Managing a Liquidity Trap: Monetary and Fiscal Policy*

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Abstract

I study monetary and fiscal policy in liquidity trap scenarios, where the zero bound on the nominal interest rate binds. I work with a continuous-time version of the standard New Keynesian model. Without commitment the economy suffers from deflation and depressed output. Surprisingly, both are exacerbated with greater price flexibility. The optimum commits to keeping the interest rate at zero past the liquidity trap. Monetary policy promotes both inflation and an output boom. I show that the prolongation of zero interest rates is related to the latter, not the former. I then turn to fiscal policy and show that, regardless of the value of “fiscal multipliers”, optimal spending increases at the start of the trap. However, it later declines below its natural level. I propose a decomposition of spending according to “opportunistic” and “stimulus” motives, where the former is defined as the static, cost-benefit optimum. Opportunistic spending is countercyclical, while stimulus spending may be zero. Finally, I consider a situation where monetary policy is discretionary, but fiscal policy has commitment and show that this increases opportunistic spending and makes stimulus spending positive.

1 Introduction

The 2007-8 crisis in the U.S. led to a steep recession, followed by aggressive policy responses. Monetary policy went full tilt, cutting interest rates rapidly to zero, where they have remained since the end of 2008. With conventional monetary policy seemingly exhausted, fiscal stimulus worth $787 billion was enacted by early 2009 as part of the American Recovery and Reinvestment Act. Unconventional monetary policies were also pursued, starting with “quantitative easing”, purchases of long-term bonds and other assets. In August 2011, the Federal Reserve’s FOMC statement signaled the intent to keep interest rates at zero until at least mid 2013 (later extended to mid 2014). Similar policies have been followed, at

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least during the peak of the crisis, by many advanced economies. Fortunately, the kind of crises that result in such extreme policy measures have been relatively few and far between. Perhaps as a consequence, the debate over whether such policies are appropriate remains largely unsettled. The purpose of this paper is to make progress on these issues.

To this end, I reexamine monetary and fiscal policy in a liquidity trap, where the zero bound on nominal interest rate binds. I work with a standard New Keynesian model that builds on the same model Eggertsson and Woodford (2003) used to study monetary policy in a liquidity trap.\(^1\) In these models a liquidity trap is defined as a situation where negative real interest rates are needed to obtain the first-best allocation. I adopt a deterministic continuous time formulation that turns out to have several advantages. It is well suited to focus on the dynamic questions of policy, such as the optimal exit strategy, whether spending should be front- or back-loaded, etc. It also allows for a simple graphical analysis and delivers several new results. Most importantly, continuous time is particularly helpful when one needs to solve for an endogenous switching time, as I do here due to the bang-bang nature of the solution. A similar benefit from casting things in continuous time is well appreciated in Krugman’s balance of payment crisis model, the Baumol-Tobin model of inventory money demand, or in sS menu cost models.

I first study monetary policy in the absence of government spending. When monetary policy lacks commitment, deflation and depression ensue. These features are commonly associated with liquidity traps. Less familiar is that both outcomes are exacerbated by price flexibility, as I show here. As a result, a large degree of price stickiness is not required for the problems created by a liquidity trap to be severe. Indeed, quite the contrary, the depression becomes unbounded as we converge to fully flexible prices. The intuition for these results is as follows. The main problem in a liquidity trap is that the real interest rate is too high. This leads to excessive savings, which depresses spending and output, which, in turn, creates deflationary pressures. Price flexibility accelerates deflation, raising the real interest rate still further and making matters worse.

As first argued by Krugman (1998), optimal monetary policy can improve on this dire outcome by committing to future policy in a way that affects current expectations favorably. Krugman highlights the role of positive inflation, since this lowers the real rate of interest. Indeed, in his model, featuring prices that are fixed for only one period, positive inflation can attain the first best. This is not the case in the New Keynesian model, yet I show that the second best optimum does feature positive inflation, at some point in time. Indeed, for

\(^1\)See also Eggertsson and Woodford (2006), which studies monetary policy together with tax policy, but does not consider government spending. Eggertsson (2001, 2006) study government spending with distortionary taxation, but focus on the case without commitment showing how spending provides implicit commitment to inflate after debt rises. Christiano et al. (2011), Woodford (2011) and Eggertsson (2011) consider the effects of spending on output, computing “fiscal multipliers”, but do not focus on optimal policy.
some parameters optimal inflation may be positive throughout the trap, so that deflation is completely avoided. Thus, the absence of deflation, far from being at odds with a liquidity trap, actually may be evidence of an optimal response to such a situation. In other cases, the solution may start with deflation but reach inflation at some later date.

Although inflation has been at the center of discussion on how to manage a liquidity trap, my results show that this is not the only goal for monetary policy. It may not even be the main goal. I show that, at the optimum, output starts below its efficient level, but rises above it towards the end and after the trap. Indeed, the boom in output is larger than what would be incidentally implied by the goal to promote positive inflation. Intuitively, both inflation and a future boom in consumption help lower the desire for savings that is the heart of the problem depressing the economy. Thus, my analysis uncovers a distinct and often overlooked second goal of monetary policy of inducing a boom after the trap. The expectation of a future boom percolates to the present making the present recession less severe.\(^2\)

There are a number of ways monetary policy can promote inflation and stimulate output. For example, monetary easing to create inflation does not necessarily imply a low equilibrium interest rate path. Indeed, it may well lead to the opposite, higher equilibrium interest rates. As in most monetary models, the nominal interest rate path does not uniquely determine an equilibrium. Thus, an interest rate of zero during the trap that becomes positive immediately after the trap is consistent with positive inflation and output after the trap.\(^3\) I show, however, that the optimal policy with commitment involves keeping the interest rate down at zero longer. This result confirms findings based on discrete-time simulations using a Poisson specification for the natural rate of interest. Eggertsson and Woodford (2003), Jung et al. (2005) and Adam and Billi (2006) find that the optimal interest rate path may keep it at zero after the natural rate of interest becomes positive. Eggertsson (2001), however, reports cases where this is not the case—this is likely due to the time aggregation effects of working in discrete time. My continuous time formulation is particularly useful here because it avoids time aggregation issues that may otherwise obscure the result. It also clarifies the bang-bang nature of the solution.

To the best of my knowledge this paper delivers the first explicit results on the optimal

\(^2\)In the US interest rates reached historically low levels immediately after the events of September 11th, 2001, prompting some concern over the zero lower bound. Subsequently, some commentators have argued that Federal Reserve, under the leadership of Chairman Alan Greenspan, was slow in raising interest rates during the ensuing recovery and that this may have over stimulated the economy.

\(^3\)For example, a zero interest during the trap and an interest equal to the natural rate outside the trap. This is the same path for the interest rate that results with discretionary monetary policy. However, in that case, the outcome for inflation and output is pinned down by the requirement that they reach zero upon exiting the trap. With commitment, the same path for interest rates is consistent with higher inflation and output upon exit. These issues are discussed in Section 5.
prolongation of zero interest rates. This has several advantages, providing new insights into the result and yielding some new implications.

In particular, I show that the reason for holding the interest rate at zero is not to promote inflation, as is commonly assumed. The sole purpose is to stimulate a consumption boom after the trap—the second goal of monetary policy discussed above. I establish this by considering three separate scenarios. First, even if prices were fully rigid, so that inflation is impossible, I show that interest rates would be kept at zero. Second, for less than fully rigid prices, I show that if we impose an ad hoc constraint to avoid positive inflation, then zero interest rates should still be prolonged. Finally, I show that if one can only commit to positive inflation then interest rates are not kept at zero. It follows that the real motive for committing to zero interest rates is to create the expectations of a future consumption boom. Thus, one contribution of the paper is to clarify the importance of this second goal in monetary policy and its connection with the commitment to zero interest rates.

I also show that interest rates should jump discretely upon exiting the zero bound. Thus, even when fundamentals vary continuously, optimal policy calls for a discontinuous interest rate path. Such a property can only be fully appreciated in continuous time. Mathematically, it captures the bang-bang nature of the solution. Economically, it implies that the exit strategy should be abrupt: optimal policy not only calls for low rates, held at zero, but also for a steep return to positive rates.

Finally, I show that while optimal policy lowers medium run interest rates to zero, it actually raises long run interest rates. As a result, the implications for the yield curve are not straightforward. Indeed, yields on long-term bonds may rise. This cautions against simple assessments of monetary policy centered around the lowering of yields at long maturities.

Turning to fiscal policy, which has been relatively unexplored, I study optimal government spending alongside optimal monetary policy. I show that away from the zero lower bound, spending should be zero. However, spending should rise above its natural level at the start of a liquidity trap, eventually falling and reaching a point below its natural level. What explains this pattern? Private consumption starts out below its efficient level, but ends up above its efficient level near the end of the liquidity trap, due to the consumption boom induced by prolonged zero interest rates. The pattern for government spending is then just the opposite. In this sense, optimal government spending is countercyclical, leaning against

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4 I focus on the government spending dimension of fiscal policy, in part because this was the focus of the countercyclical efforts in the recent US downturn. Other work has emphasized the tax dimension. In the context of Japan’s situation in the early 2000s Feldstein (2002) proposed a tax policy to fight the liquidity trap. Eggertsson and Woodford (2006) study monetary and tax policy and show that a particular combination of consumption and labor taxes, similar to Feldstein’s proposal, could potentially completely undo the liquidity trap. A similar point is made in Correia et al. (2008, 2011); see also Mankiw and Weinzierl (2011). However, in general, with limited tax instruments, tax policy is likely to be part of the mix but monetary policy remains essential and qualitatively unchanged.
the wind.

This optimal pattern for government spending actually combines two motives. The most obvious is perhaps the macroeconomic, countercyclical role, where government spending attempts to affect private consumption and inflation through dynamic general equilibrium effects. In a liquidity trap, when monetary policy is limited, this may help mitigate the depression and deflation. However, a second, often ignored, motive based on a static, cost-benefit calculation is that in a slump, wages or shadow wages are low making it is an opportune time to produce government goods. During the debates over the 2009 ARRA stimulus bill, variants of this argument were put forth in the popular press, but I know of no formal attempts to capture these concept.

To this end, I propose a decomposition of spending into "stimulus" and "opportunistic" components. The latter is defined as the currently optimal choice for government spending taking current private consumption as given. This is a static calculation that amounts to the Samuelson condition for public good provision (Samuelson, 1954). The former is simply the difference between actual spending and opportunistic spending. In the model, stimulus spending captures the desire to promote inflation, influencing private spending through this mechanism.

In various ways, my results highlight the importance of the opportunistic component of spending, relative to the stimulus component. At the optimum, stimulus spending is always zero at the start of a liquidity trap. Thus, the initial positive response of spending is entirely due to the opportunistic component and the fact that the economy initially finds itself in a recession. More surprisingly, I show that for some parameter values, stimulus spending remains at zero throughout the trap. In these cases, the opportunistic motive explains all of government spending. Moreover, commitment to the path of government spending is not required: a naive fiscal agency, acting with full discretion and performing static, cost-benefit calculations implements the optimal path for spending.

By definition, the opportunistic component of spending ignores any effects on inflation and private spending. However, it is important to recognize that it does, incidentally, have effects on inflation and private spending. Thus, even if all spending is accounted for by the opportunistic component, this does not imply that spending has not stimulated the economy. The interesting point is that these effects are not part of the calculation. During the recent US recession on the debates on the appropriate response of government spending focused on the magnitudes of government spending multipliers. In other words, on the impact of government spending on private spending. My results provide a different perspective, isolating conditions where the opportunistic motive is sufficient. At a minimum, this questions featuring discussions about multipliers at center stage, while relegating cost-benefit motives (at best) to the background. If anything, my results suggest the reverse may
be more reasonable.

Although the decomposition of spending, into opportunistic and stimulus motives, that I propose here is formalized within the context of a particular model, I expect that it can transcend this setting and be useful in other related models.

The aforementioned results assume that monetary policy is optimal, but things can be quite different when it is not, due to a lack of commitment. To address this I study a mixed case, where monetary policy is discretionary but fiscal policy has the power to commit to a path for spending, at least for some horizon. I first assume the horizon of commitment is equal to the duration of the liquidity trap. Without government spending, the outcome would feature deflation and negative output throughout the trap. As a result, optimal spending is positive throughout. Indeed, both the opportunistic and stimulus components are strictly positive. Overall, the role for government spending is magnified when monetary policy is discretionary.

If the fiscal authority can commit over a longer horizon, one that goes beyond the trap, then it will promise to lower spending immediately after the trap, eventually converging back to the natural level of spending. The rationale for this austerity is as follows. After the trap the discretionary monetary authority will implement an efficient allocation. Thus, lowering government spending boosts private consumption, through the usual neoclassical channels. This consumption boom is similar to the one that would have been created with optimal monetary policy. A fiscal commitment to austerity after the trap helps mimic the missing commitment to expansionary monetary policy. Interestingly, it also induces the discretionary monetary authority to prolong zero interest rates.

2 A Liquidity Trap Scenario

The model is a continuous-time version of the standard New Keynesian model. The environment features a representative agent, monopolistic competition and Calvo-style sticky prices; it abstracts from capital investment. Since this is a common workhorse model (see Woodford, 2003, or Galí, 2008, for useful textbook expositions), for my purposes it is convenient to skip directly to a discussion of the well-known log-linear approximation of the equilibrium conditions, which I will use in the remainder of the paper.
Euler Equation and Phillips Curve. The equilibrium conditions, log linearized around zero inflation, are

$$\dot{x}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t))$$  \hspace{1cm} (1a)
$$\dot{\pi}(t) = \rho \pi(t) - \kappa x(t)$$  \hspace{1cm} (1b)
$$i(t) \geq 0$$  \hspace{1cm} (1c)

where $\rho, \sigma$ and $\kappa$ are positive constants and the path $\{r(t)\}$ is exogenous and given. We also require a solution $(\pi(t), x(t))$ to remain bounded. The variable $x(t)$ represents the output gap: the log difference between actual output and the hypothetical output that would prevail at the efficient, flexible price, outcome. Inflation is denoted by $\pi(t)$ and the nominal interest rate by $i(t)$. Finally, $r(t)$ stands for the “natural rate of interest”, i.e. the real interest rate that would prevail in an efficient, flexible price, outcome with $x(t) = 0$ throughout.

Equation (1a) represents the consumer’s Euler equation. Output growth, equal to consumption growth, is an increasing function of the real rate of interest, $i(t) - \pi(t)$. The natural rate of interest enters this condition because output has been replaced with the output gap. Equation (1b) is the New-Keynesian, forward-looking Phillips curve. It can be restated as saying that inflation is proportional, with factor $\kappa > 0$, to the present value of future output gaps,

$$\pi(t) = \kappa \int_{0}^{\infty} e^{-\rho s} x(t + s) ds.$$ 

Thus, positive output gaps stimulate inflation, while negative output gaps produce deflation. Finally, inequality (1c) is the zero-lower bound on nominal interest rates (hereafter, ZLB). This constraint is an equilibrium requirement since agents can freely choose their portfolio between money and bond balances. If money were to strictly dominate bonds in rates of return an arbitrage opportunity would be present.

As for the constants, $\rho$ is the discount rate, $\sigma^{-1}$ is the intertemporal elasticity of substitution and $\kappa$ controls the degree of price stickiness. Lower values of $\kappa$ imply greater price stickiness. As $\kappa \to \infty$ we approach the benchmark with perfectly flexible prices, where high levels of inflation or deflation are compatible with minuscule output gaps.

Quadratic Welfare Loss. I will evaluate outcomes using the quadratic loss function

$$L \equiv \frac{1}{2} \int_{0}^{\infty} e^{-\rho t} \left(x(t)^2 + \lambda \pi(t)^2\right) dt.$$  \hspace{1cm} (2)

According to this loss function it is desirable to minimize deviations from zero for both inflation and the output gap. The constant $\lambda$ controls the relative weight placed on the
inflationary objective.

The quadratic objective can be derived as a second order approximation to welfare around zero inflation when the flexible price equilibrium is efficient.\(^5\) Such an approximation implies that \(\lambda = \bar{\lambda}/\kappa\) for some constant \(\bar{\lambda}\), so that \(\lambda \to 0\) as \(\kappa \to \infty\), as prices become more flexible, price instability becomes less harmful.

