variance of returns) but are otherwise unrestricted, the most successful is the two-factor model based on consumption and inflation (model 5). The coefficients on the consumption and inflation covariances are jointly significant and this model is preferred to the C-CAPM at the 0.0846 level. We note that the models which include production and the market return (which is embodied in the conditional variance, models 6,7 and 9) are less successful.

Unconditional CAPM (model 8) is clearly dominated by the multi-factor discount factor models. The coefficient on the conditional variance in the estimate of the unconditional CAPM is significantly positive but the fit is very low (due to the smoothness of the estimate of the conditional variance of the excess return in this model). A better version of the CAPM that is consistent with our results, therefore, would be a conditional version in which beta is a function of the SDF factors as in equation (8).

To summarise for the US, we find that a two factor model based on consumption and inflation provides the best SDF model for the equity premium. This implies an estimated average equity premium of about 10% and that variations in the risk premium explain about 3% of variations in the excess return to equity.

UK Model	PU real 1	PU 2	EZ 3&4	SDF2 5	SDF3 6	SDF4 7&9	CAPM 8
$V_t(i_{t+1}^{uk})$	-0.5	-0.5	2.213 (0.45)	-0.5	-0.5	3.359 (0.74)	3.698 (3.24)
$C_t(\Delta c_{t+1}^{uk}, i_{t+1}^{uk})$	202.99 (2.54)	203.41 (2.54)	153.53 (1.38)	169.51 (1.57)	230.29 (1.85)	257.65 (1.95)	-
$C_t(\pi_{t+1}^{uk}, i_{t+1}^{uk})$	-	-1	-690.27 (2.73)	-701.88 (2.75)	-592.05 (2.06)	-490.19 (1.72)	-
$C_t(\Delta d_{t+1}^{uk}, i_{t+1}^{uk})$	-	-	-	-	-56.30 (0.60)	-114.12 (1.07)	-
Dummy, d_t	-0.314 (1.53)	-0.314 (1.53)	-0.321 (1.65)	-0.321 (1.70)	-0.319 (1.80)	-0.318 (1.86)	-0.318 (1.45)
Deg of Freedom	12.54 (3.16)	12.54 (3.16)	11.47 (3.42)	11.28 (3.43)	11.37 (3.37)	11.63 (3.32)	12.67 (3.06)
Log-likelihood	3907.85	3907.84	3912.07	3911.97	3912.04	3912.27	3908.32
Mean risk premium	7.21%	7.23%	10.81%	10.65%	10.41%	10.54%	9.72%
$ \lambda_{\max} $	0.972	0.972	0.979	0.979	0.977	0.976	0.972
$\bar{\varepsilon}_{t+1}$	0.147	0.145	-1.486	-1.170	-0.993	-1.433	-1.74
$V(\phi_{t+1})$	68.148	68.189	140.082	146.086	139.993	127.526	0.152
$\frac{V(\phi_{t+1})}{V(i_{t+1}^{uk} + \frac{1}{2}V_t(i_{t+1}^{uk}) - \hat{\alpha}D)}$	0.0211	0.0211	0.0434	0.0452	0.0433	0.0395	0.00047
Key: $ \lambda_{\max} $ =maximum eigenvalue, $\bar{\varepsilon}_{t+1}$ =mean error, $V(\phi_{t+1})$ =risk premium unconditional variance							
$\frac{V(\phi_{t+1})}{V(i_{t+1}^{us} + \frac{1}{2}V_t(i_{t+1}^{us}) - \hat{\alpha}D)}$ =proportion of variation of adjusted excess return explained by the risk premium							

Table 4. Estimates of no-arbitrage equations for the UK

UK Estimates of the various models for the UK data presented in Table 4 are similar to those for the US. Although the estimates of the CRRA are smaller, they are still far too large to be plausible. In the Epstein-Zin model the covariance of inflation with the return on equity is now significant, but the unrestricted own variance is not. A likelihood ratio test comparing C-CAPM with the Epstein-Zin model, which is distributed $\chi^2(2)$, is 8.46. This is significant even at the 0.0146 level. The implied estimate of σ in the Epstein-Zin model is 844.8 and that of γ is -4.50 . We conclude once more therefore that in empirical terms the Epstein-Zin model is a minor generalisation of the power utility model and it gives unreasonable estimates in terms of general equilibrium theory. The unconditional CAPM is also dominated by the discount factor models.

