Limitations on the creation of maximal entanglement

Pieter Kok and Samuel L. Braunstein
Informatics, University of Wales, Bangor LL57 1UT, United Kingdom
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We study a limited set of optical circuits for creating near maximal polarization entanglement *without* the usual large vacuum contribution. The optical circuits we consider involve passive interferometers, feed-forward detection, down converters, and squeezers. For input vacuum fields we find that the creation of maximal entanglement using such circuits is impossible when conditioned on two detected auxiliary photons. So far, there have been no experiments with more auxiliary photons. Thus, based on the minimum complexity of the circuits required, if near maximal polarization entanglement is possible it seems unlikely that it will be implemented experimentally with the current resources.

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Entanglement is one of the key ingredients in quantum communication and information. For instance, quantum protocols such as dense coding, quantum error correction, and quantum teleportation [1] rely on the nonclassical correlations provided by entanglement. Currently, substantial efforts are being made to use *optical* implementations for quantum communication.

The advantages of this are obvious: light travels at high speed and it weakly interacts with the environment. However, this weak interaction poses serious drawbacks. The fact that photons do not interact with each other makes it hard to manipulate them. For example, it has recently been shown that it is impossible to perform so-called complete Bell measurements on two-mode polarization states in linear quantum optics [2,3] (although theoretical schemes involving Kerr media [4] and atomic coherence [5] have been reported). Furthermore, maximally polarization-entangled two-photon states have not been produced. In this paper we investigate the possibility of creating such states with linear optics and a specific class of nonlinear elements.

The maximally polarization-entangled states that are most commonly considered are the Bell states,

$$\begin{aligned} |\Psi^{\pm}\rangle &= (|\updownarrow,\leftrightarrow\rangle \pm |\leftrightarrow,\updownarrow\rangle)/\sqrt{2}, \\ |\Phi^{\pm}\rangle &= (|\updownarrow,\updownarrow\rangle \pm |\leftrightarrow,\leftrightarrow\rangle)/\sqrt{2}, \end{aligned} \tag{1}$$

where $|\updownarrow\rangle$ and $|\leftrightarrow\rangle$ denote single-photon states with orthogonal polarizations. In practice, these states have only been produced *randomly*, using for instance parametric down conversion [6]. This process can yield a state

$$|\psi\rangle \propto |0\rangle + \xi|\Psi^{-}\rangle + O(\xi^2), \tag{2}$$

where $|0\rangle$ denotes the vacuum and $\xi \ll 1$. This means that the Bell state $|\Psi^{-}\rangle$ is only produced with a small probability of the order of $|\xi|^2$. Although $|\psi\rangle$ has a maximally entangled component, as a state it is very weakly entangled (this may be quantified by its partial von Neumann entropy [7]). Since we have no way of telling that an entangled photon-pair was produced without measuring (and hence destroying) the outgoing state, we call this randomly produced entanglement. Currently, in quantum optics we have access to this type of entanglement only.

By contrast, we would like to be able to tell that we in fact produced a maximally entangled state before it is used. That is, we wish to have a source which gives a macroscopic indication that a maximally polarization-entangled state has been produced. Such a source is said to create *event-ready* entanglement. The vacuum contribution in Eq. (2) can be eliminated by means of a polarization-independent quantum nondemolition (QND) measurement. However, this would involve higher-order nonlinearities (like the Kerr effect) which, in practice, are very noisy (especially when they are required to operate at the single-photon level). In general, the creation of event-ready entanglement can be quantified by a certain probability of "happening." When this probability is equal to one, we have a *deterministic* source of event-ready entanglement.

Random entanglement has been used to demonstrate, for example, nonlocal features of quantum teleportation and entanglement swapping [8–10]. One might, therefore, suppose that in practice we do not really need event-ready entanglement. However, on a theoretical level Bell states appear as primitive notions. This means that protocols like entanglement purification and error correction [11,12] have been designed for maximally entangled states, rather than for random entanglement. For quantum communication to become a mature technology, one most certainly needs the ability to perform entanglement purification and error correction. It is not at all clear how these protocols can be convincingly implemented with random entanglement. One approach would be to try and investigate such protocols. However, that is not our aim here.

