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QUANTUM RULES: AN EFFECT CAN HAVE MORE THAN ONE OPERATION

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We review the formalism of Effects and Operations in order to demonstrate that this formalism is equivalent to the standard rules of quantum mechanics. D'Espagnat has studied an example where he finds a discrepancy between an analysis based on the standard quantum rule for calculating probabilities and an analysis that uses a particular Operation taken from the work of Barchielli and his collaborators. We use the formalism of Effects and Operations to explore and explain this discrepancy.

Key words: quantum measurement theory, effect, operation

1. INTRODUCTION

Measurements in quantum mechanics are usually described in the following way: the system to be measured is coupled to a "measuring apparatus"; after the interaction one obtains information about the system by observing some property of the apparatus. We shall refer to such a description as a "measurement model."

Quantum-mechanical measurements can also be described by a formalism that uses mathematical objects called Effects and Operations.[1–3] The formalism of Effects and Operations is equivalent to a description in terms of a measurement model. Recently, however, there has been some question about the usage of this formalism.[4,5] The purpose of this brief note is to clear up this question.

The formalism of Effects and Operations can be obtained by applying the standard rules of quantum mechanics to the observation of the apparatus in a measurement model. What the formalism does is to provide a very convenient, compact notation for describing a measurement directly in terms of the system state (density operator). The properties of a particular model—i.e., the type of apparatus and its quantum state and the type of interaction with the system—are incorporated in the Effects and Operations.

A set of Effects generates from the system state the statistics of the possible outcomes. Given a particular result, an Operation maps the system state before the measurement to a new (unnormalized) system state after the measurement. Thus one obtains a complete quantum-mechanical description of the measurement. Different models can give rise to the same measurement statistics, yet lead to different system states after the measurement. Thus one says that an Effect can have more than one Operation. The difference between two such models shows up in the statistics of a second measurement.

Barchielli, Lanz, and Prosperi[4] have used Effects and Operations to develop an elegant formal description of a continuous measurement of position. To generate the measurement statistics, they use a simple set of Effects. More importantly, they adopt the simplest Operations that are consistent with their Effects. In particular, they use Operations, called pure Operations, that map pure system states to pure system states, though even within the class of pure Operations their choice is special.

Barchielli *et al.* do not give a measurement model that realizes their choice of Effects and Operations (the reader can find such a model in Ref. 6), although there is a theorem that any consistent set of Effects and Operations can be realized.[1] Even a careful reading of their paper might lead one to believe that they are proposing their particular choice of Operations as a fundamental constituent in a new theory of measurement. Only in their Appendix B do they make it clear that their choice of Operations is not unique; it corresponds to some measurement model that they do not specify.

In a recent paper d'Espagnat[5] takes the pure Operation of Barchielli *et al.* as part of a new theory, which he dubs the "Milano theory." Seeking to compare this theory with standard quantum mechanics, he analyzes two consecutive measurements of spin by two different methods. In the first method he formulates a model for measurements of spin and applies to the model the standard quantum rule for calculating probabilities. In the second method he applies to two measurements of spin a pure Operation analogous to the pure Operation for position measurements of Barchielli *et al.* He finds that the two methods do not agree and concludes that the "Milano theory" is different from standard quantum theory. In view of the above discussion, however, the discrepancy he finds is not at all surprising. The pure Operation of the "Milano theory" corresponds to *some* model for measurements of spin, but it is not the measurement model that d'Espagnat has formulated. D'Espagnat knocks down a straw man—the "Milano theory." The formalism of Effects and Operations remains untouched.

In this note we explore and explain the discrepancy found by d'Espagnat. In Sect. 2 we review briefly the formalism of Effects and Operations, showing how it arises naturally from applying the standard quantum rules to a measurement model. In Sect. 3 we study d'Espagnat's model in detail and discuss how it is handled within the framework of Effects and Operations. In Sect. 4 we consider the pure Operation for a spin measurement (the "Milano theory"), and we formulate a measurement model, different from d'Espagnat's, that realizes this pure Operation.

