Multi-dimensional Hermite polynomials in quantum optics

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Abstract
We study a class of optical circuits with vacuum input states consisting of
Gaussian sources without coherent displacements such as down-converters
and squeezers, together with photo-detectors and passive interferometry
(beamsplitters, polarization rotations, phase-shifters, etc). We show that the
outgoing state leaving the optical circuit can be expressed in terms of so-
called multi-dimensional Hermite polynomials and give their recursion and
orthogonality relations. We show how quantum teleportation of single-photon
polarization states can be modelled using this description.

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Linear optics is widely used in experimental quantum information and communication. In
particular, optical circuits using parametric down-conversion are used to study Bell states and
their applications. Recently, Lütkenhaus *et al* [1] and Vaidman and Yoran [2] have established
limits on the recognition of Bell states using linear optics. Furthermore, Knill *et al* [3] and
Kok and Braunstein [4] have studied the creation of Bell states with linear optical circuits. In
this paper, we present a tool for further analysis of these optical circuits. We give a description
of the outgoing state for a special class of optical circuits with a special class of input states.

First, in section 1, we define this class of optical circuits and show that they can be
described by so-called multi-dimensional Hermite polynomials (MDHPs). In section 2, we
give an example of this description. Section 3 discusses the Hermite polynomials, and finally,
in section 4, we briefly consider the effect of imperfect detectors on the outgoing state.

1. The optical circuit

What do we mean by an optical circuit? We can think of a black box with incoming and
outgoing modes of the electromagnetic field. The black box transforms a state of the incoming
modes into a (different) state of the outgoing modes. It is what we call an *optical circuit*.

We can now take a more detailed look inside the black box. We will consider three types of components.

First, the modes might be mixed by beamsplitters, or they may pick up a relative phase shift or polarization rotation. These operations all belong to a class of optical components which do not mix the creation and annihilation operators of the modes: $\hat{b}_j = \sum_k U_{jk}\hat{a}_k$, with U an arbitrary unitary matrix. We call them *passive* optical components.

Second, we may find optical components such as lasers, down-converters or (optical) parametric amplifiers in the black box. These components include photon sources, since they do not necessarily leave the photon number invariant. We will call these components *active* optical components (an example of an active component which does leave the total photon number invariant is a Kerr cell).

And finally, the box will generally include measurement devices, the outcomes of which may modify optical components on the remaining modes, depending on the detection outcomes. This is called *feed-forward* detection. Here we shall simplify optical circuits using feed-forward detection by considering the family of fixed circuits corresponding to the set of measurement outcomes (see also [4]). In addition, we can postpone the measurement to the end, where all the optical components have 'acted' on the modes.

These three component types have their own characteristic mathematical description. A passive component yields a unitary evolution U_i, which can be written as

$$U_i = \exp\left(-i\kappa \sum_{jk} c_{jk}\hat{a}_j\hat{a}_k^\dagger - \text{H.c.}\right) \tag{1}$$

where H.c. denotes the Hermitian conjugate. This unitary evolution commutes with the total number operator $\hat{n} = \sum_j \hat{a}_j^\dagger \hat{a}_j$. The creation and annihilation operators $\hat{a}_i^\dagger$ and $\hat{a}_i$ for mode i satisfy the standard canonical commutation relations:

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \qquad \text{and} \qquad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \tag{2}$$

with $i, j = 1, \ldots, N$.

Active components also correspond to unitary transformations, which can be written as $\exp(-it_j H_I^{(j)})$. Here $H_I^{(j)}$ is the interaction Hamiltonian associated with the jth active component in a sequence and t_j is the *duration* of the interaction. This Hamiltonian does not necessarily commute with the total number operator. To make a typographical distinction between passive and active components, we denote the ith passive component by U_i and the jth active component by its evolution in terms of the interaction Hamiltonian.

The mathematical description of the (ideal) measurement will correspond to taking the inner product of the outgoing state prior to the measurement with the eigenstate corresponding to the measurement.

Now that we have the components of an optical circuit of N modes, we have to combine them into an actual circuit. Mathematically, this corresponds to applying the unitary evolutions of the successive components to the input state. Let $|\psi_{\text{in}}\rangle$ be the input state and $|\psi_{\text{prior}}\rangle$ the output state *prior* to the measurement. We then have (with $K > 0$ some integer)

$$|\psi_{\text{prior}}\rangle = U_K e^{-it_K H_I^{(K)}} \ldots U_1 e^{-it_1 H_I^{(1)}} U_0 |\psi_{\text{in}}\rangle \tag{3}$$

where it should be noted that U_i might be the identity operator $\mathbb{I}$ or a product of unitary transformations corresponding to passive components:

$$U_i = \prod_k U_{i,k}. \tag{4}$$

When the (multi-mode) eigenstate corresponding to the measurement outcome for a limited set of modes labelled $1, \ldots, M$, with $M < N$, is given by $|\gamma\rangle = |n_1, n_2, \ldots, n_M\rangle$ with M the

number of detected modes out of a total of N modes and n_i the number of photons found in mode i, the state leaving the optical circuit in the undetected modes is given by

$$|\psi_{\text{out}}\rangle_{M+1,\dots,N} = {}_{1,\dots,M}\langle\gamma|\psi_{\text{prior}}\rangle_{1,\dots,N}. \tag{5}$$