In this paper we give limitations to the creation of near maximal entanglement with linear optics and some nonlinear optical components (such as down converters and squeezers). First we present the tools with which we will attempt to produce event-ready entanglement. Then we derive a general condition for an optical setup, which should be satisfied in order to yield event-ready entanglement. We subsequently examine this condition in the context of several types of photon-sources.

Given a pair of photons in one maximally polarization-entangled state, we can obtain any other such state by a combination of a polarization rotation and a polarization-dependent phase shift. When we study the creation of maxi-

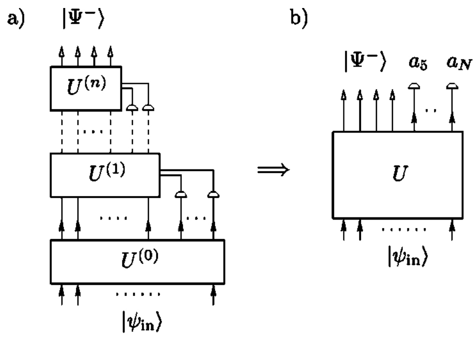

FIG. 1. If an optical circuit with feed-forward detection (a) produces a specific state, the same output can be obtained by an optical circuit where detection of the *auxiliary* modes takes place at the end (b). The efficiency of the latter, however, will generally be smaller.

mal polarization entanglement we shall therefore restrict ourselves to the $|\Psi^-\rangle$ Bell state without loss of generality.

In order to make $|\Psi^-\rangle$, we will assume that we have several resources at our disposal. *In this paper, the class of reasonable elements will consist of beam splitters, phase shifters, photodetectors, and nonlinear components, such as down converters, squeezers, etc.* These elements are then arranged to give a specific optical circuit [see Fig. 1(a)]. Part of this setup might be so-called *feed-forward* detection. In this scheme the outcome of the detection of a number of modes dynamically chooses the internal configuration of the subsequent optical circuit based on the interim detection results (see also Ref. [2]). Conditioned on these detections we want to obtain a freely propagating $|\Psi^-\rangle$ Bell state in the remaining undetected modes.

We now introduce two simplifications for such an optical circuit. First, we will show that we can discard feed-forward detection. Second, we will see that we only have to consider the detection of modes with at most one photon.

Theorem 1. In order to show that it is *possible* to produce a specific outgoing state, any optical circuit with feed-forward detection can be replaced by a *fixed* optical circuit where detection only takes place at the end.

Proof. Suppose a feed-forward optical circuit [like the one depicted in Fig. 1(a)] giving $|\Psi^-\rangle$ exists. That means that the circuit creates $|\Psi^-\rangle$ conditioned on one of potentially many patterns of detector responses. It is sufficient to consider a single successful pattern. We can then take every interferometer to be fixed and postpone all detections of the auxiliary modes to the very end [Fig. 1(b)]. Note that this procedure selects generally only *one* setup in which entanglement is produced, whereas a feed-forward optical circuit potentially allows more setups. It therefore might reduce the efficiency of the process. However, since we are only interested in the *possibility* of creating $|\Psi^-\rangle$, the efficiency is irrelevant. □

Theorem 2. Suppose an optical circuit produces a specific outgoing state conditioned on n_1 detected photons in mode 1, n_2 detected photons in mode 2, etc. (with $n_i = 0,1,2,\ldots$). The same output can be obtained by a circuit where in every detected mode *at most* one photon is found.