2. REVIEW OF EFFECTS AND OPERATIONS

Consider a measurement model in which the system, initially with density operator $\hat{\rho}$, interacts with a measuring apparatus, initially with density operator $\hat{\rho}_{\mathcal{A}}$. The interaction leaves the system and the apparatus with joint density operator

$$\hat{\rho}_{\text{tot}} = \hat{U}\hat{\rho} \otimes \hat{\rho}_{\mathcal{A}}\hat{U}^{\dagger}, \tag{2.1}$$

where $\hat{U}$ is the joint evolution operator including the interaction.

After the interaction one observes the apparatus in order to learn something about the system. Suppose, for example, that one measures an apparatus observable $\hat{B}$, which has eigenvalues B. Let $\hat{P}_B$ be the projector onto the space spanned by the eigenvectors with eigenvalue B. (An eigenvalue B can be degenerate, in which case $\hat{P}_B$ projects into a multi-dimensional subspace of the apparatus Hilbert space.) The standard quantum rule for calculating probabilities gives the probability to obtain result B:

$$P(B) = \text{tr}\,(\hat{P}_B\hat{\rho}_{\text{tot}}) = \text{tr}_{\mathcal{S}}\,(\hat{F}_B\hat{\rho}), \tag{2.2}$$

where

$$\hat{F}_B \equiv \text{tr}_{\mathcal{A}}(\hat{U}^{\dagger}\hat{P}_B\hat{U}\hat{\rho}_{\mathcal{A}}). \tag{2.3}$$

Here $\text{tr}_{\mathcal{S}}$ and $\text{tr}_{\mathcal{A}}$ denote traces over system and apparatus variables respectively, and tr denotes a trace over both sets of variables. The system operators $\hat{F}_B$ are called Effects;[1-3] they provide a compact notation for generating measurement statistics from the system density operator.

Suppose now that the measurement yields the particular result B. The standard quantum rule for state reduction gives the system density operator $\hat{\rho}_B$ after the measurement: project the joint density operator into the subspace corresponding to eigenvalue B, trace out the apparatus, and normalize. The resulting state is

$$\hat{\rho}_B = \frac{\text{tr}_{\mathcal{A}}(\hat{P}_B\hat{\rho}_{\text{tot}}\hat{P}_B)}{P(B)} = \frac{\mathcal{F}_B\hat{\rho}}{P(B)}, \tag{2.4}$$

where $\mathcal{F}_B$ is a linear mapping on system density operators, defined by

$$\mathcal{F}_B\hat{\rho} \equiv \text{tr}_{\mathcal{A}}(\hat{P}_B\hat{\rho}_{\text{tot}}\hat{P}_B) = \text{tr}_{\mathcal{A}}(\hat{P}_B\hat{U}\hat{\rho} \otimes \hat{\rho}_{\mathcal{A}}\hat{U}^{\dagger}). \tag{2.5}$$

The map $\mathcal{F}_B$ is called an Operation;[1,3] it maps the system density operator before the measurement into the (unnormalized) system density operator after the measurement. An immediate consequence of the above is that

$$\text{tr}_{\mathcal{S}}\,(\mathcal{F}_B\hat{\rho}) = \text{tr}_{\mathcal{S}}\,(\hat{F}_B\hat{\rho}) = P(B). \tag{2.6}$$

We emphasize again that Effects and Operations arise from applying the standard quantum rules to a measurement model—i.e., from applying projection operators to the measuring apparatus. Indeed, Operations generalize the usual state reduction to "indirect measurements," where a projection operator is applied to the apparatus rather than directly to the system.