Again, the best performing model is the two factor model with consumption and inflation, with inflation seemingly more important than consumption. This model is preferred over the C-CAPM at the 0.0040 level. Output has little explanatory power in models 6,7 and 9, which is similar to the results for the US. This two factor model implies an estimated average equity premium

of about 10.8% and that variations in the risk premium explain about 4.5% of variations in the excess return to equity.

4.2.3 Estimates of the time-varying risk premium

An important additional feature of our econometric methodology is that we are able to estimate how the equity premium varies over time and we are able to measure the individual contribution to the risk premium of the different macroeconomic factors. These results are particularly interesting. Previously we have noted that Tables 3 and 4 give the average risk premia and the proportion of the variance of excess returns explained by variation in the risk premium (adjusted for the Jensen effect and the October 1987 dummy). There are small differences across models and countries but the estimated mean risk premium is about 10%, and the proportion of the excess return explained by the various conditional models is about 3% for the US, and 4.5% for the UK. This compares favourably with the 0.05% explained by unconditional CAPM. The fit of these models is not expected to be high as the error term is dominated by unexpected changes in the equity return.

Turning to the variation over time in the estimated risk premia, Figures 1 and 2 report estimates of the US and UK risk premia for three models: power utility, Epstein-Zin and the two factor model. They show that there is considerable variation in the risk premia over time but that the estimated risk premia for the three models are similar. For both countries, in the main, the risk premia are positive, but in the period 1975-1977 there is a tendency for the risk premia to be negative. For the US the Epstein-Zin and two-factor model risk premia are negative in 2001.

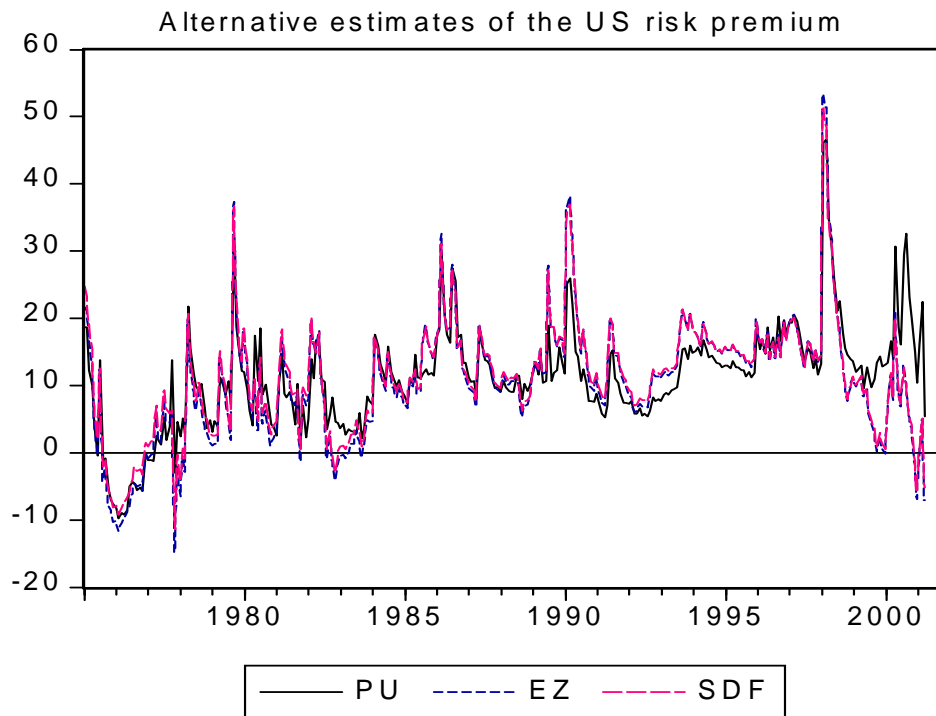


Figure 1.