In this paper, we study the outgoing states $|\psi_{\text{out}}\rangle$ for a special class of optical circuits. First, we assume that the input state is the vacuum on all modes. Thus, we effectively study optical circuits as a special class of state preparation devices. Secondly, our class of optical circuits include all possible passive components, but *only* active components with quadratic interaction Hamiltonians:

$$H_I^{(j)} = \sum_{kl} \hat{a}_k^\dagger R_{kl}^{(j)} \hat{a}_l^\dagger + \sum_{kl} \hat{a}_k R_{kl}^{(j)*} \hat{a}_l \tag{6}$$

where $R^{(j)}$ is some complex symmetric matrix. This matrix determines the behaviour of the jth active component, which can be any combination of down-converters and squeezers. Finally, we consider ideal photo-detection, where the eigenstate corresponding to the measurement outcome can be written as $|\gamma\rangle = |n_1, \dots, n_M\rangle$.

The class of optical circuits we consider here is not the most general class, but it still includes important experiments like quantum teleportation [5], entanglement swapping [6] and the demonstration of GHZ correlations [7]. In section 2 we show how teleportation can be modelled using the methods presented here.

The state $|\psi\rangle$ prior to the photo-detection can be written in terms of the components of the optical circuit as (t_j is again the duration of the interaction given by $H_I^{(j)}$)

$$|\psi\rangle = U_K e^{-it_K \mathcal{H}_I^{(K)}} \dots U_1 e^{-it_1 H_I^{(1)}} |0\rangle. \tag{7}$$

Considering that we only include interaction Hamiltonians which are quadratic in the creation and annihilation operators, equation (7) can be written as

$$|\psi\rangle = \exp\left[-\tfrac{1}{2} \sum_{i,j=1}^N (\hat{a}_i^\dagger A_{ij}^\dagger \hat{a}_j^\dagger - \hat{a}_i A_{ij} \hat{a}_j + \hat{a}_i^\dagger C_{ij} \hat{a}_j) \right]|0\rangle \tag{8}$$

where A and C are complex matrices. Furthermore, A can always be chosen symmetric and $C = C^\dagger$. We will now simplify this expression by normal-ordering this evolution.

Define $(\vec{a}, A\vec{a}) \equiv \sum_{ij} \hat{a}_i A_{ij} \hat{a}_j$. As shown by Braunstein [8], we can rewrite equation (8) using two passive unitary transformations U and V as

$$|\psi\rangle = U e^{-(\vec{a}^\dagger, \Lambda^\dagger \vec{a}^\dagger)/2 + (\vec{a}, \Lambda\vec{a})/2} V^T |0\rangle \tag{9}$$

where Λ is a diagonal matrix with real non-negative eigenvalues λ_i. This means that, starting from vacuum, the class of optical circuits we consider here is equivalent to a set of single-mode squeezers, followed by a unitary transformation U, corresponding to a set of passive optical components, and photo-detection. Since Λ is diagonal, we can write equation (9) as

$$|\psi\rangle = U\left(\prod_{i=1}^N \exp\left[-\frac{\lambda_i^*}{2}(\hat{a}_i^\dagger)^2 + \frac{\lambda_i}{2}\hat{a}_i^2 \right] \right) V^T |0\rangle. \tag{10}$$

Since V^T is a unitary transformation associated with a set of passive optical components, we can also write $V^T |0\rangle = |0\rangle$.

We now determine the normal ordering of every factor $\exp[-\frac{\lambda_i^*}{2}(\hat{a}_i^\dagger)^2 + \frac{\lambda_i}{2}\hat{a}_i^2]$ separately. According to [9–11], this is given by

$$e^{-\lambda_i^*/2(\hat{a}_i^\dagger)^2 + \lambda_i/2\hat{a}_i^2} = e^{-\hat{\lambda}_i^* \tanh|\lambda_i/2|(\hat{a}_i^\dagger)^2} e^{-2\ln(\cosh|\lambda_i/2|)\hat{a}_i^\dagger \hat{a}_i} e^{\hat{\lambda}_i \tanh|\lambda_i/2|\hat{a}_i^2} \tag{11}$$

where $\hat{\lambda}_i = \lambda_i/|\lambda_i|$. When we now apply this operator to the vacuum, the annihilation operators will vanish, leaving only the exponential function of the creation operators. We thus have (bearing in mind that $U V^T |0\rangle = |0\rangle$)

$$|\psi\rangle = U \mathrm{e}^{-(\vec{a}^\dagger, \Lambda^* \vec{a}^\dagger)/2} V^T |0\rangle = \mathrm{e}^{-(\vec{a}^\dagger, B \vec{a}^\dagger)/2} |0\rangle \tag{12}$$

with $B \equiv U \Lambda^* U^\dagger$. This is the state of the interferometer prior to photo-detection. It corresponds to *multi-mode squeezed vacuum.*