An important special case arises when the apparatus is initially in a pure state $\hat{\rho}_{\mathcal{A}} = |\Upsilon\rangle\langle\Upsilon|$ and the projection operator $\hat{P}_B = |B\rangle\langle B|$ projects into a one-dimensional subspace of the apparatus Hilbert space. Then, defining a system operator $\hat{\Upsilon}_B \equiv \langle B|\hat{U}|\Upsilon\rangle$, one finds that $\hat{F}_B = \hat{\Upsilon}_B^\dagger \hat{\Upsilon}_B$ and $\mathcal{F}_B\hat{\rho} = \hat{\Upsilon}_B \hat{\rho} \hat{\Upsilon}_B^\dagger$. In this case the Operation maps pure states to (unnormalized) pure states, and it is called a pure Operation.[3] Still more special is the case where $\hat{\Upsilon}_B$ is Hermitian, so that $\mathcal{F}_B\hat{\rho} = \hat{F}_B^{1/2}\hat{\rho}\hat{F}_B^{1/2}$.

An Operation determines the corresponding Effect via Eq. (2.6), but the reverse is not true. Many Operations lead to the same Effect. Physically we would say that many measurement models yield the same measurement statistics for a single measurement even though their internal workings are different. The Effect $\hat{F}_B$ is consistent with the pure Operation $\hat{F}_B^{1/2}\hat{\rho}\hat{F}_B^{1/2}$, but many other Operations—pure and impure—produce the same Effect.

How could one distinguish different Operations that lead to the same Effect? Not by the statistics of a single measurement, because Effects determine those statistics. One looks instead at the statistics of two or more successive measurements. Different Operations—i.e., different measurement models—lead to different post-measurement system states and thus in general to different statistics for a second measurement.

Barchielli, Lanz, and Prosperi[4] use the pure Operation $\hat{F}_B^{1/2}\hat{\rho}\hat{F}_B^{1/2}$ in their formal description of continuous position measurements. This choice corresponds to a particular measurement model,[6] which they do not specify. D'Espagnat[5] dubs this choice the "Milano theory" and seeks to compare it with standard quantum mechanics. He formulates a model for measurements of spin and shows that the "Milano theory" yields different statistics for two consecutive measurements of spin than he obtains when he applies to his model the standard quantum rule for calculating probabilities. The discrepancy he finds, while correct, has a simple explanation: the pure Operation $\hat{F}_B^{1/2}\hat{\rho}\hat{F}_B^{1/2}$ corresponds to a different measurement model than the model d'Espagnat has formulated.

We proceed in Sect. 3 to show how d'Espagnat's model is treated in terms of Effects and Operations.

3. D'ESPAGNAT'S MODEL

In this section we consider the measurement model discussed by d'Espagnat,[5] with attention paid to its description in terms of Effects and Operations. The model also makes explicit the general formalism sketched in Sect. 2.

The system is a spin-$\frac{1}{2}$ particle, whose free evolution is described by a Hamiltonian $\hat{H}_0$; the corresponding free evolution operator is $\hat{U}_0(t) = \exp(-i\hat{H}_0 t)$. The measuring apparatus is a one-dimensional quantum system; its position x serves as a pointer that provides information about the z-component of the particle's spin. The apparatus is coupled to the particle strongly for a short time; we idealize the interaction as a δ-function in time. Thus the total Hamiltonian is

$$\hat{H} = \hat{H}_0 + \alpha\delta(t - t_1)\hat{\sigma}_z\hat{p}, \tag{3.1}$$

where t_1 is the time when the apparatus is coupled to the particle, $\alpha > 0$ is a coupling constant, $\hat{\sigma}_z$ is the z-component of the particle's spin (in units of $\frac{1}{2}\hbar$), and $\hat{p}$ is the momentum conjugate to the pointer position $\hat{x}$. We assume that the apparatus has large enough mass that its free evolution can be neglected.