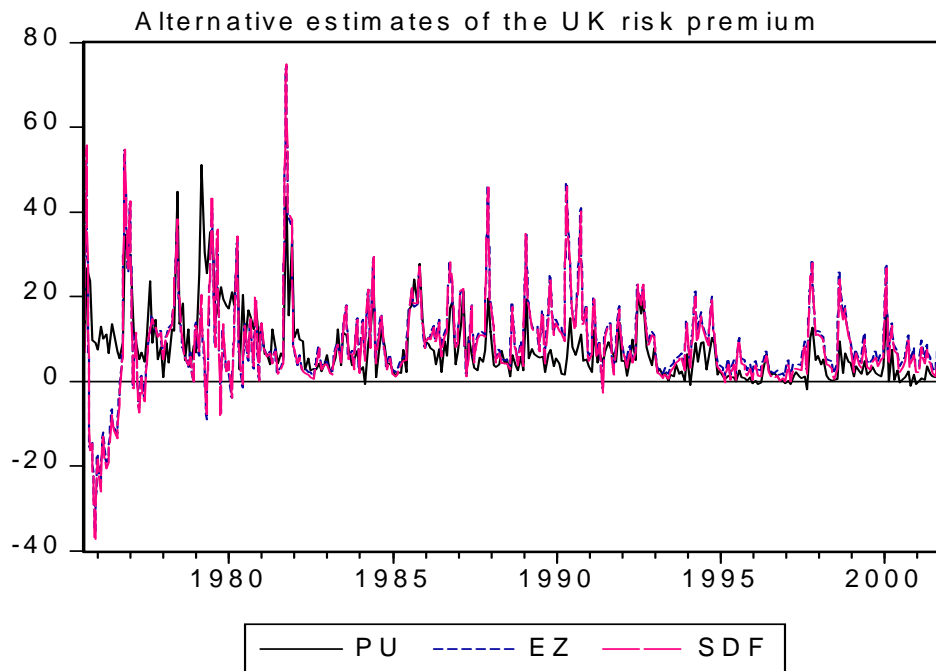


Figure 2.

Figures 3 and 4 enable a comparison to be made between the risk premia and the excess returns (adjusted for the Jensen effect and the October 1987 dummy). Taking the two-factor risk premium (model 5) as an example, it is interesting to observe that the risk premium tends to jump upwards when excess returns fall sharply and become strongly negative. As excess returns recover, the risk premia fall but more slowly. In contrast, large positive excess returns are not associated with sharp drops in the risk premium; but again they fall slowly.

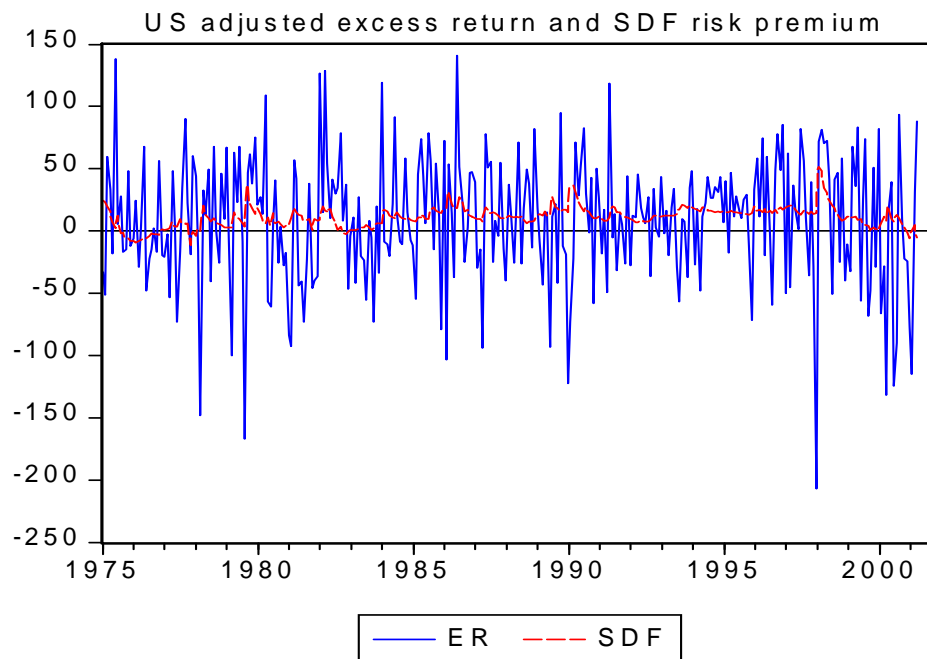


Figure 3.