The photo-detection itself can be modelled by successive application of annihilation operators. Every annihilation operator $\hat{a}_i$ removes a photon in mode i from the state $|\psi\rangle$. Suppose the optical circuit employs N distinct modes. We will now detect M modes, finding $n_1 + \cdots + n_M = N_{\mathrm{tot}}$ photons (with $M < N$). These modes can be relabelled 1 to M. The vector $\vec{n}$ denotes the particular detector 'signature': $\vec{n} = (n_1, \ldots, n_M)$ means that n_1 photons are detected in mode 1, n_2 in mode 2, and so on. The freely propagating outgoing state $|\psi_{\vec{n}}\rangle$ can then be described as

$$|\psi_{\vec{n}}\rangle =_{1\ldots M} \langle n_1, \ldots, n_M | \psi \rangle_{1\ldots N} = c_{\vec{n}} \langle 0 | \hat{a}_1^{n_1} \cdots \hat{a}_M^{n_M} | \psi \rangle \tag{13}$$

with $c_{\vec{n}} = (n_1! \cdots n_M!)^{-1/2}$.

At this point we find it convenient to introduce the N-mode Bargmann representation [12]. The creation and annihilation operators obey the commutation relations given in equation (2). We can replace these operators with c-numbers and their derivatives according to

$$\hat{a}_i^\dagger \to \alpha_i \qquad \text{and} \qquad \hat{a}_i \to \partial_i \equiv \frac{\partial}{\partial \alpha_i}. \tag{14}$$

The commutation relations are then

$$[\partial_i, \alpha_j] = \delta_{ij} \qquad \text{and} \qquad [\partial_i, \partial_j] = [\alpha_i, \alpha_j] = 0. \tag{15}$$

Note that the actual values of α_i are irrelevant (the creation and annihilation operators do not have numerical values either); what matters here is the functional relationship between α_i and ∂_{α_i}.

The state created by the optical circuit in this representation (prior to the detections, analogous to equation (12)) in the Bargmann representation is

$$\psi(\vec{\alpha}) = \exp\left[-\tfrac{1}{2}(\vec{\alpha}, B\vec{\alpha})\right] = \exp\left[-\tfrac{1}{2}\sum_{ij} \alpha_i B_{ij} \alpha_j\right]. \tag{16}$$

Returning to equation (13), we can write the freely propagating state after detection of the auxiliary modes in the Bargmann representation as

$$\psi_{\vec{n}}(\vec{\alpha}) \propto c_{\vec{n}} \, \partial_1^{n_1} \cdots \partial_M^{n_M} \mathrm{e}^{-(\vec{\alpha}, B\vec{\alpha})/2}|_{\vec{\alpha}'=0} \tag{17}$$

up to some normalization factor, where $\vec{\alpha}' = (\alpha_1, \ldots, \alpha_M)$. By setting $\vec{\alpha}' = 0$ we ensure that *no more* than n_i photons are present in mode i. It plays the role of the vacuum bra in equation (13).

Now that we have an expression for the freely propagating state emerging from our optical setup after detection, we seek to simplify it. We can multiply $\psi_{\vec{n}}(\vec{\alpha})$ by the identity operator I, written as

$$I = (-1)^{2N_{\mathrm{tot}}} \exp\left[-\tfrac{1}{2}(\vec{\alpha}, B\vec{\alpha})\right] \exp\left[\tfrac{1}{2}(\vec{\alpha}, B\vec{\alpha})\right] \tag{18}$$

where N_{tot} is the total number of detected photons. We then find the following expression for the unnormalized freely propagating state created by our optical circuit:

$$\psi_{\vec{n}}(\vec{\alpha}) \propto c_{\vec{n}} (-1)^{N_{\mathrm{tot}}} H_{\vec{n}}^B(\vec{\alpha}) \, \mathrm{e}^{-(\vec{\alpha}, B\vec{\alpha})/2}|_{\vec{\alpha}'=0} \tag{19}$$
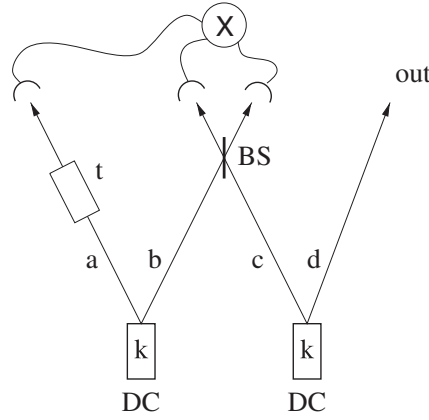
Figure 1. Schematic setup of the teleportation experiment. Two parametric down-converters (DC) with coupling strength τ create polarization-entangled photon pairs in modes a, b, c and d. Mode a is subjected to a polarization rotation θ and modes b and c are mixed in a beamsplitter (BS). The output state $\psi_{\text{out}}(\vec{\alpha})$ is conditioned on a three-fold coincidence in the detectors.

where

$$H_{\vec{n}}^{B}(\vec{\alpha}) \equiv (-1)^{N_{\text{tot}}} e^{(\vec{\alpha}, B\vec{\alpha})/2} \frac{\partial^{n_1}}{\partial \alpha_1^{n_1}} \cdots \frac{\partial^{n_M}}{\partial \alpha_M^{n_M}} e^{-(\vec{\alpha}, B\vec{\alpha})/2} \tag{20}$$

is the so-called MDHP, or MDHP for short. The outgoing state can now be found immediately by making the substitution $\alpha_i \to \hat{a}_i^\dagger$ and $\partial_{\alpha_i} \to \hat{a}_i$.

