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Interpretation for a positive P representation

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We show that a "canonical" form of the positive P representation has a simple interpretation as the statistics of four detectors, two of which make redundant position measurements, while the other two simultaneously make redundant momentum measurements. This interpretation allows us to understand the additional degrees of freedom for the canonical positive P representation.

I. INTRODUCTION

A time-honored and powerful technique for formulating and solving quantum-mechanical problems relies on c-number phase-space representations.[1–6] By fixing the operator ordering, an arbitrary operator can be mapped uniquely to a c-number function on phase space. In this way, a density operator can be mapped to a phase-space quasiprobability distribution, examples being the Wigner function,[1] the Q function,[2] and the Glauber-Sudarshan P function.[3] Further, the evolution equation for the density operator gets mapped to a partial differential equation resembling a Fokker-Planck equation, for which much classical intuition is available.

These distributions have various problems. First, some of them become too singular when used to represent certain quantum states; for example, the Glauber-Sudarshan P function becomes more singular than a δ function for squeezed states and number eigenstates. Second, some distributions—for example, the Wigner function—can take on negative values and so must be thought of as quasiprobability distributions, rather than proper probability distributions, thus weakening their interpretational value. Third, the partial differential equations that describe evolution in time are not always true Fokker-Planck equations: the equation can have derivatives higher than second order, and even it if it does not, it need not have a positive-definite diffusion matrix. This third problem is the most critical for performing calculations, since a true Fokker-Planck equation can be transcribed to an equivalent set of first-order stochastic ordinary differential equations,[7] which can be simulated numerically.

The positive P representation of Drummond and Gardiner[6] overcomes most of these problems. If its evolution equation is of second order, then one can always arrange the diffusion matrix to be positive definite, the result being a true Fokker-Planck equation. When this is done, the distribution remains positive for all times, if it is given a well-behaved, positive initial condition, as can always be done. Unfortunately, the positive P representation achieves this considerable success by doubling the number of degrees of freedom of the system, i.e., doubling the number of dimensions of phase space. With these extra degrees of freedom, it has been generally assumed that the positive P representation carries no physical interpretation. We show here how to interpret a "canonical" form of the positive P representation as the statistics of a particular set of measurements.

A distribution for which such an interpretation is known is the Q function, which for a state $\hat{\rho}$ is defined in terms of coherent states $|\alpha\rangle$ as

$$Q(\alpha) \equiv \langle\alpha|\hat{\rho}|\alpha\rangle \ . \tag{1.1}$$

The Q function, which is positive by definition, can always be thought of as a probability distribution on phase space. More than just a theoretical curiosity, however, it is *the* probability distribution for the statistics in a particular model of simultaneous measurements of position and momentum.[8,9] It is in this sense that we interpret the canonical form of the positive P representation.

In Sec. II we rederive the Arthurs-Kelly[8] measurement model for the Q function in a pedagogical manner that uses the language of effects[10] and generalizes the Arthurs-Kelly result. In Sec. III we consider a simple generalization of this model that yields the canonical form of the positive P representation as its measurement statistics. Finally, in Sec. IV we show how the measurement model for the canonical form leads naturally to normally ordered statistics.

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II. SIMULTANEOUS MEASUREMENTS AND THE QUANTUM Q FUNCTION

We rederive and generalize here the result implicit in the work of Arthurs and Kelly.[8] Consider a simultaneous measurement of the position x and momentum p of a one-dimensional quantum-mechanical system. The measurement model is specified by (i) an interaction Hamiltonian

$$\hat{H}_{\rm int} \equiv \delta(t)(\hat{x}\hat{p}_1 + \hat{p}\hat{p}_2) , \tag{2.1}$$

where p_1 and p_2 are the momenta of the pointers of the two detectors, and (ii) a choice of initial state for the detectors. After the interaction, information about x and p is encoded in the pointers' positions, x_1 and x_2. The Arthurs-Kelly result is that the measurement statistics—i.e., the joint statistics of x_1 and x_2—are given by the Q function, namely,

$$\text{Prob}(x_1, x_2) = \frac{1}{2\pi}Q(\alpha) , \quad \alpha \equiv \frac{1}{\sqrt{2}}(x_1 + ix_2) , \tag{2.2}$$

when the detectors' initial state is chosen to be a particular Gaussian pure state. What is novel about our derivation is that the choice of the detectors' wave function is left until the very last step; hence, we can display what happens for other initial detector wave functions.

