Entanglement entropy in all dimensions

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ABSTRACT: It has long been conjectured that the entropy of quantum fields across boundaries scales as the boundary area. This conjecture has not been easy to test in spacetime dimensions greater than four because of divergences in the von Neumann entropy. Here we show that the Rényi entropy provides a convergent alternative, yielding a quantitative measure of entanglement between quantum field theoretic degrees of freedom inside and outside hypersurfaces. For the first time, we show that the entanglement entropy in higher dimensions is proportional to the higher dimensional area. We also show that the Rényi entropy diverges at specific values of the Rényi parameter q in each dimension, but this divergence can be tamed by introducing a mass to the quantum field.

Entanglement, a term first coined by Schrödinger, is an intriguing and quintessentially quantum mechanical property, which correlates microscopic systems in a precise way, even if they are separated by large distances. On the one hand it gives rise to apparent contradictions (such as the EPR paradox) and on the other, hides enormous untapped resources for computation and communication (e.g., via teleportation). A mathematically precise way of measuring entanglement has remained elusive however, except in the simplest cases where the combined system is in a pure state, i.e., for which all quantum numbers are known. Usually, the entanglement entropy is computed as the von Neumann entropy associated with ρ: $S_{\mathrm{vN}} = -\mathrm{tr}\,(\rho \ln \rho)$. For recent reviews, see [1, 2].

Entanglement as a source of entropy has also been examined in an entirely different context, that of the microscopic origin of black hole entropy. The basic idea is that the long-range entanglement of the quantum fields across a black-hole horizon can leave its mark on the reduced density matrix of the external degrees of freedom (DOF) which, in turn, accounts for the black-hole entropy. Starting with [3, 4], it was demonstrated that to leading order, and for a scalar field in its global ground state, the entanglement entropy between its DOF inside and outside a black hole horizon is proportional to the area of the horizon. Recently, it was demonstrated that the DOF at or near the horizon contribute to most of this entropy [5] and that excited states of the field lead to sub-leading order power law corrections [6, 7].

However, entanglement as a source of black-hole entropy has a couple of drawbacks: (i) The proportionality constant depends on the ultra-violet cut-off and the number of fields present (also these are in general independent of each other, although it was recently heuristically argued that the requirement of the stability of the cosmos relates the two and naturally relates the cut-off to the Planck mass [8]), and (ii) it is not evident that the area proportionality holds for higher spacetime dimensions, say $D + 2 > 4$ (throughout $D + 1$ denotes the number of *spatial* dimensions)?

Although there have been attempts in the literature [9–12], however, it is yet to be shown from first principles. For instance, in refs. [9–11], the authors attempt to obtain universal expressions for higher dimensions from two-dimensional (conformal field theory) entropy c-functions. It is also interesting to point that in all these references [9–11], it is implicitly assumed that Srednicki's analysis is extendable to all dimensions (see page 23 of ref. [9]). Following Srednicki [4], if one regularizes the entropy function by introducing a radial lattice, the sum of partial waves does not converge and the entropy turns out to be infinite in higher dimensions [13]. In ref. [12], the convergence of the eigenvalues (and, hence, the entropy) assumes that the parameter c is less than $1/(2d)$ where d is the number of space dimensions. For higher dimensional black-holes, the brick-wall entropy contains extra divergent terms other than from the ultra-violet modes [14]. By using a different measure of entropy, we show that a *non-divergent* entropy-area relation can be obtained for all dimensions, and furthermore, that the divergences are similar in nature to the infrared divergences in QED, which can be tamed by introducing a mass to the field.

While the von Neumann entropy is the most common measure of entanglement, it is neither the most general, nor unique. There are other measures, such as the Rényi and Tsallis entropies, which under certain limits reduce to the von Neumann entropy. In this

work we study entanglement via the Rényi entropy defined as

$$S^{(q)} \equiv \frac{1}{1-q} \ln\left(\sum_{i=1}^{n} p_i^q\right). \tag{1}$$

In the limit that $q \to 1$, the Rényi entropy reduces to its von Neumann counterpart. Also, like von Neumann entropy (and unlike Tsallis entropy) Rényi entropy is additive and has maximum value $\ln(n)$ for $p_i = 1/n$. In the framework of statistical mechanics, Rényi entropy may be physically interpreted as the q-derivative of the Free energy with-respect-to temperature [15].