**The Natural Rate of Interest.** The path for the natural rate \(\{r(t)\}\) plays a crucial role in the analysis. Indeed, if the natural rate were always positive, so that \(r(t) \geq 0\) for all \(t \geq 0\), then the flexible price outcome with zero inflation and output gap, \(\pi(t) = x(t) = 0\) for all \(t \geq 0\), would be feasible and obtained by letting \(i(t) = r(t)\) for all \(t \geq 0\). This outcome is also optimal, since it is ideal according to the loss function in equation (2). This result is sometimes referred to as the “divine coincidence”, since both zero inflation and zero output gaps are possible and optimal.

The situation described in the previous paragraph amounts to the case where the ZLB constraint (1c) is always slack. The focus of this paper is on situations where the ZLB constraint binds. Thus, I am interested in cases where \(r(t) < 0\) for some range of time. For a few results it is useful to further assume that the the economy starts in a liquidity trap that it will eventually and permanently exit at some date \(T > 0\):

\[
\begin{align*}
    r(t) &< 0 & t < T, \\
    r(t) &\geq 0 & t \geq T.
\end{align*}
\]

I call such a case a *liquidity trap scenario*. A simple example is the step function

\[
r(t) = \begin{cases} 
    r & t \in [0, T) \\
    \bar{r} & t \in [T, \infty) 
\end{cases}
\]

where \(r < 0 < \bar{r}\). I use this step function in some figures and simulations, but it is not required for any of the results in the paper.

Finally, I also make a technical assumption: that \(r(s)\) is bounded and that the integral \(\int_0^t r(s) ds\) be well defined and finite for any \(t \geq 0\).

### 3 Monetary Policy without Commitment

Before studying optimal policy with commitment, it is useful to consider, as a benchmark, the situation without commitment, where the central bank is benevolent but cannot credibly

\(^5\)This requires a constant subsidy to production to undo the monopolistic markup.
announce plans for the future. Instead, it acts opportunistically at each point in time, with absolute discretion.

### 3.1 Deflation and Depression

To isolate the problems created by a complete lack of commitment, I rule out explicit rules as well as reputational mechanisms that bind or affect the central bank’s actions directly or indirectly. I construct the unique equilibrium as follows.\(^6\)

For \( t \geq T \) the natural rate is positive, \( r(t) = \bar{r} > 0 \), so that, as mentioned above, the ideal outcome \((\pi(t), x(t)) = (0, 0)\) is attainable. I assume that the central bank can guarantee this outcome so that \((\pi(t), x(t)) = (0, 0)\) for \( t \geq T \).\(^7\) Taking this as given, at all earlier dates \( t < T \) the central bank will find it optimal to set the nominal interest rate to zero. The resulting no-commitment outcome is then uniquely determined by the ODEs (1a)–(1b) with \( i(t) = 0 \) for \( t \leq T \) and the boundary condition \((\pi(T), x(T)) = (0, 0)\).

This situation is depicted in Figure 1 which shows the dynamical system (1a)–(1b) with \( i(t) = 0 \) and depicts a path leading to \((0, 0)\) precisely at \( t = T \). Output and inflation are both

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\(^6\)In this section, I proceed informally. With continuous time, a formal study of the no-commitment case requires a dynamic game with commitment over vanishingly small intervals.

\(^7\)Although this seems like a natural assumption, it presumes that the central bank somehow overcomes the indeterminacy of equilibria that plagues these models. A few ideas have been advanced to accomplish this, such as adhering to a Taylor rule with appropriate coefficients, or the fiscal theory of the price level. However, both assume commitment on and off the equilibrium path. Although this issue is interesting, it seem completely separate from the zero lower bound. Thus, supposing that \((\pi(t), x(t)) = (0, 0)\) can be guaranteed for \( t \geq T \) allows us to focus on the interaction between no commitment and a liquidity trap scenario.
negative for $t < T$ as they approach $(0,0)$. Note that the loci on which $(\pi(t), x(t))$ must travel towards $(0,0)$ is independent of $T$, but a larger $T$ requires a starting point further away from the origin. Thus, initial inflation and output are both decreasing in $T$. Indeed, as $T \to \infty$ we have that $\pi(0), x(0) \to -\infty$.

**Proposition 1.** Consider a liquidity trap scenario, with $r(t) < 0$ for $t < T$ and $r(t) \geq 0$ for $t \geq T$. Let $\pi^{nc}(t)$ and $x^{nc}(t)$ denote the equilibrium outcome without commitment. Then inflation and output are zero after $t = T$ and strictly negative before that:

$$
\pi^{nc}(t) = x^{nc}(t) = 0 \quad t \geq T
$$

$$
\pi^{nc}(t) < 0 \quad x^{nc}(t) < 0 \quad t < T.
$$

Moreover, $\pi(t)$ and $x(t)$ are strictly increasing in $t$ for $t < T$. In the limit as $T \to \infty$, if the natural rate satisfies $\int_0^T r(t; T) ds \to -\infty$, then

$$
\pi^{nc}(0, T), x^{nc}(0, T) \to -\infty.
$$

The equilibrium features deflation and depression. The severity of both depend, among other things, on the duration $T$ of the liquidity trap. Both become unbounded as $T \to \infty$. In this sense, discretionary policy has very adverse welfare implications.

How can the outcome be so dire? The main distortion is that the real interest rate is set too high during the liquidity trap. This depresses consumption. Importantly, this effect accumulates over time. Even if inflation were zero, then consumption would be depressed by $\sigma^{-1} \int_t^T r(t) ds$. For example, with log utility $\sigma = 1$, if the natural rate is -4% and the trap lasts two years, then the loss in output is at least 8%. Matters are just made worse by deflation, which raises the real interest rate even more, further depressing output, leading to even more deflation, in a vicious cycle.

Note that it is the lack of commitment during the liquidity trap $t < T$ to policy actions and outcomes after the liquidity trap $t \geq T$ that is problematic. Policy commitment during the liquidity trap $t < T$ is not useful. Neither is the ability to announce a credible plan once $t = T$ is reached regarding the remaining future $t > T$. Indeed, if we solve for an optimum imposing $(\pi(T), x(T)) = (0,0)$ as an additional constraint, then the solution coincides with the discretionary equilibrium outcome. The key thing is to commit during the liquidity trap to policy actions after the liquidity trap. In particular, to something other than $(\pi(T), x(T)) = (0,0)$.
3.2 Harmful Effects from Price Flexibility

How is this bleak outcome affected by the degree of price stickiness? One might expect things to improve when prices are more flexible. After all, the main friction in New Keynesian models is price rigidities, suggesting that outcomes should improve as prices become more flexible. The next proposition shows, perhaps counterintuitively, that the reverse is actually the case.

**Proposition 2.** Without commitment higher price flexibility leads to more deflation and lower output: if $\kappa < \kappa'$ then

$$\pi^{nc}(t, \kappa') < \pi^{nc}(t, \kappa) < 0 \quad \text{and} \quad x^{nc}(t, \kappa') < x^{nc}(t, \kappa) < 0 \quad \text{for all } t < T.$$  

Indeed, for given $T > 0$ and $t < T$ in the limit as $\kappa \to \infty$ then

$$\pi(t, \kappa), x(t, \kappa) \to -\infty \quad \text{and} \quad L(\kappa) \to \infty.$$  

Sticky prices are beneficial because they dampen deflation, this in turn mitigates the depression. In fact, the most favorable outcome is obtained when prices are completely rigid, $\kappa = 0$. At the other end of the spectrum, in the limit of perfectly flexible prices, as $\kappa \to \infty$, the depression and deflation become unbounded.

To see this more clearly, note that the Phillips curve equation (1b) implies that, for a given negative output gap, a higher $\kappa$ creates more deflation. More deflation, in turn, increases the real interest rate $i - \pi$. By the Euler equation (1a), this requires higher growth in the output gap, but since $x(T) = 0$ this translates into a lower level of $x(t)$ for earlier dates $t < T$. In words, flexible prices lead to more vigorous deflation, raising the real interest rate, increasing the desire for saving, lowering consumption and depressing output. Lower output reinforces the deflationary pressures, creating a vicious cycle. The proof in the appendix echoes this intuition closely.

A similar result is reported in the analysis of fiscal multipliers by Christiano et al. (2011). They compute the equilibrium when monetary policy follows a Taylor rule and the natural rate of interest is a Poisson process taking two values. They show numerically that output may be more depressed when prices are more flexible (similar Poisson calculations appear also in Woodford (2011) and Eggertsson (2011), but they do not discuss the effects of greater price flexibility). They do not pursue a limiting result towards full flexibility. My result is also somewhat distinct, since it applies to a situation with optimal discretionary monetary policy, instead of a given Taylor rule. Also, it holds for any deterministic path for the natural rate. Another difference is that with Poisson uncertainty an equilibrium fails to exist when prices are sufficiently flexible. Despite these differences, the logic for the effect is the same.
in both cases. De Long and Summers (1986) is an earlier contribution that make the point that, for given monetary policy rules, price flexibility may be destabilizing, even away from a liquidity trap, in the sense of increasing the variance of output.

The zero lower bound and the lack of commitment are not critical for this result. The same conclusions follow for any path of the natural rate \( \{ r(t) \} \) if we assume the central bank sets the nominal interest rate above the natural rate \( i(t) = r(t) + \Delta \) with \( \Delta > 0 \) for some period of time \( t \in [0, T] \) and then returns to implementing the first-best outcome \( x(t) = \pi(t) = 0 \) and \( i(t) = r(t) \) for \( t > T \). The zero lower bound and the lack of commitment serve to motivate such a scenario. However, another justification is policy mistakes of this particular form, where interest rates are set too high (or too low) for a fixed amount of time—such as the time between FOMC meetings, or longer if relevant information arrives with longer lags.

As I discuss later, when the central bank commits to an optimal policy, price flexibility may still be harmful. Whether it is beneficial or harmful depends on parameters.

**Is there a Discontinuity at Full Flexibility? Not really.**

The result that price flexibility makes matters worse may seem puzzling, especially in the limit, since it seems to contradict the notion that perfectly flexible prices lead to zero output gaps. That is, at \( \kappa = \infty \) we expect \( x(t) = 0 \) for all \( t \geq 0 \), but, paradoxically, as \( \kappa \to \infty \) we obtain \( x(t) \to -\infty \) instead. Does this reveal an inherent discontinuity in the New Keynesian model?

No. The result is best seen as arising from a discontinuity in monetary policy at \( \kappa = \infty \), not a discontinuity in the model itself. The equilibria obtained for finite \( \kappa \) described in Proposition 2 satisfy \( \pi(t) \leq 0 \) and \( \pi(T) = 0 \). If one takes these two features as a requirement then there is no equilibrium with \( \kappa = \infty \). Economically, this suggests a form of continuity: an explosive outcome converges to a situation where an equilibrium seizes to exist. In any case, when \( \kappa = \infty \), monetary policy must allow strictly positive inflation for an equilibrium to exist. In this sense, the discontinuity in outcomes is produced by a discontinuity in monetary policy regarding inflation at \( \kappa = \infty \).

A more practical lesson from this discussion is that the benefits of flexible prices only accrue with monetary policies that accept higher inflation. Price flexibility can only work its magic if we use it, allowing for inflation.
3.3 Elbow Room with a Higher Inflation Target: The Value of Commitment

Before studying optimal policy it is useful to consider commitments to simple, non-optimal policies.

Consider a plan that keeps inflation and the output gap constant at
\[ \pi(t) = -\bar{r} > 0 \quad x(t) = -\frac{1}{\kappa}r > 0 \quad \text{for all} \quad t \geq 0. \]

Since \( \dot{x}(t) = 0 \), it follows that \( i(t) = r(t) + \pi(t) \). Hence, \( i(t) = 0 \) for \( t < T \) while \( i(t) = \bar{r} + \bar{\pi} > \bar{r} > 0 \) for \( t \geq T \).

This idea is more general. For any path for the natural interest rate \( \{r(t)\} \), set a constant inflation rate given by
\[ \pi(t) = \bar{\pi} = -\min_{t \geq 0} r(t) \]
and an output gap of \( x(t) = \bar{x} = \kappa \bar{\pi} \). This plan is feasible with a non-negative nominal interest rates \( i(t) \geq 0 \).

These simple policy capture the main idea behind calls to tolerate higher inflation targets that leave more “elbow room” for monetary policy during liquidity traps (e.g. Summers, 1991; Blanchard et al., 2010). However, given the forward looking nature of inflation in this model, what is crucial is the commitment to higher inflation after the liquidity trap. This contrasts with the conventional argument, where a higher inflation rate before the trap serves as a precautionary sacrifice for future liquidity traps.

Although this policy is not optimal, it approximates the optimum as prices become fully flexible. In this limit as \( \kappa \to \infty \) the output gap converges uniformly to zero while inflation remains constant. Thus, if we adopt the natural case where \( \lambda = \bar{\lambda}/\kappa \to 0 \), the loss function converges to its ideal value of zero, \( L(\kappa) \to 0 \). Compare this to the dire outcome without commitment in Proposition 2, where the output gap and losses converge to \( -\infty \).

Just as in the case without commitment, this simple policy sets the nominal interest rate to zero during the liquidity trap, for \( t < T \). Note that after the trap, for \( t > T \), the nominal interest rate is actually set to a higher level than in the case without commitment. Thus, the advantages of this simple policy do not hinge on lower nominal interest rates. Quite the contrary, higher inflation here coincides with higher nominal interest rates, due to the Fischer effect. One may still describe the outcome as resulting from looser monetary policy, but the point is that the kind of monetary easing needed to promote inflation does not imply lower equilibrium nominal interest rates. Thus, a commitment to higher inflation may raise the yield curve, relative to the discretionary case. This foreshadow a similar conclusion under optimal policy taken up in the next section.

It is perhaps surprising that commitment to a simple policy can avoid both deflation...
and depressed output altogether. However, they do so at the expense of inflation and over-stimulated output. If the required inflation target $\bar{\pi}$ or output gap $\bar{x}$ are large, or if the duration of the trap $T$ is small, these policies may be far from optimal, because they require a permanent sacrifice to resolve a temporary problem. This motivates the study of optimal monetary policy in the next section.

4 Optimal Monetary Policy with Commitment

I now turn to optimal monetary policy with commitment. The central bank’s problem is to minimize the objective (2) subject to (1a)–(1c) with both initial values of the states, $\pi(0)$ and $x(0)$, free. The problem seeks the most preferable outcome, across all those compatible with an equilibrium. In what follows I focus on characterizing the optimal path for inflation, output and the nominal interest rate.

4.1 Optimal Interest Rates, Inflation and Output

The problem can be analyzed as an optimal control problem with state $(\pi(t), x(t))$ and control $i(t) \geq 0$. The associated Hamiltonian is

$$H \equiv \frac{1}{2} x^2 + \frac{1}{2} \lambda \pi^2 + \mu_x \sigma^{-1} (i - r - \pi) + \mu_\pi (\rho \sigma - \kappa x).$$

The maximum principle implies that the co-state for $x$ must be non-negative throughout and zero whenever the nominal interest rate is strictly positive

$$\mu_x(t) \geq 0, \quad i(t) \mu_x(t) = 0. \tag{3a} \tag{3b}$$

The law of motion for the co-states are

$$\dot{\mu}_x(t) = -x(t) + \kappa \mu_\pi(t) + \rho \mu_x(t), \quad \tag{3c}$$

$$\dot{\mu}_\pi(t) = -\lambda \pi(t) + \sigma^{-1} \mu_x(t). \tag{3d}$$

8I do not dedicate much discussion to the question of implementation, in terms of a choice of (possibly time varying) policy functions that would make the optimum a unique equilibrium. It is well understood that, once the optimum is computed, a time varying interest rate rule of the form $i(t) = i^*(t) + \psi_\pi(\pi(t) - \pi^*(t)) + \psi_x(x(t) - x^*(t))$ ensures that this optimum is the unique local equilibrium for appropriately chosen coefficients $\psi_\pi$ and $\psi_x$. Eggertsson and Woodford (2003) propose a different policy, described in terms of an adjusting target for a weighted average of output and the price level, that also implements the equilibrium uniquely.
Finally, because both initial states are free, we have

\[
\mu_x(0) = 0, \quad (3e)
\]
\[
\mu_\pi(0) = 0. \quad (3f)
\]

Taken together, equations (1a)–(1c) and (3a)–(3f) constitute a system for \(\{\pi(t), x(t), i(t), \mu_\pi(t), \mu_x(t)\}_{t \in [0,\infty)}\). Since the optimization problem is strictly convex, these conditions, together with appropriate transversality conditions, are both necessary and sufficient for an optimum. Indeed, the optimum coincides with the unique bounded solution to this system.