The system's position and momentum are treated as those of a harmonic oscillator with unit mass and unit frequency ($\hbar = 1$ throughout), so that we may avail ourselves of the apparatus of displacement operators and oscillator coherent states. Thus we define an "annihilation operator"

$$\hat{a} \equiv \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) . \tag{2.3}$$

An immediate consequence of the definition (1.1) of the Q function is that antinormally ordered moments of $\hat{a}$ and $\hat{a}^\dagger$ can be calculated as expectations of α and α^* over $Q(\alpha)/\pi$:

$$\langle \hat{a}^l(\hat{a}^\dagger)^k \rangle = \int \frac{d^2\alpha}{\pi}\, \alpha^l(\alpha^*)^k Q(\alpha) \equiv E(\alpha^l(\alpha^*)^k) . \tag{2.4}$$

We also define a displacement operator,

$$\hat{D}(\sigma,\tau) \equiv \exp[-i(\sigma\hat{p} - \tau\hat{x})] = \exp(\gamma\hat{a}^\dagger - \gamma^*\hat{a}) \equiv \hat{D}(\gamma) , \tag{2.5}$$

where $\hat{D}(\sigma,\tau)$ is written in terms of position and momentum shifts, σ and τ, and $\hat{D}(\gamma)$ is written conventionally in terms of the complex displacement in phase space, $\gamma \equiv (\sigma + i\tau)/\sqrt{2}$.

We assume that before they interact, the system and the detectors are uncorrelated, having an initial density operator $\hat{\rho}_s\hat{\rho}_d$, where $\hat{\rho}_s$ and $\hat{\rho}_d$ are the system and detector density operators. The evolution operator for the interaction (2.1),

$$\hat{U} = \exp[-i(\hat{x}\hat{p}_1 + \hat{p}\hat{p}_2)] , \tag{2.6}$$

shifts the pointer positions of the two detectors, the first detector recording the system's position x and the second detector recording the system's momentum p. To read off the results, we measure the pointer positions x_1 and x_2. If we let

$$|x_1, x_2\rangle \equiv |x_1\rangle \otimes |x_2\rangle \tag{2.7}$$

denote the δ-function-normalized simultaneous eigenstates of $\hat{x}_1$ and $\hat{x}_2$, then we can write the probability density of finding pointer positions x_1 and x_2 as

$$\begin{aligned}\text{Prob}(x_1, x_2) &= \text{tr}_s\text{tr}_d\Big(\hat{U}\hat{\rho}_s\hat{\rho}_d\hat{U}^\dagger |x_1,x_2\rangle\langle x_1,x_2|\Big) \\ &= \text{tr}_s(\hat{\rho}_s\hat{F}_{x_1,x_2}) ,\end{aligned} \tag{2.8}$$

where the system operator

$$\hat{F}_{x_1,x_2} \equiv \text{tr}_d\Big(\hat{\rho}_d\hat{U}^\dagger |x_1,x_2\rangle\langle x_1,x_2|\hat{U}\Big) \tag{2.9}$$

is called an effect density (or an effect-valued measure).[10] For the Q function the effect density is simply $|\alpha\rangle\langle\alpha|/2\pi$ [see Eq. (2.2)]. By completeness the effect density must satisfy

$$\int dx_1 dx_2\, \hat{F}_{x_1,x_2} = \hat{1}_s , \tag{2.10}$$

where $\hat{1}_s$ is the identity operator in the system Hilbert space.

If we assume further that the detectors are initially in a pure state $\hat{\rho}_d = |\psi_d\rangle\langle\psi_d|$, we can put the effect density in the form

$$\hat{F}_{x_1,x_2} = \hat{\Upsilon}^\dagger_{x_1,x_2}\hat{\Upsilon}_{x_1,x_2} , \tag{2.11}$$

where the system operator

$$\hat{\Upsilon}_{x_1,x_2} \equiv \langle x_1,x_2|\hat{U}|\psi_d\rangle \tag{2.12}$$

is a "resolution operator." Using completeness in the momentum basis, we can cast the resolution operator in the form

$$\begin{aligned}\hat{\Upsilon}_{x_1,x_2} &= \int dp_1 dp_2\, \langle x_1,x_2|p_1,p_2\rangle \exp[-i(\hat{x}p_1 + \hat{p}p_2)]\langle p_1,p_2|\psi_d\rangle \\ &= \int \frac{dp_1 dp_2}{2\pi}\, \langle p_1,p_2|\psi_d\rangle \exp\{-i[p_2(\hat{p}-x_2) + p_1(\hat{x}-x_1)]\} = \hat{D}(x_1,x_2)\hat{\Upsilon}_{0,0}[\hat{D}(x_1,x_2)]^\dagger ,\end{aligned} \tag{2.13}$$

with

$$\hat{\Upsilon}_{0,0} = \int \frac{dp_2 dp_1}{2\pi}\, \langle p_1,-p_2|\psi_d\rangle[\hat{D}(p_2,p_1)]^\dagger . \tag{2.14}$$

