Wringing Out Better Bell Inequalities*

SAMUEL L. BRAUNSTEIN[†]

*Theoretical Astrophysics 130-33, California Institute of Technology,
Pasadena, California 91125*

AND

CARLTON M. CAVES

*Center for Laser Studies, University of Southern California,
Los Angeles, California 90089-1112*

Local realism implies constraints on the statistics of two physically separated systems. These constraints, known collectively as Bell inequalities, can be violated by quantum mechanics. The standard Bell inequalities apply to a pair of two-state systems and constrain the value of some linear combination of correlation functions between the two systems. We generalize these standard Bell inequalities in two ways. First, we "chain" the Clauser–Horne–Shimony–Holt Bell inequality to obtain chained correlation Bell inequalities for two-state systems; we model a real experiment to show that these chained Bell inequalities lead to stronger quantum violations. Second, we formulate information-theoretic Bell inequalities, which are written in terms of the average information obtained in several measurements on a pair of physically separated systems (not just two-state systems); these information Bell inequalities have an appealing interpretation: if local realism holds, the two systems must carry an amount of information consistent with the inequality. © 1990 Academic Press, Inc.

1. INTRODUCTION

Local realism holds that physical systems posses local objective properties that are independent of observation. Bell's [1] genius was to realize that local realism, far from being just an appealing world view, has experimentally testable consequences. Specifically, local realism implies constraints on the statistics of two or more physically separated systems. These constraints, known collectively as Bell inequalities (for reviews and extensive reference lists see [2–7]), can be violated by the statistical predictions of quantum mechanics. Thus emerges the possibility of an experimental confrontation between the predictions of quantum mechanics and the requirements of local realism.

The standard Bell inequalities apply to a pair of physically separated two-state

† Present address: Research House, 2639 East Fourth Street, Tucson, Arizona 85716-4417.

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systems (e.g., two spin-$\frac{1}{2}$ particles) and are written in terms of correlations between measurable quantities (observables) associated with the two systems. A prime example of such a correlation Bell inequality is the Clauser–Horne–Shimony–Holt (CHSH) inequality [8], which deals with four measurable quantities, two for each two-state system (e.g., spin components in different directions). Associated with these four quantities are four correlation functions that involve one quantity from each system. The CHSH inequality constrains the value of a certain linear combination of these four correlation functions. It is an immediate consequence of assuming that the four quantities are local objective properties of the two systems.

Since observables associated with one system commute with those associated with the other system, quantum mechanics does predict values for the four correlation functions. When the two systems are prepared in an appropriate correlated quantum-mechanical state (e.g., a state of zero total spin for two spin-$\frac{1}{2}$ particles), the quantum-mechanical predictions violate the CHSH inequality for certain choices of the four measurable quantities. Thus quantum mechanics is inconsistent with the dictates of local realism. The violation has been confirmed experimentally in two-photon atomic cascade experiments [9–11], where the photons have correlated polarizations and the measurable quantities are polarizations along various directions.

In this paper we explore two ways to generalize the usual correlation Bell inequalities. The first kind of generalization relies on a technique that we call "chaining" [12–14], because the resulting Bell inequalities follow from iterating a simple Bell inequality, such as the CHSH inequality. Although chaining should apparently lead to weaker conditions for local realism, chained Bell inequalities, in fact, have stronger quantum violations, because chaining makes more onerous the requirements of local realism by attributing to each of the two systems a larger number of local objective properties. We apply chaining to the CHSH inequality, and we then consider a simple model of an experimental setup, in order to quantify the strength of quantum violations.

The second generalization we consider is to formulate information-theoretic Bell inequalities [12, 13], which are written in terms of the average information obtained in several measurements on two physically separated systems.The motivation for introducing information comes from the idea that if these systems have local objective properties, they must carry information about those properties— information that we obtain in measurements. Unlike correlation Bell inequalities, our information Bell inequalities apply to any pair of physically separated systems—not just two-state systems. As expressions of local realism, these information Bell inequalities have a particularly appealing interpretation: if local realism holds, the two systems must carry an amount of information consistent with the inequality. The simplest information Bell inequality is analogous to the CHSH inequality in that it involves four measurable quantities, two from each system, and four types of measurements, in which one quantity from each system is measured. The quantum statistics of two spin-s particles in a state of zero total spin violate this simplest information Bell inequality for all values of s.

In Section 2 we review the conflict between quantum mechanics and local realism and present a simple derivation [15, 16] of the CHSH inequality. The derivation highlights the roles of objectivity and locality in establishing the CHSH inequality; generalizations of the same derivation are used to establish all the other Bell inequalities in this paper. In Section 3 we sketch briefly the status of experimental tests of the CHSH inequality. In Section 4 we chain the CHSH inequality and explore the implications of such chained correlation inequalities for experimental tests of local realism. In Section 5 we review pertinent elements of information theory as a prelude to formulating information Bell inequalities in Section 6. Finally, in Section 7 we use information Bell inequalities as a tool to explore the meaning of Bell inequalities. We suggest that there is a hierarchy of information Bell inequalities that apply to a single system, to two systems, and to more than two systems. Within this hierarchy the two-system Bell inequalities play a special role in that they lead to the simplest direct tests of local realism.

2. Local Realism and Quantum Mechanics

What is the problem with quantum mechanics? Brash though it is to ask this question, it is brasher still to attempt an answer. We take the plunge anyway, with assurance to the reader that in the end we deal not with the whole problem, but only with a small, rather well-defined piece.

Anyone who has taken an introductory course in quantum mechanics knows that the wave function plays an essential role in the theory. One might be tempted, therefore, to regard the wave function as an objective quantity, which exists "out there," independent of our own existence. This view rolls merrily along so long as the wave function evolves according to the deterministic Schrödinger equation, but it hits a brick wall when it encounters the collapse of the wave function—the sudden, unpredictable change that occurs when one observes a system and obtains information about it. Wave-function collapse keeps those parts of the wave function that are consistent with the newly obtained information and throws away the rest. It raises some difficult questions. Why should an objective wave function change at all when one obtains information? Where do the discarded parts of the wave function "go" when one observes a system?

A logical way out of this conundrum is to adopt the "many-worlds" interpretation [17] of quantum mechanics, in which the Schrödinger equation is never suspended and there is no wave-function collapse. Unfortunately, this interpretation retrieves the discarded parts of the wave function only to place them in other branches of the universe where they remain inaccessible from "our" branch forever. This interpretation, then, leaves us with an untestable proposition.

Try again. Suppose we regard the wave function not as an objective quantity, but "merely" as a mathematical tool—a repository of information about a physical system. Then there is no problem with wave-function collapse, which simply

instructs us to throw away those parts of the wave function that are inconsistent with new information obtained when we observe a system. The situation is similar to the Bayesian view [18] of probabilities, in which a probability is a subjective repository of information about objective properties of some system. If we learn something about the system, we naturally change the probability to be consistent with our new knowledge. In probability theory this "collapse of the probability" goes under the name of Bayes's theorem. Now that we are happy about wave-function collapse, however, we face another tough question: Is the information stored in the wave function information about objective properties? If so, why can we not get our hands on those properties? More important, why can we not use ordinary probability logic to describe these objective properties, instead of being forced to use the probability amplitudes of quantum mechanics?

Bell's great contribution [1] was to see that these questions can be put to experimental test. He realized that if systems have local objective properties, this fact alone places constraints on the statistics of two physically separated systems, and he further realized that under certain circumstances the statistical predictions of quantum mechanics violate these constraints. Thus quantum mechanics is inconsistent with the notion that systems have local objective properties. The real problem, then, is this: how do objective properties—what we actually see in experiments—arise from a quantum-mechanical description that is manifestly not objective? We do not attempt to answer this deep question, our goal in this paper being more modest: to wring out variations on the theme of Bell, with the hope of gaining insight into the conflict of quantum mechanics with objectivity.

We start by reviewing the argument of Einstein, Podolsky, and Rosen (EPR) [19] that systems do have objective properties and that quantum mechanics must be extended to describe those properties. We use Bohm's version [20] of the original EPR paradox, in which a pair of spin-$\frac{1}{2}$ particles, $\mathcal{A}$ and $\mathcal{B}$, with spin vectors Σ_A and Σ_B (in units of $\frac{1}{2}\hbar$), are formed by decay in a spin-singlet state, so that their spins are perfectly anticorrelated. If a measurement of the spin component $\Sigma_\mathcal{A} \cdot \mathbf{b}$ of particle $\mathcal{A}$ yields the value $+1$ (here $\mathbf{b}$ is a unit vector that specifies the orientation of a detector), then a subsequent measurement of the spin component $\Sigma_\mathcal{B} \cdot \mathbf{b}$ of particle $\mathcal{B}$ must yield -1. If the detectors are separated by a large distance, so that they cannot communicate during these measurements, then the result of measuring $\Sigma_\mathcal{B} \cdot \mathbf{b}$ must have been predetermined, since we could have predicted it from a measurement of $\Sigma_\mathcal{A} \cdot \mathbf{b}$ without having in any way disturbed particle $\mathcal{B}$. Its being predetermined, however, should not depend on our first measuring $\Sigma_\mathcal{A} \cdot \mathbf{b}$, so we conclude that $\Sigma_\mathcal{B} \cdot \mathbf{b}$ is a "concrete," objective property of particle $\mathcal{B}$.

This argument does not depend on the direction of $\mathbf{b}$, so we conclude that any spin component of particle $\mathcal{B}$ is an objective property of particle $\mathcal{B}$. Reversing the roles of particles $\mathcal{A}$ and $\mathcal{B}$, we conclude similarly that any spin component of particle $\mathcal{A}$ is an objective property of particle $\mathcal{A}$. A quantum-mechanical description via a wave function allows predictions for pairs of spin components, one for each particle, but it has no notion of more than one spin component for each

particle. Two different spin components for a single particle do not commute, so we cannot talk about them quantum mechanically. Thus the argument of EPR is that quantum mechanics is an incomplete description of reality that must be extended in some way to describe all these objective properties.

The notion that systems have local objective properties is called local realism. Bell [1] took seriously the argument of EPR and explored the consequences of assuming local realism. He derived a constraint that must be satisfied by any local realistic description of the two particles, thus turning local realism into an experimentally testable proposition. Bell's constraint took the form of an inequality satisfied by a certain linear combination of correlation functions between the two particles. Bell further showed that the quantum statistics of the two particles in a spin-singlet state violate this inequality, thus placing quantum mechanics in direct conflict with local realism.

Since Bell's original work many other constraints imposed by local realism have been identified (see [2–6]). These constraints, known collectively as Bell inequalities, generally apply to pairs of two-state systems, just as in Bohm's version of the EPR paradox. We turn now to a Bell inequality first formulated by Clauser, Horne, Shimony, and Holt (CHSH) [8], and we review a simple derivation of this inequality [15, 16]. The derivation is instructive in that it highlights the roles of objectivity and locality and it serves as the basis for deriving generalized Bell inequalities in subsequent sections.

Return to the two spin-$\frac{1}{2}$ particles considered above, and pick out two spin components for each particle. Let the spin components for particle $\mathscr{A}$ be specified by unit vectors $\mathbf{a}$ and $\mathbf{a}'$ and the spin components for particle $\mathscr{B}$ be specified by unit vectors $\mathbf{b}$ and $\mathbf{b}'$. We introduce a shorthand notation for the spin components: $\Sigma_a \equiv \Sigma_{\mathscr{A}} \cdot \mathbf{a}$, $\Sigma_{a'} \equiv \Sigma_{\mathscr{A}} \cdot \mathbf{a}'$, $\Sigma_b \equiv \Sigma_{\mathscr{B}} \cdot \mathbf{b}$, and $\Sigma_{b'} \equiv \Sigma_{\mathscr{B}} \cdot \mathbf{b}'$. Each of these spin components takes on values ± 1; we use the corresponding lower-case letters—σ_a, $\sigma_{a'}$, σ_b, and $\sigma_{b'}$—to denote these possible values. In a quantum-mechanical description the two spin components for each particle do not commute and hence cannot be determined simultaneously. Thus we have in mind a series of experimental runs, in each of which one measures two spin components, one from each particle.