Let $|\phi_1\rangle$ and $|\phi_2\rangle$ be the eigenstates of $\hat{\sigma}_z$ corresponding to spin up and spin down respectively:

$$\hat{\sigma}_z|\phi_j\rangle = (-1)^{j+1}|\phi_j\rangle, \qquad j = 1, 2. \tag{3.2}$$

Suppose that at $t = 0$ the particle is in a pure state $|\psi_0\rangle$ and that the apparatus is prepared in a state with wave function $g(x)$. Then just after the interaction the joint state of the particle and apparatus becomes

$$\sum_j g(x - a_j)|\phi_j\rangle\langle\phi_j|\psi_1\rangle, \tag{3.3}$$

where $|\psi_1\rangle \equiv \hat{U}_0(t_1)|\psi_0\rangle$ is the particle's state just before the interaction at time t_1 and

$$a_j \equiv (-1)^{j+1}\alpha, \qquad j = 1, 2. \tag{3.4}$$

The interaction displaces the pointer a distance α—to the right if the particle's z-spin is up, to the left if the particle's z-spin is down. The probability density to find the pointer at position x is

$$\begin{aligned} P(x) &= \sum_j |g(x - a_j)|^2 |\langle\phi_j|\psi_1\rangle|^2 \\ &= \left\langle\psi_1\middle|\left(\sum_j |\phi_j\rangle|g(x - a_j)|^2\langle\phi_j|\right)\middle|\psi_1\right\rangle. \end{aligned} \tag{3.5}$$

Suppose now that at time t_2 the particle interacts in the same way with a second, identical apparatus, whose position is labeled by y. Just after the second interaction the joint state of the system and the two apparatuses is

$$\sum_{i,j} g(y - a_i)g(x - a_j)|\phi_i\rangle\langle\phi_i|\hat{U}_{21}|\phi_j\rangle\langle\phi_j|\psi_1\rangle, \tag{3.6}$$

where $\hat{U}_{21} \equiv \hat{U}_0(t_2 - t_1)$. The joint probability density for the two pointer positions is

$$\begin{aligned} P(x, y) = \sum_{i,j,k} &|g(y - a_i)|^2 g^*(x - a_k)g(x - a_j) \\ &\times\langle\psi_1|\phi_k\rangle\langle\phi_k|\hat{U}_{21}^\dagger|\phi_i\rangle\langle\phi_i|\hat{U}_{21}|\phi_j\rangle\langle\phi_j|\psi_1\rangle. \end{aligned} \tag{3.7}$$

D'Espagnat considers the case where one asks only whether the pointer lies to the right of the origin or to the left of the origin. This corresponds to measuring an apparatus observable $\hat{P}_{(+)} - \hat{P}_{(-)}$, which has eigenvalues ± 1; the projection operators $\hat{P}_{(+)}$ and $\hat{P}_{(-)}$ are defined by

$$\hat{P}_I \equiv \int_I dx\, |x\rangle\langle x|, \tag{3.8}$$

where I can be either the interval $(+) \equiv (0,\infty)$ or the interval $(-) \equiv (-\infty, 0)$. If the initial apparatus wave function is centered at the origin and if the interaction displaces the pointer a distance somewhat larger than the width of the wave function, then knowing on which side of the origin the pointer lies gives a good indication of whether the particle's z-spin is up or down.

The probability that the first pointer lies in the interval I is

$$P(I) = \int_I dx\, P(x) = \mathrm{tr}_S(\hat{F}_I \hat{\rho}_1). \tag{3.9}$$

Here $\hat{\rho}_1 \equiv |\psi_1\rangle\langle\psi_1|$ is the particle's density operator just before the interaction at time t_1, and

$$\hat{F}_I \equiv \sum_j |\phi_j\rangle \left(\int_I dx\, |g(x-a_j)|^2 \right) \langle\phi_j| \tag{3.10}$$

is an Effect [cf. Eq. (3.5)]. Similarly, the joint probability that the first pointer lies in the interval I_1 and the second lies in the interval I_2 is

$$\begin{aligned} P(I_1, I_2) &= \int_{I_1} dx \int_{I_2} dy\, P(x,y) \\ &= \sum_{i,j,k} \left(\int_{I_2} dy\, |g(y-a_i)|^2 \right) \left(\int_{I_1} dx\, g^*(x-a_k) g(x-a_j) \right) \\ &\qquad \times \langle\psi_1|\phi_k\rangle\langle\phi_k|\hat{U}_{21}^\dagger|\phi_i\rangle\langle\phi_i|\hat{U}_{21}|\phi_j\rangle\langle\phi_j|\psi_1\rangle. \end{aligned} \tag{3.11}$$

