PHYSICAL REVIEW LETTERS

VOLUME 69 | 12 OCTOBER 1992 | NUMBER 15

Maximum-Likelihood Analysis of Multiple Quantum Phase Measurements

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(Received 1 June 1992)

Shapiro, Shepard, and Wong [Phys. Rev. Lett. **62**, 2377 (1989)] suggested that a scheme of multiple phase measurements, using quantum states with minimum "reciprocal peak likelihood," could achieve a phase sensitivity scaling as $1/N_{\rm tot}^2$, where $N_{\rm tot}$ is the mean number of photons available for all measurements. We have simulated their scheme for as many as 240 measurements and have found optimum phase sensitivities for $3 \leq N_{\rm tot} \leq 120$. A power-law fit to the simulated data yields a phase sensitivity that scales as $1/N_{\rm tot}^{0.85\pm 0.01}$. We conclude that reciprocal peak likelihood is not a good measure of sensitivity.

PACS numbers: 03.65.Bz, 42.50.Dv, 89.70.+c

Precise determination of small changes in the phase of light has been an important tool for physics and measurement science for over a hundred years [1] and continues in that role today [2]. Even with this long history, we have begun only recently to understand the ultimate bounds on how efficiently we can measure phase.

For interferometers using coherent states of light, the Poisson counting statistics produced by the independent arrivals of photons at the detector yields an optimum phase sensitivity $\Delta\Phi = 1/2\sqrt{N_{\rm tot}}$, where $N_{\rm tot}$ is the mean total number of photons available during the time the phase to be determined is stable. The $1/2\sqrt{N_{\rm tot}}$ behavior, called the "shot-noise" limit, is not fundamental. It has been shown experimentally [3] that interferometry with squeezed states can beat the shot-noise limit, as was predicted theoretically [4]. The optimum sensitivity for squeezed-state interferometry is believed [5] to scale as $\Delta\Phi \sim 1/N_{\rm tot}$, with a multiplicative constant of order unity.

The theoretical basis for handling phase measurements quantum mechanically has generated controversy. Although there have been many approaches to the quantum-mechanical phase [6–10], a common feature has been the introduction of a probability distribution $P(\phi|\Phi)$ for the phase ϕ, which is derived from the quantum state $|\psi_\Phi\rangle$. Here Φ, the phase shift to be determined, can be thought of as a parameter that gives the "location" of $|\psi_\Phi\rangle$—on a phase-space diagram or, more generally, along a curve in Hilbert space. The goal of phase measurements is to determine Φ as precisely as possible; the $\Delta\Phi$ quoted above for interferometric measurements are precise statements of the statistical confidence in an experimentally determined value of Φ.

In order to address fundamental limits on phase measurements, Shapiro, Shepard, and Wong (SSW) [11] (see also Ref. [12]) assumed and analyzed *ideal* phase measurements, whose statistics are given by $P(\phi|\Phi)$. The question of how well we can determine Φ then reduces to finding the optimal shape of $P(\phi|\Phi)$, given a constraint on the number of photons. Using their familiarity with ordinary parameter estimation, SSW sought the quantum state whose phase distribution has the highest peak. To ensure normalizability, they introduced a cutoff in the number-state basis and so found a family of states

$$|\psi_\Phi\rangle_{\rm SSW} = A\sum_{n=0}^{M}\frac{e^{in\Phi}}{n+1}|n\rangle\ , \tag{1}$$

with normalization constant $A = \sqrt{6}/\pi + O(M^{-1})$. The phase distribution for the SSW state (1) is

$$P_{\rm SSW}(\phi|\Phi) = \frac{A^2}{2\pi}\left|\sum_{n=0}^{M}\frac{e^{-in(\phi-\Phi)}}{n+1}\right|^2\ . \tag{2}$$

These states have a mean photon number

$$\langle n\rangle = -1 + 6\left[\gamma + \ln(M+1)\right]/\pi^2 + O(M^{-1})\ , \tag{3}$$

determined solely by the cutoff M, where $\gamma \simeq 0.577\,21$ is

Euler's constant.