Proof. If there are more photons in a mode, we can replace the corresponding detector by a so-called detector *cascade* [13]. This device splits the mode into many modes which are all detected. For a sufficiently large cascade there is always a nonvanishing probability to have at most one photon in each outgoing mode. In that case, the same state is created while at most one photon enters each detector. Note that this again yields a lower efficiency. □

Applying these results to the creation of $|\Psi^-\rangle$, it is sufficient to consider a single *fixed* interferometer acting on an incoming state, followed at the end by detection of the so-called *auxiliary* modes. $|\Psi^-\rangle$ is signaled by at least one fixed detection pattern with at most one photon in each detector.

How do we proceed in trying to make the $|\Psi^-\rangle$ Bell state? Let the time-independent interaction Hamiltonian $\mathcal{H}_I$ incorporate both the interferometer U and the creation of $|\psi_{\rm in}\rangle$ [see Fig. 1(b)]. The outgoing state *prior to the detection* can be formally written as

$$|\psi_{\rm out}\rangle = U|\psi_{\rm in}\rangle \equiv \exp(-it\mathcal{H}_I/\hbar)|0\rangle, \tag{3}$$

with $|0\rangle$ the vacuum. This defines an effective Hamiltonian $\mathcal{H}_I$ which is generally not unique.

At this point we find it useful to change our description. Since the creation and annihilation operators satisfy the same commutation relations as c numbers and their derivatives, we can make the substitution $a_i^\dagger \to \alpha_i$ and $a_i \to \partial_i$, where $\partial_i \equiv \partial/\partial\alpha_i$. Furthermore, we define $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$. Quantum states are then represented by functions of c numbers and their derivatives. This is called the Bargmann representation [14].

Furthermore, suppose we can normal order the operator $\exp(-it\mathcal{H}_I/\hbar)$ in Eq. (3). This would leave us with a function of only the creation operators, acting on the vacuum. In the Bargmann representation we then obtain a function of complex numbers without their derivatives. In particular, when we have an optical circuit consisting of N distinct modes (for notational convenience we treat distinct polarizations like, for instance, x and y as separate modes), we obtain the function $\psi_{\rm out}(\vec{\alpha})$ after the unitary evolution U and normal ordering. The normal ordering of the evolution operator in conjunction with the vacuum input state is crucial, since it allows us to simplify the problem significantly.

We now treat the (ideal) detection of the auxiliary modes in the Bargmann representation. Suppose the outgoing state after the detection of M photons emerges in modes a_1, a_2, a_3, and a_4. After a suitable reordering of the detected modes the state which is responsible for the detector coincidence indicating success can be written as $|1_5, \ldots, 1_{M+4}, 0_{M+5}, \ldots\rangle$ (possibly on a countably infinite number of modes). We then have the postselected state $|\psi_{\rm post}\rangle$

$$|\psi_{\rm post}\rangle_{1\cdots4} \propto \langle 1_5, \ldots, 1_{M+4}, 0_{M+5}, \ldots|\psi_{\rm out}\rangle$$
$$= \langle 0|a_5 \cdots a_{M+4}|\psi_{\rm out}\rangle. \tag{4}$$

In the Bargmann representation the right-hand side of Eq. (4) is

$$\partial_5 \cdots \partial_{M+4}\psi_{\rm out}(\vec{\alpha})|_{\vec{\alpha}'=0}, \tag{5}$$

where we write $\vec{\alpha}' = (\alpha_5, \ldots, \alpha_{M+4}, \ldots)$.

Writing out the entanglement explicitly in the four modes (treating the polarization implicitly), we arrive at the following condition for the creation of two photons in the antisymmetric Bell state:

$$\partial_5 \cdots \partial_{M+4} \psi_{\text{out}}(\vec{\alpha})|_{\vec{\alpha}'=0} \propto \alpha_1\alpha_2 - \alpha_3\alpha_4 + O(\xi). \tag{6}$$

The term $O(\xi)$ will allow for a small pollution ($\xi \ll 1$) in the outgoing state. We will show that for certain special classes of interaction Hamiltonians this condition is very hard (if not impossible) to satisfy. This renders the experimental realization of two maximally polarization entangled photons at least highly impractical.