The use of MDHPs and Hermite polynomials of two variables have previously been used to describe N-dimensional first-order systems [13, 14] and photon statistics [15–17]. Here, we have shown that the lowest order of the *outgoing* state of optical circuits with quadratic components (as described by equation (8)) and conditional photo-detection can be expressed directly in terms of an MDHP.

In physical systems, the coupling constants (λ_i) are usually very small (i.e., $\lambda_i \ll 1$ or possibly $\lambda_i \lesssim 1$). This means that, for all practical purposes, only the first order term in equation (19) is important (i.e., for small λ_i we can approximate the exponential by 1). Consequently, studying the MDHPs yields knowledge about the typical states we can produce using Gaussian sources without coherent displacements. In section 3 we take a closer look at these polynomials, but first we consider the description of quantum teleportation in this representation.

2. Example: quantum teleportation

As an example of how to determine the outgoing state of an optical circuit, consider the teleportation experiment of Bouwmeester *et al* [5,18,19] (see also figure 1). The optical circuit corresponding to this experiment consists of four spatial modes a, b, c and d, each of which consists of two polarization modes x and y. There are therefore eight optical modes, initially in the vacuum state. Two down-converters create entangled polarization states; they belong to the class of active Gaussian components without coherent displacements. Mode a undergoes a polarization rotation over an angle θ and modes b and c are mixed in a 50 : 50 beamsplitter. Finally, modes b and c emerging from the beamsplitter are detected with polarization *insensitive* detectors and mode a is detected using a polarization *sensitive* detector. The state which is to

be teleported is therefore given by

$$|\Psi\rangle = \cos\theta\,|1_x, 0_x\rangle_b + \sin\theta\,|0_x, 1_y\rangle_b \tag{21}$$

where $|1_x, 0_y\rangle_b$ and $|0_x, 1_y\rangle_b$ denote single-photon states in mode b with polarizations x and y, respectively.

The state prior to the detection and normal ordering (corresponding to equation (3)) is given by

$$|\psi_{\text{prior}}\rangle = U_{\text{BS}}U_\theta e^{\tau(\vec{u}^\dagger, L\vec{u}^\dagger)/2 + \tau^*(\vec{u}, L\vec{u})/2 + \tau(\vec{v}^\dagger, L\vec{v}^\dagger)/2 + \tau^*(\vec{v}, L\vec{v})/2}|0\rangle \tag{22}$$

where τ is a possibly complex coupling constant denoting the strength of the down-converter, and

$$L = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{23}$$

and $\vec{u}^\dagger = (\hat{a}_x^\dagger, \hat{a}_y^\dagger, \hat{b}_x^\dagger, \hat{b}_y^\dagger)$, $\vec{v}^\dagger = (\hat{c}_x^\dagger, \hat{c}_y^\dagger, \hat{d}_x^\dagger, \hat{d}_y^\dagger)$. This can be written as

$$|\psi_{\text{prior}}\rangle = \exp\left[\frac{\tau}{2}(\vec{a}^\dagger, A\vec{a}^\dagger) + \frac{\tau^*}{2}(\vec{a}, A\vec{a})\right]|0\rangle \tag{24}$$

with $\vec{a} \equiv (\hat{a}_x, \ldots, \hat{d}_y)$ and A the (symmetric) matrix:

$$A = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 0 & -\sin\theta & \cos\theta & -\sin\theta & \cos\theta & 0 & 0 \\ 0 & \cos\theta & \sin\theta & \cos\theta & \sin\theta & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & -1 \\ & & 0 & 0 & 0 & -1 & 0 \\ & & & 0 & 0 & 0 & 1 \\ & & & & 0 & 1 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}. \tag{25}$$

This can be easily calculated from $A = U_{\text{BS}}U_\theta\,L\,U_\theta^{-1}U_{\text{BS}}^{-1}$.

We now have to find the normal ordering of equation (24). Since A is unitary, the polynomial $(\vec{a}^\dagger, A\vec{a}^\dagger)$ is a generator of an $su(1, 1)$ algebra. According to Truax [11], the normal ordering of the exponential thus yields a state

$$|\psi_{\text{prior}}\rangle = \exp\left[\frac{\xi}{2}(\vec{a}^\dagger, A\vec{a}^\dagger)\right]|0\rangle \tag{26}$$

with $\xi = (\tau\tanh|\tau|)/|\tau|$. The lowest order contribution after three detected photons is due to the term $\xi^2(\vec{a}^\dagger, A\vec{a}^\dagger)^2/8$. However, first we write equation (26) in the Bargmann representation:

$$\psi_{\text{prior}}(\vec{\alpha}) = \exp\left[\frac{\xi}{2}(\vec{\alpha}, A\vec{\alpha})\right] \tag{27}$$

where $\vec{\alpha} = (\alpha_{a_x}, \ldots, \alpha_{d_y})$ and $\vec{\alpha}' = (\alpha_{a_x}, \ldots, \alpha_{c_y})$. That is, $\vec{\alpha}'$ denotes the detected modes.