Transforming the initial detector state to the position basis yields another form,

$$\hat{\Upsilon}_{0,0} = \int dx_1 dx_2 \frac{\langle -x_1, -x_2|\psi_d\rangle}{2\pi} \int \frac{dp_2 dp_1}{2\pi} [\hat{D}(p_2, p_1)]^\dagger e^{-i(p_2 x_2 - p_1 x_1)} . \tag{2.15}$$

Suppose now that $|\phi\rangle$ is a system state vector that has the Wigner function $W(x,p)$, and suppose that we wish to make $\hat{\Upsilon}_{0,0} \propto |\phi\rangle\langle\phi|$. The brief description of symmetrically ordered phase-space representations in the Appendix shows that we achieve our desire by choosing the initial detector wave function to be

$$\langle x_1, x_2|\psi_d\rangle = \frac{1}{\sqrt{\mathcal{N}}} W(-x_1, -x_2) , \tag{2.16}$$

where $\mathcal{N}$ is a normalization constant given by Eq. (A7). This choice of detector wave function implies that

$$\hat{\Upsilon}_{0,0} = \frac{1}{2\pi\sqrt{\mathcal{N}}} |\phi\rangle\langle\phi| . \tag{2.17}$$

Putting together Eqs. (2.11), (2.13), and (2.17), one finds that this choice leads to an effect density

$$\hat{F}_{x_1,x_2} = \frac{1}{4\pi^2\mathcal{N}} \hat{D}(x_1,x_2)|\phi\rangle\langle\phi|[\hat{D}(x_1,x_2)]^\dagger \tag{2.18}$$

and, hence, to measurement statistics

$$\text{Prob}(x_1,x_2) = \frac{1}{4\pi^2\mathcal{N}} \left\langle \phi \left| [\hat{D}(x_1,x_2)]^\dagger \hat{\rho}_s \hat{D}(x_1,x_2) \right| \phi \right\rangle . \tag{2.19}$$

The Arthurs-Kelly result follows from letting $|\phi\rangle$ be the harmonic-oscillator vacuum $|0\rangle$. This choice requires the initial detector wave function

$$\langle x_1, x_2|\psi\rangle = \sqrt{\frac{2}{\pi}} \exp[-(x_1^2 + x_2^2)] \tag{2.20}$$

($\mathcal{N} = 1/2\pi$) and leads to

$$\hat{\Upsilon}_{0,0} = \frac{1}{\sqrt{2\pi}} |0\rangle\langle 0| . \tag{2.21}$$

Hence, for detector wave function (2.20), we arrive at the Arthurs-Kelly result (2.2),

$$\begin{aligned}\text{Prob}(x_1,x_2) &= \frac{1}{2\pi} \left\langle 0 \left| [\hat{D}(x_1,x_2)]^\dagger \hat{\rho}_s \hat{D}(x_1,x_2) \right| 0 \right\rangle \\ &= \frac{1}{2\pi} Q(\alpha) ,\end{aligned} \tag{2.22}$$

that the measurement statistics are given by the Q function.

The Q-function measurement statistics mean that the two detectors add noise, over and above the intrinsic uncertainties in x and p. Such added noise is an unavoidable feature of a simultaneous measurement of position and momentum.[8] We discuss this added noise further in Sec. IV.

III. REDUNDANT MEASUREMENTS AND THE POSITIVE P REPRESENTATION

A positive P representation[6] can be defined for a large class of operators. We restrict ourselves here to those that are built up from the standard annihilation and creation operators of a harmonic oscillator. In particular, our work does not apply to generalizations of the positive P representation that include spin or pseudospin operators often used to describe a two-level atom.

A positive P representation $P(\alpha,\beta^*)$ for a density operator $\hat{\rho}$ allows the density operator to be expanded as

$$\hat{\rho} = \int d^2\alpha\, d^2\beta \frac{|\alpha\rangle\langle\beta|}{\langle\beta|\alpha\rangle} P(\alpha,\beta^*) . \tag{3.1}$$

Here we write β^* in place of the conventional β for the convenience of our argument below. In general Eq. (3.1) cannot be inverted uniquely; i.e., a density operator does not have a unique positive P representation. This is in contrast to the Glauber-Sudarshan P function,[3] which, if it exists, is unique.