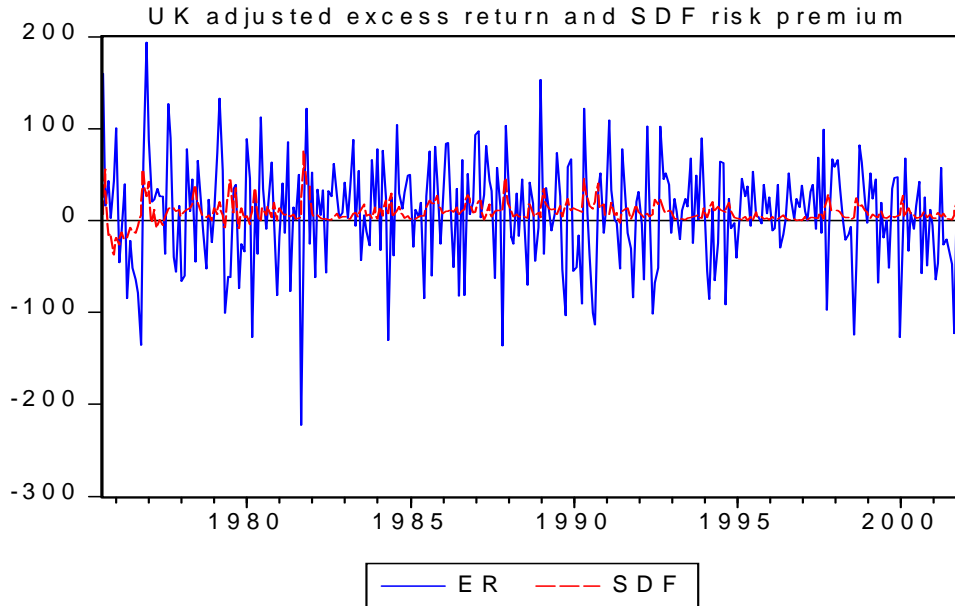


Figure 4.

We now turn to the individual contribution to the risk premium of the different macroeconomic factors. We examine the two-factor risk premium where the factors are consumption and inflation. Figures 5 and 6 plot the risk premium and the conditional covariances of consumption growth and inflation with the excess return on equity. The most striking point is that the consumption covariance tends to be almost entirely positive, while the inflation covariance tends to be largely negative, thereby echoing the unconditional correlations shown in Table 2. This explains why the coefficients on the conditional covariance with inflation in the estimated no-arbitrage models are negative. For the US, the inflation covariance increases from 1998 and becomes positive from 1999. This coincides with falling inflation over this period. For the UK, both covariances have converged towards zero in the 1990's. As a result, the risk premium is generally lower in this period than in earlier periods.

Our estimates also allow us to comment on the relationship between macroeconomic volatility and the risk premium. Figures 7 and 8 display the conditional standard deviations of consumption growth and inflation, together with the risk premium for the two-factor model. Figure 7, which is for the US, shows a clear reduction in macroeconomic volatility from the mid-1970's to the late 1990's; both consumption and inflation variability fell. According to our estimates, however, the risk premium did not fall over this period. The reason is that the risk premium is a linear function of the covariances of the two factors with the excess return. These are shown in Figures

5 and 6. In contrast, the conditional covariances of consumption growth with excess returns have risen over the period and the conditional covariances with inflation have fallen. Thus, given our parameter estimates, the risk premium has shown no clear trend. This is different from the findings of Lettau, Ludvigson and Wachter (2006) as our estimates show that the trend in volatility has reversed towards the end of the sample. The recent fall in the risk premium is due mostly to an increase in the covariance of inflation and excess returns. Such trends are not so pronounced in the UK and there is a little evidence that the risk premium has drifted lower in recent years in line with lower macroeconomic volatility. The risk premium itself and its determinants also appear to have reduced in volatility for the UK.