Suppose the zero-bound constraint is not binding over some interval \(t \in [t_1, t_2]\). Then it must be the case that \(\mu_x(t) = \dot{\mu}_x(t) = 0\) for \(t \in [t_1, t_2]\), so that condition (3c) implies \(x(t) = \kappa \mu_\pi(t)\), while condition (3d) implies \(\dot{\mu}_\pi(t) = -\lambda \pi(t)\). As a result,

\[
\dot{x}(t) = \kappa \dot{\mu}_\pi(t) = -\kappa \lambda \pi(t) = \sigma^{-1}(i(t) - r(t) - \pi(t)).
\]

Solving for \(i(t)\) gives

\[
i(t) = I(r(t), \pi(t)),
\]

where

\[
I(r, \pi) \equiv r(t) + (1 - \kappa \sigma \lambda) \pi,
\]

is a function that gives the optimal nominal rate whenever the zero-bound is not binding. This is the interest rate condition derived in the traditional analysis that assumes the ZLB never binds (see e.g. Clarida, Gali and Gertler, 1999, pg. 1683). Note that this rate equals the natural rate when inflation is zero, \(I(r, 0) = r\). Thus, it encompasses the well-known price stability result from basic New-Keynesian models. Away from zero inflation, the interest rate generally departs from the natural rate, unless \(\sigma \lambda \kappa = 1\).

Given this result, it follows that \(I^*(r(t), \pi^*(t)) \geq 0\) is a necessary condition for the zero-bound not to bind. The converse, however, is not true.

**Proposition 3.** Suppose \(\{\pi^*(t), x^*(t), i^*(t)\}\) is optimal. Then at any point in time either \(i^*(t) = I(r(t), \pi^*(t))\) or \(i^*(t) = 0\). Moreover if

\[
I(r(t), \pi^*(t)) < 0 \quad \forall t \in [t_0, t_1)
\]

then

\[
i^*(t) = 0 \quad \forall t \in [t_0, \hat{t}_1].
\]

for \(t_1 < \hat{t}_1\). Likewise, if \(t_0 > 0\) then \(i^*(t) = 0\) for \(t \in [\hat{t}_0, t_0]\) for \(\hat{t}_0 < t_0\).

According to this result, the nominal interest rate should be held down at zero longer
than what current inflation warrants. That is, the optimal path for the equilibrium path for
the nominal interest rate is not equal to the upper envelope of \( I(r(t), \pi^*(t)) \) and zero:

\[
i^*(t) \neq \max\{0, I(r(t), \pi^*(t))\}.
\]

Instead, the nominal interest rate path must be lower than this upper envelope, strictly so
over some intervals where \( i(t) = 0 \) but \( I(r(t), \pi^*(t)) > 0 \).

The notion that committing to future monetary easing is beneficial in a liquidity trap
was first put forth by Krugman (1998). His model featured prices that cannot adjust within
a period, but are fully flexible between periods. In his setting, the first best is obtained
by committing to money growth, inducing higher future inflation. Inflation is costless and
easily obtained in the model. Eggertsson and Woodford (2003) work with the same New
Keynesian model as I do here. They report numerical simulations where a prolonged period
of zero interest rates are optimal. My result provides the first formal explanation for these
patterns. It also clarifies that the relevant comparison for the nominal interest rate
\( i^*(t) \) is the unconstrained optimum \( I(r(t), \pi^*(t)) \), not the natural rate \( r(t) \); the two are not equivalent,
unless \( \kappa \sigma \lambda = 1 \). The continuous time framework employed here helps capture the bang-
bang nature of the solution. In contrast, a discrete-time setting can obscure things due to
time aggregation. Indeed, Eggertsson (2001), in a discrete time setting, reports simulations
where it is not optimal to prolong zero interest rates.

One interesting implication of my result is that the optimal exit strategy features a dis-
crete jump in the nominal interest rate. Whenever the zero-bound stops binding the nom-
inal interest must equal \( I(r(t), \pi^*(t)) \), which given Proposition 3, will generally be strictly
positive. Thus, optimal policy requires a discrete upward jump, from zero, in the nomi-
nal interest rate. Even when economic fundamentals vary smoothly, so that \( I(r(t), \pi^*(t)) \) is
continuous, the best exit strategy calls for a discontinuous hike in the nominal interest rate.

Does commitment to the optimal policy, with prolonged zero interest rates, lower the
yield on long-term bonds? Not necessarily. Consider a liquidity trap scenario. The optim-
um keeps interest rates at zero up to some date \( \hat{T} > T \). This does lower the yield on
medium term bonds, those with maturity between \( T \) and \( \hat{T} \). However, upon exit from the
zero lower bound at \( \hat{T} \), the short-term interest rate is set to \( I(r(\hat{T}), \pi(\hat{T})) \) and one can show
that \( I(r(\hat{T}), \pi(\hat{T})) \geq r(\hat{T}) \) with strict inequality as long as \( \kappa \sigma \lambda \neq 1 \) and \( \kappa \neq 0 \). Once again, as
in Section 3.3, looser monetary policy does not necessarily lead to lower long-term interest
rates. As a result, the yield on a bond of sufficiently long maturity may be higher, not lower.

The previous result characterizes nominal interest rates, but what can be said about the
paths for inflation and output? This question is important for a number of reasons. First,
output and inflation are of direct concern, since they determine welfare. In contrast, the
nominal interest rate is merely an instrument to influence output and inflation. Second, as in most monetary models, the equilibrium outcome is not uniquely determined by the equilibrium path for the nominal interest rate. A central bank wishing to implement the optimum needs to know more than the path for the nominal interest rate. For example, the central bank may employ a Taylor rule centered around the target path for inflation

\[ i(t) = i^*(t) + \psi(\pi(t) - \pi^*(t)) \]

with \( \psi > 1 \). Finally, understanding the outcome for inflation and output sheds light on the kind of policy commitment required.

The next proposition characterizes optimal inflation and output.

**Proposition 4.** Suppose the first-best outcome is not attainable and that \( \{\pi^*(t), x^*(t), i^*(t)\} \) is optimal:

1. Inflation must be strictly positive at some point in time: \( \pi(t) > 0 \) for some \( t \geq 0 \).
2. Output is initially negative \( x(0) \leq 0 \), but becomes strictly positive at some point, \( x(t) > 0 \) for some \( t > 0 \).
3. Furthermore, if (a) \( \kappa \sigma \lambda = 1 \) then inflation is initially zero and is nonnegative throughout, \( \pi^*(0) = 0 \) and \( \pi^*(t) \geq 0 \) for all \( t \geq 0 \); (b) \( \kappa \sigma \lambda < 1 \) then \( \pi^*(t) > 0 \) for all \( t \); (c) \( \kappa \sigma \lambda > 1 \) then \( \pi^*(0) \leq 0 \) with strict inequality if \( x(0) < 0 \).

Inflation must be positive at some point. Depending on parameters, initial inflation may positive or negative. In some cases, inflation may be positive throughout. Output, on the other hand, must switch signs. The initial recession is never completely avoided.

According to this proposition, there are two things optimal monetary policy accomplishes. First, it promotes inflation to mitigate or reverse the deflationary spiral during the liquidity trap. This lowers the real rate of interest, which lessens the recession. Second, it stimulates future output to create a boom after the trap. This percolates back in time, making consumers, who anticipate a boom, lower their desired savings. In other words, the root problem during a liquidity trap is that desired savings and the real interest rate are too high. Optimal policy addresses both.

In this model the two goals are related: inflation requires a boom in output. Thus, pursuing the first goal already leads, incidentally, to the second, and vice versa. However, the nominal interest rate path implied by Proposition 3 stimulates a larger boom than what is required by the inflation promise alone. To see this, suppose that along the optimal plan

\[ I(r(t), \pi^*(t)) \geq 0 \quad \text{for} \quad t \geq t_1, \quad \text{and} \quad I(r(t), \pi^*(t)) < 0 \quad \text{otherwise}. \]

The optimal plan then calls for \( i^*(t) = 0 \) over some interval \( t \in [t_1, t_2] \). However, consider an alternative plan that has the same inflation at \( t_1 \), so that \( \pi(t_1) = \pi^*(t_1) \), but, in contradiction with Proposition 3,
features \( i(t) = I(r(t), \pi(t)) \) for all \( t \geq t_1 \). Suppose also that, for both plans, the long-run output gap is zero: \( \lim_{t \to \infty} x(t) = \lim_{t \to \infty} x^*(t) = 0 \). It then follows that \( x(t_1) < x^*(t_1) \). In this sense, holding down the interest rate to zero stimulates a boom that is greater than the one implied by the inflation promise.

Proposition 4 singles out a case with \( \kappa \sigma \lambda = 1 \) where inflation starts and ends at zero and is positive throughout. This case occurs when the costate \( \mu_\pi(t) \) on the Phillips curve is zero for all \( t \geq 0 \). This case turns out to be an interesting benchmark with interesting implications for government spending.

Figure 2 plots the equilibrium paths for a numerical example. The parameters are set to \( T = 2, \sigma = 1, \kappa = .5 \) and \( \lambda = 1/\kappa \). These choices are made for illustrative purposes and to ensure that \( \kappa \sigma \lambda = 1 \). The black line represents the equilibrium with discretion; the blue line, the optimum with commitment. With discretion output is initial depressed by about 11%, at the optimum this is reduced to just under 4%. The optimum features a boom which peaks at about 3% at \( t = T \). The discretionary case features significant deflation. In contrast, because \( \kappa \sigma \lambda = 1 \) optimal inflation starts at zero and is always positive. Both paths end at origin, which represents the ideal first-best outcome. However, although the optimum reaches it later at \( \hat{T} = 2.7 \), it circles around it, managing to stay closer to it on average. This improves welfare.

One implication of Proposition 4 is that, whenever the first best is unattainable, optimal monetary policy requires commitment. Output is initially negative \( x^*(0) \leq 0 \), but must turn strictly positive \( x^*(t') > 0 \) at some future date for \( t' > 0 \). This implies that, if the planner

---

9Note that, depending on the value of \( \kappa \sigma \lambda \), the interest rate may even be greater than the natural rate \( r(t) \). The fact that this policy is consistent with positive inflation and output after the trap even though it may have higher interest rates than the discretionary solution underscores, once again, that monetary easing does not necessarily manifest itself in lower equilibrium interest rates.
can reoptimize and make a new credible plan at time \( t' \), then this new plan would involve initially negative output \( x^* (t') \leq 0 \). Hence, it cannot coincide with the original plan which called for positive output.

Note that the kind of commitment needed in this model involves more than a promise for future inflation, at time \( T \), as in Krugman (1998). Indeed, my discussion here emphasizes commitment to an output boom. More generally, the planning problem features both \( \pi \) and \( x \) as state variables, so commitment to deliver promises for both inflation and output are generally required.

Liquidity traps are commonly associated with deflation, but these results suggest that the optimum completely avoids deflation in some cases. This is more likely to be the case if prices are less flexible (low \( \kappa \)), if the intertemporal elasticity of substitution is high (low \( \sigma \)), or if the central bank is not too concerned about inflation (low \( \lambda \)).

It is worth noting that prolonged zero nominal interest rates are not needed to promote positive inflation and stimulate output after the trap. Indeed, there are equilibria with both features and a nominal interest rate path given by \( i(t) = \max\{0, I(r(t), \pi(t))\} \). In the liquidity trap scenario, the same is true for the interest rate path considered under pure discretion, \( i(t) = 0 \) for \( t < T \) and \( i(t) = r(t) \) for \( t \geq T \). Without commitment, a unique equilibrium was obtained by adding the condition that the first best outcome \( \pi(t) = x(t) = 0 \) was implemented for \( t \geq T \). However, positive inflation and output, \( \pi(T), x(T) \geq 0 \) are also compatible with this very same interest rate path. This is possible because equilibrium outcomes are not uniquely determined by equilibrium nominal interest rates. Policy may still be described as one of monetary easing, even if this is not necessarily reflected in equilibrium nominal interest rates.\(^{10}\)

### 4.2 A Simple Case: Fully Rigid Prices

To gain intuition it helps to consider the extreme case with fully rigid prices, where \( \kappa = 0 \), so that \( \pi(t) = 0 \) for all \( t \geq 0 \). Consider the liquidity trap scenario, where \( r(t) < 0 \) for \( t < T \) and \( r(t) > 0 \) for \( t > T \), and suppose we keep the nominal interest rate at zero until some time \( \hat{T} \geq T \), and implement \( x(t) = \pi(t) = 0 \) after \( \hat{T} \). Output is then

\[
x(t; \hat{T}) \equiv \sigma^{-1} \int_t^{\hat{T}} r(s)ds.
\]

\(^{10}\)To be specific, suppose policy is determined endogenously according to a simple Taylor rule, with a time varying intercept, \( i(t) = \bar{i}(t) + \phi_{\pi} \pi(t) \) with \( \phi_{\pi} > 1 \). In the unique bounded equilibrium, a temporarily low value for \( \bar{i}(t) \) typically leads to higher inflation \( \pi(t) \), but not necessarily a lower equilibrium interest rate \( i(t) \). The outcome for the nominal interest rate \( i(t) \) depends on various parameters. Either way, the situation with temporarily low \( \bar{i}(t) \) may be described as one of “monetary easing”.
Figure 3: Fully rigid prices. The path for output with $\hat{T} = T$ and $\hat{T} > T$.

When $\hat{T} = T$, the integrand is always negative, so that output is negative: $x(t, T) < 0$ for $t < T$. In fact, the equilibrium coincides with the full discretion case isolated by Proposition 1. When $\hat{T} > T$ the integral includes strictly positive values for $r(t)$ for $t \in (T, \hat{T}]$. This increases the path for $x(t; \hat{T})$. For any date $t \leq T$ output increases by the constant $\sigma^{-1} \int_{T}^{\hat{T}} r(s)ds > 0$. Starting at $t = 0$, output rises and peaks at $T$, then falls until it reaches zero at $\hat{T}$. The boom induced at $T$ percolates to earlier dates, increasing output in a parallel fashion.

Larger values of $\hat{T}$ shrink the initially negative output gaps, but lead to larger positive gaps later. Starting from $\hat{T} = T$ an increase in $\hat{T}$ improves welfare because the loss from positive output gaps are second order, while the gain from reducing existing negative output gaps is first order. Formally, minimizing the objective $V(\hat{T}) \equiv \frac{1}{2} \int_{0}^{\infty} e^{-\rho t} x(t; \hat{T})^2 dt$ yields

$$V'(\hat{T}^*) = r(\hat{T}^*) \sigma^{-1} \int_{0}^{\hat{T}^*} e^{-\rho t} x(t; \hat{T}^*) dt = 0,$$

and it follows that $T < \hat{T}^* < T$ where $x(0, T) = 0$. According to this optimality condition, the present value of output should be zero $\int_{0}^{\infty} e^{-\rho t} x(t) dt = 0$, implying that the current recession and subsequent boom should average out, in present value.

When prices are fully rigid inflation is zero regardless of monetary policy. Hence, creating inflation cannot be the purpose of monetary easing. Instead, committing to zero nominal interest rates is useful here because it creates an output boom after the trap. This boom helps mitigate the earlier recession. The logic here is completely different from the one in Krugman (1998), which isolated the inflationary motive for monetary easing. Next I turn to a graphical analysis of intermediate cases, where both motives are present.

4.3 Stitching a Solution Together: A Graphical Representation

To see the solution graphically, consider the particular liquidity trap scenario with the step function path for the natural rate of interest: $r(t) = \bar{r} < 0$ for $t < T$ but $r(t) = \bar{r} \geq 0$
for \( t \geq T \). It is useful to break up the solution into three separate phases, from back to front. I first consider the solution after some time \( \hat{T} > T \) when the ZLB constraint is no longer binding (Phase III). I then consider the solution between time \( T \) and \( \hat{T} \) with the ZLB constraint (Phase II). Finally, I consider the solution during the trap \( t \in [0, T] \) (Phase I).