The parameter Φ gives the location of the peak of the SSW phase distribution (2). Figure 1 shows examples of the SSW distribution for small values of $\langle n\rangle$. For $\langle n\rangle \gtrsim 2$ the distribution takes on a universal shape in the domain $|\phi-\Phi| \gg 1/M$:

$$P_{\rm SSW}(\phi|\Phi) \simeq \frac{A^2}{2\pi}|\ln(1-e^{-i(\phi-\Phi)})|^2 \,. \tag{4}$$

For $|\phi-\Phi| \lesssim 1/M$ the photon-number cutoff softens the logarithmic singularity into a Gaussian peak with height

$$P_{\rm SSW}(\Phi|\Phi) = (\langle n\rangle+1)^2/2\pi A^2 \,. \tag{5}$$

For $\langle n\rangle \gtrsim 2$ the picture is that of a universal shape, consisting of broad tails and a runup to a central peak, whose height $\sim \langle n\rangle^2$ and width $\sim 1/M$ are the only features that depend on $\langle n\rangle$.

By analogy with the efficiency of maximum-likelihood (ML) estimation to locate the edge of a rectangular distribution, SSW introduced the "reciprocal peak likelihood,"

$$\delta\phi \equiv \left(\sup_{-\pi<\Phi\le\pi} P(\phi|\Phi)\right)^{-1} \underset{\rm SSW}{=} \frac{2\pi A^2}{(\langle n\rangle+1)^2} \,, \tag{6}$$

as their measure of phase sensitivity and concluded that the sensitivity of ideal phase measurements using their states scales as 1/(mean photon number)^2. A single sampling of $P_{\rm SSW}(\phi|\Phi)$, however, yields almost no information about Φ, because of the broad tails. In contrast, for mean photon number $\langle n\rangle \gg 1$, the phase probability distribution of a coherent state or a squeezed state is very nearly a Gaussian, with central peak at $\phi=\Phi$ and standard deviation $1/2\langle n\rangle^{1/2}$ for a coherent state or $\sim 1/\langle n\rangle$ for an optimized squeezed state. A Gaussian has no broad tails, almost all of the probability being concentrated within a few standard deviations of the central peak; a single sampling provides an estimate of Φ, which is good to within a few standard deviations.

SSW argued that many samplings of $P_{\rm SSW}(\phi|\Phi)$ could determine the peak's location accurately. To perform N measurements, however, requires N copies of the SSW state and thus a mean *total* number of photons $N_{\rm tot}=\langle n\rangle N$. We *interpret* SSW's proposal as a suggestion that the phase sensitivity of multiple SSW measurements scales as $\Delta\Phi \sim 1/N_{\rm tot}^2$.

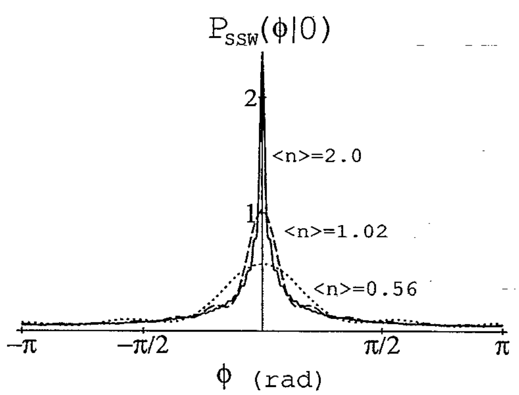

FIG. 1. SSW distribution $P_{\rm SSW}(\phi|\Phi)$, with peak at $\Phi=0$ and with mean photon numbers $\langle n\rangle=0.56$ (dotted line), 1.02 (dashed line), and 2.00 (solid line).