The detection of n photons in mode α_i is modelled by the nth derivative $\partial_{\alpha_i}^n$. By choosing $\alpha_i = 0$ afterwards we ensure that *no more* than n photons are detected. The polarization independent bell detection of modes α_b and α_c is then modelled by the differentiation $(\partial_{b_x}\partial_{c_y} - \partial_{b_y}\partial_{c_x})$. Given a detector hit in mode a_x, the polarization sensitive detection of mode a is modelled by ∂_{a_x}:

$$\psi_{\text{out}}(\vec{\alpha}) = \partial_{a_x}\left(\partial_{b_x}\partial_{c_y} - \partial_{b_y}\partial_{c_x}\right)\exp\left[\frac{\xi}{2}(\vec{\alpha}, A\vec{\alpha})\right]\Bigg|_{\vec{\alpha}'=0}. \tag{28}$$

Defining $\vec{n}_1 = (1_{a_x}, 0_{a_y}, 1_{b_x}, 0_{b_y}, 0_{c_x}, 1_{c_y})$ and $\vec{n}_2 = (1_{a_x}, 0_{a_y}, 0_{b_x}, 1_{b_y}, 1_{c_x}, 0_{c_y})$ we find that the outgoing state in the Bargmann representation is given by

$$
\begin{aligned}
\psi_{\text{out}}(\vec{\alpha}) &= [H_{\vec{n}_1}^A(\vec{\alpha}) - H_{\vec{n}_2}^A(\vec{\alpha})] e^{\xi(\vec{\alpha}, A\vec{\alpha})/2}|_{\vec{\alpha}'=0} \\
&= (\cos\theta\,\alpha_{d_x} + \sin\theta\,\alpha_{d_y}) e^{\xi(\vec{\alpha}, A\vec{\alpha})/2}|_{\vec{\alpha}'=0}.
\end{aligned}
\tag{29}
$$

We can now leave the Bargmann representation by making the substitution $\alpha_i \rightarrow \hat{a}_i^\dagger$. Equation (29) then becomes

$$
|\psi_{\text{out}}\rangle = \cos\theta\,|1_x, 0_y\rangle_d + \sin\theta\,|0_x, 1_y\rangle_d
\tag{30}
$$

which is the state teleported from mode a to mode d in the Bargmann representation. This procedure essentially amounts to evaluating the MDHPs $H_{\vec{n}_1}^A(\vec{\alpha})$ and $H_{\vec{n}_2}^A(\vec{\alpha})$. Note that the polarization independent detection of modes b and c yields a *superposition* of the MDHPs. One recognizes the algebraic form of the singlet state in the differential operator $(\partial_{b_x}\partial_{c_y} - \partial_{b_y}\partial_{c_x})$, that is, the Bell measurement projects the incoming modes onto a Bell state.

The above example demonstrates how knowledge of the matrix A (which determines the transformation of the creation operators) can be used to calculate the output state. In other words, the transformation properties of the operators and the outgoing state of the optical circuits are related through MDHPs.

3. The Hermite polynomials

The one-dimensional Hermite polynomials are, of course, well known from the description of the linear harmonic oscillator in quantum mechanics. These polynomials may be obtained from a generating function G. Furthermore, there exist two recursion relations and an orthogonality relation between them. The theory of MDHPs with real variables has been developed by Appell and Kempé de Fériet [20] and in the Bateman project [21]. Mizrahi derived an expression for real MDHPs from an n-dimensional generalization of the Rodriguez formula [22]. We will now give the generating function for the complex MDHPs given by equation (20) and consecutively derive the recursion relations and the orthogonality relation (see also [14]).

Define the generating function $G_B(\vec{\alpha}, \vec{\beta})$ to be

$$
G_B(\vec{\alpha}, \vec{\beta}) = e^{(\vec{\alpha}, B\vec{\beta}) - (\vec{\beta}, B\vec{\beta})/2} = \sum_{\vec{n}} \frac{\beta_1^{n_1}}{n_1!} \cdots \frac{\beta_M^{n_M}}{n_M!} H_{\vec{n}}^B(\vec{\alpha}).
\tag{31}
$$

$G_B(\vec{\alpha}, \vec{\beta})$ gives rise to the MDHP in equation (20), which determines this particular choice. Note that the inner product $(\vec{\alpha}, B\vec{\beta})$ does not involve any complex conjugation. If complex conjugation was involved, we would have obtained different polynomials (which we could also have called MDHPs, but they would not bear the same relationship to optical circuits).