Consider a free massless real scalar field propagating in a $(D + 2)$-dimensional flat spacetime with action

$$\frac{1}{2} \int dt\, dr\, r^D d\Omega_D (\eta^{ab}\partial_a \Phi\, \partial_b \Phi + g^{\theta_n \theta_n}\, \partial_{\theta_n}\Phi\, \partial_{\theta_n}\Phi), \tag{2}$$

where the Latin indices (a and b) take the values t and r, Ω_D is the D-dimensional solid angle, η^{ab} is the Minkowski metric, and $g^{\theta_n \theta_n}$ are the metric coefficients for the angular coordinates. Decomposing the scalar field in terms of real hyper-spherical harmonics one has

$$\Phi(x^\mu) = \sum_{\ell m_i} r^{D/2}\, \varphi_{\ell m_i}(t,r)\, Z_{\ell m_i}(\theta, \phi_i), \quad 1 \le i \le D-1.$$

The Hamiltonian is given by $H = \sum_{\ell, m_i} H_{\ell, m_i}$ where

$$H_{\ell, m_i} = \frac{1}{2} \int_0^\infty dr \left\{ \pi_{\ell m_i}^2 + \frac{\ell(\ell + D - 1)}{r^2}\, \varphi_{\ell m_i}^2 + r^D \left[\partial_r \left(\frac{1}{r^{D/2}}\, \varphi_{\ell m_i}\right)\right]^2 \right\}, \tag{3}$$

and where $\pi_{\ell m_i} = \partial_t \varphi_{\ell m_i}$ is the canonical momentum. Note that the Hamiltonian of a massless scalar field propagating in a general (i.e., not necessarily flat) spherically symmetric spacetime at fixed Lemaitre time coordinate reduces to the above Hamiltonian [7].

For simplicity, we assume that the scalar field is in the ground (vacuum) state. As mentioned earlier, our interest is in determining the quantity of information shared by modes across a "horizon." This may be found by eliminating (tracing out) the quantum degrees of freedom associated with the scalar field inside a spherical region of radius $\mathcal{R}$ (the location of our "horizon"). The resulting reduced density matrix can then be used to determine the strength of quantum correlations across the horizon.

It is not possible to obtain a closed form analytic expression for the density matrix and hence, we need to resort to numerical methods. In order to do that we take a spatially uniform radial grid, $\{r_j\}$, with $a = r_{j+1} - r_j$. To achieve better precision, we adopt the middle-prescription in discretizing the terms containing the derivatives, so

$$A(r)\, \partial_r[G(r)] \longrightarrow A\left(j + \frac{1}{2}\right) [G(j+1) - G(j)]/a, \tag{4}$$

Discretizing the Hamiltonian (3), and suppressing the subscripts l, m_i, leads to

$$H = \sum_j H_j = \frac{1}{2a} \sum_{i,j}^{N} (\delta_{ij}\pi_j^2 + \varphi_j\, K_{ij}\, \varphi_i), \tag{5}$$

where the interaction matrix, K_{ij}, is given by

$$
\frac{1}{j^D} \left[\left(j+\frac{1}{2}\right)^D + j^{D-2}\ell(\ell+D-1) \right] \delta_j^i \delta_i^1
$$
$$
+ \frac{1}{j^D} \left[\left(j+\frac{1}{2}\right)^D + \left(j-\frac{1}{2}\right)^D + j^{D-2}\ell(\ell+D-1) \right] \delta_j^i \delta_i^k
$$
$$
- \frac{\left(j+\frac{1}{2}\right)^D}{j^{D/2}\, i^{D/2}}\, \delta_i^{j+1} - \frac{\left(i+\frac{1}{2}\right)^D}{i^{D/2}\, j^{D/2}}\, \delta_j^{i+1}
$$
$$
+ \frac{1}{j^D} \left[\left(N-\frac{1}{2}\right)^D + j^{D-2}\ell(\ell+D-1) \right] \delta_j^i \delta_i^N, \tag{6}
$$