For any positive P representation, normally ordered moments of $\hat{a}$ and $\hat{a}^\dagger$ can be calculated as expectations of α and β^* over $P(\alpha,\beta^*)$:

$$\langle(\hat{a}^\dagger)^k \hat{a}^l\rangle = \int d^2\alpha\, d^2\beta\, \alpha^l (\beta^*)^k P(\alpha,\beta^*) \equiv E(\alpha^l(\beta^*)^k) . \tag{3.2}$$

These normally ordered moments are physical moments, determined by the density operator $\hat{\rho}$; other moments of α and β are unphysical, since they are determined not by the density operator, but only by a particular choice of positive P representation.

Given the nonuniqueness of positive P representations, it seems unlikely that they can be given any kind of general interpretation, in the sense of the Arthurs-Kelly measurement model for the Q function. Indeed, some choices for $P(\alpha,\beta^*)$ are not positive so, at least for those cases, no such interpretation is possible. To avoid this problem, we focus here on a specific choice[6] of positive P representation, which we call the "canonical" form and which is always well defined:

$$P_{\text{can}}(\alpha,\beta^*) \equiv \frac{1}{4\pi^2} \exp(-\tfrac{1}{4}|\alpha-\beta|^2) \left\langle \tfrac{1}{2}(\alpha+\beta) \right| \hat{\rho} \left| \tfrac{1}{2}(\alpha+\beta) \right\rangle = \frac{1}{4\pi^2} \exp(-\tfrac{1}{4}|\alpha-\beta|^2) Q(\tfrac{1}{2}(\alpha+\beta)) . \tag{3.3}$$

The canonical form is clearly positive, and because it is essentially the Q function, we might expect to find a simple interpretation in terms of a measurement model. Because of the integral relation (3.1), there is a transformation from an arbitrary choice of positive P representation, $P(\alpha,\beta^*)$, to the canonical form, $P_{\text{can}}(\alpha,\beta^*)$, namely,

$$P_{\rm can}(\alpha,\beta^*) = \frac{1}{4\pi^2}\exp(-\tfrac{1}{4}|\alpha-\beta|^2)\int d^2\alpha' d^2\beta' \frac{\langle\frac{1}{2}(\alpha+\beta)|\alpha'\rangle\langle\beta'|\frac{1}{2}(\alpha+\beta)\rangle}{\langle\beta'|\alpha'\rangle} P(\alpha',\beta'^*) . \tag{3.4}$$

Thus, even though the canonical form is not in general preserved by a Fokker-Planck equation, we can take the time-evolved solution for $P(\alpha,\beta^*)$ and convert it to canonical form.

Let us now consider the structure (3.3) of the canonical form. The second factor is the Q function in the collective variable $\frac{1}{2}(\alpha+\beta)$, which could be obtained as the probability density for simultaneous measurements of position and momentum. The first term is a Gaussian in the collective variable $\frac{1}{2}(\alpha-\beta)$, which could be obtained from a Gaussian wave function for detectors that are not coupled to the system. Hence, the interaction Hamiltonian for the model whose measurement statistics reproduce $P_{\rm can}(\alpha,\beta^*)$ is

$$\hat{H}_{\rm int} \equiv \delta(t)[\hat{x}(\hat{p}_1+\hat{p}_2)+\hat{p}(\hat{p}_3+\hat{p}_4)] , \tag{3.5}$$

where there are now four detectors, with pointer momenta p_1, p_2, p_3, and p_4 and with corresponding pointer positions x_1, x_2, x_3, and x_4.

The pertinence of this interaction is brought out by changing to "center-of-mass" coordinates,

$$X_1 \equiv \tfrac{1}{2}(x_1+x_2) , \quad \Pi_1 \equiv p_1+p_2 , \tag{3.6}$$

$$X_3 \equiv \tfrac{1}{2}(x_3+x_4) , \quad \Pi_3 \equiv p_3+p_4 , \tag{3.7}$$

and "relative" coordinates,

$$X_2 \equiv \tfrac{1}{2}(x_1-x_2) , \quad \Pi_2 \equiv p_1-p_2 , \tag{3.8}$$

$$X_4 \equiv \tfrac{1}{2}(x_3-x_4) , \quad \Pi_4 \equiv p_3-p_4 . \tag{3.9}$$

In terms of these collective coordinates, the interaction reduces to the familiar form

$$\hat{H}_{\rm int} = \delta(t)(\hat{x}\hat{\Pi}_1+\hat{p}\hat{\Pi}_3) . \tag{3.10}$$

The center-of-mass coordinates play the role of the coordinates of the two detectors in the measurement model of Sec. II, whereas the relative coordinates do not participate in the interaction. This allows us to apply the results of Sec. II directly to the present model.