In the rest of the paper we use the following notation: by $\vec{n} - e_j$ we mean that the jth entry of the vector $\vec{n} = (n_1, \ldots, n_M)$ is lowered by one, thus becoming $n_j - 1$. By differentiation of both sides of the generating function in equation (31) we can thus show that the first recursion relation becomes

$$
\frac{\partial}{\partial \alpha_i} H_{\vec{n}}^B(\vec{\alpha}) = \sum_{j=1}^M B_{ij} n_j H_{\vec{n}-e_j}^B(\vec{\alpha}).
\tag{32}
$$

The second recursion relation is given by

$$
H_{\vec{n}+e_i}^B(\vec{\alpha}) - \sum_{j=1}^M B_{ij}\alpha_j H_{\vec{n}}^B(\vec{\alpha}) + \sum_{j=1}^M B_{ij} n_j H_{\vec{n}-e_j}^B(\vec{\alpha}) = 0
\tag{33}
$$

which can be proved by mathematical induction using

$$\sum_{k=1}^{M} B_{ik} n_k H_{\vec{n}-e_k+e_i}^{B}(\vec{\alpha}) - B_{ii} H_{\vec{n}}^{B}(\vec{\alpha}) = \sum_{k=1}^{M} B_{ik} m_k H_{\vec{m}+e_i}^{B}(\vec{\alpha}). \tag{34}$$

Here, we have chosen $\vec{m} = \vec{n} - e_k$.

The orthogonality relation is somewhat more involved. Ultimately, we want to use this relation to determine the normalization constant of the states given by equation (19). To find this normalization we have to evaluate the integral

$$\int_{\mathbb{C}^N} d\vec{\alpha} \; \psi_{\vec{n}}^*(\vec{\alpha}) \psi_{\vec{m}}(\vec{\alpha}). \tag{35}$$

The state $\psi_{\vec{n}}$ includes $|_{\vec{\alpha}'=0}$, which translates into a delta function $\delta(\vec{\alpha}')$ in the integrand. The relevant integral thus becomes

$$\int_{\mathbb{C}^N} d\vec{\alpha} \; e^{-\mathrm{Re}\,(\vec{\alpha}, B\vec{\alpha})} [H_{\vec{n}}^{B}(\vec{\alpha})]^* H_{\vec{m}}^{B}(\vec{\alpha}) \, \delta(\vec{\alpha}'). \tag{36}$$

From the orthonormality of different quantum states we know that this integral must be proportional to $\delta_{\vec{n},\vec{m}}$.

Since in the Bargmann representation we are only concerned with the *functional* relationship between α_i and ∂_{α_i}, and not the actual values, we can choose α_i to be real. To stress this, we write $\alpha_i \to x_i$. The orthogonality relation is thus derived from

$$\int_{\mathbb{R}^N} d\vec{x} \; \psi_{\vec{n}}^*(\vec{x}) \psi_{\vec{m}}(\vec{x}) = \int_{\mathbb{R}^N} d\vec{x} \; e^{-(\vec{x}, \mathrm{Re}\,(B)\vec{x})} H_{\vec{n}}^{B*}(\vec{x}) H_{\vec{m}}^{B}(\vec{x}) \, \delta(\vec{x}') \tag{37}$$

where $\delta(\vec{x}')$ is the real version of $\delta(\vec{\alpha}')$. Following Klauderer [14] we find that

$$\int d\vec{x} \; e^{-(\vec{x}, \mathrm{Re}\,(B)\vec{x})} H_{\vec{n}}^{B*}(\vec{x}) H_{\vec{m}}^{B}(\vec{x}) = (-1)^{N_{\mathrm{tot}}} \int d\vec{x} \; e^{-(\vec{x}, B\vec{x})/2} \partial_{\vec{x}}^{\vec{n}} [e^{-(\vec{x}, B^*\vec{x})/2}] H_{\vec{m}}^{B}(\vec{x}) \tag{38}$$

where $\partial_{\vec{x}}^{\vec{n}}$ is the differential operator $\partial_{x_1}^{n_1} \cdots \partial_{x_M}^{n_M}$ acting solely on the exponential function. We now integrate the right-hand side by parts, yielding

$$(-1)^{N_{\mathrm{tot}}} \int d\vec{x} \; e^{-(\vec{x}, B\vec{x})/2} \partial_{\vec{x}}^{\vec{n}} e^{-(\vec{x}, B^*\vec{x})/2} H_{\vec{m}}^{B}(\vec{x})$$

$$= (-1)^{N_{\mathrm{tot}}} \int d'\vec{x} \; e^{-(\vec{x}, B\vec{x})/2} \partial_{\vec{x}}^{\vec{n}-e_i} e^{-(\vec{x}, B^*\vec{x})/2} H_{\vec{m}}^{B}(\vec{x}) |_{x_i=-\infty}^{+\infty}$$

$$- (-1)^{N_{\mathrm{tot}}} \int d\vec{x} \; e^{-(\vec{x}, B\vec{x})/2} \partial_{\vec{x}}^{\vec{n}-e_i} e^{-(\vec{x}, B^*\vec{x})/2} \partial_{x_i} H_{\vec{m}}^{B}(\vec{x}) \tag{39}$$

with $d'\vec{x} = dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_N$. The left-hand term is equal to zero when $\mathrm{Re}\,(B)$ is positive definite, i.e., when $(\vec{x}, \mathrm{Re}\,(B)\vec{x}) > 0$ for all non-zero $\vec{x}$. Repeating this procedure n_i times yields

$$\int d\vec{x} \; e^{-(\vec{x}, \mathrm{Re}\,(B)\vec{x})} H_{\vec{n}}^{B*}(\vec{x}) H_{\vec{m}}^{B}(\vec{x})$$

$$= (-1)^{N_{\mathrm{tot}}+n_i} \int d\vec{x} \; e^{-(\vec{x}, B\vec{x})/2} \partial_{\vec{x}}^{\vec{n}-n_i e_i} e^{-(\vec{x}, B^*\vec{x})/2} \partial_{x_i}^{n_i} H_{\vec{m}}^{B}(\vec{x}). \tag{40}$$