US SDF risk premium and conditional covariances for consumption and inflation

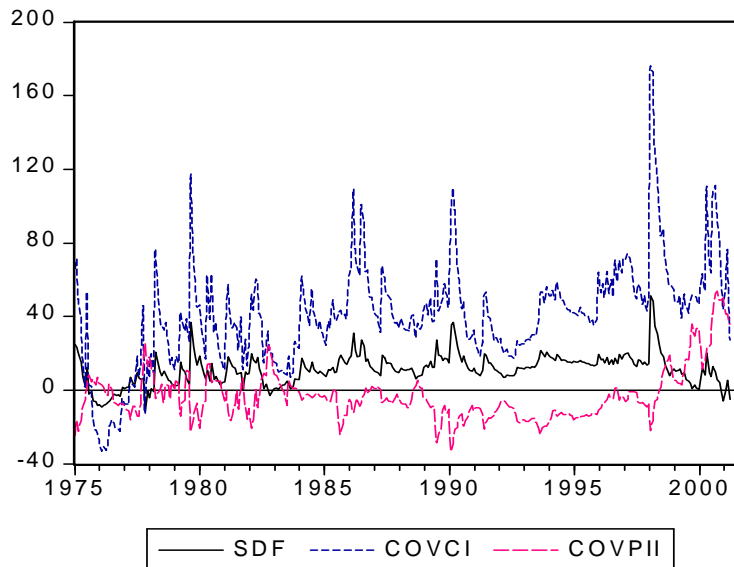


Figure 5.

UK SDF risk premium and conditional covariances for consumption and inflation

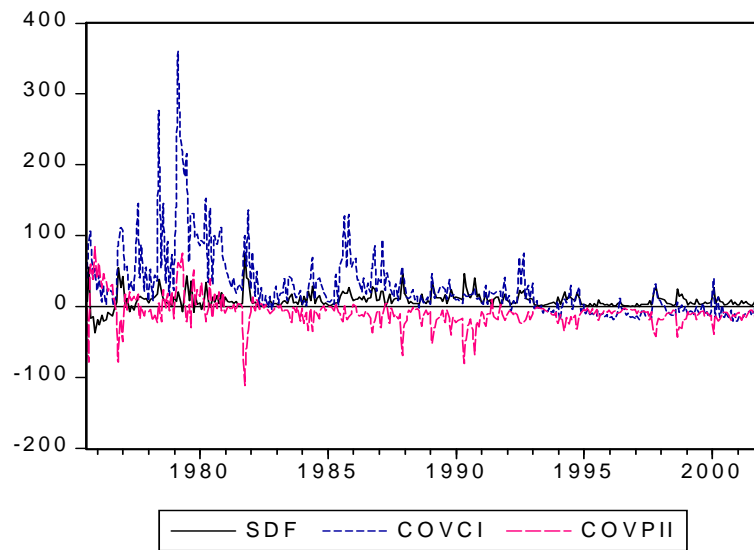


Figure 6.

US Sharpe Ratio and Shares of Macro Factors

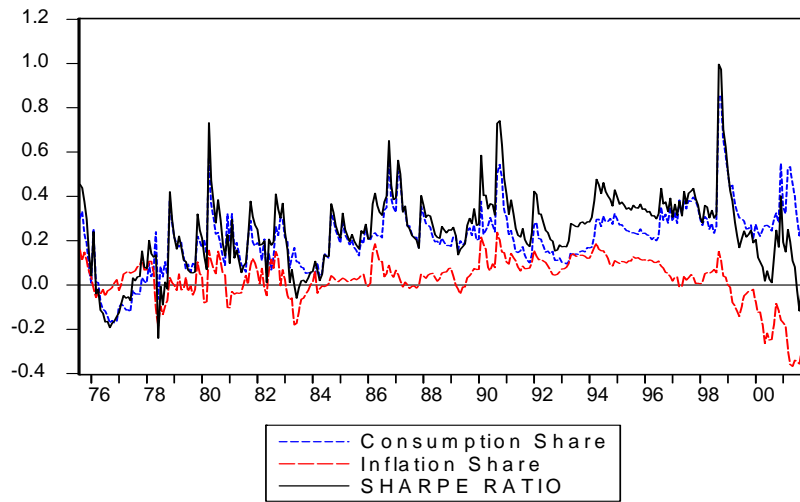


Figure 7.