The case of interest occurs when the detectors' initial wave function is chosen to be

$$\langle X_1,X_2,X_3,X_4|\psi_d\rangle = \frac{2}{\pi}\exp[-(X_1^2+X_2^2+X_3^2+X_4^2)] ; \tag{3.11}$$

in terms of the original position variables, the wave function is

$$\langle x_1,x_2,x_3,x_4|\psi_d\rangle = \frac{1}{\pi}\exp[-\tfrac{1}{2}(x_1^2+x_2^2+x_3^2+x_4^2)] . \tag{3.12}$$

With this choice the analysis in Sec. II implies immediately that the probability density for the center-of-mass and relative positions is given by

$$\mathrm{Prob}(X_1,X_2,X_3,X_4) = \frac{1}{\pi^2}\exp[-2(X_2^2+X_4^2)] \times Q\left(\frac{1}{\sqrt{2}}(X_1+iX_3)\right) . \tag{3.13}$$

Identifying

$$\frac{1}{2}(\alpha+\beta) \equiv \frac{1}{\sqrt{2}}(X_1+iX_3) , \tag{3.14}$$

$$\frac{1}{2}(\alpha-\beta) \equiv \sqrt{2}(X_2+iX_4) \tag{3.15}$$

puts the probability density (3.13) into the form (3.3) of the canonical positive P representation.

We now invert the transformations (3.14) and (3.15) and write them in terms of the original detector position variables:

$$\alpha = \frac{1}{\sqrt{2}}[X_1+2X_2+i(X_3+2X_4)] = \frac{1}{2\sqrt{2}}[3x_1-x_2+i(3x_3-x_4)] , \tag{3.16}$$

$$\beta = \frac{1}{\sqrt{2}}[X_1-2X_2+i(X_3-2X_4)] = \frac{1}{2\sqrt{2}}[-x_1+3x_2+i(-x_3+3x_4)] . \tag{3.17}$$

This leads to our final result,

$$\mathrm{Prob}(x_1,x_2,x_3,x_4) = \tfrac{1}{4}\mathrm{Prob}(X_1,X_2,X_3,X_4) = P_{\rm can}(\alpha,\beta^*) , \tag{3.18}$$

that the probability density of the original detector positions is given by the canonical form of the positive P representation.

The canonical positive P representation has now acquired a simple interpretation. The real and imaginary parts of the variable $\frac{1}{2}(\alpha+\beta)$ [Eq. (3.14)] encode information about x and p, contaminated by added noise from the center-of-mass variables. The variable $\frac{1}{2}(\alpha-\beta)$ is unaffected by the interaction, but carries added noise from the relative variables. Thus the real and imaginary parts of α and β encode *redundant* information about x and p, but they are contaminated by different sources of added noise. The added noise in α is anticorrelated with the added noise in β, and this anticorrelation gives rise to normal ordering for the physical moments, as we now show.

IV. OPERATOR ORDERING FOR SYSTEM MOMENTS

Our interpretation of the canonical positive P representation in terms of a measurement model leads naturally to normally ordered statistics for the physical moments. To warm up, we turn again to the Q function.

Consider an expectation over the Q function of some arbitrary (analytic) function of x_1 and x_2,

$$E(f(x_1,x_2)) \equiv \int dx_1 dx_2\, f(x_1,x_2)\mathrm{Prob}(x_1,x_2)\ , \quad (4.1)$$

with $\mathrm{Prob}(x_1,x_2)$ given by Eq. (2.2). Using Eq. (2.8), we can write this expectation as

$$E(f(x_1,x_2)) = \mathrm{tr}_s\mathrm{tr}_d\left(\hat{U}\hat{\rho}_s\hat{\rho}_d\hat{U}^\dagger \int dx_1 dx_2\, f(x_1,x_2)|x_1,x_2\rangle\langle x_1,x_2|\right) = \langle \hat{U}^\dagger f(\hat{x}_1,\hat{x}_2)\hat{U}\rangle \equiv \left\langle f(\hat{M}(x_1),\hat{M}(x_2))\right\rangle\ , \quad (4.2)$$

where the unitary operator $\hat{U}$ is given by Eq. (2.6), and where the bracket average is taken with respect to the initial system state $\hat{\rho}_s$ and the initial detector state given by the wave function (2.20). The map from classical phase-space variables x_i to operators $\hat{M}(x_i)$ is defined by