When there is at least one $n_i > m_i$, differentiating the MDHP n_i times to x_i will yield zero. Thus we have

$$\int d\vec{x} \; e^{-(\vec{x}, \mathrm{Re}\,(B)\vec{x})} H_{\vec{n}}^{B*}(\vec{x}) H_{\vec{m}}^{B}(\vec{x}) = 0 \qquad \text{for} \quad \vec{n} \neq \vec{m} \tag{41}$$

when Re (B) is positive definite and $n_i \neq m_i$ for any i. The case where $\vec{n}$ equals $\vec{m}$ is given by

$$\int \mathrm{d}\vec{x}\, \mathrm{e}^{-(\vec{x},\mathrm{Re}\,(B)\vec{x})/2} H_{\vec{n}}^{B^*}(\vec{x}) H_{\vec{m}}^{B}(\vec{x}) = \delta_{\vec{n}\vec{m}}\mathcal{N} \tag{42}$$

where $\delta_{\vec{n}\vec{m}}$ denotes the product of $\delta_{n_i m_i}$ with $1 \leqslant i \leqslant N$. Here, $\mathcal{N}$ is equal to

$$\mathcal{N} \equiv 2^{N_{\mathrm{tot}}} B_{11}^{n_1} \cdots B_{NN}^{n_N}\, n_1! \cdots n_N!\, |\pi^{-1} B|^{-1/2}. \tag{43}$$

For the proof of this identity we refer to [14].

4. Imperfect detectors

So far, we have only considered the use of ideal photo-detection. That is, we assumed that the detectors tell us exactly and with unit efficiency how many photons were present in the detected mode. However, in reality such detectors do not exist. In particular, we have to incorporate losses (non-perfect efficiency) and dark counts. Furthermore, we have to take into account the fact that most detectors do not have a single-photon resolution (i.e., they cannot distinguish a single photon from two photons) [17]. We will now show that imperfections like reduced detection efficiency and lack of single-photon resolution can be taken into account.

First of all, one should note that this model might not be suitable for the description of dark counts. In single-shot experiments, however, dark counts can be neglected, for example, when the detectors operate only within narrow gated time intervals.

We can model the quantum efficiency η^2 of a detector by placing a beamsplitter with transmission amplitude η in front of a perfect detector [23, 24] (see figure 2). The part of the signal which is reflected by the beamsplitter into the environment (and which will therefore never reach the detector) is the loss due to the imperfect detector. Since beamsplitters are part of the set of optical devices we allow, the model allows us to make this generalization.

There is, however, one subtlety which needs to be addressed before we can incorporate inefficient detectors. When we trace out the environment (i.e., the part which is inaccessible to the experimenter), the outgoing state will in general be a *mixed* state ρ. Imperfections in our detectors will therefore generally yield a mixed state. The question is now what a mixed state looks like in terms of Hermite polynomials, and how we trace out the environment.

Suppose M modes are detected and the losses are reflected into the environment. These lost modes can formally be included as $\vec{\beta} = (\beta_1, \ldots, \beta_M)$ in the Bargmann representation. The lost photons then determine a vector $\vec{m} = (m_1, \ldots, m_M)$, where m_i denotes the number of lost photons in mode β_i. Secondly, we can write the detector efficiencies as a vector $\vec{\eta} = (\eta_1, \ldots, \eta_M)$. The transition matrix B in equation (17) then transforms according to $\alpha_j \rightarrow \eta\alpha_j + \sqrt{1 - \eta^2}\beta_j$ for each detected mode. We write the resulting matrix as $\tilde{B}(\vec{\eta})$.
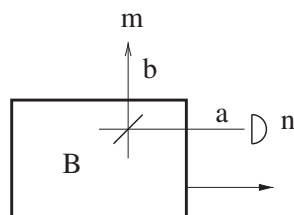

Figure 2. The optical circuit defined by $\tilde{B}(\vec{\eta})$. The modes denoted by $\vec{\beta}$ are lost into the environment. The configuration in the black box represents a set of modes which are inefficiently detected. The set of detectors gives a coincidence denoted by $\vec{n}$, whereas the lost photons are given by $\vec{m}$.