The probabilities (3.9) and (3.11) come directly from the standard quantum rule for calculating probabilities; in particular, to obtain $P(I_1, I_2)$ there is no need to invoke state reduction after the first measurement.

How would the same two measurements be described using Effects and Operations? Equation (3.9) gives the probability for the first measurement in terms of an Effect $\hat{F}_I$. If the first pointer is found to lie in the interval I, then the state of the particle just after the first measurement is

$$\hat{\rho}_I = \mathcal{F}_I \hat{\rho}_1 / P(I), \tag{3.12}$$

where the Operation $\mathcal{F}_I$ is defined by

$$\mathcal{F}_I \hat{\rho}_1 \equiv \sum_{j,k} \int_I dx\, g(x-a_j) g^*(x-a_k) |\phi_j\rangle\langle\phi_j|\hat{\rho}_1|\phi_k\rangle\langle\phi_k|. \tag{3.13}$$

Notice that in general $\hat{\rho}_I$ is a mixed state. Physically this is because one throws away information when one asks only on which side of the origin the pointer lies. Formally it is because $\hat{P}_{(+)}$ and $\hat{P}_{(-)}$ are not one-dimensional projectors.

The second measurement is identical to the first. Thus the conditional probability that the second pointer lies in the interval I_2, given that the first lay in I_1, is

$$P(I_2|I_1) = \mathrm{tr}_S(\hat{F}_{I_2}\hat{U}_{21}\hat{\rho}_{I_1}\hat{U}_{21}^{\dagger}). \tag{3.14}$$

The joint probability for the two measurements can now be derived as

$$P(I_1, I_2) = P(I_2|I_1)P(I_1) = \mathrm{tr}_S(\hat{F}_{I_2}\hat{U}_{21}\mathcal{F}_{I_1}\hat{\rho}_1\hat{U}_{21}^{\dagger}). \tag{3.15}$$

One easily checks that Eq. (3.15) gives the same joint probability as Eq. (3.11). The quantum rule for calculating probabilities goes directly to this joint probability. The formalism of Effects and Operations proceeds more indirectly: a probability for the first measurement, followed by a state reduction described by an Operation, then a conditional probability for the second measurement, and finally the joint probability for the two measurements. Direct or indirect, the results are the same, because they are constructed to be the same.

After formulating and analyzing his model, d'Espagnat goes on to analyze two consecutive measurements of spin using the special pure Operation $\hat{F}_I^{1/2}\hat{\rho}\hat{F}_I^{1/2}$, which he takes from Barchielli *et al.* and which he calls the "Milano theory." We now proceed in Sect. 4 to consider this pure Operation and to formulate a measurement model that corresponds to it.

4. THE PURE OPERATION AND A MODEL FOR IT

Consider now the special pure Operation $\mathcal{K}_I$ discussed in Sect. 2:

$$\begin{aligned}\mathcal{K}_I\hat{\rho}_1 &\equiv \hat{F}_I^{1/2}\hat{\rho}_1\hat{F}_I^{1/2}\\ &= \sum_{j,k}\left(\int_I dx\,|g(x-a_j)|^2\right)^{1/2}\left(\int_I dx\,|g(x-a_k)|^2\right)^{1/2}\\ &\qquad\times|\phi_j\rangle\langle\phi_j|\hat{\rho}_1|\phi_k\rangle\langle\phi_k|.\end{aligned} \tag{4.1}$$

This pure Operation clearly reproduces the Effect (3.10) for a single measurement in d'Espagnat's model—i.e., $\mathrm{tr}_S(\mathcal{K}_I\hat{\rho}_1) = \mathrm{tr}_S(\hat{F}_I\hat{\rho}_1)$; thus it leads to the same measurement statistics for a single measurement. It should be compared and contrasted, however, with the actual Operation $\mathcal{F}_I$ [Eq. (3.13)] for the model.