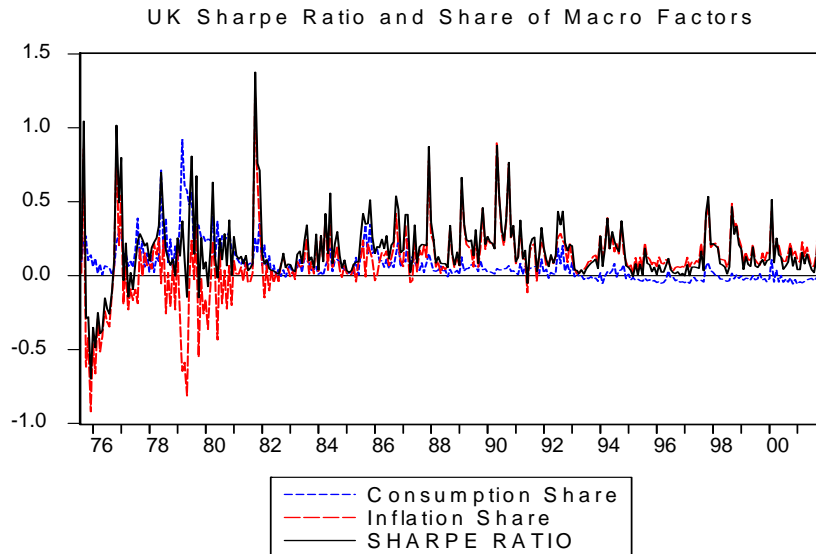


Figure 8.

Finally, we comment on the consequences of our results for the behaviour of the Sharpe ratio. As is clear from equation (8), there are two elements driving the Sharpe ratio. First is the conditional correlation between returns and the discount factor $\rho_t(m_{t+1}, r_{t+1})$; second is the variability of the discount factor $SD_t(m_{t+1})$. Figures 9 and 10 show this decomposition for the two-factor model. For the US, we find that consumption growth is the key driver of variations in the Sharpe ratio, whilst, for the UK inflation is much more important. This is especially evident in the period at the end of the 1970's and the start of the 1980's when inflation in the UK was high and variable. The pattern at the end of the 1990's is also rather different in the two countries where the pick-up in the importance of inflation in the US is not found in the results for the UK. This serves to emphasise the differences between the estimates for the two countries. The fact that the two-factor model delivers such a high degree of variation in the Sharpe ratio for the US echoes the results in Lettau and Ludvigson (2003) who demonstrate the difficulty that general equilibrium models have in generating the observed volatility in the Sharpe ratio in the US.

U S Standard deviations of macro factors and the risk premium (RHS)