Let the outgoing state of the optical circuit (including the modes which are lost into the environment) be given by $\psi_{\vec{n}}(\vec{\alpha})$ with $\vec{\alpha} = (\beta_1, \ldots, \beta_M, \alpha_1, \ldots, \alpha_N)$. Alternatively, we can write this as

$$\rho_{\vec{n}}(\vec{\alpha}, \partial_{\bar{\alpha}}) = \psi_{\vec{n}}(\vec{\alpha}) \cdot \psi_{\bar{n}}^*(\partial_{\bar{\alpha}}). \tag{44}$$

When the state lost in the environment is given by

$$\phi_{\vec{m}}(\vec{\beta}) \propto H_{\vec{m}}^{\tilde{B}(\vec{\eta})}(\vec{\beta})e^{-(\vec{\beta}, \tilde{B}(\vec{\eta})\vec{\beta})/2}|_{\vec{\beta}=0} \tag{45}$$

we can then trace out the environment by writing

$$\rho_{\vec{n}} = \sum_{\vec{m}} \phi_{\vec{m}}^*(\partial_{\vec{\beta}}) \cdot \psi_{\vec{n}}(\vec{\alpha}) \cdot \psi_{\bar{n}}^*(\partial_{\bar{\alpha}}) \cdot \phi_{\vec{m}}(\vec{\beta})$$

$$\propto \sum_{\vec{m}} [(H_{\vec{m}}^{\tilde{B}(\vec{\eta})}(\partial_{\vec{\beta}}))^* H_{\vec{n}}^{\tilde{B}(\vec{\eta})}(\vec{\alpha}) (H_{\bar{n}}^{\tilde{B}(\vec{\eta})}(\partial_{\bar{\alpha}}))^* H_{\vec{m}}^{\tilde{B}(\vec{\eta})}(\vec{\beta})$$

$$\times e^{-(\vec{\alpha}, \tilde{B}(\vec{\eta})\vec{\alpha})/2 - (\partial_{\bar{\alpha}}, \tilde{B}^*(\vec{\eta})\partial_{\bar{\alpha}})/2}]_{\vec{\alpha}', \vec{\beta}=0}. \tag{46}$$

In principle, we need to sum over all possible $\vec{m}$, but when higher order contributions are small we can truncate this sum. To retrieve an expression for the mixed state leaving the optical circuit, we can make the substitution $\alpha_i \to \hat{a}_i^\dagger$ and $\partial_{\alpha_i} \to \hat{a}_i$. In a similar fashion we can model the detector's lack of single-photon resolution [17].

As an example of this lack of single-photon resolution, consider again the teleportation experiment by Bouwmeester *et al* [5]. It was shown in [18] and [19] that a three-fold coincidence in the detectors (modes a, b and c) can either come from one entangled pair in both down-converters or two pairs in modes a and b and vacuum in the others. When the detector in mode a lacks single-photon resolution, it cannot distinguish between these two cases, and the outgoing state will be a mixture due to these possibilities. More explicitly, the outgoing state will be a mixture of the intended incoming state $|\psi_{\text{out}}\rangle$ and the vacuum.

The differential operators corresponding to these two possibilities are then given by

$$\begin{aligned} \nabla_1 &= \partial_{\alpha_{a_x}} (\partial_{\alpha_{b_x}} \partial_{\alpha_{c_y}} - \partial_{\alpha_{b_y}} \partial_{\alpha_{c_x}}) \\ \nabla_2 &= \partial_{\alpha_{a_x}}^2 (\partial_{\alpha_{b_x}} \partial_{\alpha_{c_x}} + \partial_{\alpha_{b_x}} \partial_{\alpha_{c_y}} + \partial_{\alpha_{b_y}} \partial_{\alpha_{c_x}} + \partial_{\alpha_{b_y}} \partial_{\alpha_{c_y}}). \end{aligned} \tag{47}$$

To lowest order these give the outgoing state $|\psi_{\text{out}}\rangle$ and the vacuum $|0\rangle$, respectively. The outgoing state then becomes

$$\rho = \tfrac{1}{2} |\psi_{\text{out}}\rangle\langle\psi_{\text{out}}| + \tfrac{1}{2} |0\rangle\langle 0|. \tag{48}$$

Note that the detector efficiencies are assumed to be perfect. Furthermore, by detecting mode a_y and rejecting the event when the detector clicks, we can exclude the contribution due to $\partial_{\alpha_{a_x}} \partial_{\alpha_{a_y}}$.

Finally, we should note that our description of this class of optical circuits (in terms of MDHPs) is essentially a one-way function. Given a certain setup, it is relatively straightforward to determine the outgoing state of the circuit. The other way around, however, is very difficult. As exemplified by our efforts in [4], it is highly nontrivial to obtain the matrix B associated with an optical circuit which produces a particular predetermined state from a Gaussian source using this method.

5. Conclusions

Linear optics is an important resource for quantum information and communication. In this paper, we have derived the general form of a special class of linear optical circuits. This includes squeezed multi-mode vacuum states conditioned on photo-detection of a subset of the

modes. To lowest order, the outgoing states in the Bargmann representation are proportional to MDHPs. Determining these states is thus reduced to a series of differential operations. Furthermore, we showed that several aspects of imperfect detection (such as poor detection efficiency and lack of single-photon resolution) can be incorporated in this model. As an example, we described the teleportation of a single-photon polarization state.

Acknowledgments

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