**After the Storm: Slack ZLB Constraint (Phase III).** Consider the problem where the ZLB constraint is ignored, or no longer binding. If this were true for all time \( t \geq 0 \) then the solution would be the first best \( \pi(t) = x(t) = 0 \). However, here I am concerned with a situation where the ZLB constraint is slack only after some date \( \hat{T} > T > 0 \), at which point the state \( (\pi(\hat{T}), x(\hat{T})) \) is given and no longer free, so the first best is generally not feasible.

The relevant continuation planning problem now ignores the ZLB constraint but takes the initial state there as given. Because the ZLB constraint is absent, the constraint representing the Euler equation is not binding. Thus, it is appropriate to ignore this constraint and drop the output gap \( x(t) \) as a state variable, treating it as a control variable instead. The only remaining state is inflation \( \pi(t) \).

I seek a solution for output \( x \) as a function of inflation \( \pi \). Using the optimality conditions with \( \mu_x(t) = 0 \) one can show that \( i(t) = I(\pi(t), r(t)) \) as discussed earlier, with output satisfying

\[
x(t) = \phi \pi(t)
\]

and costate \( \mu_{\pi}(t) = \frac{\phi}{\kappa} \pi(t) \), where \( \phi \equiv \frac{\rho + \sqrt{\rho^2 + 4\lambda x^2}}{2x} \) so that \( \phi > \rho / \kappa \). The last inequality implies that the ray \( x = \phi \pi \) is steeper than that for \( \dot{\pi} = 0 \). Thus, starting with any initial value of \( \pi \) the solution converges over time along the locus \( x = \phi \pi \) to the origin \( (\pi(t), x(t)) \to (0,0) \). These dynamics are illustrated in Figure 4.

**Just out of the Trap (Phase II).** Consider next the problem for \( t \geq T \) incorporating the ZLB constraint for any arbitrary starting point \( (\pi(T), x(T)) \). The problem is stationary since \( r(t) = \bar{r} > 0 \) for \( t \geq T \).

If the initial state lies on the locus \( x = \phi \pi \), then the solution coincides with the one above. This is essentially also the case when the initial state satisfies \( x < \phi \pi \), since one can engineer an upward jump in \( x \) to reach the locus \( x = \phi \pi \). After this jump, one proceeds with the

---

11One can pick any absolutely continuous path for \( x(t) \) and solve for the required nominal interest rate as a residual: \( i(t) = \sigma \dot{x}(t) + \pi(t) + r(t) \). Discontinuous paths for \( x(t) \) can be approximated arbitrarily well by continuous ones. Intuitively, it is as if discontinuous paths for \( \{x(t)\} \) are possible, since upward or downward jumps in \( x(t) \) can be engineered by setting the interest rate to \( \infty \) or \( -\infty \) for an infinitesimal moment in time. Formally, the supremum for the problem that ignores the ZLB constraint, but carries both \( \pi(t) \) and \( x(t) \) as states, is independent of the current value of \( x(t) \). Since the current value of \( x(t) \) does not meaningfully constrain the planning problem, it can be ignored as a state variable.
solution that ignores the ZLB constraint. In contrast, the optimum features an initial state that satisfies $x > \phi \pi$. Intuitively, the optimum attempts to reach the red line as quickly as possible, by setting the nominal interest rate to zero until $x = \phi \pi$.

These dynamics are illustrated in Figure 4 using the phase diagram implied by the system (1a)–(1b) with $i(t) = 0$. The steady state with $\dot{x} = \dot{\pi} = 0$ involves deflation and a negative output gap: $\pi = -\bar{r} < 0$ and $x = -\frac{\rho}{\kappa} \bar{r} < 0$. As a result, for inflation rates near zero the output gap falls over time. As before, the red line denotes the locus $x = \phi \pi$, for the solution to the problem ignoring the ZLB constraint. For two initial values satisfying $x > \phi \pi$, the figure shows the trajectories in green implied by the system (1a)–(1b) with $i(t) = 0$. Along these paths $x(t)$ and $\pi(t)$ fall over time, eventually reaching the locus $x = \phi \pi$. After this point, the state follows the solution ignoring the ZLB constraint, staying on the $x = \phi \pi$ line and converges towards the origin.

### During the Liquidity Trap (Phase I)

During the liquidity trap $t \leq T$ the ZLB constraint binds and $i(t) = 0$. The dynamics are illustrated in Figure 6 using the phase diagram implied by equations (1a) and (1b) setting $i(t) = 0$. For reference, the red line denoting the optimum ignoring the ZLB constraint is also show.

Unlike the previous case, the steady state $\dot{x} = \dot{\pi} = 0$ for this system now has positive inflation and a positive output gap: $\pi = -\bar{r} > 0$ and $x = -\frac{\rho}{\kappa} \bar{r} > 0$. In contrast to the previous phase diagram, also featuring $i(t) = 0$, for inflation rates near zero the output gap rises over time. Two trajectories are shown in green. In one case, inflation is initially negative; in the other it is positive. Both cases have an the output gap initially negative and turning positive some time before $t = T$. Both trajectories start below the red line and end
Figure 5: The solution for $t > T$ with the ZLB constraint.

up above it at $t = T$.

Figure 7 puts the three phases together to display two possible optimal paths for all $t \geq 0$. The two trajectories illustrated in the figure are quite representative and illustrate the possibilities described in Proposition 4.

As these figures suggest one can prove that the nominal interest rate should be kept at zero past $T$. The following proposition follows from Proposition 3 and elements of the dynamics captured by the phase diagrams.

**Proposition 5.** Consider the liquidity trap scenario with $r(t) = r < 0$ for $t < T$ and $r(t) = \bar{r} > 0$ for $t \geq T$. Suppose the path $\{\pi^*(t), x^*(t), i^*(t)\}$ is optimal. Then there exists a $\hat{T} > T$ such that

$$i(t) = 0 \quad \forall t \in [0, \hat{T}].$$

There are two ways of summarizing the optimal plan. In the first, the central bank commits to a zero nominal interest rate during the liquidity trap, for $t \in [0, T]$. It also makes a commitment to an inflation rate and output gap target $(\pi^*(T), x^*(T))$ after the trap. However, note that here

$$x^*(T) > \phi \pi^*(T)$$

so that the promised boom in output is higher than that implied by the inflation promise. Commitment to a target at time $T$ is needed not just in terms of inflation, but also in terms of the output gap.

Another way of characterizing policy is as follows. The central bank commits to setting a zero interest rate at zero for longer than the liquidity trap, so that $i(t) = 0$ for $t \in [0, \hat{T}]$ with
Figure 6: The solution for $t \leq T$ and $r(t) = -\bar{r} < 0$ with the ZLB constraint binding.

Figure 7: Two possible paths of the solution for $t \geq 0$. 
Figure 8: Commitment to inflation but not an additional boom in output.

\[ T > \hat{T}. \] It also commits to implementing an inflation rate \( \pi(\hat{T}) \) upon exit of the ZLB, at time \( \hat{T} \). In this case, no further commitment regarding \( x(\hat{T}) \) is required, since \( x(\hat{T}) = \phi \pi(\hat{T}) \) is ex-post optimal given the promised \( \pi(\hat{T}) \). Note that the level of inflation promised in this case may be positive or negative, depending on the sign of \( 1 - \kappa \sigma \lambda \). A commitment to positive inflation once interest rates become positive is not necessarily a feature of all optimum.

5 Zero Interest Rates: Inflation or Boom?

It is widely believed that the main purpose of monetary easing in a liquidity trap is to promote inflation. The model confirms that an optimum features positive inflation at some point and that it commits to prolonging zero interest rates. Are the two connected?

I will argue that they are not. The optimum also calls for an output boom after the trap. I will argue that keeping interest rates at zero has everything to do with stimulating this boom and little, or nothing, to do with generating inflation. Three different special cases of the model will help cement this conclusion.

Fully Rigid Prices. When prices are fully rigid, so that \( \kappa = 0 \), inflation is just not in the cards, so only the output boom motive can be present. Yet I have shown in Section 4.2 that prolonging zero interest rates is still optimal in this case. Thus, promoting inflation is not necessary for a commitment to prolonging zero interest rates. The next example argues that it is also not sufficient.
Commitment to Inflation Promises Only. Consider a central bank in a liquidity trap scenario. Optimal policy can be summarized by a commitment to keeping the interest rate at zero up to time \( T \) together with a commitment to inflation and output upon exit, \((\pi(T), x(T))\). One can then imagine the central bank at \( t = T \) re-optimizing and committing over the continuation plan \( t \geq T \), subject to fulfilling its prior commitments to inflation and output \( \pi(T) \) and \( x(T) \).

Now consider stripping the central bank of its commitment to output \( x(T) \). Instead, suppose that at \( t = 0 \) it can announce a commitment to keeping interest rates at zero up to time \( T \) and also some promised exit inflation rate \( \pi(T) \). It can make no binding promises regarding output \( x(T) \). As before, at \( t = T \) the central bank is allowed to optimize and commit over the continuation \( t \geq T \), but it must honor its prior commitment, in this case inflation \( \pi(T) \) only.

Then, for any \( \pi(T) \), as long \( I(r(T), \pi(T)) > 0 \) (which is guaranteed, for example, if \( \kappa \sigma \lambda = 1 \)) the optimum will feature \( i(t) = I(r(t), \pi(t)) > 0 \) for all \( t \geq T \). Index the resulting equilibrium by the choice of exit inflation \( \pi(T) \) and note that setting \( \pi(T) = 0 \) leads to an outcome that is identical to that of the discretionary equilibrium. Things are only made worse by committing to negative exit inflation, \( \pi(T) < 0 \). Thus, some positive exit inflation, \( \pi(T) > 0 \), is desirable and strictly improves on the discretionary outcome, although it falls short of the full optimum.

This shows that a commitment to inflate does not lead to a commitment to prolong zero interest rates. Promising inflation is not sufficient for prolonged zero interest rates. Indeed, interest rates may be above or below the natural rate after \( t = T \), depending on the sign of \( \kappa \sigma \lambda - 1 \). As pointed out in Section 3.3, once again a commitment to looser future monetary policy does not necessarily translate into lower long term interest rates or, equivalently, lower yields at \( t = 0 \) on long term bonds.

Of course, the discussion here is about equilibrium interest rates. One interpretation is that the equilibrium with \( \pi(T) > 0 \) is implemented by some form of loose monetary policy, even if it does not necessarily lead to lower interest rates in equilibrium. For example, one popular interpretation is that after time \( T \) policy is conducted according to a Taylor rule with a time varying intercept: \( i(t) = \bar{i}(t) + \psi \pi(t) \) and \( \psi > 1 \). Then \( \bar{i}(t) = -\psi \pi^*(t) \) which is indeed negative at \( t = T \) and rises over time, converging to zero. In this sense, the central bank is committing to loose monetary policy. However, even in this case, it does not involve

\[ \text{Commitment to Inflation Promises Only. Consider a central bank in a liquidity trap scenario. Optimal policy can be summarized by a commitment to keeping the interest rate at zero up to time } T \text{ together with a commitment to inflation and output upon exit, } (\pi(T), x(T)). \text{ One can then imagine the central bank at } t = T \text{ re-optimizing and committing over the continuation plan } t \geq T, \text{ subject to fulfilling its prior commitments to inflation and output } \pi(T) \text{ and } x(T). \]

\[ \text{Now consider stripping the central bank of its commitment to output } x(T). \text{ Instead, suppose that at } t = 0 \text{ it can announce a commitment to keeping interest rates at zero up to time } T \text{ and also some promised exit inflation rate } \pi(T). \text{ It can make no binding promises regarding output } x(T). \text{ As before, at } t = T \text{ the central bank is allowed to optimize and commit over the continuation } t \geq T, \text{ but it must honor its prior commitment, in this case inflation } \pi(T) \text{ only.} \]

\[ \text{Then, for any } \pi(T), \text{ as long } I(r(T), \pi(T)) > 0 \text{ (which is guaranteed, for example, if } \kappa \sigma \lambda = 1) \text{ the optimum will feature } i(t) = I(r(t), \pi(t)) > 0 \text{ for all } t \geq T. \text{ Index the resulting equilibrium by the choice of exit inflation } \pi(T) \text{ and note that setting } \pi(T) = 0 \text{ leads to an outcome that is identical to that of the discretionary equilibrium. Things are only made worse by committing to negative exit inflation, } \pi(T) < 0. \text{ Thus, some positive exit inflation, } \pi(T) > 0, \text{ is desirable and strictly improves on the discretionary outcome, although it falls short of the full optimum.} \]

\[ \text{This shows that a commitment to inflate does not lead to a commitment to prolong zero interest rates. Promising inflation is not sufficient for prolonged zero interest rates. Indeed, interest rates may be above or below the natural rate after } t = T, \text{ depending on the sign of } \kappa \sigma \lambda - 1. \text{ As pointed out in Section 3.3, once again a commitment to looser future monetary policy does not necessarily translate into lower long term interest rates or, equivalently, lower yields at } t = 0 \text{ on long term bonds.} \]

\[ \text{Of course, the discussion here is about equilibrium interest rates. One interpretation is that the equilibrium with } \pi(T) > 0 \text{ is implemented by some form of loose monetary policy, even if it does not necessarily lead to lower interest rates in equilibrium. For example, one popular interpretation is that after time } T \text{ policy is conducted according to a Taylor rule with a time varying intercept: } i(t) = \bar{i}(t) + \psi \pi(t) \text{ and } \psi > 1. \text{ Then } \bar{i}(t) = -\psi \pi^*(t) \text{ which is indeed negative at } t = T \text{ and rises over time, converging to zero. In this sense, the central bank is committing to loose monetary policy. However, even in this case, it does not involve} \]

\[ \text{It is true, however, that the commitment to zero interest rates up to time } T \text{ is binding. To see this suppose the central bank could only commit to exit inflation, but not to the interest rate path before } T. \text{ As before, suppose at } T \text{ it can bind itself to an optimal continuation plan given } \pi(T). \text{ The resulting equilibrium, for a given promise } \pi(T) > 0, \text{ has } \lim_{t \to T} x(t) = 0 \text{ and then jumps discontinuously to } x(T) = \phi \pi(T). \text{ Intuitively, } x(t) > 0 \text{ is not possible without commitment. The interest rate path is } i(t) = 0 \text{ before } T \text{ but includes an "instant" with an "infinite" interest rate at, or immediately before, } t = T \text{ that allows } x(t) \text{ to jump upward. Although peculiar, realistic values of } \phi \text{ are small, so this difference in the equilibrium is minor.} \]

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Figure 9: The optimum with the added constraint that $\pi(t) \leq 0$ for all $t \geq 0$.

a commitment to keeping interest rates at zero.

**An Outside Constraint to Avoid Inflation.** A third useful exercise is to consider imposing an arbitrary restriction to avoid positive inflation: $\pi(t) \leq 0$ for all $t \geq 0$. This restriction cannot be motivated within the model laid out here. The costs from inflation are already included in the loss function. However, one may still want to account for political or economic constraints outside the model that make an increase in inflation more costly. I consider the extreme case where inflation is simply ruled out.

The optimum in this restricted case is illustrated in Figure 9. The solution cannot be on arcs that go through the quadrants with $\pi \geq 0$. At the same time, to minimize losses, it is best to stay close to the origin $(0,0)$. Thus, the optimal path goes along the arc that goes through the origin, the same arc as the no-commitment solution shown in Figure 1. However, instead of reaching the origin at $t = T$ it now goes through the origin earlier and reaches a strictly positive output level at $t = T$. To minimize the quadratic objective it is best for output to take on both signs: the boom in output at later dates helps mitigate the recession early on. Positive inflation is avoided here by promising to approach, in the long run, the origin from the bottom-left quadrant, with deflation and negative output.

To sum up, if inflation is to be avoided because of some outside imposition, then the optimum still calls for a commitment to prolonged zero interest rates.
6 Government Spending: Opportunistic and Stimulus

I now introduce government spending as an additional instrument. I first consider the full optimum, with commitment, over both fiscal and monetary policy.

As in Woodford (2011), the basic New Keynesian model is augmented with public goods provided by the government that enter the representative agent’s utility function \( U(c, g, n) \) and are produced by combining varieties in the exact same way as final consumption goods. As before, we focus on the linearized equilibrium conditions and a quadratic approximation to welfare. The planning problem becomes

\[
\min_{c, \pi, i, g} \frac{1}{2} \int_{0}^{\infty} e^{-\rho t} \left( (c(t) + (1 - \Gamma)g(t))^2 + \lambda \pi(t)^2 + \eta g(t)^2 \right) dt
\]

subject to

\[
\dot{c}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t)) \\
\dot{\pi}(t) = \rho \pi(t) - \kappa (c(t) + (1 - \Gamma)g(t)) \\
i(t) \geq 0
\]

\( x(0), \pi(0) \) free.