$$\hat{M}(x_i) \equiv \hat{U}^\dagger \hat{x}_i \hat{U}\ . \quad (4.3)$$

It follows from the unitarity of $\hat{U}$ that

$$[\hat{M}(x_1),\hat{M}(x_2)] = 0\ , \quad (4.4)$$

so the representation preserves the Abelian character of classical phase space. One easily shows that

$$\hat{M}(x_1) = \hat{x} + \hat{x}_1 + \tfrac{1}{2}\hat{p}_2 \equiv \hat{x} + \hat{\mathcal{X}}_1\ , \quad (4.5)$$

$$\hat{M}(x_2) = \hat{p} + \hat{x}_2 - \tfrac{1}{2}\hat{p}_1 \equiv \hat{p} + \hat{\mathcal{X}}_2\ . \quad (4.6)$$

Given the detector wave function (2.20), the "added-noise operators" $\hat{\mathcal{X}}_j$ are zero-mean uncorrelated quantum Gaussian random variables with variances

$$\langle \hat{\mathcal{X}}_j^2\rangle = \tfrac{1}{2}\ ,\ \ j = 1,2\ . \quad (4.7)$$

We have thus expressed the classical phase-space expectation (4.1) in terms of a quantum expectation on an extended Hilbert space. This construction is strongly suggestive of the Gelfand-Naimark-Segel representation in a c^* algebra.[11] A similar construction called the "thermofield technique" has also been used to write classical thermal expectation values in terms of a pure-state Hilbert-space average.[12]

In terms of the complex phase-space variable $\alpha = (x_1 + ix_2)/\sqrt{2}$, the representation (4.3) is

$$\hat{M}(\alpha) = \hat{a} + \frac{1}{\sqrt{2}}(\hat{\mathcal{X}}_1 + i\hat{\mathcal{X}}_2) = \hat{a} + \frac{1}{\sqrt{2}}\hat{a}_1^\dagger + \frac{i}{\sqrt{2}}\hat{a}_2^\dagger\ , \quad (4.8)$$

where $\hat{a}$ is the system annihilation operator (2.3) and the operators

$$\hat{a}_j \equiv \hat{x}_j + \tfrac{1}{2}i\hat{p}_j\ ,\ \ j = 1,2\ , \quad (4.9)$$

are detector "annihilation operators," relative to which the detector state (2.20) is the harmonic-oscillator vacuum.

In calculating moments of α and α^*, no questions of ordering arise at the level of $\hat{M}(\alpha)$ and $\hat{M}(\alpha^*) = [\hat{M}(\alpha)]^\dagger$, because these two operators commute [Eq. (4.4)]. A formal ordering of $\hat{M}(\alpha)$ and $\hat{M}(\alpha^*)$, however, leads naturally to an ordering of system and detector operators, which can suggest an interpretation or make the moment calculation particularly simple. Suppose, for example, that we are interested in the Q-function moment $E(\alpha\alpha^*)$. Choosing symmetric ordering for $\hat{M}(\alpha)$ and $\hat{M}(\alpha^*)$ and working with the added-noise operators, we write

$$\begin{aligned} E(\alpha\alpha^*) &= \tfrac{1}{2}\left\langle \hat{M}(\alpha)\hat{M}(\alpha^*) + \hat{M}(\alpha^*)\hat{M}(\alpha)\right\rangle \\ &= \tfrac{1}{2}\langle \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\rangle + \tfrac{1}{2}(\langle\hat{\mathcal{X}}_1^2\rangle + \langle\hat{\mathcal{X}}_2^2\rangle) \\ &= \tfrac{1}{2}\langle \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\rangle + \tfrac{1}{2} = \langle \hat{a}\hat{a}^\dagger\rangle\ . \end{aligned} \quad (4.10)$$

This calculation shows explicitly that antinormal ordering of the Q-function moments is a consequence of the added noise. We can get at the antinormal ordering directly—but we miss the interpretation—by starting with antinormal ordering of $\hat{M}(\alpha)$ and $\hat{M}(\alpha^*)$ and remembering that the operators $\hat{a}_j$ annihilate the detector state. Indeed, in this way it is easy to demonstrate trivially the antinormal ordering for arbitrary moments:

$$E(\alpha^l(\alpha^*)^k) = \left\langle [\hat{M}(\alpha)]^l[\hat{M}(\alpha^*)]^k\right\rangle = \langle \hat{a}^l(\hat{a}^\dagger)^k\rangle\ . \quad (4.11)$$