and $2 \leq k \leq (N-1)$. The procedure to obtain the entanglement entropy in higher dimensions is similar to the one discussed in refs. [4, 16] (chapter 6). In this work we assume that the quantum state corresponding to the Hamiltonian of the N-Harmonic oscillator system (5) is the ground state with wave-function $\Psi_{GS}(x_1,\ldots,x_n;t_1,\ldots,t_{N-n})$. The reduced density matrix $\rho(\vec{t},\vec{t}')$ is obtained by tracing over the first n of the N oscillators

$$
\int (\Pi_{i=1}^n dx_i)\, \Psi_{GS}^*(x_1,\ldots,x_n;\vec{t})\Psi_{GS}(x_1,\ldots,x_n;\vec{t}'). \tag{7}
$$

The Rényi entanglement entropy in $(D+2)$-dimensional spacetime is then given by

$$
S^{(q)}(n,N) = \sum_\ell (2\ell+D-1)\, \mathcal{W}(\ell)\, S_\ell^{(q)}(n,N), \tag{8}
$$
$$
\mathcal{W}(\ell) = \frac{(\ell+D-2)!}{(D-1)!\,\ell!}, \tag{9}
$$

is the angular degeneracy factor [14, 17] and $S_\ell^{(q)}(n,N)$ is the Rényi entropy for partial waves with total angular momentum ℓ.

To highlight the advantage of Rényi entropy as the measure of the entanglement entropy as compared to von Neumann entropy, we determine the Rényi entropy in the large $\ell \gg N$ limit. In this limit, eq. (8) becomes

$$
S^{(q)}(n,N) \simeq \sum_\ell (2\ell+D-1)\frac{\mathcal{W}(\ell)}{1-q}\, \xi_\ell^{\,q}, \tag{10}
$$

where the asymptotic eigenvectors and angular degeneracy factors, respectively, are

$$
\xi_\ell \simeq \frac{1}{2^{2(D+1)}} \frac{(2n+1)^{2(D+1)}[n(n+1)]^{3-D}}{\ell^2(\ell+D-1)^2}
$$
$$
\mathcal{W}(\ell) \simeq \frac{1}{(D-1)!} \left(\frac{\ell}{e}\right)^{D-2}. \tag{11}
$$

Substituting these expressions into eq. (10) yields

$$
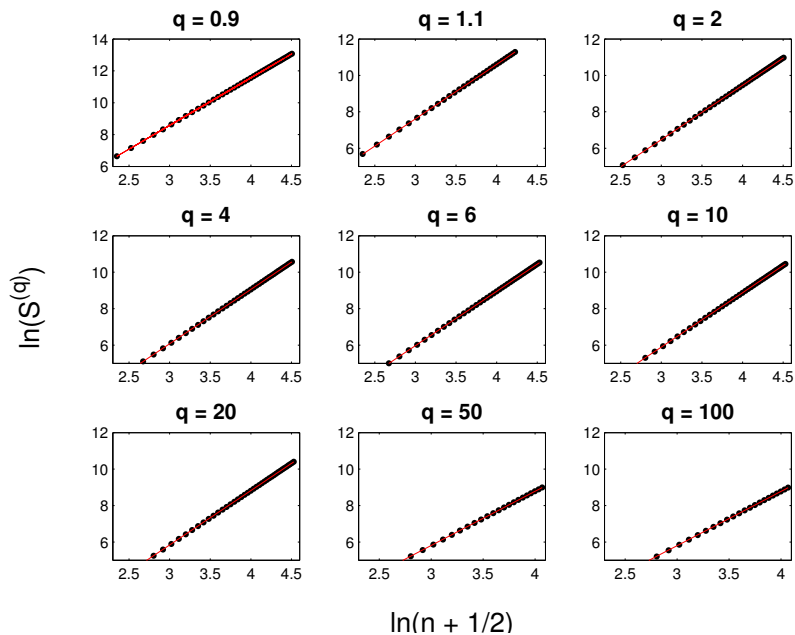S^{(q)}(n,N) \sim \sum_\ell \ell^{D-1-4q}. \tag{12}
$$