Suppose now that these four spin components are local objective properties of the two particles; then, in each run of the experiment, all four have definite values, independent of observation. One does not know these values, and one determines only two values—one from each particle—in each run. Nonetheless, in any local realistic theory, what is known is described by a joint probability $p(\sigma_a, \sigma_{a'}, \sigma_b, \sigma_{b'})$ for the values of the four components. From this joint probability follows a pair probability for each measurable pair of spin components—e.g.,

$$p(\sigma_a, \sigma_b) = \sum_{\sigma_{a'}, \sigma_{b'}} p(\sigma_a, \sigma_{a'}, \sigma_b, \sigma_{b'}).$$

If the two particles are physically separated, a measurement of a local property of one should not disturb a local property of the other. This no-disturbance

assumption, justified on the basis of locality, means that the statistics of runs that measure a particular pair of spin components are described by the appropriate pair probability. More precisely, for runs that measure Σ_a and Σ_b, the probability of obtaining the value σ_b for Σ_b is $p(\sigma_b) = \sum_{\sigma_a} p(\sigma_a, \sigma_b)$, and the probability of obtaining the value σ_a for Σ_a, given value σ_b for Σ_b, is $p(\sigma_a \mid \sigma_b) = p(\sigma_a, \sigma_b)/p(\sigma_b)$. How would a disturbance manifest itself? The conditional statistics of Σ_a, given that Σ_b has been determined to have value σ_b, would not be those predicted by the conditional probability $p(\sigma_a \mid \sigma_b)$. It should be noted that quantum mechanics satisfies this locality assumption if Σ_a and Σ_b are commuting observables, as indeed they are since spin components for different particles commute.

In the preceding argument objectivity is called on to establish the existence of the joint probability $p(\sigma_a, \sigma_{a'}, \sigma_b, \sigma_{b'})$, and locality is invoked to justify a no-disturbance assumption, which establishes the relevance of the joint probability to the statistics of the planned experimental runs. Note that both objectivity and locality are needed to relate the joint probability to the statistics of an actual experiment.

If the spin components are local objective properties, then we can write expressions involving their values in each experimental run (i.e., for each pair of particles). Consider, for instance, the expression [15, 16]

$$\sigma_a(\sigma_{b'} - \sigma_b) + \sigma_{a'}(\sigma_{b'} + \sigma_b) \equiv \pm 2, \tag{2.1}$$

which is trivially true for each run since each spin component takes on only the values ± 1. This expression contains values that cannot be determined simultaneously in quantum mechanics, but we can average it over the joint probability $p(\sigma_a, \sigma_{a'}, \sigma_b, \sigma_{b'})$ to obtain an inequality [(absolute value of average) $\leqslant$ (average of absolute value)],

$$|C(\mathbf{a}, \mathbf{b'}) + C(\mathbf{b'}, \mathbf{a'}) + C(\mathbf{a'}, \mathbf{b}) - C(\mathbf{b}, \mathbf{a})| \leqslant 2. \tag{2.2}$$

Here the average

$$C(\mathbf{a}, \mathbf{b}) \equiv \langle \sigma_a \sigma_b \rangle = \sum_{\sigma_a, \sigma_b} \sigma_a \sigma_b \, p(\sigma_a, \sigma_b) \tag{2.3}$$

is the correlation between spin components Σ_a and Σ_b. Inequality (2.2) is the CHSH Bell inequality. The correlation functions in (2.2) can be determined from the statistics of the four types of experimental runs.

By writing the four unit vectors as a list $\mathbf{a}$, $\mathbf{b'}$, $\mathbf{a'}$, $\mathbf{b}$, one can easily remember the form of the CHSH inequality (2.2); we shall say that Eq. (2.2) is the CHSH inequality corresponding to the list $\mathbf{a}$, $\mathbf{b'}$, $\mathbf{a'}$, $\mathbf{b}$. For any four unit vectors there are four independent CHSH inequalities (actually eight inequalities since each CHSH inequality is a double inequality), corresponding to the four lists $\mathbf{a}$, $\mathbf{b'}$, $\mathbf{a'}$, $\mathbf{b}$; $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a'}$, $\mathbf{b'}$; $\mathbf{a'}$, $\mathbf{b'}$, $\mathbf{a}$, $\mathbf{b}$; and $\mathbf{a'}$, $\mathbf{b}$, $\mathbf{a}$, $\mathbf{b'}$.

What does quantum mechanics have to say about the CHSH inequality? The

quantum-mechanical correlation function $C(\mathbf{a}, \mathbf{b})$ for two spin-$\frac{1}{2}$ particles in a spin-singlet state depends only on the angle θ between $\mathbf{a}$ and $\mathbf{b}$ and is given by [2]

$$C(\mathbf{a}, \mathbf{b}) \equiv C(\theta) = -\cos \theta. \tag{2.4}$$

Suppose we let the unit vectors $\mathbf{a}$, $\mathbf{b}'$, $\mathbf{a}'$, $\mathbf{b}$ be coplanar, with successive vectors in the list separated by angle $\theta/3$, so that $\mathbf{a}$ and $\mathbf{b}$ are separated by angle θ. Then quantum mechanics violates the CHSH inequality (2.2) whenever the quantity

$$S \equiv |3C(\theta/3) - C(\theta)| - 2, \tag{2.5}$$

the "signal for violation," is positive. So long as θ is small enough that the term within the absolute value is not positive ($|\theta| \leqslant 3\pi/2$), S can be written as

$$S = [1 + C(\theta)] - 3[1 + C(\theta/3)]. \tag{2.6}$$

The non-negative quantity $1 + C(\theta)$ goes to zero at $\theta = 0$ because of the perfect anticorrelation of two spin components in the same direction. To satisfy the CHSH inequality (i.e., $S \leqslant 0$), however, $1 + C(\theta)$ cannot go to zero faster than linearly in θ. Thus the quantum-mechanical correlation function (2.4), with its quadratic dependence on θ for small angles, obviously violates the CHSH inequality:

$$S = \theta^2/3 \quad \text{for} \quad |\theta| \ll 1. \tag{2.7}$$

The maximum value of S, which occurs at $\theta = 3\pi/4 = 135°$, is $2(\sqrt{2} - 1) \simeq 0.8284$.

The quadratic dependence of the quantum-mechanical correlation function expresses the fact that the spins of the two particles are more tightly correlated for small angles than the CHSH inequality allows. This quadratic dependence arises because the calculational tools of quantum mechanics are probability amplitudes instead of probabilities. The probability amplitudes have a linear dependence on θ for small angles; thus the probabilities, which appear in the correlation function, have a quadratic dependence.

At this point it is instructive to consider the meaning and significance of the CHSH inequality and other Bell inequalities. The assumption that the statistics of a set of measurable quantities follow from a "grand" joint probability captures, we claim, two important notions: first, that the quantities under consideration are objectively real physical properties and, second, that in measurements of certain subsets of these quantities, a measurement of one quantity does not disturb the remaining quantities in the subset. For physically separated systems, locality provides a natural justification for this no-disturbance assumption: a measurement on one system should not disturb quantities of the other physically separated systems. Thus we claim that the assumption of a grand probability for a set of measurable quantities associated with physically separated systems provides a general framework for elucidating the requirements set by objectivity and locality—i.e., by local realism. This point of view has been championed by Garg

and Mermin [21], who formulate it precisely and investigate its consequences for a pair of spin-s particles.

In the above derivation the CHSH inequality does follow from assuming that a joint probability $p(\sigma_a, \sigma_{a'}, \sigma_b, \sigma_{b'})$ gives the observable pair statistics of the four spin components Σ_a, $\Sigma_{a'}$, Σ_b, and $\Sigma_{b'}$. The question of how the CHSH inequality is related to the joint probability was answered definitively by Fine [22, 23], who showed that the observable pair statistics of the four spin components can be derived from a joint probability if and only if the observable pair probabilities satisfy a set of eight inequalities called the Clauser–Horne [24] inequalities. The Clauser–Horne inequalities are, in turn, precisely equivalent to the four CHSH inequalities for the four spin components.

Fine's theorem sheds light on a potential restriction in the above derivation of the CHSH inequalities. That derivation is couched in the language of a "local deterministic hidden-variable theory" [2], although we are careful not to mention hidden variables. In such a theory the two particles have actual values for their respective spin components, which are regarded as determined by some unknown hidden variables. The CHSH inequalities also hold for "local stochastic hidden-variable theories" [2], where the hidden variables determine not the actual values of the spin components, but only their probabilities. Fine's theorem shows that the observable pair statistics in a stochastic theory can be derived from a joint probability, and, indeed, the assumptions of a stochastic theory lead directly to a joint probability [22, 23]. From the Bayesian view [18] this result is not surprising, because there is no difference between deterministic and stochastic theories: the function of a (classical) probability is to describe insufficient knowledge of the actual value of objective quantities; it matters not whether one invokes hidden variables that are sufficient or not sufficient to determine those actual values.

When quantum mechanics violates the CHSH inequality, it means, strictly speaking, only that the quantum statistics of the four spin components cannot be derived from a grand joint probability. Since the quantum formalism never contemplates introducing such a joint probability, the violation might seem neither surprising nor significant. Indeed, Fine [22, 23] uses precisely this line of reasoning to question the physical significance of the CHSH inequality—and, more generally, of all Bell inequalities. Significance is rescued, we believe, by arguing that local realism—the assumption of local objective properties—ensures the existence and relevance of the joint probability. Violation of the CHSH inequality is thus interpreted to mean not just that quantum mechanics fails to yield some joint probability, but more importantly, that it conflicts either with objectivity or locality. Our view is that quantum mechanics is local and that it conflicts with objectivity, there being no nonlocal disturbance in the sense defined above.

The significance and meaning of the CHSH and other Bell inequalities is a contentious question; the reader should be warned that our views are by no means universally shared. To illustrate further the range of views, consider that de Muynck [25], following Fine [22, 23], argues that the joint probability—and, hence, the

CHSH inequality—does not involve a locality assumption. We have argued above that the relevance of the joint probability *does* rely on a no-disturbance assumption, which can be justified by locality. In contrast to de Muynck, Stapp [26, 27], in a very subtle argument, endeavors to show that the CHSH inequality follows from locality, with no considerations of objectivity. As we read his derivation, however, it is the same as the derivation given above, except that Stapp refuses to refer to the values in Eq. (2.1) as the actual values of the spin components, insisting instead on calling them values that "would appear" in different experiments. The distinction is lost on us, but the reader should not be prejudiced that it would be lost on him.

For the remainder of this paper we adopt the view [21] that Bell inequalities arise from assuming a grand joint probability. If so, why not take a direct approach? Start with marginal probabilities predicted by quantum mechanics, and ask if they can be derived from higher-order "grand" probabilities. This approach has been pursued by Garg and Mermin [21], who formulate it mathematically and investigate it for pairs of spin-*s* particles for several values of *s*. The Garg–Mermin approach is, we believe, the right way to ferret out all the consequences of local realism for arbitrary systems, but it is not simple to deal with mathematically, nor does it yield clear-cut constraints for experimental test. We say more about the Garg–Mermin aproach and about the meaning of Bell inequalities in Section 7.

3. EXPERIMENTAL STATUS OF LOCAL REALISM

In this section we review briefly the experimental status of the confrontation between quantum theory and local realism. The review sets the stage for chaining the CHSH inequality in Section 4.

A series of experiments performed in the mid-seventies tested quantum mechanics against the requirements of local realism. All these experiments were of the same basic type, along the lines of Bohm's version of the EPR experiment. Pairs of correlated particles were generated from some source, and the spin projections or polarizations of the two particles were measured along various directions in separate experimental runs. Various kinds of sources were used to generate the pairs of correlated particles: an atomic cascade producing pairs of photons, annihilation of positronium, and proton–proton scattering. None of these experiments was able to test directly the CHSH inequality; they tested instead other consequences of local realism which require additional assumptions. Although these experiments tended to support quantum mechanics, they could not be regarded as conclusive. Clauser and Shimony [2] have reviewed this series of experiments, and the reader is referred to their review for a detailed description and for references.

The experimental situation was improved decisively by a new set of two-photon atomic cascade experiments performed by Aspect and various coworkers [9–11, 28] in the early eighties. In these experiments a pair of photons is created by cascade decay in a zero-angular-momentum, even-parity state. Polarization analyzers are

used to measure the polarizations of two such photons moving in nearly opposite directions.

Suppose the polarization analyzer for the left-moving photon, which we dub photon $\mathscr{A}$, is oriented along unit vector **a**, and the polarization analyzer for the right-moving photon, which we dub photon $\mathscr{B}$, is oriented along unit vector **b**. We let σ_a and σ_b denote the possible results of the polarization measurements in the following way: $\sigma_a = +1(-1)$ if photon $\mathscr{A}$ passes through (reflects from) the analyzer—i.e., if it has polarization along (perpendicular to) **a**; similarly, $\sigma_b = +1(-1)$ if photon $\mathscr{B}$ is found to be polarized along (perpendicular to) **b**.