We investigate this suggestion by performing Monte Carlo simulations of multiple SSW phase measurements. We assume that N field modes, which we call "pulses," are prepared in the same SSW state and are shifted in phase by the same amount Φ. An ideal phase measurement is performed on each. The measurements yield N results $\phi_1,\ldots,\phi_N$ (our N "samples"), from which we must estimate the actual phase shift Φ. Because SSW's work is based on ML estimation and because ML estimation is optimal asymptotically in N, we use the ML estimator for Φ,

$$\Phi_{\rm MLE} = \arg\max_{-\pi<\Phi\le\pi} \mathcal{L}(\phi_1,\ldots,\phi_N|\Phi) \,, \tag{7}$$

where

$$\mathcal{L}(\phi_1,\ldots,\phi_N|\Phi) \equiv \sum_{j=1}^{N} \ln P_{\rm SSW}(\phi_j|\Phi) \tag{8}$$

is the log-likelihood. This entire process—preparing and phase shifting the pulses, collecting the phase data, and estimating Φ—is a single "experiment"—an idealization of a multiple-measurement laboratory experiment. The ultimate result of an experiment is a single number, the ML estimate $\Phi_{\rm MLE}$.

An experimenter wants to know how much confidence should be placed in the estimate $\Phi_{\rm MLE}$. Thus we seek to determine the probability distribution $P(\Phi_{\rm MLE}|\Phi)$ of the ML estimator, given an actual phase shift Φ. This "estimator distribution" contains *all* statistical information about the estimator. To summarize important information in a single number, we give the one-sided 68.26% confidence interval $(\Delta\Phi)_{68\%}$, defined by saying that the estimator has a 68.26% probability to lie within an interval of width $2(\Delta\Phi)_{68\%}$, which brackets symmetrically the actual phase shift Φ. Such a confidence interval is unquestionably a good measure of phase sensitivity. We use the one-sided 68.26% confidence interval because it corresponds to 1 standard deviation of a Gaussian.

The asymptotic behavior of ML estimation is known from Fisher's theorem [13]: As $N\to\infty$ the estimator distribution approaches a Gaussian with standard deviation

$$\Delta\Phi = 1/\sqrt{NF} \,. \tag{9}$$

Here F, the Fisher information [13], is determined by the phase distribution. For the SSW distribution, F is a function only of $\langle n\rangle$ (or M) and increases roughly linearly with M. ML estimation is also asymptotically efficient: The Cramér-Rao lower bound [13] states that for a very broad class of estimation techniques, $\Delta\Phi \ge 1/\sqrt{NF}$ for

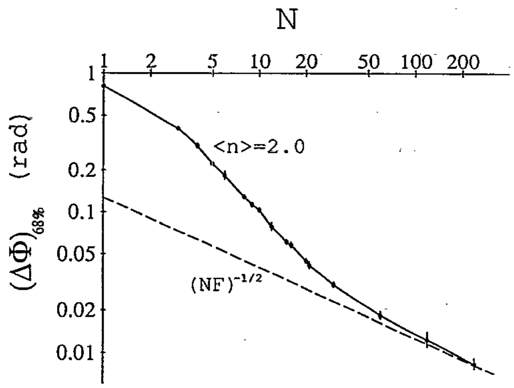

FIG. 2. Log-log plot of 68% confidence intervals $(\Delta\Phi)_{68\%}$ (in radians) vs number of pulses N for SSW states with $\langle n \rangle = 2.00$ (error bars represent 95% confidence in the plotted confidence intervals). The dashed line gives the $(NF)^{-1/2}$ Cramér-Rao lower bound on the standard deviation $\Delta\Phi$ of any estimator, with the Fisher information F calculated for $\langle n \rangle = 2.00$.