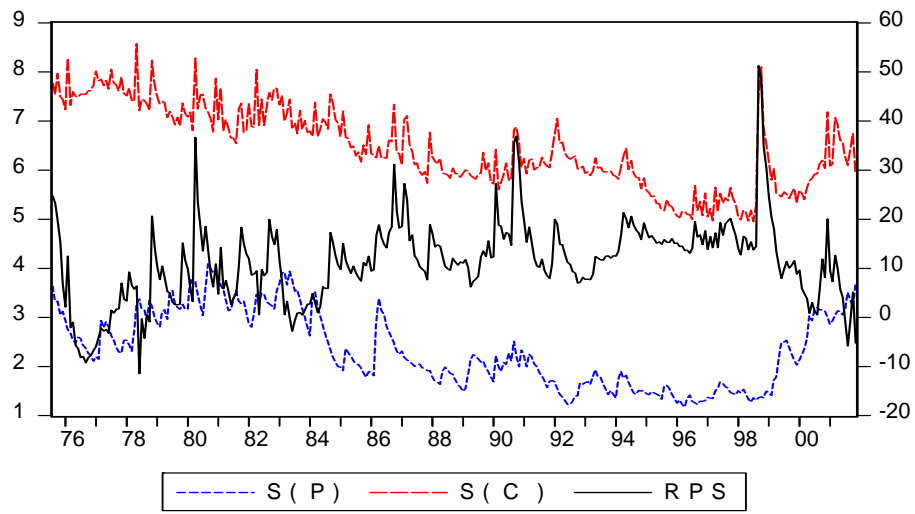


Figure 9

U K Standard deviations of macro factors and the risk premium (RHS)

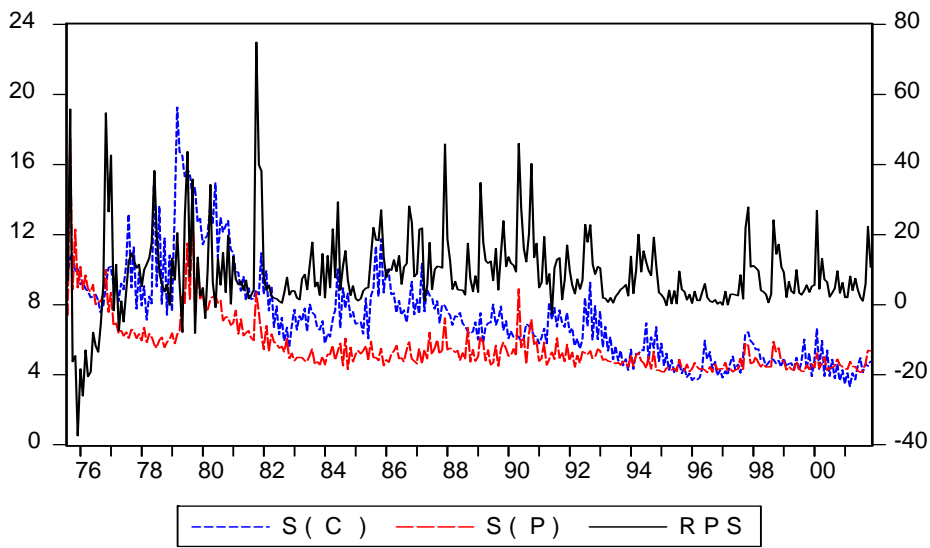


Figure 10.

5 Conclusions

This paper fills a surprising gap in the literature. Using a new econometric methodology which permits the estimation of the no-arbitrage condition arising from models with observable factors, it provides perhaps the first estimates of a time-varying equity risk premium implied by two key theoretical general equilibrium models, consumption CAPM with time separable power utility and with time non-separable Epstein-Zin preferences.

Our theoretical framework is the stochastic discount factor approach. This encompasses general equilibrium asset pricing models such as the C-CAPM together with CAPM and no-arbitrage models are based on such considerations.

Using monthly data for the US and UK from 1975 to 2001, we confirm the findings of other empirical approaches that the coefficient of relative risk aversion is too large to be accounted for by either C-CAPM with power utility or by Epstein-Zin preferences. We also show that CAPM can be rejected as it ignores significantly priced sources of risk. We find that the best model, both for the US and the UK, is a two-factor model based on the observable factors, consumption growth and inflation. Our results indicate that a two-factor model in which there is a significant role for inflation as a priced factor in addition to consumption growth is to be preferred to purely statistical models that relate equity returns to their volatility.

An important contribution of our paper is that we are able to show how the equity premium has varied over time. We find that equity risk premia are predominately positive; the exceptions being at the start of our data period in the mid 1970's and, for the US, since 1999. An interesting feature of the estimated risk premia is that they tend to increase sharply during periods of negative excess returns before slowly declining. No corresponding jump is discernible when excess returns are high.

Our conclusion is that this methodology offers a new way to test asset pricing theories, it provides new insights into the sources of equity risk and new estimates of the behaviour of the equity risk premium. We have obtained corresponding results for FOREX in Wickens and Smith (2001) and for bonds in Balfoussia and Wickens (2007).

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