The ideal quantum-mechanical prediction for the correlation

$$C(\mathbf{a}, \mathbf{b}) \equiv \langle \sigma_a \sigma_b \rangle = \sum_{\sigma_a, \sigma_b} \sigma_a \sigma_b \, p(\sigma_a, \sigma_b) \tag{3.1}$$

between the polarizations depend only on the angle θ between the analyzers and is given by [2]

$$C(\mathbf{a}, \mathbf{b}) \equiv C(\theta) = \cos 2\theta. \tag{3.2}$$

In practice this ideal correlation cannot be achieved because of imperfections in the experimental setup. Imperfections in the experiments of Aspect and coworkers can be taken into account by changing the quantum-mechanical prediction for the correlation function to be

$$C(\theta) = \eta \cos 2\theta, \tag{3.3}$$

where $\eta \leqslant 1$ is a coefficient that characterizes the experimental imperfections. The coefficient η can be modeled by a "data flipping error," represented by a probability q that either photon flips its polarization from $+1$ to -1 or from -1 to $+1$. This model yields a coefficient $\eta = (1 - 2q)^2$. The model describes well the experiments of Aspect and coworkers, with the main causes of data flipping being imperfections in the polarization analyzers and the use of detectors that accept photons from a nonzero solid angle. In [10] Aspect, Dalibard, and Roger achieved $\eta \simeq 0.955$.

The major innovations in the experiments of Aspect and coworkers are worth mentioning. Aspect, Grangier, and Roger [28] started with a high-intensity source of pairs of low-energy photons emitted in a cascade decay of calcium. This gave them good statistics in short running times and allowed them to have large source-polarizer separations (up to 6.5 m). The stability of their source reduced problems from drifts in the source intensity between runs with the polarization analyzers set at different orientations. These first experiments—like all previous experiments—did not test directly the CHSH inequality. Aspect, Grangier, and Roger [9] next incorporated two-channel polarization analyzers in the experiment. This allowed them to measure directly polarization correlations—and thus to test directly the CHSH inequality—without resorting to assumptions about photons not detected behind the analyzer. The results of these experiments confirmed quantum mechanics and

violated the CHSH inequality by more than 40 standard deviations. Aspect, Dalibard, and Roger [10] subsequently included acoustic switches in front of the polarization analyzers. Each switch could choose the orientation of its polarization analyzer while the photons were in flight from the source, thus apparently ruling out any information transfer between the photons. The use of the acoustic switches degraded the statistics, but the results of these final experiments nonetheless agreed with quantum mechanics and violated the CHSH inequality by five standard deviations, showing empirically that a local realistic description of two-photon decays is not possible.

It is worth mentioning two objections to the experiments of Aspect and coworkers. The most serious is that the photomultipliers used as detectors had low quantum efficiency and thus detected only a small fraction of the pairs of photons. Thus one must make an additional assumption that the pairs observed constituted an unbiased sample of all pairs. There are several papers [24, 29–31] that give models for systematic effects that could "enhance" the correlations in such a way as to mock the predictions of quantum mechanics. The second objection [32] questions whether locality was strictly enforced with the source-polarizer separations used in the final experiments of Aspect, Dalibard, and Roger [10]. In these experiments one of the photons comes from an atomic transition with a lifetime exceeding 40 nsec—longer than it takes light to travel the maximum separations used—so subluminal information exchanges are not entirely ruled out.

In atomic cascade tests of the CHSH inequality there is a tradeoff between a desire to keep the correlation function nearly ideal, which pushes the experimenter to reduce the solid angle accepted by his detectors, and the desire for good statistics, which drives the experimenter to open up the solid angle. An experiment by Alley and Shih [33, 34] holds the promise of avoiding this tradeoff. In the Alley–Shih experiment a pair of photons is created by parametric down conversion. The two photons have the same polarization, and they are correlated spatially and temporally. They travel in diverging but well defined directions. The polarization of one of the photons is first rotated by 90°, and then the two photons are combined at a 50/50 beam splitter. For each pair of photons there are four possible outcomes: two outcomes in which both photons end up on the same side of the beam splitter—one transmitted, the other reflected—and two outcomes in which the photons end up on opposite sides—both transmitted or both reflected. If one records only events in which there are photons on both sides of the beam splitter, the effective quantum state becomes a simple superposition of the latter two possibilities. This state mimics the state produced by a two-photon cascade decay, except that the photons travel in well-defined directions, so each detector can intercept the entire beam without suffering any reduction in the ideal quantum-mechanical correlation function.

The current versions of the Alley–Shih experiment [33–36] cannot test directly the CHSH inequality, because they do not use two-channel polarization analyzers. Instead they test a different inequality, which relies on additional assumptions about the stability of the parametric down converter and the unbiased nature of the

detectors. Given these assumptions, the most recent experiment [36] finds that the Bell inequality tested is violated by six standard deviations.

By using two-channel analyzers an Alley–Shih experiment could test directly the CHSH inequality. In this sort of experiment, in which the photons travel in well-defined directions, one can achieve good statistics without introducing data error (no solid-angle problem); thus a very small data flipping error should be achievable. We now proceed to Section 4, where we show that small data flipping error is precisely the requirement for using chained Bell inequalities to give stronger quantum violations.

4. CHAINING THE CHSH INEQUALITY

In this section we chain the CHSH Bell inequality (2.2) and explore the experimental consequences of the resulting chained correlation Bell inequalities. The CHSH inequality applies to a pair of two-state systems and follows from attributing to the two systems a set of local objective properties. Examples include (i) a pair of spin-$\frac{1}{2}$ particles, where the objective properties are spin components in various directions, and (ii) a pair of correlated photons, where the objective properties are polarizations in various directions. This section is couched in the language of polarization measurements on a pair of correlated photons, as in the experiments discussed in Section 3.

Chaining, or iterating, an inequality typically leads to a condition which is weaker mathematically than the original inequality. Thus chaining the CHSH inequality should apparently lead to weaker conditions for local realism. We show in this section, however, that chaining the CHSH inequality actually leads to inequalities that have stronger quantum violations (in a sense we make precise below) over a larger range of angles. The reason for these stronger violations is that chaining attributes to the two-state systems a greater number of objective properties, thus making more onerous the requirements of local realism.

Return now to the two correlated photons, $\mathscr{A}$ and $\mathscr{B}$, discussed in Section 3. The CHSH inequality deals with four polarization directions—a and a′ for photon $\mathscr{A}$, b and b′ for photon $\mathscr{B}$. We imagine these four unit vectors as making up the list a, b′, a′, b. As is shown in Section 2, the CHSH inequality follows from assuming that the polarizations defined by these four directions are local objective properties of the photons (local realism), and it constraints the value of a linear combination of the four correlation functions between the two photons:

$$-2 \leqslant C(\mathbf{a}, \mathbf{b}') + C(\mathbf{b}', \mathbf{a}') + C(\mathbf{a}', \mathbf{b}) - C(\mathbf{b}, \mathbf{a}) \leqslant 2 \qquad (4.1)$$

[Eq. (2.2)]. Consider now two additional polarization directions—a″ for photon $\mathscr{A}$ and b″ for photon $\mathscr{B}$. We can write a CHSH inequality for the list a, b″, a″, b′:

$$-2 \leqslant C(\mathbf{a}, \mathbf{b}'') + C(\mathbf{b}'', \mathbf{a}'') + C(\mathbf{a}'', \mathbf{b}') - C(\mathbf{b}', \mathbf{a}) \leqslant 2. \qquad (4.2)$$

Adding Eqs. (4.1) and (4.2) yields the first chained correlation Bell inequality,

$$-4 \leqslant C(\mathbf{a}, \mathbf{b}'') + C(\mathbf{b}'', \mathbf{a}'') + C(\mathbf{a}'', \mathbf{b}') + C(\mathbf{b}', \mathbf{a}') + C(\mathbf{a}', \mathbf{b}) - C(\mathbf{b}, \mathbf{a}) \leqslant 4, \quad (4.3)$$

which applies to the six directions in the list $\mathbf{a}$, $\mathbf{b}''$, $\mathbf{a}''$, $\mathbf{b}'$, $\mathbf{a}'$, $\mathbf{b}$. Clearly, we can iterate this procedure to obtain more and more complicated chained correlation Bell inequalities. We prefer, however, to derive the general form directly by using a generalization of the derivation given in Section 2, because this derivation draws attention again to the roles of objectivity and locality.

Consider, then, N polarization directions (N even), $N/2$ for each photon. We adopt a notation in which the polarization directions specified by odd numbers, $\mathbf{a}_1, \mathbf{a}_3, ..., \mathbf{a}_{N-1}$, apply to photon $\mathscr{A}$, and the polarization directions specified by even numbers, $\mathbf{b}_2, \mathbf{b}_4, ..., \mathbf{b}_N$, apply to photon $\mathscr{B}$. Furthermore, we let $\sigma_j = +1(-1)$ for j odd correspond to polarization of photon $\mathscr{A}$ along (perpendicular to) $\mathbf{a}_j$, and we let $\sigma_j = +1(-1)$ for j even correspond to polarization of photon $\mathscr{B}$ along (perpendicular to) $\mathbf{b}_j$. Just as for the CHSH inequality, we have in mind a series of experimental runs, in each of which one uses a pair of polarization analyzers, one for each photon. The experimental runs are of N types: $N-1$ types in which the two analyzer orientations are adjacent vectors from the list $\mathbf{a}_1$, $\mathbf{b}_2$, $\mathbf{a}_3$, $\mathbf{b}_4$, ..., $\mathbf{a}_{N-1}$, $\mathbf{b}_N$, and one type in which the analyzer orientations are $\mathbf{b}_N$ and $\mathbf{a}_1$.

Suppose now that the polarizations defined by all these directions are local objective properties of the two photons. Then, just as in Section 2, we can use objectivity and locality—local realism—to argue that the statistics of the N polarizations follow from a joint probability $p(\sigma_1, \sigma_2, ..., \sigma_{N-1}, \sigma_N)$. Furthermore, just as in Section 2, we can write expressions involving the values of all these properties in each experimental run (i.e., for each pair of photons). Consider, for instance, the inequality

$$|\sigma_1(\sigma_2 - \sigma_N) + \sigma_3(\sigma_2 + \sigma_4) + \sigma_5(\sigma_4 + \sigma_6) + \cdots + \sigma_{N-1}(\sigma_{N-2} + \sigma_N)| \leqslant N - 2. \quad (4.4)$$

That this inequality does indeed hold for each and every pair of photons we see in the following way. The expression within the absolute value is the sum of $N/2$ terms, each of which can take on values 0, ± 2. At least one of these terms must vanish (if σ_2 and σ_N have the same sign, the first term vanishes; if σ_2 and σ_N have opposite signs, at least one of the other terms must vanish). Hence the maximum value of the absolute value is $2 \times (N/2 - 1) = N - 2$.

Inequality (4.4) contains values that cannot be determined simultaneously in quantum mechanics, but by averaging it over the joint probability $p(\sigma_1, \sigma_2, ..., \sigma_{N-1}, \sigma_N)$, we obtain an inequality

$$|C(\mathbf{a}_1, \mathbf{b}_2) + C(\mathbf{b}_2, \mathbf{a}_3) + C(\mathbf{a}_3, \mathbf{b}_4) + \cdots + C(\mathbf{a}_{N-1}, \mathbf{b}_N) - C(\mathbf{b}_N, \mathbf{a}_1)| \leqslant N - 2, \quad (4.5)$$

which involves only correlation functions that can be determined from the statistics of the N types of experimental runs. Inequality (4.5) is the chained correlation Bell inequality. It could be obtained by taking linear combinations of the CHSH

inequality (4.1) (which corresponds to $N = 4$), as was done to obtain inequality (4.3) (which corresponds to $N = 6$).

The general chained correlation Bell inequality has been derived previously [3, 29, 37], but there has been very little discussion of it. There does exist a theorem [3, 37], which states that any Bell inequality that restricts the value of a linear combination of correlation functions can be derived from linear combinations of the CHSH inequality. We present the chained correlation Bell inequalities again here in order to explore their experimental consequences.

Before considering experimental implications, however, we can gain insight into the meaning of inequality (4.5) by considering the quantum-mechanical predictions. The ideal quantum-mechanical correlation function for a pair of photons emitted by a two-photon atomic cascade is given by Eq. (3.2). The best geometry to choose is illustrated in Fig. 1: successive vectors in the list $\mathbf{a}_1$, $\mathbf{b}_2$, $\mathbf{a}_3$, $\mathbf{b}_4$, ..., $\mathbf{a}_{N-1}$, $\mathbf{b}_N$ are separated by angle $\theta/(N-1)$, so that $\mathbf{a}_1$ and $\mathbf{b}_N$ are separated by angle θ. With this geometry quantum mechanics violates inequality (4.5) whenever the quantity

$$S_N \equiv |(N-1)\, C(\theta/(N-1)) - C(\theta)| - (N-2), \tag{4.6}$$

the "signal for violation," becomes positive [cf. Eq. (2.5)]. So long as θ is small enough that the term within the absolute value is non-negative ($|\theta| \leqslant (N-1)\,\pi/4$), S_N can be written as

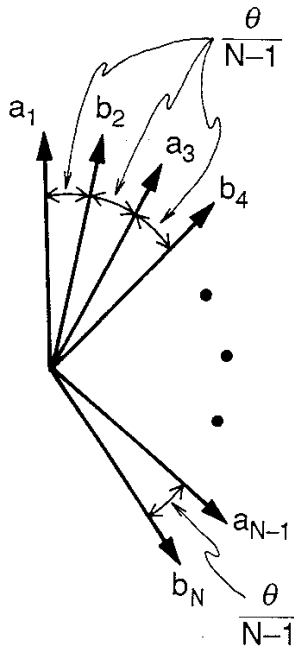$$S_N = [1 - C(\theta)] - (N-1)[1 - C(\theta/(N-1))]. \tag{4.7}$$

FIG. 1. Geometry leading to maximal violation of the chained Bell inequality (4.5). Successive vectors in the list $\mathbf{a}_1$, $\mathbf{b}_2$, $\mathbf{a}_3$, $\mathbf{b}_4$, ..., $\mathbf{a}_{N-1}$, $\mathbf{b}_N$ are separated by angle $\theta/(N-1)$, so that the outermost vectors $\mathbf{a}_1$ and $\mathbf{b}_N$ are separated by angle θ.