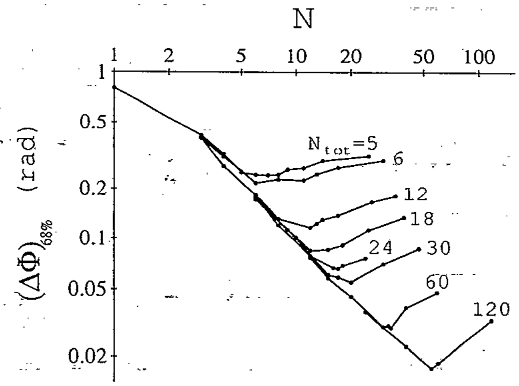

FIG. 3. Log-log plot of 68% confidence intervals $(\Delta\Phi)_{68\%}$ (in radians) vs number of pulses N for selected values of $N_{\rm tot}$. Error bars are suppressed, and points with the same $N_{\rm tot}$ are joined by line segments. For each $N_{\rm tot}$ there is an optimum number of pulses which gives the best phase sensitivity.

all sample sizes N.

Monte Carlo simulation proceeds by doing a sufficient number of experiments for particular values of $\langle n \rangle$ and N to map out the estimator distribution $P(\Phi_{\rm MLE}|0)$ for those values. Since the simulations are demanding of computer time, we have only been able to do a sufficient number of numerical experiments for representative values in the range $\langle n \rangle \leq 5$, $N \leq 240$, and $N_{\rm tot} \leq 480$.

Figure 2 shows the convergence of the 68% confidence interval with increasing numbers of pulses N for the SSW state with $\langle n \rangle = 2.00$. For $N \lesssim 3$ the ML estimator distribution $P(\Phi_{\rm MLE}|0)$ is so wide that it is sensitive to the finite 2π range of phases. At the other extreme, for $N \gtrsim 50$ the asymptotic $1/\sqrt{NF}$ behavior of Fisher's theorem takes over as the confidence intervals become small enough to probe the $1/M$ width of the central peak. Between these two regions lies a region of "universal convergence," where ML estimation is unable to "see" the narrow central peak. In this region the confidence intervals are determined by the universal shape of the SSW distribution, and hence the convergence of the confidence intervals must follow some universal form *independent of* $\langle n \rangle$. Unfortunately, we have been unable to discover the scale-invariant shape that characterizes this preasymptotic universal convergence. A power-law fit to our data for $\langle n \rangle = 3$, 4, and 5, with $3 \leq N \leq 40$—all of which data are comfortably within the preasymptotic region—yields a $1/N^{1.13}$ scaling for the preasymptotic 68% confidence intervals.

The universal convergence in the preasymptotic region is easier to see in Fig. 3, where we plot 68% confidence intervals for selected values of $N_{\rm tot}$. Points with the same $N_{\rm tot}$ are joined by straight line segments. By following the confidence intervals with $N_{\rm tot}$ fixed, we can see the effect of changing the split between the number of pulses, N, and the mean number of photons per pulse, $\langle n \rangle$. As we split up the photon-number resources into pulses, the preasymptotic universal convergence gives initially an improvement in the 68% confidence intervals, until they approach the Cramér-Rao lower bound. Then the curves turn up into the asymptotic $1/\sqrt{NF}$ behavior of Fisher's theorem, which here has an upward slope because $N_{\rm tot}$ is fixed. The optimal split occurs at the "knee" between the preasymptotic universal convergence and Fisher's asymptotic behavior.

Figure 4 plots the optimized 68% confidence intervals, obtained by using the optimal split of resources from Fig. 3, against the mean total number of photons available, i.e., against the constrained quantity $N_{\rm tot}$. A straight-line fit yields