We are now ready to shape ψ_{out} in more detail. In this paper we consider two distinct classes of interaction Hamiltonians.

First, suppose $\mathcal{H}_I$ is linear in the creation operators. This means that the optical circuit consists of coherent inputs, linear operations, and no squeezing. The state prior to the detection can be written as $\exp(\Sigma_i d_i \alpha_i)$. We immediately see that the detection in condition (6) only yields constant factors. This can never give us the $|\Psi^-\rangle$ Bell state.

By contrast, we consider optical circuits including mode-mixing, squeezers, and down converters. The corresponding interaction Hamiltonians $\mathcal{H}_I$ are *quadratic* in the creation operators. There are no linear terms, so there are no coherent displacements. More formally

$$\mathcal{H}_I = \sum_{i,j=1}^{N} a_i^\dagger A_{ij}^{(1)} a_j^\dagger + \sum_{i,j=1}^{N} a_i^\dagger A_{ij}^{(2)} a_j + \text{H.c.}, \tag{7}$$

with $A^{(1)}$ and $A^{(2)}$ complex matrices. According to Braunstein [15], such an active interferometer is equivalent to a passive interferometer V, followed by a set of single-mode squeezers and another passive interferometer U'. We can view the photon source described by Eq. (7) as an active bilinear component of an interferometer. For vacuum input and after normal ordering [16], the optical setup then gives rise to

$$\psi_{\text{out}} = \exp[(\vec{\alpha}, B\vec{\alpha})], \tag{8}$$

with $(\vec{\alpha}, B\vec{\alpha}) = \Sigma_{ij}^N \alpha_i B_{ij} \alpha_j$. Such an optical setup would correspond to a collection of single-mode squeezers acting on the vacuum, followed by a passive optical interferometer U'. Here, B is a complex symmetric matrix determined by the interaction Hamiltonian $\mathcal{H}_I$ and the interferometer U'. We take B to be proportional to a common coupling constant ξ. The outgoing auxiliary modes a_5 to a_N are detected (see Fig. 2). We will now investigate whether we can produce $|\Psi^-\rangle$ conditioned on a given number of detected photons.

In the case of a bilinear interaction Hamiltonian, photons are always created in pairs. In addition, we seek to create *two* maximally entangled photons. An odd number of detected photons can never give $|\Psi^-\rangle$ and the number of detected photons should therefore be even. The lowest even number is zero. In this case no photons are detected and ψ_{out} in Eq. (8) is proportional to $1 + O(\xi)$, which corresponds to the vacuum state.

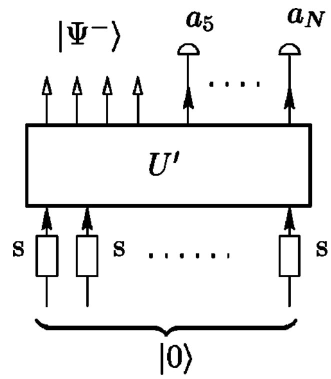

FIG. 2. The unitary interferometer U' with conditional photodetection and single-mode squeezers which should transform $|0\rangle$ into $|\Psi^-\rangle$.

The next case involves two detected photons. To have entanglement in modes α_1 to α_4 after detecting *two* photons requires

$$\partial_5 \partial_6 e^{(\vec{\alpha}, B\vec{\alpha})}|_{\vec{\alpha}'=0} \propto \alpha_1\alpha_2 - \alpha_3\alpha_4 + O(\xi). \tag{9}$$

The left-hand side of Eq. (9) is equal to

$$\left(B_{56} + \sum_{i,j=1}^{4} \alpha_i B_{i5} B_{j6} \alpha_j \right) e^{(\vec{\alpha}, B\vec{\alpha})}|_{\vec{\alpha}'=0}. \tag{10}$$

To satisfy Eq. (9), the vacuum contribution B_{56} should be negligible. We now ask whether the second term can give us entanglement.