The non-negative quantity $1 - C(\phi)$ goes to zero at $\phi = 0$ because of the perfect correlation between polarizations when the two polarization analyzers are aligned. To satisfy the chained Bell inequality (4.5) (i.e., $S_N \leqslant 0$), $1 - C(\theta)$ must lie below a straight line drawn from the origin through the value of $1 - C(\theta/(N-1))$. Because of the quadratic dependence of the quantum-mechanical correlation function for small angles, this straight line asymptotes to the x axis as $N \to \infty$. Indeed, in the limit of large N, for fixed θ, the signal for violation becomes

$$S_N = [1 - C(\theta)] - 2\theta^2/(N-1) \xrightarrow[N \to \infty]{} 1 - C(\theta), \qquad (4.8)$$

which shows a violation at every angle except multiples of π, since perfect correlation occurs only when the analyzers are aligned.

The violation occurs, just as for the CHSH inequality, because the quantum-mechanical polarizations are more tightly correlated for small angles than local realism allows. Chaining allows us to take full advantage of this tight correlation: the greater the number of analyzer orientations between fixed outer vectors $\mathbf{a}_1$ and $\mathbf{b}_N$, the smaller the combined effect of the quantum correlations summed over the intermediate pairs of orientations. This is precisely analogous to the quantum Zeno effect [38, 39], where it is found that repeated measurements on a system at closer and closer times "stop" the quantum evolution of the system's state.

It is worth noting here that S_N attains a maximum value $2 + N[\cos(\pi/N) - 1]$ at $\theta = [(N-1)/N] \pi/2$; in the limit $N \to \infty$, this gives a maximum value of 2 at $\theta = 90°$, compared with a maximum value $\simeq 0.8284$ at $\theta = 67.5°$ for the CHSH inequality.

The above considerations suggest that chaining leads to stronger quantum violations. To investigate this suggestion, however, we must model at least two aspects of a real experiment. First, any increase in the strength of the quantum violation with increasing N relies on the tight quantum-mechanical correlation at small angles; thus we need to model the reduction in correlation that comes from imperfections in a real experiment. Second, to compare meaningfully the strength of the violation for different values of N, we need to measure the signal for violation in terms of an appropriate "unit"; that unit is provided by the size of the noise, which we assume to be statistical uncertainty that comes from estimating the correlation functions from a finite number of experimental runs.

We model imperfections in a real experiment by a "data flipping error," represented by a probability q that either photon flips its polarization from $+1$ to -1 or from -1 to $+1$. With this data flipping error the quantum-mechanical prediction for the correlation function becomes

$$C(\mathbf{a}, \mathbf{b}) \equiv C(\theta) = \eta \cos 2\theta, \qquad \eta \equiv (1 - 2q)^2. \qquad (4.9)$$

The quantity $1 - \eta$ characterizes the size of the data flipping error. As is discussed in Section 3, this model for experimental imperfections describes well the atomic cascade experiment of Aspect and collaborators, with η coming from two sources—

imperfections in the polarization analyzers and the use of detectors that accept photons from a nonzero solid angle. In [10] Aspect, Dalibard, and Roger achieved a value $\eta \simeq 0.955$.

We turn now to the statistical noise. We assume that it is good experimental practice to do the entire experiment in as short a time as possible, because, for example, the two-photon source or other aspects of the experiment might be stable only over a finite amount of time. Thus we restrict ourselves to a total running time that yields on average $\mathcal{N}$ experimental runs. To test the chained Bell inequality (4.5), we must split up this total time suitably to estimate N correlation functions. As N increases we can afford to spend a progressively smaller amount of time on each correlation function. This raises the possibility that there might be no benefit to chaining; the noise might increase so rapidly with increasing N that it would swamp the increasing size of the signal for violation. That this possibility does not materialize—at least for sufficiently small data flipping error—we now show.

Focus attention on a particular pair of analyzer orientations **a** and **b**. One spends some fraction of the total running time on this pair of orientations—a fraction that yields on average $\mathcal{M}$ experimental runs. The correlation function for this pair of orientations can be written as

$$C(\mathbf{a}, \mathbf{b}) = \sum_{\sigma_a, \sigma_b} \sigma_a \sigma_b \, p(\sigma_a, \sigma_b)$$

$$= p(+, +) + p(-, -) - p(+, -) - p(-, +) = \sum_{\alpha=1}^{4} \lambda_\alpha p_\alpha. \quad (4.10)$$

Here we introduce a shorthand notation in which the subscript α denotes the four possible outcomes of an experimental run: $\alpha = 1$ corresponds to $(+, +)$, i.e., to both photons having polarization $+1$; $\alpha = 2$ corresponds to $(-, -)$; $\alpha = 3$ to $(+, -)$; and $\alpha = 4$ to $(-, +)$. We also define $\lambda_1 = \lambda_2 \equiv +1$ and $\lambda_3 = \lambda_4 \equiv -1$. Suppose now that in the allotted running time one accumulates precisely M runs for this pair of analyzer orientations and that the number of these runs that yield outcome α is $R_\alpha (M = \sum_\alpha R_\alpha)$. The experimental estimate of the probability p_α is the frequency $g_\alpha = R_\alpha/M$ of occurrence of outcome α. Given these estimated probabilities, one estimates the correlation function to be

$$C_{\exp}(\mathbf{a}, \mathbf{b}) = g(+, +) + g(-, -) - g(+, -) - g(-, +) = \sum_{\alpha=1}^{4} \lambda_\alpha g_\alpha. \quad (4.11)$$

It is the mean and variance of the experimental estimate $C_{\exp}(\mathbf{a}, \mathbf{b})$ which are now of interest.

For a fixed number of runs M, the numbers of counts R_α for the four outcomes are distributed according to the multinomial distribution,

$$p(R_1, R_2, R_3, R_4 \mid M) = \frac{M!}{R_1! \, R_2! \, R_3! \, R_4!} \, p_1^{R_1} p_2^{R_2} p_3^{R_3} p_4^{R_4}. \quad (4.12)$$

Hence the means and second moments for the numbers of counts (with M fixed) are

$$(\overline{R_\alpha})_{M\text{ fixed}} = Mp_\alpha, \tag{4.13}$$

$$(\overline{R_\alpha R_\beta})_{M\text{ fixed}} = M(M-1)p_\alpha p_\beta + Mp_\alpha \delta_{\alpha\beta}, \tag{4.14}$$

where an overbar denotes a statistical average. In calculating means involving the frequencies, we must take a further average over the number of runs M; when doing so we assume that the average number of runs, $\mathcal{M}$, is large enough that we can approximate the mean of M^{-1} by $\mathcal{M}^{-1}$. From the above results we find the standard results that the mean of the frequency is

$$\overline{g_\alpha} = p_\alpha \tag{4.15}$$

and that the covariance matrix of the frequencies is given by

$$\overline{\Delta g_\alpha \, \Delta g_\beta} = \mathcal{M}^{-1}(p_\alpha \delta_{\alpha\beta} - p_\alpha p_\beta), \tag{4.16}$$

where $\Delta g_\alpha \equiv g_\alpha - \overline{g_\alpha} = g_\alpha - p_\alpha$. The average of the experimentally estimated correlation function is simply the quantum-mechanical prediction,

$$\overline{C_{\exp}(\mathbf{a}, \mathbf{b})} = C(\mathbf{a}, \mathbf{b}) = C(\theta) = \eta \cos 2\theta, \tag{4.17}$$

and the variance of $C_{\exp}(\mathbf{a}, \mathbf{b})$ is given by

$$\overline{[\Delta C_{\exp}(\mathbf{a}, \mathbf{b})]^2} = \sum_{\alpha, \beta} \lambda_\alpha \lambda_\beta \, \overline{\Delta g_\alpha \, \Delta g_\beta} = \Delta^2(\theta)/\mathcal{M}, \tag{4.18}$$

where

$$\Delta^2(\theta) \equiv 1 - C^2(\theta). \tag{4.19}$$

Note that this variance is strongly dependent on angle. When the data flipping error is small, there is almost perfect correlation near $\theta = 0$, of which we can be very confident after comparatively few experimental runs. Chaining succeeds because of this ability to estimate the correlation function for small angles with comparatively few runs. There is a similar reduction in the noise near $\theta = 90°$ because of the almost perfect anticorrelation there.

Recall that we are using the geometry depicted in Fig. 1. The chained correlation Bell inequality (4.5) is violated whenever the quantity

$$|C(\mathbf{a}_1, \mathbf{b}_2) + C(\mathbf{b}_2, \mathbf{a}_3) + \cdots + C(\mathbf{a}_{N-1}, \mathbf{b}_N) - C(\mathbf{b}_N, \mathbf{a}_1)| - (N-2) \tag{4.20}$$

is greater than zero. The experimental estimate of this signal for violation,

$$S_{N,\exp} \equiv |C_{\exp}(\mathbf{a}_1, \mathbf{b}_2) + C_{\exp}(\mathbf{b}_2, \mathbf{a}_3) + \cdots$$
$$+ C_{\exp}(\mathbf{a}_{N-1}, \mathbf{b}_N) - C_{\exp}(\mathbf{b}_N, \mathbf{a}_1)| - (N-2), \tag{4.21}$$

has mean value

$$\overline{S_{N,\,exp}} \equiv S_N = |(N-1)\,C(\theta/(N-1)) - C(\theta)| - (N-2) \qquad (4.22)$$

[cf. Eq. (4.6)]. (The absolute value causes a problem in taking the mean as the absolute value approaches zero; there being no violation in this region, however, it is of no interest, so we ignore this problem.)

We now want to calculate the variance of the estimated signal for violation. One has available a total running time which yields on average $\mathcal{N}$ experimental runs. The variance of $S_{N,\,exp}$ depends on how the total time is apportioned among the measurements of the various correlation functions. Suppose a fraction f of the total time is devoted to measuring the one correlation function at the large angle θ (polarizer orientations $\mathbf{a}_1$ and $\mathbf{b}_N$); for this correlation function we let $\mathcal{M} = f\mathcal{N}$ in Eq. (4.18). The remaining time is divided equally among measurements of the $N-1$ correlation functions at the small angle $\theta/(N-1)$; for these correlation functions we take $\mathcal{M} = (1-f)\,\mathcal{N}/(N-1)$. The variance of the estimated signal (4.21), which comes from adding the variances of all the estimated correlation functions, tells us the size of the "noise":

$$N_N^2 \equiv \overline{(\Delta S_{N,\,exp})^2} = \frac{1}{\mathcal{N}} \left[\frac{(N-1)^2\,\Delta^2(\theta/(N-1))}{1-f} + \frac{\Delta^2(\theta)}{f} \right] \qquad (4.23)$$

[Eq. (4.18)]. We want to minimize this noise with respect to the division of the total running time. The minimum occurs with f given by

$$f = \frac{\Delta(\theta)}{(N-1)\,\Delta(\theta/(N-1)) + \Delta(\theta)}, \qquad (4.24)$$

in which case the noise becomes

$$N_N = \frac{1}{\sqrt{\mathcal{N}}}\,[(N-1)\,\Delta(\theta/(N-1)) + \Delta(\theta)]. \qquad (4.25)$$

We can now define a signal-to-noise ratio $(S_N/N_N)_{\mathcal{N}}$ for an experiment that yields on average $\mathcal{N}$ experimental runs. This signal-to-noise ratio has a trivial $\sqrt{\mathcal{N}}$ dependence on the total number of runs, so it is useful to introduce a "single-run signal-to-noise ratio"

$$\left(\frac{S_N}{N_N}\right)_{\mathcal{N}=1} \equiv \frac{1}{\sqrt{\mathcal{N}}}\left(\frac{S_N}{N_N}\right)_{\mathcal{N}}$$

$$= \frac{|(N-1)\,C(\theta/(N-1)) - C(\theta)| - (N-2)}{(N-1)\,\Delta(\theta/(N-1)) + \Delta(\theta)}. \qquad (4.26)$$

The single-run signal-to-noise ratio $(S_N/N_N)_{\mathcal{N}=1}$ gives the size of the violation of the chained Bell inequality (4.5) in units of the statistical noise; we claim that it is

the appropriate way to characterize the strength of the violation. (We define this single-run signal-to-noise ratio only when S_N is positive—i.e., when there is, in fact, a violation.)