Figure 1. Log-log plot of Rényi entropy, $S^{(q)}$, versus the scaled radius of the sphere, $s \equiv \mathcal{R}/a = n + \frac{1}{2}$, for $D = 3$, $N = 300$ and $100 \leq n \leq 200$. The black dots represent the numerical output and straight lines denote lines of best fit. These fits are: $S^{(0.9)} = 0.69\,s^{2.99}$, $S^{(1.1)} = 0.26\,s^{2.99}$, $S^{(2)} = 0.068\,s^{2.99}$, $S^{(4)} = 0.051\,s^{2.99}$, $S^{(6)} = 0.050\,s^{2.98}$, $S^{(10)} = 0.047\,s^{2.98}$, $S^{(20)} = 0.045\,s^{2.98}$, $S^{(50)} = 0.045\,s^{2.98}$ and $S^{(100)} = 0.045\,s^{2.97}$.

In the limit $q \to 1$, eq. (12) is identical to the asymptotic limit of the von Neumann entropy, and diverges for all $D > 2$ (as also noted in refs. [4, 13]). By contrast, if that limit is not taken, eq. (12) converges for all $q > D/4$. For instance, in 4-dimensional spacetime this implies that for all values of $q > \frac{1}{2}$ entanglement entropy converges, while in 6-dimensions, this entropy converges only for $q > 1$. Herein lies the main advantage of using Rényi entropy. The above asymptotic expression also provides an understanding as to why von Neumann entropy converges for 4-dimensions and not for 6-dimensions.

We will now show that the Rényi entropy indeed provides a good measure for the entanglement entropy in all dimensions for $q > D/4$, and also reproduces the area proportionality in each case. We compute the Rényi entropy numerically for the discretized Hamiltonian (5). The computations are done using Matlab for the lattice size $N = 300$, $100 \leq n \leq 200$ and the relative error in the computation of Rényi entropy is 10^{-6}. The computations were done for 4-, 5-, 6- and 10-dimensional spacetimes. However, in this manuscript we only present the results for 5 and 10 dimensions.