Suppose one uses the pure Operation $\mathcal{K}_I$ in place of the actual Operation $\mathcal{F}_I$ to analyze two consecutive measurements of spin. Using $\mathcal{K}_I$ to generate a new system state after the first measurement, one finds a joint probability for two measurements,

$$P'(I_1, I_2) = \mathrm{tr}_S(\hat{F}_{I_2}\hat{U}_{21}\mathcal{K}_{I_1}\hat{\rho}_1\hat{U}_{21}^{\dagger})$$

$$
\begin{aligned}
&= \sum_{i,j,k} \left(\int_{I_2} dx \, |g(x-a_i)|^2 \right) \\
&\quad \times \left(\int_{I_1} dx \, |g(x-a_k)|^2 \right)^{1/2} \left(\int_{I_1} dx \, |g(x-a_j)|^2 \right)^{1/2} \\
&\quad \times \langle \psi_1 | \phi_k \rangle \langle \phi_k | \hat{U}_{21}^\dagger | \phi_i \rangle \langle \phi_i | \hat{U}_{21} | \phi_j \rangle \langle \phi_j | \psi_1 \rangle
\end{aligned} \tag{4.2}
$$

[cf. Eq. (3.15)]. As d'Espagnat points out, the joint probability (4.2) is different in general from the joint probability (3.11), which comes directly from applying the standard quantum rules to his model. D'Espagnat concludes that the "Milano theory" is different from standard quantum mechanics. Having learned about Effects and Operations, we can reach a different conclusion: the "Milano theory" is a straw man, because the pure Operation $\mathcal{K}_I$ is not meant to apply to all measurement models for spin; it is only one of many Operations that correspond to the Effect $\hat{F}_I$, and it is not the Operation that applies to d'Espagnat's model.

The two descriptions coincide when the initial apparatus state and the coupling constant are chosen so that the measurements are "ideal"—i.e., when $g(x) = 0$ for $|x| > \alpha$. In this case knowing the pointer lies to the right (left) of the origin tells one unambiguously that the particle's z-spin is up (down). The Effect $\hat{F}_I$ reduces to a projection operator—$\hat{F}_{(+)} = |\phi_1\rangle\langle\phi_1|$ and $\hat{F}_{(-)} = |\phi_2\rangle\langle\phi_2|$—and the corresponding Operations $\mathcal{F}_I = \mathcal{K}_I$ project onto $|\phi_1\rangle$ and $|\phi_2\rangle$.

More generally, it is interesting to inquire what measurement model does realize the pure Operation $\mathcal{K}_I$. Since the Operations $\mathcal{K}_{(+)}$ and $\mathcal{K}_{(-)}$ take a pure state $\hat{\rho}_1 = |\psi_1\rangle\langle\psi_1|$ to (unnormalized) pure states $\mathcal{K}_{(+)}\hat{\rho}_1$ and $\mathcal{K}_{(-)}\hat{\rho}_1$, it is convenient to represent their action in terms of state vectors instead of density operators:

$$
|\psi_1\rangle \overset{\mathcal{K}_{(+)}}{\rightarrow} \Big(|\phi_1\rangle A \langle\phi_1| + |\phi_2\rangle B \langle\phi_2| \Big) |\psi_1\rangle, \tag{4.3a}
$$

$$
|\psi_1\rangle \overset{\mathcal{K}_{(-)}}{\rightarrow} \Big(|\phi_1\rangle (1-A^2)^{1/2} \langle\phi_1| + |\phi_2\rangle (1-B^2)^{1/2} \langle\phi_2| \Big) |\psi_1\rangle, \tag{4.3b}
$$

where

$$
A \equiv \left(\int_0^\infty dx \, |g(x-\alpha)|^2 \right)^{1/2}, \tag{4.4a}
$$

$$
B \equiv \left(\int_0^\infty dx \, |g(x+\alpha)|^2 \right)^{1/2}. \tag{4.4b}
$$