When there is no data flipping error [$\eta = 1$, $C(\theta) = \cos 2\theta$, $\Delta(\theta) = |\sin 2\theta|$], the single-run signal-to-noise ratio simplifies to

$$\left(\frac{S_N}{N_N}\right)_{\mathcal{N}=1,\eta=1} = \frac{|(N-1)\cos[2\theta/(N-1)] - \cos 2\theta| - (N-2)}{(N-1)|\sin[2\theta/(N-1)]| + |\sin 2\theta|}$$

$$\xrightarrow[N\to\infty]{} \frac{2\sin^2\theta}{2|\theta| + |\sin 2\theta|}.$$

$$(4.27)$$

In Fig. 2 we plot $(S_N/N_N)_{\mathcal{N}=1,\eta=1}$ vs. angle θ in degrees for $N = 4, 6, 8, 12, 20$, and ∞. The noticeable cusp at $\theta = 90°$ comes from the $|\sin 2\theta|$ term in the noise N_N. As
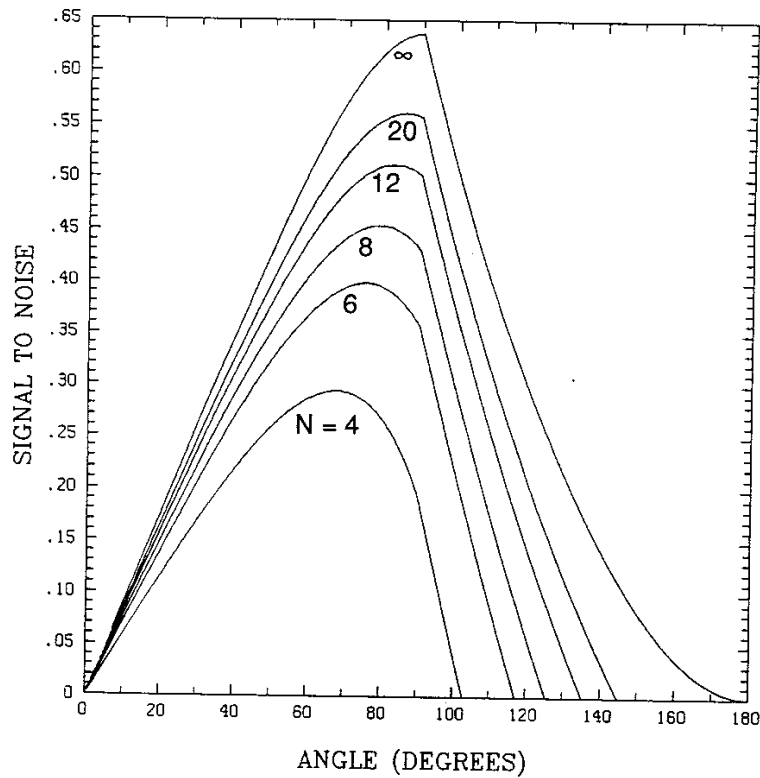

FIG. 2. Single-run signal-to-noise ratio $(S_N/N_N)_{\mathcal{N}=1,\eta=1}$ [Eq. (4.27)] vs. angle θ in degrees for the case of no data error ($\eta = 1$) and for chainings $N = 4, 6, 8, 12, 20$, and ∞. The single-run signal-to-noise ratio gives the strength of the violation of the chained correlation Bell inequality (4.5) in units of statistical noise; for $\eta = 1$ it increases with increasing amounts of chaining. (The part of the $N = 4$ curve that follows from reflection symmetry through $\theta = 135°$ is omitted for visual clarity.)

expected, when there is no data flipping error, the strength of the violation increases with increasing N.

The single-run signal-to-noise ratio $(S_N/N_N)_{\mathcal{N}=1,\eta=1}$ attains a maximum at $\theta = [(N-1)/N]\,\pi/2$—the angle that simultaneously maximizes S_N and minimizes N_N. The maximum value of $(S_N/N_N)_{\mathcal{N}=1,\eta=1}$,

$$\frac{2+N[\cos(\pi/N)-1]}{N\sin(\pi/N)},$$

varies from $(\sqrt{2}-1)/\sqrt{2}\simeq 0.2929$ at $\theta = 67.5°$ for $N=4$ (CHSH inequality) to $2/\pi \simeq 0.6366$ at $\theta = 90°$ as $N\to\infty$. In principle, then, chaining can lead to a violation $\simeq 2.174$ times stronger than for the CHSH inequality. Interestingly, the maximum value is achieved when the total running time is apportioned equally among all the correlation functions—i.e., $f = 1/N$. The $N-1$ correlation functions at the small angle $\theta/(N-1)$ take advantage of the noise reduction due to the near perfect *correlation* at small angles, while the one correlation function at angle θ takes advantage of the noise reduction due to the near perfect *anticorrelation* for angles near 90°. For angles below the maximum, more time is devoted to the one large angle, whereas for angles above the maximum, more time is devoted to each of the small angles.

The effects of data error are illustrated in Fig. 3, where we plot the single-run signal-to-noise ratio (4.26) vs. angle θ in degrees for $N=4, 6, 8, 12,$ and 20 and for two values of η: (a) $\eta = 0.99$ and (b) $\eta = 0.955$. The beneficial effects of chaining decrease, as expected, with increasing data error, but for the achievable [10] data error of Fig. 3(b), the $N=6$ and $N=8$ chained Bell inequalities both yield a stronger violation than does the CHSH inequality ($N=4$). For $N=6$ the increase in signal-to-noise ratio is about 20%.

We gain insight into the effects of data error by expanding S_N [Eq. (4.22)] and N_N [Eq. (4.25)] for large N,

$$S_N = (1-\eta\cos 2\theta) - 2\eta\theta^2/(N-1) - (1-\eta)(N-1),\tag{4.28}$$

$$N_N = \frac{1}{\sqrt{\mathcal{N}}}\left\{[4\eta^2\theta^2 + (1-\eta^2)(N-1)^2]^{1/2} + (1-\eta^2\cos^2 2\theta)^{1/2}\right\},\tag{4.29}$$

where we must assume that $|\theta| \ll (N-1)$. There are two new terms due to data error here, one in S_N and one in N_N, both of which reduce the strength of the violation. These terms limit the improvement in signal-to-noise ratio that can be achieved by chaining. Indeed, when η is near 1, one can show that for fixed θ the signal-to-noise ratio is maximized when N is near a critical value given by

$$N_c - 1 = (1-\eta)^{-1/3}\,|\theta|\left(\frac{2\,|\theta|+|\sin 2\theta|}{\sin^2\theta}\right)^{1/3}.\tag{4.30}$$

This estimate for the best value of N gives an excellent account for the examples plotted in Fig. 3.
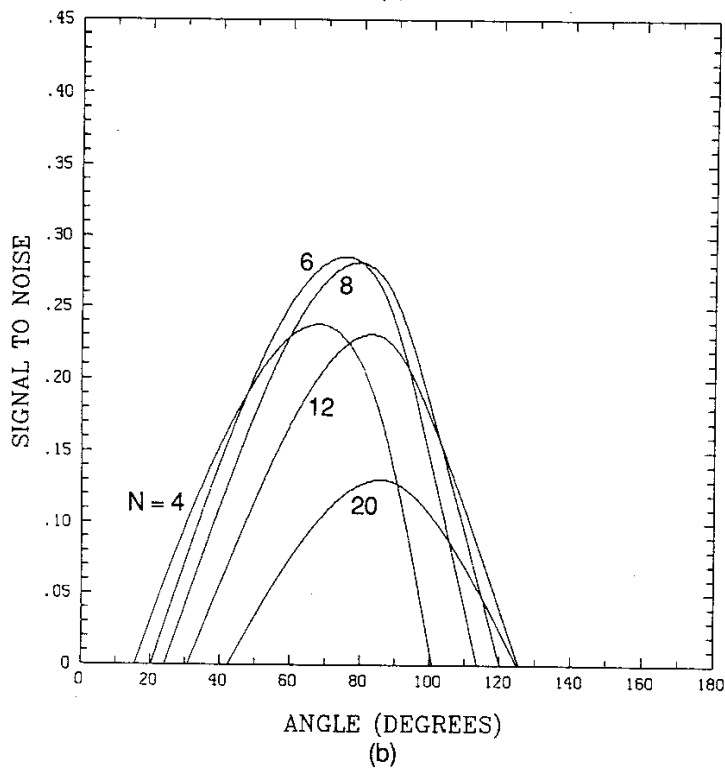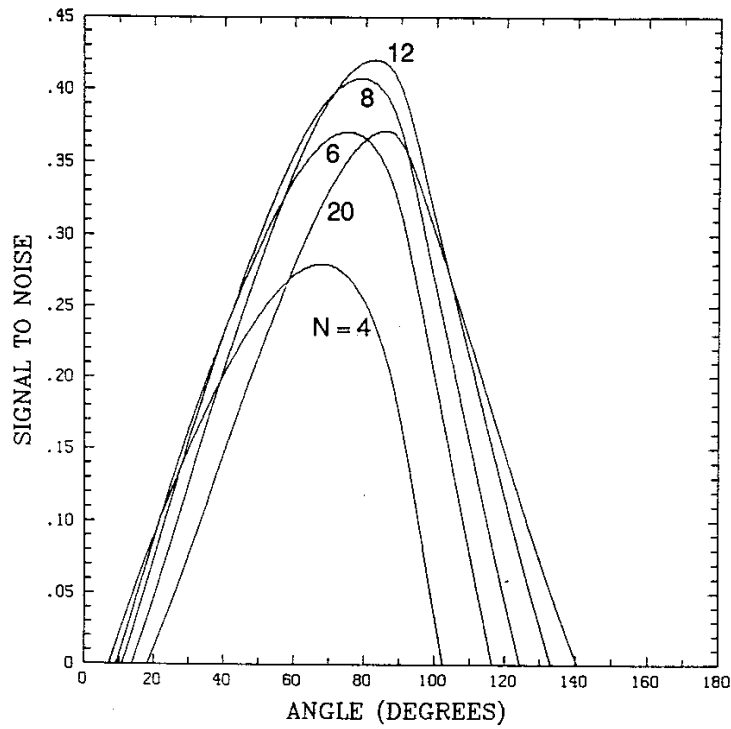
FIG. 3. Single-run signal-to-noise ratio $(S_N/N_N)_{\nu=1}$ [Eq. (4.26)] vs. angle θ in degrees for chainings $N = 4, 6, 8, 12,$ and 20 and for two cases of data error: (a) $\eta = 0.99$ and (b) $\eta = 0.955$. Data error decreases the beneficial effects of chaining, but even in case (b) the $N = 6$ and $N = 8$ chainings yield stronger violations than the CHSH inequality $(N = 4)$. (As in Fig. 2 part of the $N = 4$ curve is omitted.)

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5. Information Theory and Realism

Why introduce information concepts into a discussion of local realism? Suppose that, following the argument of Einstein, Podolsky, and Rosen [19] (Section 2), we believe that physical systems have local objective properties and that quantum mechanics must be extended to describe those properties. If we "know" the values assumed by all these properties, then conventional information theory tells us that the system carries no information. If, however, we believe in the statistical predictions of quantum mechanics—i.e., that however quantum mechanics is extended, the resulting theory should agree with the statistical predictions of quantum mechanics—then we certainly do not know the values assumed by all these properties. Then the system must carry information about the actual values—information that we obtain when we make measurements. In essence, the system must carry with it information about the results of all possible measurements whose results are not definite. Thus local realism, together with the statistical predictions of quantum mechanics, requires physical systems to carry an enormous baggage of information. Where and how is this information to be stored? Where in quantum theory is there any notion of this vast quantity of information?

In Section 6 we formulate information-theoretic Bell inequalities, which quantify these ideas for any pair of physically separated systems. If local realism holds, the two systems must carry an amount of information consistent with the information Bell inequality. We show that the quantum statistics of a pair of spin-s particles in a state of zero total spin violate our information Bell inequalities for all values of s. Quantum statistics simply do not allow systems to carry enough information to be consistent with the requirements of local realism.

As preparation for Section 6, we devote the rest of this section to a brief review of information theory [40, 41]. Consider two measurable quantities, denoted by capital letters A and B. In quantum theory these two quantities would be commuting observables; in a local realistic theory they could be any two objective properties. We label the (discrete) possible values of A and B by the corresponding lower-case letters, a and b. Based on one's knowledge about the quantities A and B, one assigns a joint probability $p(a, b)$ that A has value a and B has value b. Bayes's theorem,

$$p(a, b) = p(a \mid b) \, p(b) = p(b \mid a) \, p(a), \tag{5.1}$$

relates the joint probability to the conditional probabilities $p(a \mid b)$ (the probability that A has value a, given that B has value b) and $p(b \mid a)$ and to the single-quantity probabilities $p(b)$ (the probability that B has value b) and $p(a)$.