In figures 1 and 2 we have plotted $\ln S^{(q)}$ versus $\ln(R/a)$ for 5- and 10-dimensional spacetimes, respectively. From the best fit curves in figure 1 we see that for different values of q the Rényi entropy generically scales with an approximate power law $S^{(q)} \sim (\mathcal{R}/a)^3$. Although as we increase q, the prefactor decreases, nonetheless the power remains close to 3. Similarly, in the case of 10-dimensional spacetime (figure 2), for all q, the power always remains close to 8. It is important to note that, although it has long been
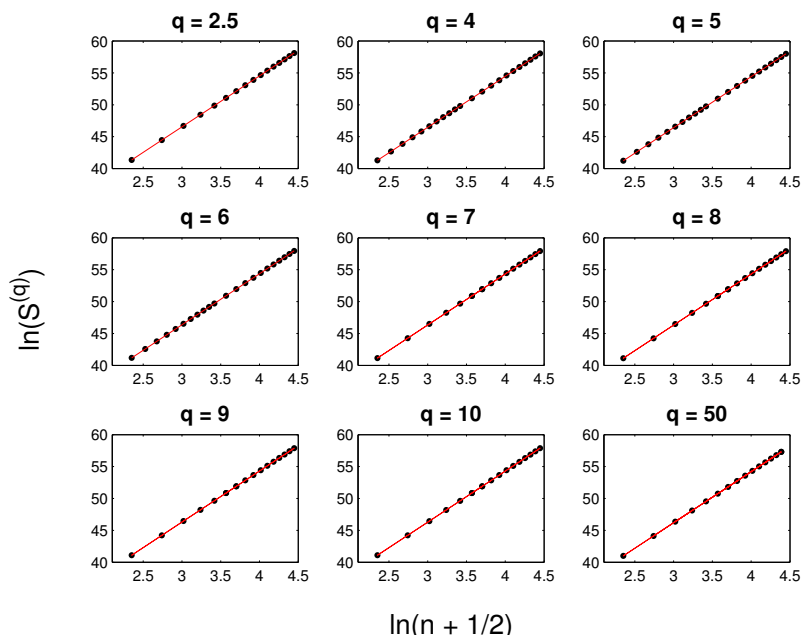
Figure 2. Log-log plot of Rényi entropy, $S^{(q)}$, versus the scaled radius of the sphere, $s \equiv \mathcal{R}/a = n + \frac{1}{2}$, for $D = 8$, $N = 300$ and $100 \leq n \leq 200$. The fits are: $S^{(2.5)} = 6.08 \times 10^9\, s^{8.00}$, $S^{(4)} = 5.71 \times 10^9\, s^{8.00}$, $S^{(5)} = 5.35 \times 10^9\, s^{8.00}$, $S^{(6)} = 5.14 \times 10^9\, s^{8.00}$, $S^{(7)} = 5.01 \times 10^9\, s^{8.00}$, $S^{(8)} = 4.89 \times 10^9\, s^{8.00}$, $S^{(9)} = 4.81 \times 10^9\, s^{8.00}$, $S^{(10)} = 4.76 \times 10^9\, s^{8.00}$, and $S^{(50)} = 4.37 \times 10^9\, s^{8.00}$.
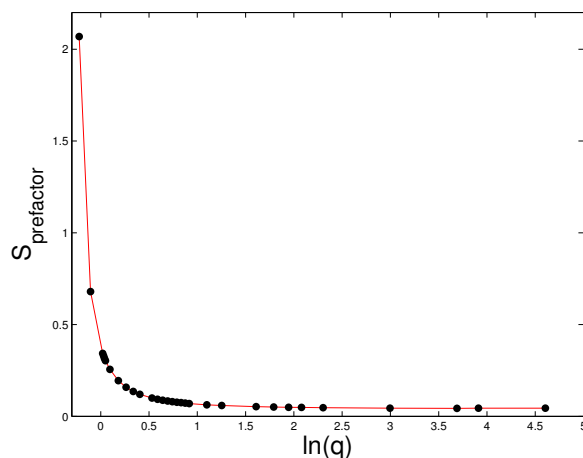

Figure 3. Plot of the prefactor, $S_{\mathrm{prefactor}}$, of the Rényi entropy in 5-dimensional spacetime versus $\ln(q)$. The small circles represent the numerical output for each run; the red line shows a spline interpolation. The divergence is apparent as $q \to \frac{3}{4}$.

conjectured in the literature that the entanglement entropy is proportional to the area of the D-dimensional surface (for $D \geq 3$), this is the first time this has been explicitly shown in higher spatial dimensions.