Here the constants satisfy \( \Gamma \in (0, 1) \) and \( \eta > (1 - \Gamma)\Gamma > 0 \); the variable \( c(t) = (C(t) - C^*(t))/C^*(t) \approx \log(C(t)) - \log(C^*(t)) \) represents the private consumption gap, while \( g(t) = (G(t) - G^*(t))/C^*(t) \) represents the government consumption gap, normalized by private consumption.

Note that this problem coincides with the previous one if one imposes \( g(t) = 0 \) for all \( t \geq 0 \). Government spending appears in the objective function here for two reasons: public goods are valued in the utility function and, through the resource constraint, they also affect the required amount of labor, for any given private consumption \( c \). Spending does not affect the consumer’s Euler equation, but does affect the Phillips curve. Intuitively, both private and public spending increase the wage, which creates inflationary pressure.

The coefficient \( \Gamma \in (0, 1) \) represents the first best, or flexible-price equilibrium, government spending multiplier, i.e. for each unit increase in spending, output increases by \( \Gamma \) units, consumption is reduced by \( 1 - \Gamma \) units. The loss function captures this, because given spending \( g \), the ideal consumption level is \( c = -(1 - \Gamma)g \). The Phillips curve shows that \( c = -(1 - \Gamma)g \) also corresponds to a situation with zero inflation, replicating the flexible-price equilibrium.
6.1 A Non-Optimal Policy of Filling in the Gap

The potential usefulness of the additional spending instrument \( g \) can be easily seen noting that spending can zero out the first two quadratic terms in the loss function, ensuring \( c(t) + (1 - \Gamma)g(t) = \pi(t) = 0 \) for all \( t \geq 0 \). This requires a particular path for spending satisfying

\[
\dot{g}(t) = \sigma^{-1} \frac{1}{1 - \Gamma} \left( r(t) - i(t) \right).
\]

For simplicity, suppose we set \( i(t) = 0 \) for \( t < T \) and \( i(t) = r(t) \) for \( t \geq T \). Then spending is declining for \( t < T \) and given by

\[
g(t) = \frac{\sigma^{-1}}{1 - \Gamma} \int_0^t r(s) ds + g(0).
\]

After this, spending is flat \( g(t) = g(T) \) for \( t \geq T \). Since this policy ensures that the first two terms in the objective are zero, to minimize the quadratic loss from spending, the optimal initial value \( g(0) \) is set to ensure that \( g(t) \) takes on both signs: \( g(0) \) is positive and \( g(T) \) is negative. As a result, the same is true for consumption \( c(t) = -(1 - \Gamma)g(t) \).

Although this plan is not optimal, it is suggestive that optimal spending may take on both positive and negative values during a liquidity trap. We prove this result in the next subsection.

6.2 The Optimal Pattern for Spending

It will be useful to transform the planning problem by a change variables. In fact, I will use two transformations. Each has its own advantages. For the first transformation, define the output gap \( x(t) \equiv c(t) + (1 - \Gamma)g(t) \). The planning problem becomes

\[
\min_{x, \pi, i, g} \frac{1}{2} \int_0^\infty e^{-\rho t} \left( x(t)^2 + \lambda \pi(t)^2 + \eta g(t)^2 \right) dt
\]

subject to

\[
\dot{x}(t) = (1 - \Gamma)\dot{g}(t) + \sigma^{-1}(i(t) - r(t) - \pi(t))
\]

\[
\dot{\pi}(t) = \rho \pi(t) - \kappa x(t)
\]

\[
i(t) \geq 0
\]

\[
x(0), \pi(0) \text{ free.}
\]
This is an optimal control problem with $i$ and $\dot{g}$ as controls and $x$, $\pi$ and $g$ as states. According to the objective, the ideal level of government spending, given current state variables $x(t)$ and $\pi(t)$ is always zero. However, spending also appears in the constraints, so it may help relax them. In particular, spending enters the constraint associated with the consumer’s Euler equation. Indeed, the change in spending, $\dot{g}$, plays a role that is analogous to the nominal interest rate, but unlike the interest rate, the change in spending is not restricted by a zero lower bound.

Since government spending relaxes the Euler equation, it should be zero whenever the zero-bound constraint is not binding, which is the case when the zero lower bound is not binding. Conversely, if the zero-bound constraint binds and $i(t) = 0$ then government spending is generally non-zero.

**Proposition 6.** The conclusions in Proposition 3, regarding the nominal interest rate $i(t)$, extend to the model with government spending. In addition, whenever the zero lower bound is not binding government spending is zero: $g(t) = 0$. Suppose the zero lower bound binds over the interval $(t_0, t_1)$ and is slack in a neighborhood to the right of $t_1$, then $g(t) < 0$ in a neighborhood to the left of $t_1$. Similarly, if $t_0 > 0$ and the zero lower bound is slack to the left of $t_0$ then $g(t) > 0$ in a neighborhood to the right of $t_0$. Government spending is always initially nonnegative $g(0) \geq 0$ and strictly so if $x(0) < 0$.

In a liquidity trap scenario, we have $t_0 = 0$ and by implication government spending should jump above its natural level. The proposition then suggests a typical pattern, confirmed by a number simulations, where spending is initially positive, then declines and becomes negative and finally returns to zero. In this sense, optimal government spending is front loaded.

It may seem surprising that spending takes on both positive and negative values. The intuition is as follows. Initially, higher spending helps compensate for the negative consumption gap at the start of a liquidity trap. However, recall that optimal monetary policy eventually engineers a consumption boom. If government spending leans against the wind, we should expect lower spending. The next subsection refines this intuition by decomposing spending into an opportunistic and a stimulus component.

Figure 10 provides a numerical example, following the same parametrization used for the example in Section 4, with the additional parameters $\Gamma = 0.5$ and $\eta = .5$. The figure shows both consumption and output. As we see from the figure consumption is not as affected as output is in this case.
6.3 Opportunistic vs. Stimulus Spending

Even a shortsighted government that ignores dynamic general equilibrium effects on the private sector finds reasons to increase government spending during a slump. When the economy is depressed, the wage, or shadow wage, is lowered. This provides a cheap opportunity for government consumption.

To capture this idea, I define an opportunistic component of spending, the level that is optimal from a simple, static, cost-benefit calculation. To see what this entails, take the utility function $U(C, G, N)$ and take some level of private consumption $C$ as given. Then opportunistic spending is defined as the solution to max$_G U(C, G, N)$ subject to the resource constraint relating $C$, $G$ and $N$. For example, if the resource constraint were simply $N = C + G$, then one maximizes $U(C, G, C + G)$ taking private spending $C$ as given. The first order condition $U_G = -U_N$ can be thought of as a Samuelson condition $U_G/U_C = -U_N/U_C$, equating the marginal utility of public goods relative to consumption goods to the marginal cost, equal here to the real wage (Samuelson, 1954). Private spending will generally affect the optimal level of public goods, by affecting the marginal benefit or marginal cost. With separable utility, opportunistic spending is decreasing in private spending, because a fall in private consumption, holding the consumption of public goods constant, lowers the marginal disutility of labor.

I need to make this concept operational using the quadratic approximation to the welfare function and for variables expressed in terms of gaps. They translate to minimizing the loss function

$$g^*(c) \equiv \arg \max_g \left\{ (c + (1 - \Gamma)g)^2 + \eta g^2 \right\}.$$
Define stimulus spending as the difference between actual and opportunistic spending,
\[
\hat{g}(t) \equiv g(t) - g^*(c(t)).
\]

Note that
\[
g^*(c) = -\frac{1 - \Gamma}{\eta} \psi c,
\]
\[
c + (1 - \Gamma)g^*(c) = \psi c,
\]
with the constant \(\psi \equiv \eta / (\eta + (1 - \Gamma)^2) \in (0, 1)\). Thus, opportunistic spending leans against the wind, \(\psi < 1\), but does not close the gap, \(\psi > 0\).

Using these transformations, I rewrite the planning problem as
\[
\min_{\hat{x}, \pi, i, \hat{g}} \frac{1}{2} \int_0^\infty e^{-\rho t} \left(c(t)^2 + \hat{\lambda} \pi(t)^2 + \hat{\eta} \hat{g}(t)^2\right) dt
\]
such that
\[
\dot{c}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t))
\]
\[
\dot{\pi}(t) = \rho \pi(t) - \kappa (\psi c(t) + (1 - \Gamma)\hat{g}(t))
\]
\[
i(t) \geq 0,
\]
\[
c(0), \pi(0),
\]
where \(\hat{\lambda} = \lambda / \psi\) and \(\hat{\eta} = \eta / \psi^2\). According to the loss function, the ideal level of stimulus spending is zero. However, stimulus may help relax the Phillips curve constraint.

This problem is almost identical to the problem without spending. The only new optimality condition is
\[
\hat{g}(t) = \frac{\kappa (1 - \Gamma)}{\hat{\eta} \mu \pi(t)}.
\]

This leads to the following result.

**Proposition 7.** Stimulus spending is always initially zero: \(\hat{g}(0) = 0\). (a) If \(\kappa = 0\) or \(\kappa \sigma \lambda = 1\) then stimulus spending is zero, \(\hat{g}(t) = 0\) for all \(t \geq 0\); (b) if \(\kappa \sigma \lambda > 1\) then stimulus spending turns positive initially; (c) if \(0 < \kappa \sigma \lambda < 1\) then stimulus spending turns negative initially.

The proposition provides three instances where we should expect stimulus spending to be zero. First, unlike total spending, stimulus spending is always initially zero, so that total spending is entirely opportunistic.\(^\text{13}\) This result is especially relevant in a liquidity

\(^\text{13}\)Another implication of equation (4) is that stimulus spending, unlike total spending, may be nonzero even when the zero lower bound constraint is not currently binding and will never bind in the future. This occurs
trap scenario, since then \( t = 0 \) coincides with the start of the liquidity trap and it implies that the entire initial increase in spending can be attributed to the opportunistic motive. Second, when prices are completely rigid, so that \( \kappa = 0 \), spending cannot affect inflation so that stimulus spending is zero and spending is always opportunistic. This highlights the somewhat indirect role that stimulus spending plays during a liquidity trap in this model as a promoter of inflation.\(^{14}\). Third, even for \( \kappa > 0 \) the costate for the Phillips curve, unlike the costate for the Euler equation, can take on both signs or remain at zero, depending on parameters. The third result exploits this fact to provide another benchmark, with \( \kappa \sigma \lambda = 1 \), where stimulus spending is always zero.

Whenever stimulus spending is zero it is as if spending were chosen according to a purely static, cost-benefit calculation that ignores any dynamic general equilibrium feedback effects on private spending. By implication, government spending can be chosen by a naive agency, lacking commitment, that performs such a static cost-benefit calculation. The fact that, in these special cases, government spending can be chosen without commitment contrasts sharply with the importance of commitment to monetary policy.

Figure 11 displays the optimal paths for total, opportunistic and stimulus spending for our numerical example (with the same parameters as those behind Figure 10). Spending starts at 2% of output above its efficient level. It then falls at a steady state reaching almost 2% below its efficient level of output. In this example, spending is virtually all opportunistic. Stimulus spending is virtually zero.

Away from this benchmark, numerical simulations show that stimulus starts at zero, it has a sinusoidal shape, switching signs once. When \( \kappa \sigma \lambda > 1 \) it first becomes positive, then turns negative, eventually converging to zero from below; when \( \kappa \sigma \lambda < 1 \) the reverse pattern obtains: first negative, then turns positive, eventually converging to zero from above. In most cases, stimulus spending is a small component of total spending.

The results highlight that positive stimulus spending is just not a robust feature of the optimum for this model. Opportunistic spending does affect private consumption, by affecting the path for inflation. In particular, by leaning against the wind, it promotes price stability, mitigates both deflations and inflations. However, the effects are incidental, in that they would be obtained by a policy maker choosing spending that ignores these effects.

Whenever inflation is nonzero. Indeed, since total spending must be zero, stimulus spending must be canceling out opportunistic spending. This makes sense. If we have promised positive inflation, for example, then we require a positive gap. Opportunistic spending would call for lower spending, but doing so would frustrate stimulating the promised inflation.

\(^{14}\)Although the model has Keynesian features and conclusions in the sense of price stickiness, private spending is forward looking and satisfies Ricardian equivalence, so it is not directly affected by government spending’s effect on current income as in standard ISLM discussions.
7 Spending with Discretionary Monetary Policy

I now relax the assumption of full commitment and consider the mixed case where monetary policy is assumed to be discretionary, as in Section 3, while fiscal policy can commit to a path for government spending, at least over some intermediate horizon.

One motivation for these assumptions is the fact that government spending decisions feature both legislative and technical lags in implementation over the medium run. Spending may be planned and legislated over a horizon of at least a year or two. Reversing these decisions may be politically difficult or simply impractical from a technical point of view, e.g. if infrastructure construction is under way. In sharp contrast, monetary policy is determined almost instantaneously, at frequent meetings, and the nominal interest rate can easily react to the current state of the economy. This may tend to make monetary policy discretionary over the short and medium run—the relevant time frame for countercyclical policies, which is the present focus.

Of course, this comparison may be inverted when one considers the long run determinants of inflation and spending. At least for advanced economies, it may be more accurate to assume monetary policy can commit to avoid inflationary bias pressures. On the other hand, fiscal policy may not overcome time inconsistency or other political economy problems that lead to excess government spending and debt. The medium-run lags in fiscal policy are not helpful in resolving these biases in the level of spending. Likewise, the monetary authorities commitment to avoid inflationary pressures may be irrelevant (or even unhelpful) in solving the extraordinary kind of commitments needed during a liquidity trap. Thus, there is no contradiction in assuming that the relative commitment powers are different in the long run, than over the medium term required to respond to a liquidity trap.

Figure 11: Total spending (blue), opportunistic spending (green) and stimulus spending (red) for a numerical example.
7.1 Commitment to spending during the liquidity trap

Formally, the situation amounts to a dynamic game, where the fiscal authority acts as a Stackelberg leader.

I will consider a liquidity trap scenario, where \( r(t) < 0 \) for \( t < T \) and \( r(t) \geq 0 \) for \( t \geq T \). The central bank sets nominal interest rates with full discretion. In contrast, government spending can be credibly announced, at least until time \( T \). I first assume that spending after \( T \) is chosen with discretion. This implies that \( g(t) = 0 \) for \( t \geq T \). I then turn to the case where commitment is possible for the entire path of government spending.

Under these assumptions, at time \( t = T \) the first best is attainable. I suppose then that the monetary and fiscal authorities jointly reoptimize and implement the first best: \( c(t) = \pi(t) = g(t) = 0 \) for all \( t \geq T \). Next, I characterize the equilibrium for \( t < T \).

For \( t < T \), the central bank only takes into account the short run effects of its policy choices. At any point in time, it takes as given the behavior of future output and inflation. In continuous time, this implies that it only considers current output and inflation. However, the latter is essentially out if its control, since inflation is proportional to an integral of future output gaps. This makes the central bank’s behavior easy to characterize.

At any point in time \( t \) its choice for the nominal interest rate \( i(t) \) depends entirely on the current output gap \( x(t) = c(t) + (1 - \Gamma)g(t) \). Since there is no upper bound on the nominal interest rate, \( x(t) > 0 \) is not possible: if \( x(t) > 0 \) occurred then the central bank would increase \( i(t) \) without bound. In the limit, this would bring \( x(t) \) immediately down to zero. If \( x(t) < 0 \) the unique optimum is to set \( i(t) = 0 \). When \( x(t) = 0 \) the central bank is content to keep \( x(t) = 0 \) by setting \( i(t) \) so that \( \dot{x}(t) = 0 \), unless this requires \( i(t) < 0 \) in which case it sets \( i(t) = 0 \).