The information obtained when one discovers values a for A and b for B is

$$I(a, b) \equiv -\log p(a, b). \tag{5.2}$$

The base of the logarithm determines the units of the information (base 2 for bits, base e for nats). In the same way

$$I(b) \equiv -\log p(b) \tag{5.3}$$

is the information obtained when one discovers value b for B, and

$$I(a \mid b) \equiv -\log p(a \mid b) \qquad (5.4)$$

is the further information obtained when one discovers value a for A, provided one already knows the value b of B. Bayes's theorem, rewritten in terms of information, becomes

$$I(a, b) = I(a \mid b) + I(b) = I(b \mid a) + I(a). \qquad (5.5)$$

A crucial role is played by the mean information obtained when one finds values for A and B:

$$H(A, B) = \sum_{a, b} p(a, b) I(a, b). \qquad (5.6)$$

This mean information is the entropy of the probability $p(a, b)$ (up to a multiplicative constant, which is equivalent to a choice of units for information). It can also be thought of as the total information carried by the quantities A and B, defined relative to the knowledge about A and B that is used to make the probability assignment $p(a, b)$. In the same way

$$H(B) = \sum_{b} p(b) I(b) \qquad (5.7)$$

is the information carried by B, and

$$H(A \mid b) = \sum_{a} p(a \mid b) I(a \mid b) \qquad (5.8)$$

is the information carried by A, given the value b of B. It is useful to average $H(A \mid b)$ over B to obtain a conditional information carried by A,

$$H(A \mid B) = \sum_{b} p(b) H(A \mid b) = \sum_{a, b} p(a, b) I(a \mid b). \qquad (5.9)$$

An immediate consequence of Bayes's theorem is the relation

$$H(A, B) = H(A \mid B) + H(B) = H(B \mid A) + H(A). \qquad (5.10)$$

We require one further ingredient, the mutual information

$$I(a; b) \equiv I(a) - I(a \mid b) = I(b) - I(b \mid a) = I(b; a). \qquad (5.11)$$

This mutual information can be either positive or negative, but its mean,

$$H(A; B) = \sum_{a, b} p(a, b) I(a; b)$$

$$= \sum_{b} p(b) \left[\sum_{a} p(a \mid b) \log \left(\frac{p(a \mid b)}{p(a)} \right) \right] \geq 0, \qquad (5.12)$$

is non-negative [40] (Gibbs's theorem [41]). Equality holds in Eq. (5.12) if and only if A and B are statistically independent, i.e., $p(a, b) = p(a)\, p(b)$. The mean mutual information,

$$H(A; B) = H(A) - H(A \mid B) = H(B) - H(B \mid A) = H(B; A), \qquad (5.13)$$

is the information carried in common (mutually) by A and B; i.e., it is the average information one obtains about $A(B)$ when one finds a value for $B(A)$.

To establish our information-theoretic Bell inequalities, the only results we need from information theory are the inequalities

$$H(A \mid B) \leqslant H(A) \leqslant H(A, B). \qquad (5.14)$$

The left-hand inequality, a consequence of the non-negativity of the mean mutual information $H(A; B)$, means that removing a condition never decreases the information carried by a quantity. The right-hand inequality, a consequence of Eq. (5.10), means that two quantities never carry less information than each quantity carries separately.

6. Information Bell Inequalities

In this section we derive information-theoretic Bell inequalities [12, 13] for a pair of physically separated systems. These information Bell inequalities apply to any pair of physically separated systems—not just two-state systems—so we frame the derivations in a general language applicable to any pair of systems.

Consider, then, two physically separated systems, $\mathscr{A}$ and $\mathscr{B}$, and four measurable quantities—A and A' associated with $\mathscr{A}$, B and B' associated with $\mathscr{B}$. The (discrete) possible values of these quantities are denoted by a, a', b, and b'. In a quantum-mechanical description the two observables associated with each system would not commute and hence could not be determined simultaneously. Thus we have in mind a series of experimental runs, in each of which one measures two quantities, one from each system (as in a test of the CHSH inequality).

An example of this general formulation is the spin-s generalization [42] of Bohm's version [20] of the Einstein–Podolsky–Rosen [19] paradox. Two counter-propagating spin-s particles, $\mathscr{A}$ and $\mathscr{B}$, having spins $S_{\mathscr{A}}$ and $S_{\mathscr{B}}$ (in units of $\hbar$), are emitted by the decay of a zero angular-momentum particle and thus have zero total spin. Each particle is sent through a Stern–Gerlach apparatus, which measures a component of the particle's spin along one of two possible directions. For particle $\mathscr{A}$ the two observables are spin components $A = S_{\mathscr{A}} \cdot \mathbf{a}$ and $A' = S_{\mathscr{A}} \cdot \mathbf{a}'$, where unit vectors $\mathbf{a}$ and $\mathbf{a}'$ specify orientations of the Stern–Gerlach apparatus. Similarly, for particle $\mathscr{B}$, the two observables, $B = S_{\mathscr{B}} \cdot \mathbf{b}$ and $B' = S_{\mathscr{B}} \cdot \mathbf{b}'$, are specified by unit vectors $\mathbf{b}$ and $\mathbf{b}'$.

Suppose now that the four quantities A, A', B, B' are local objective properties of the two systems; i.e., in each run of the experiment, all four have definite values,

independent of observation. Suppose further that the two systems are physically separated, so that a measurement on one does not disturb the other. Then we can repeat in this general language the argument given in Section 2 and conclude that objectivity and locality—i.e., local realism—ensure that the statistics of the four quantities follow from a joint probability $p(a, a', b, b')$.

In terms of this joint probability we can define the total information

$$H(A, A', B, B') \equiv - \sum_{a, a', b, b'} p(a, a', b, b') \log p(a, a', b, b') \tag{6.1}$$

carried by the four quantities. An obvious generalization of the right-hand inequality in (5.14) yields

$$H(A, B) \leqslant H(A, A', B, B')$$

$$= H(A \mid B', A', B) + H(B' \mid A', B) + H(A' \mid B) + H(B), \tag{6.2}$$

where we use a generalization of Eq. (5.10) to expand the right-hand side. The right-hand side involves probabilities of noncommuting observables and hence could not be defined in quantum mechanics. We can, however, use a slight generalization of the left-hand inequality in (5.14) to eliminate conditions, i.e.,

$$H(A \mid B', A', B) \equiv - \sum_{a, a', b, b'} p(a, a', b, b') \log p(a \mid a', b, b') \leqslant H(A \mid B'), \tag{6.3}$$

$$H(B' \mid A', B) \equiv - \sum_{a', b, b'} p(a', b, b') \log p(b' \mid a', b) \leqslant H(B' \mid A'). \tag{6.4}$$

Subtracting $H(B)$ from both sides of Eq. (6.2), we obtain the desired information Bell inequality [12, 13]

$$H(A \mid B) \leqslant H(A \mid B') + H(B' \mid A') + H(A' \mid B). \tag{6.5}$$

The four pieces of conditional information in this Bell inequality involve pair probabilities that are defined in quantum mechanics; they can be determined from the statistics of the four types of experimental runs. Note that the information Bell inequality (6.5) has a form closely analogous to the form of the CHSH correlation Bell inequality (2.2).

The information Bell inequality (6.5) follows directly from the assumption of a joint probability $p(a, a', b, b')$; it applies to any four quantities whose statistics can be derived from such a joint probability. Objectivity and locality are meant to compel belief in the existence and relevance of this joint probability. The content of the information Bell inequality lies in the first step (6.2) of the derivation: four objective quantities cannot carry less information than any two of them.

Zurek [43] has recently introduced an information distance

$$\delta(A, B) \equiv H(A \mid B) + H(B \mid A), \tag{6.6}$$

which satisfies the conditions—in particular, the triangle inequality—to be called a distance between (equivalence classes for) A and B. Schumacher [44] has used the information distance to derive information Bell inequalities. Indeed, as Schumacher notes, the "quadrilateral inequality" for this distance,

$$\delta(A, B) \leqslant \delta(A, B') + \delta(B', A') + \delta(A', B), \qquad (6.7)$$

which follows directly from the triangle inequality, is a consequence of symmetrizing the information Bell inequality (6.5).

We return now to the two spin-s particles introduced above to explore whether quantum systems carry the requisite amount of information. We use conventional notation in which the $2s + 1$ possible values of $A = \mathbf{S}_{\mathscr{A}} \cdot \mathbf{a}$ and $A' = \mathbf{S}_{\mathscr{A}} \cdot \mathbf{a}'$, labeled above by a and a', are denoted by quantum numbers m_a and $m_{a'}$, which take on values $-s$, $-s + 1$, ..., $s - 1$, s. The eigenstate of spin component $\mathbf{S}_{\mathscr{A}} \cdot \mathbf{e}$ with eigenvalue m, where $\mathbf{e}$ is an arbitrary unit vector, is denoted by $|s, m\rangle_{\mathscr{A}, \mathbf{e}}$. Similar notation applies to particle $\mathscr{B}$.

The quantum statistics are derived from the state of zero total spin [42]

$$|\phi\rangle = (2s + 1)^{-1/2} \sum_{m = -s}^{s} (-1)^{s - m} |s, m\rangle_{\mathscr{A}, \mathbf{e}} \otimes |s, -m\rangle_{\mathscr{B}, \mathbf{e}}, \qquad (6.8)$$

where the quantization axis $\mathbf{e}$ is arbitrary. This state describes a situation in which spin components of the two particles in the same direction are perfectly anticorrelated. Quantum mechanics predicts the probability

$$p(m_a, m_b) = |_{\mathscr{A}, \mathbf{a}}\langle s, m_a| \otimes _{\mathscr{B}, \mathbf{b}}\langle s, m_b| |\phi\rangle|^2$$

$$= \frac{1}{2s + 1} |_{\mathscr{A}, \mathbf{a}}\langle s, m_a | s, -m_b\rangle_{\mathscr{A}, \mathbf{b}}|^2 \qquad (6.9)$$

that $A = \mathbf{S}_{\mathscr{A}} \cdot \mathbf{a}$ has value m_a and $B = \mathbf{S}_{\mathscr{B}} \cdot \mathbf{b}$ has value m_b. Rotational invariance guarantees that $p(m_a, m_b)$ depends only on the angle θ between $\mathbf{a}$ and $\mathbf{b}$. We can put $p(m_a, m_b)$ in the form [42]

$$p(m_a, m_b) = \frac{1}{2s + 1} |D_{m_a, -m_b}(R_{\mathbf{n}}(\theta))|^2, \qquad (6.10)$$

where

$$D_{m_a, -m_b}(R_{\mathbf{n}}(\theta)) \equiv _{\mathscr{A}, \mathbf{e}}\langle s, m_a| e^{-i\theta \mathbf{n} \cdot \mathbf{S}_{\mathscr{A}}} |s, -m_b\rangle_{\mathscr{A}, \mathbf{e}} \qquad (6.11)$$

is a matrix element for a rotation $R_{\mathbf{n}}(\theta)$ by angle θ about any unit vector $\mathbf{n}$ orthogonal to an arbitrary quantization axis $\mathbf{e}$. The quantum-mechanical prediction for the conditional information,

$$H(A \mid B) = H(B \mid A) \equiv H(\theta), \qquad (6.12)$$

depends only on the angle θ between **a** and **b** and takes the form

$$H(\theta) = -\frac{1}{2s+1} \sum_{m_a, m_b} |D_{m_a, -m_b}(R_n(\theta))|^2 \log |D_{m_a, -m_b}(R_n(\theta))|^2. \quad (6.13)$$

Symmetries of $p(m_a, m_b)$ imply that $H(\theta) = H(\theta + \pi) = H(-\theta) = H(\pi - \theta)$.

It is useful to note the small-angle form of the quantum-mechanical conditional information,

$$H(\theta) \simeq \frac{s(s+1)}{3} \theta^2 [\log_2(4/\theta^2) + \log_2 e - F(s)] \text{ bits}, \qquad s|\theta| \ll 1. \quad (6.14)$$

Here $H(\theta)$ is given explicitly in bits, and $F(s)$ is the series

$$F(s) \equiv \frac{3}{s(s+1)(2s+1)} \sum_{k=1}^{2s} k(2s+1-k) \log_2 k \geqslant 0 \quad (6.15)$$

[note that $F(\frac{1}{2}) = 0$]. For large s an integral approximation to the series $[F(s) \sim \log_2 4s^2 - \frac{5}{3} \log_2 e]$ yields an asymptotic form

$$H(\theta) \sim \frac{1}{3}(s\theta)^2 [\log_2(1/(s\theta)^2) + \frac{8}{3} \log_2 e] \text{ bits}, \quad (6.16)$$

valid for large s and small θ.