The only condition on A and B is that they be between zero and one inclusive. (An "ideal" measurement would correspond to the case $A = 1$ and $B = 0$.)

The form (4.3) makes easy the search for a model yielding the Operation $\mathcal{K}_I$. Since the Operation is pure and since there are only two distinguished configurations for the apparatus, we can model the apparatus as another spin-$\frac{1}{2}$ system. We choose an interaction Hamiltonian

$$
\hat{H}_{\text{int}} \equiv \hbar\beta\delta(t-t_1)\hat{\sigma}_z\hat{S}_y, \tag{4.5}
$$

where $\beta > 0$ is a coupling constant, $\hat{\sigma}_z$ is again the z-component of the particle's spin, and $\hat{S}_y$ is the y-component of the apparatus's spin (in units of $\frac{1}{2}\hbar$). The interaction causes the apparatus spin to precess about the y-axis—in the positive sense if the particle's z-spin is up, in the negative sense if the particle's z-spin is down.

The evolution operator corresponding to $\hat{H}_{\text{int}}$ is

$$\hat{U} = \exp(-i\beta\hat{\sigma}_z\hat{S}_y) = \hat{1}\cos\beta - i\hat{\sigma}_z\hat{S}_y\sin\beta. \tag{4.6}$$

Let the initial apparatus wave function be

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}; \tag{4.7}$$

here we use matrix notation in the basis where the apparatus z-spin $\hat{S}_z$ is diagonal. Just after the interaction the joint state of the particle and the apparatus becomes

$$\sum_j \begin{pmatrix} \cos(\theta+\beta_j) \\ \sin(\theta+\beta_j) \end{pmatrix} |\phi_j\rangle\langle\phi_j|\psi_1\rangle, \tag{4.8}$$

where

$$\beta_j \equiv (-1)^{j+1}\beta, \qquad j = 1, 2. \tag{4.9}$$

If the apparatus is observed to have spin up in the z-direction, then the particle's state changes according to

$$|\psi_1\rangle \to \Big(|\phi_1\rangle\cos(\theta+\beta)\langle\phi_1| + |\phi_2\rangle\cos(\theta-\beta)\langle\phi_2|\Big)|\psi_1\rangle. \tag{4.10a}$$

Similarly, if the apparatus is observed to have spin down in the z-direction, then the particle's state changes according to

$$|\psi_1\rangle \to \Big(|\phi_1\rangle\sin(\theta+\beta)\langle\phi_1| + |\phi_2\rangle\sin(\theta-\beta)\langle\phi_2|\Big)|\psi_1\rangle. \tag{4.10b}$$

These changes represent the same pure Operation as Eqs. (4.3), with the identifications $A = \cos(\theta+\beta)$ and $B = \cos(\theta-\beta)$.

Just as in d'Espagnat's model, we can obtain an "ideal" measurement by choosing the initial apparatus state and the coupling strength appropriately—i.e., by choosing the angle $\theta = -\pi/4$, so that the apparatus spin is initially oriented along the negative x-axis, and choosing the coupling angle $\beta = \pi/4$, so that the apparatus spin rotates by an angle $\pi/2$. Then, after the interaction, the apparatus z-spin is up (down) if the particle's z-spin was up (down), and the Operations represented by Eqs. (4.10) project onto the states $|\phi_1\rangle$ and $|\phi_2\rangle$.

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