This motivates the following problem

\[
\int_0^T e^{-\rho t} \left( (c(t) + (1 - \Gamma)g(t))^2 + \lambda \pi(t)^2 + \eta g(t)^2 \right) dt
\]

\[
\dot{c}(t) = \sigma^{-1}(i(t) - \pi(t) - r(t))
\]

\[
\dot{\pi}(t) = \rho \pi(t) - \kappa (c(t) + (1 - \Gamma)g(t))
\]

\[
c(t) + (1 - \Gamma)g(t) \leq 0
\]

\[
c(T) = \pi(T) = 0
\]

\[
i(t) (c(t) + (1 - \Gamma)g(t)) = 0
\]

\[
c(0), \pi(0) \text{ free}
\]
This is the same problem as before, with the new constraints \( x(t) \leq 0 \) and \( c(T) = \pi(T) = 0 \) and \( i(t)x(t) = 0 \). The latter constraint implies that \( i(t) = 0 \) whenever \( x(t) < 0 \), since this is the best response from a monetary policy. As it turns out, this latter constraint is not binding anyway, because there is no conflict of interest when \( x(t) < 0 \), since this is the best response from a monetary policy. Thus, my approach is to drop the constraint that \( i(t)x(t) = 0 \) and show that the solution to the relaxed problem satisfies it. In this formulation, \( c \) and \( \pi \) are state variables, while \( i \) and \( g \) are controls. Due to the presence of the inequality \( c(t) + (1 - \Gamma)g(t) \leq 0 \), one must invoke a generalized Maximum Principle for optimal control problems with mixed (state and control) constraints. Working with the implied necessary conditions yields the following sharp characterization.

**Proposition 8.** Consider a liquidity trap scenario and assume that fiscal policy can commit to a path for spending \( g(t) \) time \( t \in [0, T] \) but monetary policy is discretionary. Then the equilibrium outcome satisfies the following properties: (a) consumption is negative and increasing; (b) total spending is positive; (c) opportunistic spending is positive and decreasing; (d) there exists a time \( t^* \in (0, T) \) such that \( x(t) < 0 \) and \( \pi(t) < 0 \) for \( t < t^* \) and \( x(t) = \pi(t) = 0 \) for \( t \geq t^* \); (e) stimulus spending is initially zero, \( \dot{g}(0) = 0 \), but strictly increasing until \( t = t^* \); (f) for \( t \geq t^* \) total spending \( g(t) \) is positive and decreasing, reaching zero at \( t = T \).

Since the output gap cannot be positive, the same is true for inflation, \( \pi(t) \leq 0 \) for all \( t \). This then implies that consumption is negative and increasing, until it hits zero exactly at \( t = T \). As a result, opportunistic spending is positive and decreasing. As before, stimulus spending is initially zero. As long as \( x(t) < 0 \), so that the mixed constraint is slack, stimulus spending is positive and increasing over time because there is deflation and government spending can help relieve these deflationary pressures. However, due to these incentives, the output gap reaches zero at some intermediate date \( t^* \in (0, T) \) and stays at zero thereafter. During this time, total spending is constrained and given by \( g(t) = -c(t)/(1 - \Gamma) \geq 0 \). Indeed, spending fills the gap, as discussed in Section 6.1, declining towards zero at \( t = T \).

Why does the solution hit \( x(t) = 0 \) for some time? Note that with full commitment, the solution involved an output boom, with positive output gaps sometime before \( t = T \). Since this is no longer possible, the constraint that \( x(t) \leq 0 \) binds over a latter interval of time.

### 7.2 Commitment to spending after the liquidity trap

I now relax the assumption that fiscal policy cannot commit past \( T \) and spending to be chosen for the entire future \( \{g(t)\}_{t=0}^\infty \).

It is useful to split the problem into two segments, before and after \( T \). The problem before \( T \) looks just as before but dropping the constraint that \( \pi(T) = c(T) = 0 \) and including the
value function for the continuation in the objective

\[ \int_0^T e^{-\rho t} ((c(t) + (1 - \Gamma)g(t))^2 + \lambda \pi(t)^2 + \eta g(t)^2) dt + e^{-rT} L_T(c(T), \pi(T)) \]

where \( L_T \) is the value function for the continuation problem after \( T \). It is defined as the solution to

\[ L_T(c_T, \pi_T) = \int_0^\infty e^{-\rho s} ((c(T + s) + (1 - \Gamma)g(T + s))^2 + \lambda \pi(T + s)^2 + \eta g(T + s)^2) dt \]

subject to all of the constraints outlined in the planning problem from the previous subsection except with initial conditions \( c(0) = c_T \) and \( \pi(0) = \pi_T \). Note that because of the restriction that \( x(T + s) \leq 0 \), this loss function is only defined for \( \pi_T \leq 0 \).

Since the first best is attainable after \( T \) when \( c(T) = \pi(T) = 0 \), the only reason for deviating from this is to improve the solution before \( T \). Deflation is not helpful, thus the solution still entails \( \pi(T) = 0 \). However, a consumption boom \( c(T) > 0 \) is very helpful because it raises consumption at all earlier dates where consumption is negative.

I now outline the solution to \( L_T(c_T, 0) \) for positive consumption \( c_T \). The basic idea is that positive consumption is consistent with a zero output gap if government spending is negative. Thus, one feasible plan is to hold spending constant at a negative level consistent with \( c_T \). A better solution, to minimize the losses coming from the \( \eta g^2 \) term, is to start with this same level of spending, but have it increase over time to eventually reach zero. During this transition, consumption rises, which requires a lower interest rate. We are constrained then by the zero lower bound. Thus, the optimum calls for \( i(T + s) = 0 \) and

\[ \dot{g}(T + s) = -(1 - \Gamma)c_T + \frac{\sigma^{-1}}{1 - \Gamma} \int_0^s r(z) dz \]

for \( t \in [T, T + \Delta] \), where \( \Delta \) is defined so that \( \dot{g}(T + \Delta) = 0 \), and \( g(t) = 0 \) for \( t > T + \Delta \). Since \( x(t) = \pi(t) = 0 \), such a plan implies a tail cost

\[ L_T(c_T, 0) \equiv \eta \int_0^\Delta e^{-\rho s} \eta \dot{g}(T + s)^2 ds. \]

This function attains it’s minimum at \( c_T = 0 \) so that \( \frac{\partial}{\partial c_T} L_T(0, 0) = 0 \).

Turning to the problem before \( T \), the new optimality condition is the transversality condition

\[ \mu_c(T) = \frac{\partial}{\partial c_T} L_T(c(T), 0). \]

Since the solution entails \( \mu_c(T) > 0 \) it follows that \( c(T) > 0 \), which implies that \( g(T) < 0 \).
Proposition 9. Consider a liquidity trap scenario and assume that fiscal policy can commit to a path for \( g(t) \) for all time \( t \geq 0 \) but monetary policy is discretionary. Then the equilibrium outcome is as in Proposition 8 except that spending is negative at the end of the trap and consumption is positive, \( g(T) < 0 \) and \( c(T) > 0 \), and both converge to zero at some finite time \( \hat{T} > T \). The nominal interest rate is zero up to \( \hat{T} \), \( i(t) = 0 \) for \( t \leq \hat{T} \).

Optimal fiscal policy entails a commitment to austerity towards the end and after the liquidity trap, \( g(t) < 0 \) in the neighborhood of \( t = T \). Spending serves the purpose of stimulating a boom in consumption, which helps mitigate the earlier consumption slump, much in the same way as optimal monetary policy would. Indeed, it is interesting to note that the optimal fiscal commitment induces the discretionary monetary authority to prolong zero interest rates. Although monetary policy cannot commit, when fiscal policy can, it creates a temporary drop in spending that lowers the interest rate that keeps the output gap at zero. Recall that the full optimum, with commitment to both monetary and fiscal policy, also entailed negative spending towards the end and after the trap (see Proposition 6). Thus, the solution without commitment to monetary policy shares the main qualitative features as with full commitment.

7.3 A Numerical Example

Figure 12 shows the equilibrium outcome for government spending as a fraction of output when monetary policy is discretionary but fiscal policy can commit. The example uses the same parameters and is thus directly comparable to the example displayed in Figure 11, which shows the full optimum, with commitment for both monetary and fiscal policy.

When fiscal policy can only commit during the trap, optimal spending is higher than in the full commitment solution. Indeed, both opportunistic and stimulus spending are higher, since the recession and deflation worsen with discretionary monetary policy. At some point (around \( t = 1.5 \)) the solution hits the constraint requiring the output gap to be nonnegative. Spending is then constrained falling towards zero at \( t = T \).

When fiscal policy can commit past the trap, optimal spending is still higher than the full commitment solution, but lower than when spending can only commit during the trap. The solution now features a large negative level of spending at \( t = T \). This serves to create a temporary boom in consumption and also reduces the quadratic cost from spending. The austerity measure is temporary: some time after \( T = 2 \) (but before \( t = 3 \)) the solution is back to \( c(t) = \pi(t) = g(t) = 0 \).

Overall, the example illustrates that optimal government spending may be significantly affected by the lack of commitment in monetary policy and the assumed horizon of commitment for fiscal policy.
Figure 12: An example with mixed commitment. Left panel: $\{g(t)\}$ commitment up to $T$. Right panel: $\{g(t)\}$ commitment for all $t \geq 0$. Total spending (blue), opportunistic spending (green) and stimulus spending (red).

References


A Proof of Proposition 1

Recall that \( \pi(t) = x(t) = 0 \) for \( t \geq T \). In integral form the equilibrium conditions for \( t < T \) are

\[
\begin{align*}
x(t) &= \int_t^T \sigma^{-1}(r(s) + \pi(s))ds, \\
\pi(t) &= \kappa \int_t^T e^{-\rho(s-t)}x(s)ds.
\end{align*}
\]

Substituting inflation \( \pi(t) \) we can write this as a single condition for the output path \( \{x(t)\} \)

\[
x(t) = \sigma^{-1} \int_t^T \left( r(s) + \kappa \int_s^T e^{-\rho(z-s)}x(z)dz \right) ds.
\]

Define the operator associated with the right hand side of this expression:

\[
T[x](t) = \sigma^{-1} \int_t^T \left( r(s) + \kappa \int_s^T e^{-\rho(z-s)}x(z)dz \right) ds = a(t) + \kappa \int_t^T m(z-t)x(z)dz
\]

with \( a(t) \equiv \sigma^{-1} \int_t^T r(z)dz \) and \( m(s) \equiv (\sigma \rho)^{-1}(1 - e^{-\rho s}) \), note that \( m \) is nonnegative, strictly increasing, with \( m(0) = 0 \) and \( \lim_{s \to \infty} m(s) = M \equiv (\sigma \rho)^{-1} > 0 \).

The operator \( T \) maps the space of continuous functions on \( (-\infty, T] \) onto itself. An equilibrium \( x^* \) is a fixed point \( T[x^*] = x^* \). Since an equilibrium represents a solution to an initial value problem for a linear ordinary differential equation, there is a unique fixed point \( x^* \). The \( T \) operator is linear and monotone (since \( m \geq 0 \)), so that if \( x^a \geq x^b \) then \( T[x^a] \geq T[x^b] \).

Fix an interval \([\hat{t}, T]\). Although \( T \) is not a contraction, starting from any continuous function \( x_0 \) that is bounded on \([\hat{t}, T]\) and defining \( x_n \equiv T^n[x_0] \) we obtain a sequence that converges uniformly on \([\hat{t}, T]\) to the unique fixed point \( x_n \to x^* \). To prove this claim, note that since \( |x_0(t)| \leq B \) and \( |r(t)| \leq R \) then

\[
|x_1(t) - x_0(t)| \leq |a(t)| + \kappa \int_t^T m(z-t) |x_0(t)| \leq \left( \frac{R}{\kappa M} + B \right) \kappa M |T - t|.
\]

In turn

\[
|x_2(t) - x_1(t)| = |T[x_1](t) - T[x_0](t)| \leq \kappa \int_t^T m(z-t) |x_1(z) - x_0(z)| dz
\]

\[
\leq \kappa M \int_t^T |x_1(z) - x_0(z)| dz \leq \left( \frac{R}{\kappa M} + B \right) (\kappa M)^2 \frac{|T - t|^2}{2}.
\]
By induction it follows that

\[ |x_n(t) - x_{n-1}(t)| \leq \left( \frac{R}{\kappa M} + B \right) (\kappa M)^n \frac{|T - t|^n}{n!}. \]

It follows that

\[ \sum_{n=1}^{\infty} |x_n(t) - x_{n-1}(t)| \leq \left( \frac{R}{\kappa M} + B \right) e^{\kappa M|T-t|}. \]

As a consequence for any \( m \geq n \)

\[ |x_n(t) - x_m(t)| \leq \sum_{j=n+1}^{m} |x_j - x_{j-1}(t)| \leq \sum_{j=n+1}^{\infty} |x_j - x_{j-1}(t)| \]

Since the right hand converges to zero as \( n \to \infty \), for any \( \varepsilon > 0 \) and \( t' < T \) there exists an \( N \) such that for all \( n, m \geq N \) we have

\[ |x_n(t) - x_m(t)| \leq \varepsilon \]

for all \( t \geq t' \). Thus, \( \{x_n\} \) is a Cauchy sequence on a complete metric space, implying that \( x_n \to x^* \) where \( x^* \) is a bounded function. Since the operator \( T \) is continuous it follows that \( x^* \) is a fixed point \( x^* = T[x^*] \).

Note that starting from the zero function \( x_0(t) = 0 \) for all \( t \) we obtain \( x_1(t) = a(t) < 0 \) for \( t < T \). Since the operator \( T \) is monotone, the sequence \( \{x_n\} \) is decreasing \( x_0 \geq x_1 \geq \cdots \geq x_n \geq \cdots \). Thus, \( x^*(t) < x_1(t) < 0 \) for all \( t < T \). This implies \( \pi^*(t) < 0 \) for \( t < T \).

Next I establish that both inflation and output \( x^*(t) \) and \( \pi^*(t) \) are monotone. Note that \( \frac{d}{dt} T[x](t) = a'(t) - \int_t^T m'(z-t)x(z)dz \). Since \( a'(t) = -\sigma^{-1}r(t) > 0 \) and \( x^*(t) < 0 \) for \( t < T \) we have that \( T[x^*](t) = x^*(t) \) is strictly increasing in \( t \) for \( t < T \) and \( \dot{x}^*(t) > 0 \). Write

\[ \pi^*(t) = \kappa \int_0^\infty e^{-\rho z} x^*(t+z)dz \]

Hence, differentiating we find \( \dot{\pi}^*(t) = \kappa \int_0^{T-t} e^{-\rho z} \dot{x}^*(t+z)dz \), which implies \( \dot{\pi}^*(t) > 0 \) for \( t < T \) given that \( \dot{x}^*(t) > 0 \) for \( t < T \).

Finally note that

\[ x^*(t;T) = \sigma^{-1} \int_t^T (r(s;T) + \pi^*(s;T))ds \leq \sigma^{-1} \int_t^T r(s;T)ds \]

and since \( \int_0^T r(s;T)ds \to -\infty \) as \( T \to \infty \) it follows that \( \int_t^T r(s;T)ds \to -\infty \), implying
$x^*(t; T) \to -\infty$ for $t < T$. Using that $x^*(t, T) \leq 0$, this implies that

$$\pi^*(t, T) = \kappa \int_t^\infty e^{-\rho(s-t)}x^*(s; T)ds \leq \kappa \int_t^{t+1} e^{-\rho(s-t)}x^*(s; T)ds \leq \kappa x^*(t + 1; T) \int_t^{t+1} e^{-\rho(s-t)}ds$$

As $T \to \infty$ we have that $x^*(t + 1; T) \to -\infty$, so it follows that $\pi^*(t; T) \to -\infty$ for any $t$.

**B Proof of Proposition 2**

Consider two values $\kappa_0 < \kappa_1$ with associated equilibria $x_0^*$ and $x_1^*$. Let $T[\cdot; \kappa]$ be the operator defined in the proof of Proposition 1 associated with $\kappa$. Define the sequence $x_n = T^n[x_0^*; \kappa_1]$. Since $x_0^*(t) < 0$ for $t < T$, it follows that $x_1(t) = T[x_0^*; \kappa_1] < T[x_0^*; \kappa_0] = x_0(t)$ for $t < T$. Since the operator $T[\cdot; \kappa_1]$ is monotone, this implies that $\{x_n\}$ is a declining sequence. Since the sequence converges to $x_1^*$. This proves that $x_0^*(t) > x_1(t) > \cdots > x_n(t) > \cdots > x_1^*(t)$ for $t < T$. This implies that $\pi_1^*(t) < \pi_0^*(t)$ for $t < T$.