Consider now the canonical case that was applied to the CHSH inequality in Section 2: the unit vectors **a**, **b'**, **a'**, **b** are coplanar, and successive vectors in the list are separated by angle $\theta/3$, so that **a** and **b** are separated by angle θ. The information Bell inequality (6.5) is violated if the information difference

$$\mathscr{H}(\theta) \equiv H(\theta) - 3H(\theta/3) \quad (6.17)$$

becomes positive. A positive value for $\mathscr{H}(\theta)$ gives directly the deficit of information carried by the two particles, relative to the requirements of local realism for this geometry.

The conditional information $H(\theta)$, which is intrinsically non-negative, goes to zero at $\theta = 0$ because of the perfect anticorrelation of spin components in the same direction. Satisfying the information Bell inequality (6.5) (i.e., $\mathscr{H} \leqslant 0$) would require that $H(\theta)$ not go to zero faster than linearly in θ. Thus the $-\theta^2 \log \theta^2$ behavior of $H(\theta)$ at small angles violates the Bell inequality (6.5) for all values of s, as is evident from the small-angle forms of $\mathscr{H}(\theta)$:

$$\mathscr{H}(\theta) \simeq \frac{2}{9} s(s+1)\theta^2 [\log_2(4/3\theta^2) + \log_2 e - F(s)]$$

$$\sim \frac{2}{9}(s\theta)^2 [\log_2(1/3(s\theta)^2) + \frac{8}{3} \log_2 e] \text{ bits}. \quad (6.18)$$

The first form assumes only $s|\theta| \ll 1$; the second is the asymptotic form for large s and small θ. The $-\theta^2 \log \theta^2$ behavior of $H(\theta)$ at small angles reflects the tight

quantum-mechanical correlation between the spins: knowing the value of $B = S_{\mathscr{B}} \cdot \mathbf{b}$ tells one so much about $A = S_{\mathscr{A}} \cdot \mathbf{a}$ for $\mathbf{a}$ near $\mathbf{b}$ that very little information is gained by determining the value of $A = S_{\mathscr{A}} \cdot \mathbf{a}$—so little as to violate the requirements of local realism.

The biggest surprise in Eq. (6.18) is not the presence of a violation for all s, but rather the increasing size of the violation as s increases for fixed θ. We investigate this surprising feature by calculating the information difference $\mathscr{H}(\theta)$ (in bits) for $s = \frac{1}{2}$, 1, 2, 5, and 25. The matrix $D_{m_a, -m_b}(R_n(\theta))$ is obtained by using a formula due to Wigner [45, Eq. (C.72)]. The results, displayed in Fig. 4, indicate that the maximum information deficit increases with increasing s, but the range of angles over which there is a violation decreases.

Several investigators [21, 42, 46–51] have formulated Bell inequalities for a pair of spin-s particles. Notable among these inequalities is one derived by Garg and
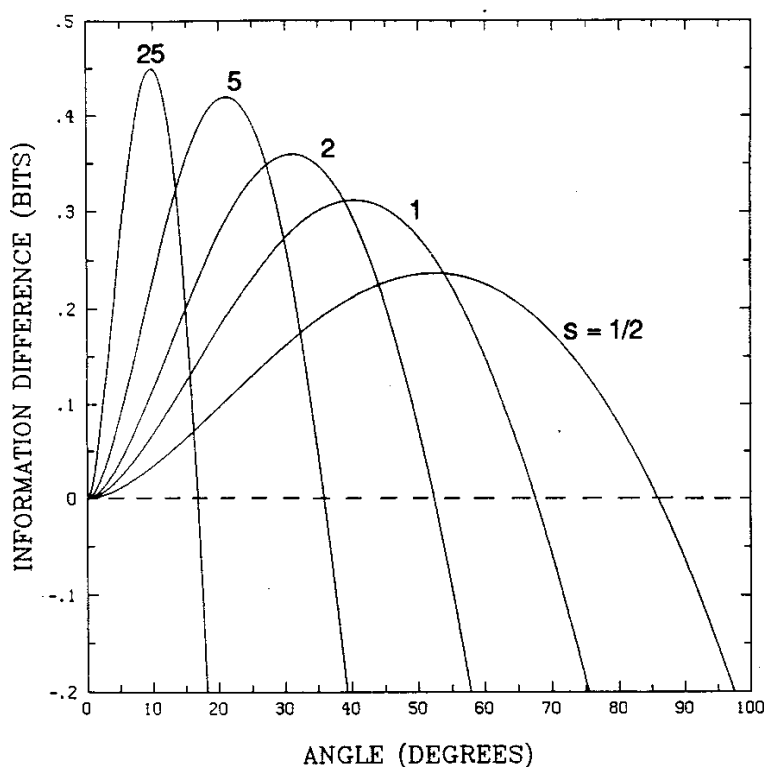

FIG. 4. Information difference $\mathscr{H}(\theta)$ [Eq. (6.17)] in bits vs. angle θ in degrees for spins $s = \frac{1}{2}$, 1, 2, 5, and 25. The information Bell inequality (6.5) is violated whenever the information difference is positive, the positive value giving the deficit of information carried by the two particles, relative to the requirements of local realism. The maximum information deficit increases with increasing spin, but the range of angles over which there is a violation decreases. The $s = \frac{1}{2}$ plot can be applied to a photon polarization experiment of the sort discussed in Sections 3 and 4 by letting the abscissa be twice the angle between the outermost polarization analyzers.

Mermin [48, 49, 51], which in the geometry considered here is violated by the quantum-mechanical predictions for a state of zero total spin for all angles $\theta \leqslant 180°$ and for all values of the spin.

We have applied chaining to the information Bell inequality (6.5) in the same way that we chained the CHSH Bell inequality in Section 4 [12]. For the case of spin-$\frac{1}{2}$ particles we have gone further to investigate whether the chained information Bell inequalities lead to stronger quantum violations than does the simple information Bell inequality (6.5). To characterize the strength of violation, we use a signal-to-noise ratio calculated using the model developed in Section 4. Although we find that chaining does lead to stronger violations when the data error is sufficiently small, we do not present the results here, because they show that information Bell inequalities lose out to correlation Bell inequalities on all experimental counts: information Bell inequalities have smaller violations over a smaller range of angles, and they garner less benefit from chaining. Hence, we believe that the most important function of information Bell inequalities is as theoretical tools for investigating the meaning of Bell inequalities—a task to which we set them in Section 7.

7. FOR WHOM DOES THE BELL INEQUALITY TOLL

It seems worthwhile trying to tie down the role of Bell inequalities. They provide a criterion for the existence of local objective properties, but are they special in this regard? Are there other, perhaps simpler tests for objectivity? Feynman [52], for instance, has argued that the two-slit experiment, in which electrons form an interference pattern on a screen, contains the essence of quantum mechanics. He argues that the interference pattern cannot be derived in general from any conditional probabilities $p(x \mid +1)$ and $p(x \mid -1)$ that the electron arrives at point x on the screen, given that it passed through slit $+1$ or slit -1. He concludes that there are no objective paths for the electron.

So what is special about Bell inequalities if the two-slit experiment already rules out objective properties? The immediate problem is that the Feynman argument does not provide a clear-cut criterion for objectivity. If one tries to formulate such a criterion, as we do below, one confronts the difficulty that the probabilities $p(x \mid +1)$ and $p(x \mid -1)$ posited by Feynman are probabilities involving noncommuting observables (electron position at the screen and electron position at the slits); such probabilities are not defined in quantum mechanics, so it is not surprising that a quantum description is inconsistent with an objective description that posits such probabilities. If one tries to *measure* the probabilities $p(x \mid +1)$ and $p(x \mid -1)$, one must modify the experimental apparatus so radically—perhaps by blocking one slit or the other—that the interference pattern is destroyed. We can never be sure that the inconsistency pointed out by Feynman is not due to our having "disturbed" the electron's final position on the screen by our measurement of which slit it went through.

The surprising feature of the standard Bell inequalities is that they avoid this

difficulty. They provide a criterion for objectivity that involves only probabilities for pairs of commuting observables—thus probabilities that are defined in quantum mechanics—even though the criterion is derived from a "grand" joint probability that is not defined in quantum mechanics. It is perhaps worth considering more fully this contrast between Feynman's argument, which attempts a criterion for objectivity for a single system, and Bell inequalities, which involve two physically separated systems. Information Bell inequalities provide a natural vehicle for such consideration.

Bell inequalities, we claim, come from assuming the existence of a grand point probability for a set of quantities that do not commute in quantum mechanics. One derives from the grand probability—using classical probability logic—an inequality (involving correlation functions, information, etc.) and shows that for some quantum state, the inequality is violated. Thus, strictly speaking, what all Bell inequalities test is whether some aspect of quantum statistics can be derived from a grand probability. This is the point of view promoted by Garg and Mermin [21], who start with marginal probabilities predicted by quantum mechanics and show that they cannot be derived from higher-order grand probabilities.

The advantage of information Bell inequalities is that they provide a straightforward, yet general framework for exploring the meaning of Bell inequalities. An information Bell inequality can be derived easily for any set of physical systems. In contrast, a correlation Bell inequality (such as the CHSH inequality) requires a separate derivation for each new set of systems. Of course, information Bell inequalities do no ferret out all the weird quantum behavior that is inconsistent with objectivity. To do that, the right approach is, we believe, the Garg–Mermin approach [21]. Information Bell inequalities are useful nonetheless, because they are mathematically straightforward compared to the Garg–Mermin approach.

One would like to think that Bell inequalities test something more cosmic than the existence of some grand probability. Thus, one tries to attach an *interpretation* to Bell inequalities. The first part of the interpretation is objectivity (or realism). If the quantities dealt with are objective, then they have definite values, independent of observation. Although we do not know these values, in any realistic description the knowledge we do have is incorporated in a probability assignment—the grand probability. The existence of the grand probability is thus interpreted as a consequence of objectivity.

If this were the whole story, then all Bell inequalities would be on the same footing. One must make further no-disturbance assumptions, however, to relate the grand probability to statistics of actual measurements. One can imagine that in a realistic theory measurements of one or more of the quantities so disturb the system that the statistics of remaining quantities are no longer those that would be inferred from the grand probability. Indeed, this is how one maintains objectivity in a naive realistic interpretation of quantum mechanics. How compelling the no-disturbance assumptions are determines how convincing the Bell inequality is as a criterion for objectivity.

To illustrate these ideas, consider a single system and two measurable quantities A and A', which do not commute in quantum mechanics. Label the possible values of these quantities by a and a'. The trivial one-system information Bell inequality,

$$H(A) = H(A \mid A') + H(A; A') \geqslant H(A \mid A'), \tag{7.1}$$

comes from assuming a joint probability $p(a, a')$ for A and A' $[H(A; A') \geqslant 0$ is the mean mutual information (5.12)]. Examples include (i) the two-slit experiment with $A' = $(position at screen with slits)$ = \pm 1$ and $A = $(discretized position at detecting screen), and (ii) a spin-$\frac{1}{2}$ particle with $A' = (y$-spin) and $A = (z$-spin). Feynman [52] claims that the two-slit experiment contains the essence of quantum mechanics, so one might think that this Bell inequality is enough. Nonetheless, as we have argued, there is a serious—indeed fatal—objection to it.

The conditional probability $p(a \mid a')$ that goes into $H(A \mid A')$ has no meaning in quantum mechanics until one specifies a procedure for measuring it. Suppose that $|\psi\rangle$ is the initial state of the system; from it one calculates a probability $p(a) = |\langle a \mid \psi \rangle|^2$, which determines $H(A)$, and a probability $p(a') = |\langle a' \mid \psi \rangle|^2$, which appears in $H(A \mid A')$. Suppose we make the natural assumption that $p(a \mid a')$, which also appears in $H(A \mid A')$, is the probability for a, given that a *measurement* of A' yielded result a'. This probability is obtained by collapsing the wave function to be $|a'\rangle$ and setting $p(a \mid a') = |\langle a \mid a' \rangle|^2$. The procedure having been specified, it is obvious that there are quantum states in the above-mentioned examples for which inequality (7.1) is violated [for instance, if $|\psi\rangle$ is an eigenstate of z-spin, $H(A) = 0$ and $H(A \mid A') = \log 2$]. Just as obvious, however, is that this violation occurs because $p(a)$ is not derivable from the constructed joint probability $p(a, a') = p(a \mid a') p(a')$ (this is essentially Feynman's argument):

$$\left| \sum_{a'} \langle a \mid a' \rangle \langle a' \mid \psi \rangle \right|^2 = p(a) \neq \sum_{a'} p(a, a') = \sum_{a'} |\langle a \mid a' \rangle|^2 \, |\langle a' \mid \psi \rangle|^2.$$

$$\tag{7.2}$$

We pursue a slightly different line of reasoning if we insist on a realistic description of A and A'. We can always introduce a joint probability $\tilde{p}(a, a')$ that yields the observed $p(a)$ and $p(a')$. Clearly no such $\tilde{p}(a, a')$ can give the conditional probability $p(a \mid a')$ obtained by the above measurement procedure—i.e., $p(a \mid a') \neq \tilde{p}(a, a')/p(a')$. We conclude that the system is disturbed by the measurement of A', so that the conditional statistics of A, given result a', are not those predicted by $\tilde{p}(a, a')/p(a')$. Indeed, this is precisely what is meant by a disturbance of A—that the conditional statistics of A are changed in some way beyond any possible correlation built into $\tilde{p}(a, a')$. We would further conclude that the violation of inequality (7.1) is a consequence of this disturbance, which means that the two sides of the inequality are calculated from different joint probabilities. In quantum mechanics the "disturbance" (quotes because one should think twice before speaking of a disturbance if there are no objective properties to disturb) comes from

using amplitude logic instead of probability logic (wave-function collapse instead of Bayes's theorem).