The right-hand side of Eq. (9) can be rewritten according to $\alpha_1\alpha_2 - \alpha_3\alpha_4 = \Sigma_{i,j=1}^4 \alpha_i E_{ij} \alpha_j$, where E_{ij} are the elements of a symmetric matrix E. It is easily seen that $\det E = 1$.

Let $M_{ij} = B_{i5} B_{j6}$. Since only the symmetric part of M contributes, we construct $\tilde{M}_{ij} = (M_{ij} + M_{ji})/2$. The condition for two detected photons now yields

$$\sum_{i,j=1}^{4} \alpha_i \tilde{M}_{ij} \alpha_j = \sum_{i,j=1}^{4} \alpha_i E_{ij} \alpha_j + O(\xi). \tag{11}$$

If this equality is to hold, we would need $\det E = \det \tilde{M} + O(\xi) = 1$. However, it can be shown that $\det \tilde{M} = 0$. $\tilde{M}$ can therefore never have the same form as E for small ξ, so it is not possible to create maximal polarization entanglement conditioned upon two detected photons.

The last case we consider here involves four detected photons. When we define $X_i = \Sigma_j B_{ij} \alpha_j$, the left-hand side of Eq. (6) for four detected photons gives

$$\begin{aligned}(B_{56}B_{78} &+ B_{57}B_{68} + B_{58}B_{67} + B_{56}X_7X_8 + B_{57}X_6X_8 \\ &+ B_{58}X_6X_7 + B_{67}X_5X_8 + B_{68}X_5X_7 + B_{78}X_5X_6 \\ &+ X_5X_6X_7X_8) e^{(\vec{\alpha}, B\vec{\alpha})}|_{\vec{\alpha}'=0}.\end{aligned} \tag{12}$$

We have not been able either to prove or disprove that $|\Psi^-\rangle$ can be made this way. The number of terms which contribute to the bilinear part in α rapidly increases for more detected photons.

We have proved that the multimode squeezed vacuum conditioned on two detected photons cannot give maximal entanglement. We conjecture that this is true for any number of detected photons. However, suppose we *could* create maximal entanglement conditioned upon four detected photons, how efficient would this process be? For four detected photons yielding $|\Psi^-\rangle$ we need at least three photon pairs. These are created with a probability of the order of $|\xi|^6$. Currently, $|\xi|^2$, the probability per mode, has a value of 10^{-4} [17]. For experiments operating at a repetition rate of 100 MHz using ideal detectors, this will amount to approximately one maximally entangled pair every few hours. For realistic detectors this is much less.

So far, there have been no experiments which exceeded the detection of more than two *auxiliary* photons (not including the actual detection of the maximally entangled state). This, and the estimation of the above efficiency appears to place strong practical limitations on the creation of maximal entanglement.

In this paper we have demonstrated strong limitations on the possibility of creating maximal entanglement with quantum optics. To this end, we introduced two simplifications to our hypothetical optical circuit: first we replaced feed-forward detection by a fixed set of detectors at the end, and second, every detector needs to detect at most one photon. Conditioned on two detected photons, the multimode squeezed vacuum fails to create maximal entanglement. This leads to our conjecture that maximal entanglement is impossible using only these sources and linear interferometry. There is a number of open questions. First of all, is our conjecture true? And second, what happens when we have a combination of squeezing and coherent displacements? In that case the approach taken here fails due to the more complex normal ordering of the interaction Hamiltonian.

Entanglement is a fascinating and important phenomenon in physics. It not only provides us with insights in the mysterious world of quantum mechanics, but it also appears as a fundamental resource in quantum information and communication theory. However, *maximal polarization* entanglement has never been produced in the laboratory. We have shown here that this type of entanglement proves to be highly elusive.

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