To prove the limit result as $\kappa_1 \to \infty$ it suffices to show that $T[x_0^*; \kappa_1](t) \to -\infty$ for all $t < T$, since $x_1^*(t) < T[x_0^*; \kappa_1](t)$. That this is the case follows from

$$T[x_0^*; \kappa_1](t) = a(t) + \kappa_1 \int_t^T m(z-t)x_0^*(z)dz \leq \kappa_1 \int_t^T m(z-t)a(z)dz$$

where I am using that $x_0^*(t) < a(t)$ for $t < T$ as established in the proof of Proposition 1. The result follows from the fact that $\int_t^T m(z-t)a(z)dz < 0$ for all $t < T$. The same implication for $\pi^*(t; \kappa)$ then follows.

The following estimate for the loss function

$$L \geq \int_0^\infty e^{-\rho t}x(t)^2dt \geq \int_0^{T/2} e^{-\rho t}x(t; \kappa)^2dt \geq x(T/2; \kappa)^2 \int_0^{T/2} e^{-\rho t}dt$$

where I have used that $x$ is nonpositive and increasing. The result then follows since $x(T/2; \kappa) \to -\infty$. 

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C Proof of Proposition 3

Differentiating the law of motion for \( \mu_x \) we obtain a linear non homogeneous second-order differential equation

\[
\ddot{\mu}_x(t) = -\dot{x}(t) + \kappa \dot{\mu}_x(t) + \rho \ddot{\mu}_x(t) \\
= -\sigma^{-1}(i(t) - r(t) - \pi(t)) - \kappa \lambda \pi(t) + \kappa \sigma^{-1} \mu_x(t) + \rho \ddot{\mu}_x(t)
\]

Equivalently, since \( I^*(\pi, t) = r(t) + (1 - \sigma \kappa \lambda) \pi \), we can write this as

\[
\ddot{\mu}_x(t) = \sigma^{-1} (I^*(\pi(t), t) - i(t)) + \kappa \sigma^{-1} \mu_x(t) + \rho \ddot{\mu}_x(t).
\]

Now suppose at the optimum we have \( i(t) > 0 \) for \( t \in (t_1, t_1 + \epsilon) \) with \( \epsilon > 0 \). Then it must be that \( \mu_x(t_1) = \dot{\mu}_x(t_1) = 0 \), which provides a convenient initial condition for the differential equation derived above. The solution for \( \mu_x \) is then

\[
\mu_x(t) = \sigma^{-1} \int_{I_1}^t \omega(s - t) (I^*(\pi(s), t) - i(s)) \, ds
\]

where

\[
\omega(x) \equiv \frac{e^{-r_2 x}}{r_1} (1 - e^{-r_1 x})
\]

where \( r_1 > 0 \) and \( r_2 < 0 \) are the two distinct real roots to the characteristic equation \( r^2 - \kappa \sigma^{-1} r - \rho = 0 \). Note that \( \omega(x) > 0 \) for \( x > 0 \). It follows that if \( i(t) = 0 \) and \( I^*(\pi(t), t) < 0 \) for all \( t \in (t_0, t_1) \), then \( \mu_x(t) < 0 \) for all \( t \in [t_0, t_1] \). A contradiction with the optimality condition that \( \mu_x(t) \geq 0 \) for all \( t \geq 0 \). A symmetric argument holds for \( t_0 \) as long as \( t_0 > 0 \). This completes the proof.

Proof that solution to ODE takes stated form. Consider a 2nd order non-homogeneous ODE of the form

\[
\alpha_1 \mu(t) + \alpha_2 \dot{\mu}(t) + \ddot{\mu}(t) = a(t)
\]

where \( a(t) \equiv \sigma^{-1} (I^*(\pi(t), t) - i(t)) \), the coefficients are \( \alpha_1 = -\kappa \sigma^{-1} \) and \( \alpha_2 = -\rho \); the initial conditions are given by \( \mu(t_1) = \dot{\mu}(t_1) = 0 \). To solve this equation postulate that

\[
-r_1 \mu(t) + \dot{\mu}(t) = e^{r_2 t} \int_{t_1}^t e^{-r_2 s} a(s) \, ds
\]

for some constants \( r_1 \) and \( r_2 \), that will turn out to be the roots of the characteristic equation.
This guess works because it implies, by differentiating both sides,

\[-r_1\dot{\mu}(t) + \ddot{\mu}(t) = r_2e^{r_2 t} \int_{t_1}^{t} e^{-r_2 s} a(s) ds + a(t) = r_2 (-r_1 \mu(t) + \dot{\mu}(t)) + a(t)\]

\[\Rightarrow r_1 r_2 \mu(t) - (r_1 + r_2) \dot{\mu}(t) + \ddot{\mu}(t) = a(t)\]

Matching coefficients gives

\[r_1 + r_2 = -\alpha_2 \quad r_1 r_2 = \alpha_1\]

\[\Rightarrow r_1 + \frac{\alpha_1}{r_1} + \alpha_2 = 0 \Rightarrow \alpha_1 + r_1 \alpha_2 + r_1^2 = 0\]

which is just the characteristic equation. In our case

\[r = -\alpha_2 \pm \sqrt{\alpha_2^2 - 4\alpha_1} = \rho \pm \sqrt{\rho^2 + 4\kappa \sigma^{-1}}\]

so that the two roots are real, distinct and one root is positive and the other negative. We can take \( r_1 > 0 \) and \( r_2 < 0 \).

Now write equation (7) as

\[\mu(t) = e^{r_1 t} \int_{t_1}^{t} e^{-r_1 s} A(s) ds\]

where \( A(t) = \int_{t_1}^{t} e^{-r_2 (s-t)} a(s) ds \). Or simply

\[\mu(t) = \int_{t_1}^{t} \int_{t_1}^{s} e^{-r_2 (z-t) - r_1 (s-t)} a(z) dz ds\]

\[= \int_{t_1}^{t} a(z) e^{r_2 (t-z)} \left( \int_{z}^{t} e^{-r_1 (s-t)} ds \right) dz\]

\[= \int_{t_1}^{t} a(z) \Gamma(t-z) dz\]

(it is useful to think of the change in the order of integration done in the second equality graphically: we are integrating over a triangle below the 45 degree line), where we have computed

\[e^{r_2 (t-z)} \int_{z}^{t} e^{-r_1 (s-t)} ds = \frac{-e^{r_2 (t-z)}}{r_1} e^{-r_1 (s-t)} \bigg|_{z}^{t} = \frac{e^{r_2 (t-z)}}{r_1} \left( e^{r_1 (t-z)} - 1 \right) \equiv \Gamma(t-z)\].

Note that \( \Gamma(0) = 0, \Gamma(x) > 0 \) for \( x > 0 \) and \( \Gamma(x) < 0 \) for \( x < 0 \). Finally when \( t < t_1 \) it is more
convenient to define
\[ \omega(x) \equiv -\Gamma(-x) \]
and write
\[ \mu(t) = \int_{t_1}^{t} a(z) \Gamma(t - z) dz = -\int_{t}^{t_1} a(z) \Gamma(-z - t) dz = \int_{t}^{t_1} a(z) \omega(z - t) dz \]
justifying the stated solution above.

D Proof of Proposition 4

The initial conditions require \( \mu_x(0) = \mu_\pi(0) = 0 \). The non-negativity requirement for \( \mu_x \) then requires \( \dot{\mu}_x(0) = -x(0) \geq 0 \), implying \( x(0) \leq 0 \).

To establish that inflation must be positive at some point, consider the perturbation \( x(t, \varepsilon) = x(t) + \varepsilon \) and
\[ \pi(t, \varepsilon) = \kappa \int_0^\infty e^{-\rho s} x(t + s, \varepsilon) ds = \kappa \int_0^\infty e^{-\rho s} x(t + s) ds + \kappa \int_0^\infty e^{-\rho s} \varepsilon ds = \pi(t) + \frac{\kappa}{\rho} \varepsilon. \]
Note that the perturbation is feasible for all \( \varepsilon > 0 \) since higher inflation relaxes the ZLB constraint \( \sigma t_x(t) \geq -r(t) - \pi(t) \). In terms of the loss function
\[ L(\varepsilon) = \frac{1}{2} \int_0^\infty e^{-\rho t} \left( x(t, \varepsilon)^2 + \lambda \pi(t, \varepsilon)^2 \right) dt \]
we have
\[ L'(0) = \int_0^\infty e^{-\rho t} \left( x(t) + \lambda \frac{\kappa}{\rho} \pi(t) \right) dt = \frac{1}{\kappa} \pi(0) + \lambda \frac{\kappa}{\rho} \int_0^\infty e^{-\rho t} \pi(t) dt \leq 0. \]
Hence, negative inflation \( \pi(t) \leq 0 \) for all \( t \geq 0 \) with strict inequality over some range, implies \( L'(0) < 0 \), a contradiction with optimality.

To establish that \( x(t) \) must be positive for some \( t \geq 0 \), proceed by contradiction. Suppose \( x(t) \leq 0 \) for all \( t \geq 0 \). This implies that \( \pi(t) \leq 0 \) for all \( t \geq 0 \), a contradiction.

**Case with \( \kappa \sigma \lambda = 1 \).** Now suppose \( \kappa \sigma \lambda = 1 \). Let us guess and verify that \( \mu_\pi(t) = 0 \) for all \( t \geq 0 \) and \( \mu_x(t) = \sigma \lambda \pi(t) \). Note that
\[ \dot{\mu}_\pi(t) = -\lambda \pi(t) + \sigma^{-1} \mu_x(t) = 0 \]
then

This implies 

While if 

Suppose that there exists a date 

Towards a contradiction, suppose 

Either way we obtain a contradiction.

Case with \( \kappa \sigma \lambda < 1 \). I first show that \( \pi(t) \geq 0 \) for all \( t \geq 0 \). Define 


Suppose that there exists a date \( \tau \) such that \( \pi(\tau) < 0 \). Then since 

it follows that \( f(\tau) < 0 \), which implies that \( \mu_\pi(t) < 0 \) for some \( t \geq \tau \). Now define a date \( \tau' \) as follows. If \( \mu_\pi(\tau) \geq 0 \), then let \( \tau' \) be the earliest date after \( \tau \) when \( \mu_\pi \) turns strictly negative: 

If \( \mu_\pi(\tau) < 0 \) then let \( \tau' \) be the latest date before \( \tau \) such that \( \mu_\pi \) turns strictly negative: 

Note that, in both cases, 

This implies \( f(\tau') < 0 \). By construction we also have \( \mu_\pi(\tau') = 0 \) and \( \dot{\mu}_\pi(\tau') \leq 0 \).

Next I argue that \( f(\tau') < 0 \), \( \mu_\pi(\tau') = 0 \) and \( \dot{\mu}_\pi(\tau') \leq 0 \) are incompatible. If \( \pi(\tau') < 0 \) then 

While if \( \pi(\tau') \geq 0 \) then 

Either way we obtain a contradiction.

This establishes that \( \pi(t) \geq 0 \) for all \( t \geq 0 \). Next I argue that \( \pi(t) > 0 \) for all \( t \geq 0 \). Towards a contradiction, suppose \( \pi(t) = 0 \) for some \( t \geq 0 \). If \( \pi(t) = 0 \) for all \( t \), then \( x(t) = 0 \) for all \( t \geq 0 \) and this is the first best allocation, which is not possible if \( r(t) < 0 \) for some positive measure of time. Thus, consider the case where \( \pi(t) \neq 0 \) for some positive measure. Let \( \tau \) denote the earliest date where \( \pi(\tau) = 0 \) and either \( \int_{\tau}^{\tau+\epsilon} \pi(t) > 0 \) or \( \int_{\tau}^{\tau+\epsilon} \pi(t) > 0 \).

By the arguments above it follows that \( f(\tau) \leq 0 \). If this inequality is strict then we can proceed as before and obtain a contradiction. Thus, suppose \( f(\tau) = 0 \). Note also that unless
\( \mu_\pi(\tau) = 0 \) then \( f(t) < 0 \) for some \( t \) in the neighborhood of \( \tau \); this also would lead to a contradiction. Thus, assume \( \mu_\pi(\tau) = 0 \). Using that \( \int_t^\infty e^{-\rho(z-t)} \mu_\pi(z)dz = \int_t^\tau e^{-\rho(z-t)} \mu_\pi(z)dz \) since \( f(\tau) = 0 \), we have

\[
\dot{\mu}_\pi(t) = (\sigma^{-1}\kappa^{-1} - \lambda) \pi(t) - \sigma^{-1}\kappa \int_t^\tau e^{-\rho(z-t)} \mu_\pi(z)dz
\]

integrating yields

\[
\mu_\pi(\tau) - \mu_\pi(t) = (\sigma^{-1}\kappa^{-1} - \lambda) \int_t^\tau \pi(s)ds - \sigma^{-1}\kappa \int_t^\tau \int_s^\tau e^{-\rho(z-s)} \mu_\pi(z)dzds
\]

Define the Piccard operator

\[
T[\mu_\pi](t) = -(\sigma^{-1}\kappa^{-1} - \lambda) \int_t^\tau \pi(s)ds + \sigma^{-1}\kappa \int_t^\tau m(z-t) \mu_\pi(z)dz
\]

where \( m(s) = (\sigma\rho)^{-1}(1 - e^{-\rho s}) \). This is similar to the operator defined in the proof of Proposition 1. As explained in more detail there, starting from the zero function, \( \mu_\pi^0(t) = 0 \) for all \( t \geq 0 \), and using that \( \pi(t) \geq 0 \) we construct a declining sequence \( T^n[\mu_\pi^0] = \mu_\pi^n(t) \) that converges to the fixed point. Indeed, if \( \int_{\tau-\epsilon}^{\tau} \pi(s)ds > 0 \), then \( \mu_\pi^n(\tau - \epsilon) < 0 \), and the fixed point has \( \mu_\pi(t) < 0 \) for \( t \in (\tau - \epsilon, \tau) \). Likewise, if \( \int_{\tau}^{\tau+\epsilon} \pi(s)ds > 0 \), then \( \mu_\pi^n(\tau + \epsilon) > 0 \), so the fixed point has \( \mu_\pi(t) > 0 \) for \( t \in (\tau, \tau + \epsilon) \). Either way, it follows that we have a point \( \tau' \), with either \( \tau' = \tau - \epsilon \) or \( \tau' = \tau + \epsilon \), such that \( f(\tau') < 0 \). This leads to a contradiction as before. We conclude that \( \pi(t) > 0 \) for all \( t \geq 0 \).

**Case with \( \kappa\sigma\lambda > 1 \).** Assume that instead \( \pi(0) > 0 \), then

\[
\mu_x(0) = \kappa^{-1}\pi(0) - \kappa f(0) = 0
\]

implies that \( f(0) > 0 \), which implies that \( \mu_\pi(t) > 0 \) for some \( t \geq 0 \). Now define a date \( \tau' \) as the earliest date when \( \mu_\pi \) turns strictly positive: \( \tau' = \inf_t \{ t : t \geq 0 \land \mu_\pi(t) > 0 \} \). Note that we have \( f(0) = e^{-\rho \tau'} f(\tau') + \int_0^{\tau'} e^{-\rho s} \mu_\pi(s)ds > 0 \) and by definition \( \int_0^{\tau'} e^{-\rho s} \mu_\pi(s)ds \leq 0 \). This implies \( f(\tau') > 0 \). By construction we also have \( \mu_\pi(\tau') = 0 \) and \( \dot{\mu}_\pi(\tau') \geq 0 \).

Next I argue that \( f(\tau') > 0, \mu_\pi(\tau') = 0 \) and \( \dot{\mu}_\pi(\tau') \geq 0 \) are incompatible. If \( \pi(\tau') \leq 0 \) then

\[
\mu_x(\tau') = \kappa^{-1}\pi(\tau') - \kappa f(\tau') < 0
\]

which is a contradiction with optimality. If \( \pi(\tau') > 0 \) then

\[
\dot{\mu}_\pi(\tau') = (\sigma^{-1}\kappa^{-1} - \lambda) \pi(\tau') - \sigma^{-1}\kappa f(\tau')ds < 0.
\]
A contradiction since by construction $\dot{\mu}_\pi(t') \geq 0$. This establishes that $\pi(0) \leq 0$.