One-system Bell inequalities fail as tests of objectivity because they rely on two assumptions: objectivity to ensure the existence of the joint probability and a no-disturbance assumption to ensure that the statistics of actual measurements follow from the joint probability. Experimental violation of a one-system Bell inequality rules out realistic theories with no measurement disturbance. One can always maintain objectivity in the face of a violation simply by blaming the violation on a disturbance. Leggett and Garg [53] have proposed a one-system Bell inequality as a test of realism; they justify a no-disturbance assumption ("non-invasive measurability") on the grounds that the system they consider—a SQUID—is "macroscopic." Ballentine [54] has argued from a simple model and Peres [55] on general grounds that there must be a disturbance—regardless of how macroscopic the SQUID is—if the SQUID is to display the quantum effects that give rise to a violation of the Leggett–Garg inequality.

To overcome this flaw in one-system Bell inequalities, one needs a criterion for objectivity in which all the probabilities that appear are defined in quantum mechanics—i.e., they are probabilities for commuting observables; then one need not specify a measuring procedure that might "disturb" the system. If one tries to modify inequality (7.1) by including an apparatus that measures A', one finds that the inequality is always satisfied by A and the apparatus observable, because A and the apparatus observable *do* commute and the joint probability *does* exist in quantum mechanics. The significance of two-system Bell inequalities is that they do provide the desired criterion.

Consider two systems and four measurable quantities—A and A' on the first system, B and B' on the second. The two-system information Bell inequality (6.5),

$$H(A \mid B) \leqslant H(A \mid B') + H(B' \mid A') + H(A' \mid B),$$ (7.3)

is a consequence of assuming a joint probability $p(a, a', b, b')$ for objective quantities A, A', B, B'. This two-system Bell inequality relies on objectivity and a no-disturbance assumption, but since it involves only measurements of pairs of commuting observables, in a quantum-mechanical description it requires no measurements in which a measurement of one observable "disturbs" the other. Thus violation by a quantum-mechanical prediction means that quantum mechanics is not objective, but an experimental violation does not rule out realistic theories because a realistic theory might have the required disturbances—a measurement of B might disturb A. This is where locality comes in—not because of quantum mechanics, but because of realistic theories. If the two systems are physically separated, so that no influence can pass between them, one is tempted to say that no measurement on one system could possibly disturb the other. Thus one ends up saying that the Bell inequality (7.3) for physically separated systems is a consequence of objectivity *and* locality; any theory that violates (7.3) for physically separated systems violates objectivity *or* locality (or both).

It is apparent from the above discussion that we believe that quantum mechanics

violates objectivity. There is no nonlocal disturbance in quantum mechanics; the only "disturbance" is local—measure A', "disturb" A. Quantum mechanics is able to violate the Bell inequality (7.3) for physically separated systems because it is not objective; realistic hidden-variable theories are able to violate it by being nonlocal. The point is that if the statistics are those of quantum mechanics, either objectivity or locality must be sacrificed.

One-system Bell inequalities are not compelling because it is necessary to disturb the system before one can obtain all the information to check them. Two-system Bell inequalities can avoid this problem, because they can be phrased solely in terms of commuting quantities. Why stop here? There must be Bell inequalities for three or more systems. We could, for example, consider a three-body decay in which each particle would be asssumed to have local objective properties. We could use our general information techniques to formulate an information Bell inequality. There is considerably more freedom in formulating information Bell inequalities for three or more systems than for two; an example of a three-system Bell inequality, which involves only probabilities that are defined in quantum mechanics, is

$$H(A_1, B_1 \mid C_1) \leqslant H(A_1, B_1 \mid C_2) + H(C_2, A_2 \mid B_2)$$

$$+ H(B_2, C_3 \mid A_3) + H(A_3, B_3 \mid C_1), \tag{7.4}$$

where A, B, and C denote properties of the three separate systems. Whether this Bell inequality or any other higher-order Bell inequality is violated by an appropriate quantum-mechanical system is an open question. At the moment this generalization apparently leads to a complicated mess (especially experimentally), with no obvious gains over two-system Bell inequalities—though it is conceivable that some special correlated many-body state could in the future yield a simplified test of local realism. Although we recognize the existence of a hierarchy of Bell inequalities for three or more systems, it seems at present that two-system Bell inequalities are special in that they yield the simplest and most compelling tests of local realism.

Undoubtedly, many will continue to regard the two-slit experiment as important evidence that quantum mechanics is inconsistent with realism, but a purist will argue that a two-system Bell inequality provides a more compelling test. Equally, a purist will demand that *all* aspects of local realism be subjected to experimental test, for to a purist local realism is not a monolith which crumbles at the discovery of the first flaw. No purist would argue that an experimental test of a Bell inequality for spin components of a two-state system proves anything about the local realistic character of other properties. Indeed, if local realism is a monolith, then surely work on Bell inequalities qualifies to be a "dehydrated elephant" (to borrow a phrase from M. Kac [56]) by which we mean a field of study whose significance has been greatly exaggerated. In this context, information Bell inequalities gain a new vitality, because they can be formulated for an arbitrary set of properties of arbitrary systems, thus expanding the range of properties for which requirements of local realism can be formulated.

ACKNOWLEDGMENTS

REFERENCES

1. J. S. BELL, *Physics (N.Y.)* 1 (1964), 195.
2. J. F. CLAUSER AND A. SHIMONY, *Rep., Prog. Phys.* 41 (1978), 1881.
3. F. SELLERI AND G. TAROZZI, *Riv. Nuovo Cimento* 4 (1981), 1.
4. B. D'ESPAGNAT, *Phys. Rep.* 110 (1984), 201.
5. W. DE BAERE, *Adv. Electron. Electron Phys.* 68 (1986), 245.
6. L. E. BALLENTINE, *Amer. J. Phys.* 55 (1987), 785.
7. N. D. MERMIN, *Phys. Today* 38, No. 4 (April 1985), 38.
8. J. F. CLAUSER, M. A. HORNE, A. SHIMONY, AND R. A. HOLT, *Phys. Rev. Lett.* 23 (1969), 880.
9. A. ASPECT, P. GRANGIER, AND G. ROGER, *Phys. Rev. Lett.* 49 (1982), 91.
10. A. ASPECT, J. DALIBARD, AND G. ROGER, *Phys. Rev. Lett.* 49 (1982), 1804.
11. A. ASPECT AND P. GRANGIER, in "Foundations of Quantum Mechanics" (S. Kamefuchi et al., Eds.), p. 214, Physical Society of Japan, Tokyo, 1983.
12. S. L. BRAUNSTEIN AND C. M. CAVES, *Phys. Rev. Lett.* 61 (1988), 662.
13. S. L. BRAUNSTEIN AND C. M. CAVES, in "Proceedings of the International Symposium on Spacetime Symmetries" (Y. S. Kim and W. W. Zachary, Eds.), *Nucl. Phys. B* (Proceedings Supplements Section) 6 (1989), 211.
14. S. L. BRAUNSTEIN AND C. M. CAVES, in "Bell's Theorem, Quantum Theory and Conceptions of the Universe" (M. Kafatos, Ed.), p. 27, Kluwer, Dordrecht, Holland, 1989.
15. P. H EBERHARD, *Nuovo Cimento B* 38 (1977), 75.
16. A. PERES AND W. H. ZUREK, *Amer. J. Phys.* 50 (1982), 807.
17. "The Many-Worlds Interpretation of Quantum Mechanics" (B. DeWitt and N. Graham, Eds.), Princeton Univ. Press, Princeton, NJ, 1973.
18. E. T. JAYNES, "Papers on Probability, Statistics and Statistical Physics" (R. D. Rosenkrantz, Ed.), Reidel, Dordrecht/Boston, 1983.
19. A. EINSTEIN, B. PODOLSKY, AND N. ROSEN, *Phys. Rev.* 47 (1935), 777.
20. D. BOHM, "Quantum Theory," pp. 611–623, Prentice–Hall, Englewood Cliffs, NJ, 1951.
21. A. GARG AND N. D. MERMIN, *Found. Phys.* 14 (1984), 1.
22. A. FINE, *Phys. Rev. Lett.* 48 (1982), 291.
23. A. FINE, *J. Math. Phys.* 23 (1982), 1306.
24. J. F. CLAUSER AND M. A. HORNE, *Phys. Rev. D* 10 (1974), 526.
25. W. M. DE MUYNCK, *Phys. Lett. A* 114 (1986), 65.
26. H. P. STAPP, *Phys. Rev. Lett.* 49 (1982), 1470.
27. H. P. STAPP, *Amer. J. Phys.* 53 (1985), 306.
28. A. ASPECT, P. GRANGIER, AND G. ROGER, *Phys. Rev. Lett.* 47 (1981), 460.
29. P. M. PEARLE, *Phys. Rev. D* 2 (1970), 1418.
30. T. W. MARSHALL, E. SANTOS, AND F. SELLERI, *Phys. Lett. A* 98 (1983), 5.
31. D. HOME AND T. W. MARSHALL, *Phys. Lett. A* 113 (1985), 183.
32. S. PASCAZIO AND J. REIGNIER, *Phys. Lett. A* 126 (1987), 163.
33. C. O. ALLEY AND Y. H. SHIH, in "Foundations of Quantum Mechanics II" (M. Namiki et al., Eds.), p. 47, Physical Society of Japan, Tokyo, 1987.
34. Y. H. SHIH AND C. O. ALLEY, *Phys. Rev. Lett.* 61 (1988), 2921.
35. C. K. HONG, Z. Y. OU, AND L. MANDEL, *Phys. Rev. Lett.* 59 (1987), 2044.
36. Z. Y. OU AND L. MANDEL, *Phys. Rev. Lett.* 61 (1988), 50.
37. A. GARUCCIO AND F. SELLERI, *Found. Phys.* 10 (1980), 209.
38. B. MISRA AND E. C. G. SUDARSHAN, *J. Math. Phys.* 18 (1977), 756.

39. A. PERES, *Amer. J. Phys.* **48** (1980), 931.
40. R. G. GALLAGER, "Information Theory and Reliable Communication," Chap. 2, Wiley, New York, 1968.
41. S. WATANABE, "Knowing and Guessing: A Quantitative Study of Inference and Information," Chap. 1, Wiley, New York, 1969.
42. N. D. MERMIN, *Phys. Rev. D* **22** (1980), 356.
43. W. H. ZUREK, *Nature (London)* **341** (1989), 119.
44. B. W. SCHUMACHER, *Phys. Rev. A*, in press.
45. A. MESSIAH, "Quantum Mechanics," Vol. II, p. 1072, North-Holland, Amsterdam, 1968.
46. N. D. MERMIN AND G. M. SCHWARZ, *Found. Phys.* **12** (1982), 101.
47. S. BERGIA AND F. CANNATA, *Found. Phys.* **12** (1982), 843.
48. A. GARG AND N. D. MERMIN, *Phys. Rev. Lett.* **49** (1982), 901.
49. A. GARG AND N. D. MERMIN, *Phys. Rev. D* **27** (1983), 339.
50. M. ÖGREN, *Phys. Rev. D* **27** (1983), 1766.
51. N. D. MERMIN, *in* "Fundamental Questions in Quantum Mechanics" (L. M. Roth and A. Inomata, Eds.), p. 7, Gordon and Breach, New York, 1986.
52. R. P. FEYNMAN, R. B. LEIGHTON, AND M. SANDS, "The Feynman Lectures on Physics," Vol. III, Chap. 1, Addison–Wesley, Reading, MA, 1965.
53. A. J. LEGGETT AND A. GARG, *Phys. Rev. Lett.* **54** (1985), 857.
54. L. E. BALLENTINE, *Phys. Rev. Lett.* **59** (1987), 1493.
55. A. PERES, *Phys. Rev. Lett.* **61** (1988), 2019.
56. M. KAC, *Q. Appl. Math.* **30** (1972), 17.