$$(\Delta\Phi)_{68\%} \sim 1/N_{\rm tot}^{0.85\pm0.01} \tag{10}$$

for the optimized 68% confidence intervals over the range

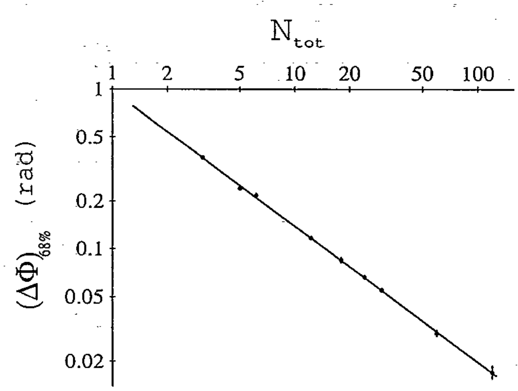

FIG. 4. Log-log plot of optimized 68% confidence intervals $(\Delta\Phi)_{68\%}$ (in radians; optimum number of pulses from Fig. 3) vs mean total number of photons $N_{\rm tot}$ (error bars represent 95% confidence in the plotted confidence intervals). The best-fit line corresponds to $(\Delta\Phi)_{68\%} = 0.97\,N_{\rm tot}^{-0.85}$.

$3 \leq N_{\text{tot}} \leq 120$ that is accessible to our simulations. Over this range the convergence rate of confidence intervals, obtained from ML estimation, is significantly slower than the $1/N_{\text{tot}}^2$ scaling which SSW extracted from the reciprocal peak likelihood. We conclude that reciprocal peak likelihood is not a good measure of sensitivity. Others [14–16] have argued that reciprocal peak likelihood is not a good measure of phase sensitivity, but our work, by finding the sensitivity of multiple SSW measurements, *demonstrates* the inadequacy of reciprocal peak likelihood. Furthermore, over the range of our simulations, we find that multiple SSW measurements do not have a phase sensitivity as good as the $1/N_{\text{tot}}$ scaling of squeezed-state interferometry.

The relevant scale in the SSW distribution is the central-peak width $1/M \sim e^{-\pi^2\langle n\rangle/6}$. This width is exponentially small in $\langle n\rangle$ [14], but many pulses are required to locate the central peak with an accuracy approaching the width, because of the small probability $\sim \langle n\rangle^2/M \sim (\ln M)^2/M$ under the peak. Indeed, the central-peak width is, crudely speaking, the phase sensitivity at the knee transition from preasymptotic universal convergence to asymptotic Fisher behavior.

Optimal SSW measurements for a fixed N_{tot} require many pulses N with only a few mean photons per pulse, $\langle n\rangle$. This behavior should be contrasted with that for coherent states and squeezed states, for which the near-Gaussian phase distribution means that the sensitivity of multiple measurements using ML estimation improves as $1/\sqrt{N}$. For coherent states, with phase standard deviation $1/2\langle n\rangle^{1/2}$, the multiple-measurement sensitivity $1/2(\langle n\rangle N)^{1/2} = 1/2\sqrt{N_{\text{tot}}}$ is independent of how the photon-number resources are split between the number of pulses and the number of photons per pulse. For optimized squeezed states, with standard deviation $\sim 1/\langle n\rangle$, the optimum sensitivity is attained by using a single pulse that carries all the photons.

The large-N_{tot} confidence-interval convergence rate cannot be obtained directly from our simulations. If the form of the preasymptotic universal convergence were known, then its approach to the $1/\sqrt{NF}$ Cramér-Rao lower bound for a particular $\langle n\rangle$ would give us the optimum number of pulses for that $\langle n\rangle$; extrapolations based on various assumptions for the universal convergence will be discussed elsewhere [17]. Alternatively, working backwards from a knowledge of where the asymptotic Gaussian approximation to the ML estimator breaks down [18] can give a direct way to attack the large-N_{tot} behavior [17, 19].

We do not know at present how to formulate the general question of quantum limits on phase sensitivity, because it becomes entangled in difficult issues of estimation theory for multiple measurements. Nonetheless, our work shows that there are surprises lurking in multiple-measurement schemes—we did not expect SSW states to achieve even the shot-noise limit—and it indicates that any investigation of the ultimate phase sensitivity must allow for a "divide-and-conquer" strategy that divides up the available photons among many measurements of the same phase datum.

This work was supported in part by the Office of Naval Research (Contract No. N00014-88-K-0042 and Grant No. N00014-91-J-1167). S.L.B. appreciated the support of the Lady Davis Fellowship Trust in writing up this work, and he thanks A. Mann for a critical reading of the manuscript.

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