Next I argue that $\pi(0) < 0$ if $x(0) < 0$. If $\pi(0) = 0$ then $f(0) = 0$. This implies that $\mu_\pi(0) = \dot{\mu}_\pi(0) = 0$. However, we also have $\pi(0) < 0$ and thus

$$\dot{\mu}_\pi(0) = (\sigma^{-1} \kappa^{-1} - \lambda) \pi(0) < 0.$$ 

This implies that $\mu_\pi(t) < 0$ in a neighborhood of $t = 0$. This implies that we can find a point $t' > 0$ such that $f(t') > 0$, $\mu_\pi(t') = 0$ and $\dot{\mu}_\pi(t') \geq 0$, which once again leads to a contradiction.

## E Proof of Proposition 5

If $\kappa \sigma \lambda \geq 1$ then Proposition 4 implies that $\pi(t) \geq 0$ for all $t$. This implies that $I(r, \pi) = r + (1 - \kappa \sigma \lambda) \pi < 0$ for $t < T$ and the result follows from Proposition 3. Similarly, if $\kappa = 0$ the result is immediate since $\pi(t) = 0$ for all $t \geq 0$.

If instead $\kappa \sigma \lambda < 1$ and $\kappa \neq 0$ then we must rule out that $\pi(t) \geq -r/ (1 - \kappa \sigma \lambda) > -r > 0$ for $t \leq T$. Towards a contradiction, suppose that $\pi(t') > -\bar{r}$ for some $t' < T$. Then since from $x(0) < 0$ it follows that $\rho \pi(t') - \kappa x(t') > 0$ (see the phase diagram). Thus, we have that $\dot{\pi}(t') > 0$ and $\dot{x}(t') < 0$. This situation is absorbing: for any $t \geq t'$ with $\rho \pi(t) - \kappa x(t) > 0$ and $\pi(t) > -\bar{r}$ we have $\dot{\pi}(t) > 0$ and $\dot{x}(t) < 0$. This is the case because at any point in time either $i(t) = 0$ which implies $\dot{x}(t) = \sigma^{-1} (-r(t) - \pi(t)) < 0$ or else $i(t) = I(r(t), \pi(t))$ in which case $\dot{x}(t) = -\kappa \lambda \pi(t) < 0$. In either case, $\dot{x}(t) \leq -\kappa \lambda \pi(t) < \kappa \underline{\lambda} r < 0$. It follows that $\pi(t) \to -\infty$ and $\pi(t) \to \infty$. This violates the present value condition $\pi(t) = \kappa \int_0^\infty e^{-\rho s} x(t + s) ds$ and is therefore not feasible.

Moreover, since $x(t) < 0$ for all $t \geq t''$ for some $t'' > t'$ it follows that $\dot{\pi}(t) > \rho \pi(t) - \kappa x(t) > \rho \pi(t)$ for all $t$ above some $t''$. This implies $e^{-\rho t} \pi(t) \to \infty$. As a result $\int e^{-\rho t} \pi(t)^2 dt = \infty$. Similarly, for all $t$ large enough we have $\dot{x}(t) \leq -\kappa \lambda \pi(t) \leq -\kappa \lambda e^{\rho t} \pi(t'')$. This implies that $x(t) \leq \frac{-\kappa \lambda}{\rho} (e^{\rho t} - 1) \pi(t'')$ (recall that $x(t'') < 0$). This implies that

$$e^{-\rho t} x(t) \leq -\frac{\kappa \lambda}{\rho} \pi(t'') + e^{-\rho t} A \leq -\frac{\kappa \lambda}{\rho} \pi(t'') + \max\{A, 0\}$$

implying that $\int e^{-\rho t} x(t)^2 dt = \infty$. As a result the loss is infinite, $L = \infty$, and this plan cannot be optimal even if we were to ignore the requirement that $\pi(t) = \kappa \int_0^\infty e^{-\rho s} x(t + s) ds$.

The argument above assumed no jumps in $x$. One may allow for upward jumps in the path for $x$, but they are not optimal. To see this, suppose an upward jump in $x$ occurs at some point $t'$. Then $\lim_{t \to t'} \dot{\mu}_x(t) > \lim_{t \to t'} \dot{\mu}_x(t)$ and since $\mu_x(t) = 0$ for $t \in [t', t' + \epsilon)$ then $\lim_{t \to t'} \dot{\mu}_x(t) < 0$. Hence, $\mu_x(t) < 0$ in a neighborhood to the left of $t'$, a contradiction with
the optimality conditions.

F Proof of Proposition 6

Write the Hamiltonian as

\[ H = \frac{1}{2} \left( x^2 + \lambda \pi^2 + \eta g^2 \right) + \mu_x((1 - \Gamma)\Delta + \sigma^{-1}(i - \pi - r)) + \mu_\pi(\rho \pi - \kappa x) + \mu_g \Delta \]

where \( \dot{g}(t) = \Delta(t) \) is the law of motion for government spending and \( \Delta(t) \in \mathbb{R} \) is a new, unrestricted, control variable. The initial state \( g(0) \) is free. The optimality conditions are just as before with the addition of

\[ (1 - \Gamma)\mu_x(t) + \mu_g(t) = 0 \]
\[ \dot{\mu}_g(t) = \rho \mu_g(t) - \eta g(t) \]

Solving for spending gives

\[ g(t) = \frac{1 - \Gamma}{\eta} (-\rho \mu_x(t) + \dot{\mu}_x(t)) = \frac{1 - \Gamma}{\eta} (x(t) + \kappa \mu_\pi(t)). \]

It follows that in regions where the zero lower bound is not binding, where \( \mu_x(t) = \dot{\mu}_x(t) = 0 \), spending is zero: \( g(t) = 0 \). It also follows that \( g(0) \geq 0 \) since \( \mu_\pi(0) = 0 \) and \( x(0) \leq 0 \) by Proposition 4. Indeed, \( g(0) > 0 \) if and only if \( x(0) < 0 \).

I now argue that the conclusions of Proposition 3 extend to the case with government spending, so that \( I^*(r(t), t) > 0 \) in a neighborhood to the left of \( t_1 \). The same is true in a neighborhood to the right of \( t_0 \) as long as \( t_0 > 0 \). Differentiating the law of motion for the costate \( \mu_x \) and using the expression for \( g(t) \) from above gives

\[ \dot{\mu}_x = -\dot{x} + \kappa \dot{\mu}_\pi + \rho \dot{\mu}_x \]
\[ = -\sigma^{-1}(i - \pi - r) - (1 - \Gamma)\dot{g} - \kappa \lambda \pi + \kappa \sigma^{-1} \mu_x + \rho \dot{\mu}_x \]
\[ = -\sigma^{-1}(i - I(\pi(t), t)) + (1 - \Gamma) \frac{1 - \Gamma}{\eta} (\rho \mu_x + \dot{\mu}_x) + \kappa \sigma^{-1} \mu_x + \rho \dot{\mu}_x. \]

Rearranging we arrive at the second order ordinary differential equation

\[ \left( 1 + \frac{(1 - \Gamma)^2}{\eta} \right) \ddot{\mu}_x = \left( 1 + \frac{(1 - \Gamma)^2}{\eta} \right) \rho \mu_x - \kappa \sigma^{-1} \mu_x = -\sigma^{-1}(i - I(\pi(t), t)). \]
The associated characteristic equation has real positive and negative roots

\[ r_1 = \rho + \sqrt{\frac{\rho^2 + \frac{4\kappa \sigma^{-1}}{1 - (1 - \Gamma)^2}}{\eta}} > \rho \quad r_2 = \frac{\rho - \sqrt{\frac{\rho^2 + \frac{4\kappa \sigma^{-1}}{1 - (1 - \Gamma)^2}}{\eta}}}{2} < 0. \]

This then implies that the solution for \( \mu_x(t) \) is given by equations (5)–(6). The rest then follows as in the proof of Proposition 3.

When \( t_0 > 0 \), applying equation (7) we can write

\[ \frac{\eta}{1 - \Gamma}g(t) = -r_1 \mu_x(t) + \mu_x(t) + (r_1 - \rho) \mu_x(t) = \sigma^{-1}e^{r_2t} \int_{t_0}^{t} e^{-r_2s} (I^*(r(s), s) - i(s)) ds + (r_1 - \rho) \mu_x(t) \]

In a neighborhood to the right of \( t_0 \) we have established that \( (I^*(r(t), s) - i(t)) > 0 \) and \( \mu_x(t) \geq 0 \) implying that \( g(t) > 0 \). A symmetric argument, but interchanging the two roots \( r_1 \) and \( r_2 \) in this expression, establishes that \( g(t) < 0 \) in a neighborhood to the left of \( t_0 \).

\section*{G Proof of Proposition 7}

The results immediately using equation (4) combined with the proof behind Proposition 4 part 3, which characterizes the sign of the multiplier \( \mu_\pi(t) \).

\section*{H Proof of Proposition 8}

An extended Maximum principle is required to attack this problem (see Seierstad and Sydsæter (1987) or Vinter (2000)). One forms the Hamiltonian

\[ H \equiv \frac{1}{2} (c + (1 - \Gamma)g)^2 + \frac{1}{2} \lambda \pi^2 + \eta g^2 + \mu_c \left( \sigma^{-1}(i - \pi - r(t)) \right) + \mu_\pi \left( \rho \pi - \kappa (c + (1 - \Gamma)g) \right) \]

and associated Lagrangian

\[ L \equiv H + \phi (c + (1 - \Gamma)g). \]

The necessary conditions are then

\[ \mu_c(t) \geq 0 = 0 \text{ if } i(t) > 0 \]

\[ g(t) = \frac{(1 - \Gamma)}{(1 - \Gamma)^2 + \eta} (-c + \kappa \mu_\pi(t) - \phi(t)) \]
where $\phi(t) \geq 0$ and
\[
\phi(t) \left(c(t) + (1 - \Gamma)g(t)\right) = 0
\]
The costates evolve according to
\[
\dot{\mu}_c = -\frac{\partial L}{\partial c} = -\left(c + (1 - \Gamma)g + \rho \mu_c + \kappa \mu_\pi - \phi\right)
\]
\[
\dot{\mu}_\pi = -\frac{\partial L}{\partial \pi} = -\lambda \pi + \sigma^{-1} \mu_c
\]
and satisfy the initial conditions
\[
\mu_c(0) = \mu_\pi(0) = 0.
\]
Since $x(t) = c(t) + (1 - \Gamma)g(t) \leq 0$ it follows that $\pi(t) \leq 0$ for all $t \geq 0$. Also, $\mu_c(t) \geq 0$. Thus, $\mu_\pi(t)$ is non-negative and monotone increasing, strictly so unless $\pi(t) = x(t) = 0$ for all $t$.

Since
\[
\dot{c}(t) = \sigma^{-1}(i(t) - \pi(t) - r(t)) \geq \sigma^{-1} \rho > 0
\]
and $c(T) = 0$ it follows that $c(t)$ is strictly negative and strictly increasing over $t \in [0, T]$.

Next we argue that $\mu_c(t) > 0$ for all $t \in (0, T)$. Whenever $x(t) < 0$ we have
\[
\dot{\mu}_c(t) = -x(t) + \rho \mu_c(t) + \kappa \mu_\pi(t)
\]
and when $x(t) = 0$ we have
\[
\dot{\mu}_c(t) = \rho \mu_c(t) - \frac{\eta}{(1 - \Gamma)^2} c(t)
\]
since $c(t) < 0$ for $t < T$ it follows that $\dot{\mu}_c(t) > 0$ and $\mu_c(t)$ is strictly increasing. Given that $\mu_c(0) = 0$, it follows that $\mu_c(t) > 0$ for $t \in (0, T)$.

Since $\mu_c(t) > 0$ it follows that $i(t) = 0$ almost everywhere for $t \in [0, T]$.

Next I argue that there is a date $t^* \in (0, T)$ such that for $t < t^*$ we have $x(t) < 0$ and for $t \geq t^*$ we have $x(t) = 0$. The optimality conditions imply that if $\phi(t) = 0$ then
\[
c(t) + (1 - \Gamma)g(t) = \frac{\eta}{(1 - \Gamma)^2 + \eta} c(t) + \frac{(1 - \Gamma)^2}{(1 - \Gamma)^2 + \eta} \kappa \mu_\pi(t)
\]
the right hand side is strictly increasing, thus, there is at most one time $t^* \in [0, T]$ where it equals zero. Note that the right hand side is strictly negative at $t = 0$ since $c(0) < 0$ and $\mu_\pi(0) = 0$. This implies that $\mu_\pi(t) > 0$ for $t > 0$. The right hand side is strictly positive
at $t = T$ since $c(T) = 0$ and we have just established that $\mu_{\pi}(T) > 0$. Thus, $t^*$ exists and $t^* \in (0, T)$.

It follows that for $t \in [0, t^*]$ we have

$$g(t) = \frac{(1 - \Gamma)}{(1 - \Gamma)^2 + \eta} \left( -c + \kappa \mu_{\pi}(t) \right)$$

and hence $g(t) > 0$. Note that $g(t)$ is the sum of the opportunistic component, which is positive and decreasing, and the stimulus component, which is positive and increasing. Total spending may be non monotone.

I’ve shown that once the output gap hits zero it stays there $c(t) + (1 - \Gamma)g(t) = 0$ and since $x(t) = \pi(t) = 0$ for $t \geq t^*$ and $c(T) = 0$, this implies that total spending equals

$$g(t) = -\frac{1}{1 - \Gamma} c(t) = -\frac{1}{1 - \Gamma} \int_t^T r(s) ds$$

for $t \in [t^*, T]$ which is positive and decreasing.

I Proof of Proposition 9

As discussed in the text, the optimality conditions are the same as in the proof of Proposition 8, except for the terminal conditions. The full set of terminal condition actually are as follows:

$$\mu_c(T) = \frac{\partial}{\partial c_T} L_T(c_T, \pi_T)$$

$$\mu_\pi(T) \geq \frac{\partial}{\partial \pi_T} L_T(c_T, \pi_T)$$

where the inequality must hold with equality if $\pi_T < 0$.

The function $L_T$ is convex and attains its global minimum at $(c_T, \pi_T) = (0, 0)$. For given $c_T$ the minimum for $L_T(c_t, \cdot)$ is attained at $\pi_T = 0$. Following the arguments in the proof of Proposition 8, the costates are nonnegative, $\mu_c(t), \mu_\pi(t) \geq 0$ for all $t \geq 0$. This implies that $\pi_T = 0$ and $c_T > 0$.

Since $i(t) \geq 0, r(t) \leq 0$ and $\pi(t) \leq 0$, consumption $c(t)$ is monotone increasing. I now argue that $c(0) \geq 0$ is never optimal. This would imply that if $\phi(t) = 0$ then

$$c(0) + (1 - \Gamma)g(0) = \frac{\eta}{(1 - \Gamma)^2 + \eta} c(0) + \frac{(1 - \Gamma)^2}{(1 - \Gamma)^2 + \eta} \kappa \mu_\pi(t)$$

since $\mu_\pi(0) = 0$ and $\mu_\pi(t) \geq 0$ this rules out $c(0) > 0$, since otherwise $x(0) > 0$ which is
not feasible. Also, if \( c(0) = 0 \) then it must be that \( x(t) = 0 \) for all \( t \in [0, T] \) since otherwise \( \phi(t) = 0 \) and \( x(t) > 0 \) for some \( t \in (0, T) \). Also, given the characterization for \( t \geq T \) we have \( x(t) = \pi(t) = 0 \) and \( c(t) \geq 0 \) for all \( t \geq 0 \). This implies that \( g(t) \leq 0 \) for all \( t \geq 0 \). But then the following plan improves welfare

\[
\hat{c}(t) = c(t) - \epsilon \\
\hat{g}(t) = g(t) + (1 - \Gamma)\epsilon
\]

for small \( \epsilon > 0 \). This is true because

\[
\int_0^\infty e^{-\rho t}(x(t)^2 + \lambda \pi(t)^2)dt = 0
\]

and

\[
\eta \frac{\partial}{\partial \epsilon} \int e^{-\rho t}(g(t) + (1 - \Gamma)\epsilon)^2 dt = \eta \int e^{-\rho t}(g(t) + (1 - \Gamma)\epsilon)^2 > 0
\]

for small enough \( \epsilon > 0 \).

The rest of the arguments now follow the proof of Proposition 8.