We now follow the same procedure for the canonical positive P representation and its measurement model. The coupling to the detectors induces a map as in Eq. (4.3), where $\hat{U}$ is now given by

$$\hat{U} = \exp[-i(\hat{x}\hat{\Pi}_1 + \hat{p}\hat{\Pi}_3)]\ . \quad (4.12)$$

It is easiest, at least initially, to work with the center-of-mass and relative positions, for which the map takes the form

$$\hat{M}(X_1) = \hat{x} + \hat{X}_1 + \tfrac{1}{2}\hat{\Pi}_3\ , \quad (4.13)$$

$$\hat{M}(X_2) = \hat{X}_2\ , \quad (4.14)$$

$$\hat{M}(X_3) = \hat{p} + \hat{X}_3 - \tfrac{1}{2}\hat{\Pi}_1\ , \quad (4.15)$$

$$\hat{M}(X_4) = \hat{X}_4\ . \quad (4.16)$$

The unitarity of $\hat{U}$ ensures that

$$[\hat{M}(X_j),\hat{M}(X_k)] = 0\ ,\ \ j,k = 1,2,3,4\ , \quad (4.17)$$

so the representation again faithfully preserves the Abelian character of the classical phase-space variables.

The Hilbert-space representations of the complex phase-space variables α and β [Eqs. (3.16) and (3.17)] are given by

$$\hat{M}(\alpha) = \hat{a} + \frac{1}{\sqrt{2}}(\hat{\mathcal{Y}}_1 + i\hat{\mathcal{Y}}_2) = \hat{a} + \hat{a}_1^\dagger + \tfrac{1}{2}\hat{a}_1 - \tfrac{1}{2}\hat{a}_2 + i(\hat{a}_3^\dagger + \tfrac{1}{2}\hat{a}_3 - \tfrac{1}{2}\hat{a}_4) , \tag{4.18}$$

$$\hat{M}(\beta) = \hat{a} + \frac{1}{\sqrt{2}}(\hat{\mathcal{Z}}_1 + i\hat{\mathcal{Z}}_2) = \hat{a} - \tfrac{1}{2}\hat{a}_1 + \hat{a}_2^\dagger + \tfrac{1}{2}\hat{a}_2 + i(-\tfrac{1}{2}\hat{a}_3 + \hat{a}_4^\dagger + \tfrac{1}{2}\hat{a}_4) . \tag{4.19}$$

Here the operators

$$\hat{\mathcal{Y}}_1 \equiv \hat{X}_1 + 2\hat{X}_2 + \tfrac{1}{2}\hat{\Pi}_3 , \tag{4.20}$$

$$\hat{\mathcal{Y}}_2 \equiv \hat{X}_3 + 2\hat{X}_4 - \tfrac{1}{2}\hat{\Pi}_1 , \tag{4.21}$$

$$\hat{\mathcal{Z}}_1 \equiv \hat{X}_1 - 2\hat{X}_2 + \tfrac{1}{2}\hat{\Pi}_3 , \tag{4.22}$$

$$\hat{\mathcal{Z}}_2 \equiv \hat{X}_3 - 2\hat{X}_4 - \tfrac{1}{2}\hat{\Pi}_1 \tag{4.23}$$

are "added-noise operators," and the operators

$$\hat{a}_j \equiv \frac{1}{\sqrt{2}}(\hat{x}_j + i\hat{p}_j) , \quad j = 1,2,3,4 , \tag{4.24}$$

are detector "annihilation operators" that annihilate the detector state (3.12).

Given the detector state (3.12), the added-noise operators (4.20)–(4.23) are zero-mean quantum Gaussian random variables with variances

$$\langle \hat{\mathcal{Y}}_j^2 \rangle = \langle \hat{\mathcal{Z}}_j^2 \rangle = \tfrac{3}{2} , \quad j = 1,2 . \tag{4.25}$$

The only other nonvanishing elements of the covariance matrix,

$$\tfrac{1}{2}\langle \hat{\mathcal{Y}}_j \hat{\mathcal{Z}}_j + \hat{\mathcal{Z}}_j \hat{\mathcal{Y}}_j \rangle = -\tfrac{1}{2} , \quad j = 1,2 , \tag{4.26}$$

characterize the anticorrelation of $\hat{\mathcal{Y}}_1$ with $\hat{\mathcal{Z}}_1$ and of $\hat{\mathcal{Y}}_2$ with $\hat{\mathcal{Z}}_2$.