In figure 3 the prefactor $S_{\mathrm{prefactor}}$, (the ratio of the Rényi entropy to 5-dimensional area) is plotted versus q, from which we infer the following: (i) Rényi entropy saturates to a constant value for large q. (ii) Consistent with the asymptotic analysis of eq. (12),
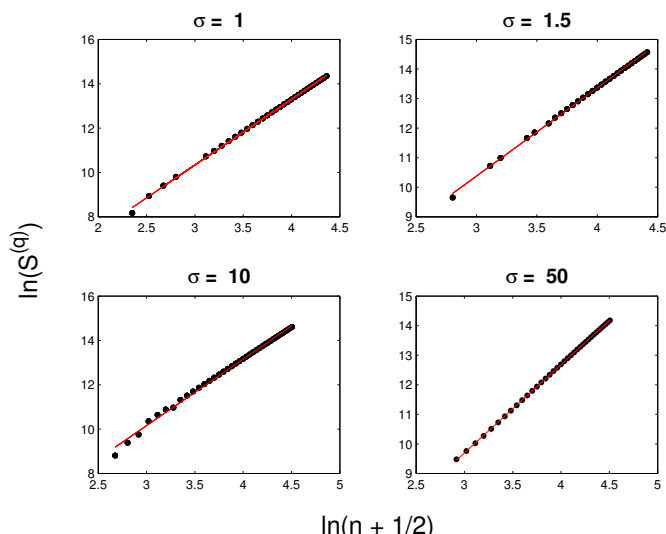
Figure 4. Log-log plot of Rényi entropy, $S^{(q)}$, versus the scaled radius of the sphere, $s \equiv \mathcal{R}/a = n + \frac{1}{2}$, in 5-dimensional spacetime for $q = \frac{3}{4}$ (which is divergent in the massless case) and different non-zero scaled masses $\sigma = ma$. Here $N = 300$ and $100 \leq n \leq 200$. The black dots represent the numerical output and straight lines are the lines of best fit. These fits are: $S^{(3/4)}(\sigma = 1) = 4.25 \, s^{2.97}$, $S^{(3/4)}(\sigma = 1.5) = 4.16 \, s^{2.99}$, $S^{(3/4)}(\sigma = 10) = 3.20 \, s^{2.99}$ and $S^{(3/4)}(\sigma = 50) = 2.07 \, s^{2.99}$.

the Rényi entropy diverges logarithmically as $q \to \frac{3}{4}$. Similarly, for $(D+2)$-dimensional spacetime, it diverges as $q \to D/4$.

To understand this divergence further, we now consider a scalar field of mass m. The new interaction matrix is

$$K_{ij}^{(m)} = K_{ij} + (m\,a)^2 \delta_{ij} \,, \tag{13}$$

where K_{ij} is given by eq. (6). In figure 4 we plot $\ln S^{(q)}$ versus $\ln(R/a)$ again for 5-dimensional spacetime *but* for $q = \frac{3}{4}$, and for different values of m, and show that the divergence disappears! We also note that as the mass increases, the prefactor $S_{\text{prefactor}}$ of the Rényi entropy modestly decreases.

To summarize, in this work we have computed Rényi entropy after tracing out the degrees of freedom inside a D-dimensional sphere of radius $\mathcal{R}$, and show that the entanglement entropy is proportional to the area of the D-dimensional surface (for $D \geq 3$). Although conjectured before, this is the first time to our knowledge, that such area proportionality has been shown explicitly for higher spatial dimensions. We have also shown that the logarithmic divergence of Rényi entropy can be tamed by considering massive fields.

We conclude with the following points:

1. Eq. (8) involves summation over partial waves ℓ that extend to infinity. Since the massless Klein-Gordon field does not have an intrinsic length scale, all angular momentum modes may contribute. However, for large enough values of ℓ, the energy corresponding to the angular momentum modes will exceed the Planck energy. One way to address this difficulty may be to introduce higher-order derivatives to the scalar field action. However, it is not clear whether the entanglement entropy-area relation would emerge in this case.

2. The entanglement entropy of a massless scalar-field is plagued by ultra-violet divergences. This is particularly disturbing since it is a free field theory without interactions, for which one normally understands how to absorb those divergences. Here on the other hand, the ultra-violet cut-off seems real, without however any clear physical meaning or knowledge about how it is implemented in the real world.

3. Over the last decade it has been shown that higher-dimensional black-holes can have a much richer topological structure than 4-dimensional black-holes. For instance, it was shown by Emparan and Reall [18] that horizons with a topology of $S^2 \times S^1$ can exist for some asymptotically flat spacetimes in 5-dimensions. It would be interesting to investigate whether the entanglement entropy-area relation holds for such topologies.

Acknowledgments

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