Consider now the physical moment $E(\alpha\beta^*)$. If we begin with symmetric ordering and work in terms of the added-noise operators, we write

$$E(\alpha\beta^*) = \tfrac{1}{2}\left\langle \hat{M}(\alpha)\hat{M}(\beta^*) + \hat{M}(\beta^*)\hat{M}(\alpha) \right\rangle = \tfrac{1}{2}\langle \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} \rangle + \tfrac{1}{2}[\tfrac{1}{2}\langle \hat{\mathcal{Y}}_1\hat{\mathcal{Z}}_1 + \hat{\mathcal{Z}}_1\hat{\mathcal{Y}}_1 \rangle + \tfrac{1}{2}\langle \hat{\mathcal{Y}}_2\hat{\mathcal{Z}}_2 + \hat{\mathcal{Z}}_2\hat{\mathcal{Y}}_2 \rangle] = \tfrac{1}{2}\langle \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} \rangle - \tfrac{1}{2} = \langle \hat{a}^\dagger\hat{a} \rangle . \tag{4.27}$$

This calculation shows explicitly that the normal ordering of the physical moments comes from the anticorrelation between the added noise in α and the added noise in β. The same anticorrelated noise appears in the unphysical moment $E(\alpha^*\beta)$, so for the canonical positive P representation, $E(\alpha^*\beta) = \langle \hat{a}^\dagger\hat{a} \rangle$, too, is normally ordered. In contrast, the unphysical moment $E(\alpha\alpha^*)$ receives even more added noise than for antinormal ordering:

$$E(\alpha\alpha^*) = \tfrac{1}{2}\langle \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} \rangle + \tfrac{1}{2}(\langle \hat{\mathcal{Y}}_1^2 \rangle + \langle \hat{\mathcal{Y}}_2^2 \rangle) = \tfrac{1}{2}\langle \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} \rangle + \tfrac{3}{2} = \langle \hat{a}\hat{a}^\dagger \rangle + 1 . \tag{4.28}$$

All the moments of the canonical positive P representation are conveniently summarized in a characteristic function,

$$E\big(\exp(\delta\beta^* - \delta^*\beta + \gamma\alpha^* - \gamma^*\alpha)\big) = \exp[-2(\delta\delta^* + \gamma\gamma^*)] \left\langle \exp[(\delta+\gamma)\hat{a}^\dagger] \exp[-(\delta^*+\gamma^*)\hat{a}] \right\rangle , \tag{4.29}$$

where δ and γ are treated formally as independent of their complex conjugates. Calculation of the characteristic function (4.29) proceeds by normal ordering both system and detector operators. For $\delta^* = \gamma = 0$, the characteristic function shows that the moments of α and β^*—the physical moments—are normally ordered; for $\delta = \gamma^* = 0$, it shows that moments of α^* and β are also normally ordered. For $\delta = \delta^* = 0$ or for $\gamma = \gamma^* = 0$, the prefactor describes the added noise in moments of α and α^* or of β and β^*.

V. CONCLUSION

We have constructed an explicit measurement model, involving four detectors, whose measurement statistics reproduce the canonical form (3.3) of the positive P representation. The model is specified by the interaction (3.5) and by the initial detector wave function (3.12). The canonical form can always be obtained from an arbitrary positive P representation by applying the integral transformation (3.4). Finally, we have investigated how the noise added by the measurement manifests itself in the physical and unphysical moments of the canonical positive P representation.

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APPENDIX: SYMMETRICALLY ORDERED PHASE-SPACE REPRESENTATIONS

A Taylor expansion of the displacement operator (2.5) defines symmetrically ordered products of $\hat{x}$ and $\hat{p}$. We can expand an arbitrary function of position and momentum, $f(\hat{x},\hat{p})$, in terms of symmetrically ordered products by writing[4,5]

$$f(\hat{x},\hat{p}) = \int \frac{d\sigma d\tau}{2\pi}\,\tilde{F}(\sigma,\tau)[\hat{D}(\sigma,\tau)]^\dagger\,, \tag{A1}$$

which can be inverted to give

$$\tilde{F}(\sigma,\tau) = \mathrm{tr}[f(\hat{x},\hat{p})\hat{D}(\sigma,\tau)]\,. \tag{A2}$$

The Fourier transform of $\tilde{F}(\sigma,\tau)$,

$$\begin{aligned} F(x,p) &= \int \frac{d\sigma d\tau}{(2\pi)^2}\,\tilde{F}(\sigma,\tau)e^{i(\sigma p-\tau x)} \\ &= \mathrm{tr}\left(f(\hat{x},\hat{p})\int \frac{d\sigma d\tau}{(2\pi)^2}\,\hat{D}(\sigma,\tau)e^{i(\sigma p-\tau x)}\right)\,, \end{aligned} \tag{A3}$$

is the symmetrically ordered associated function for $f(\hat{x},\hat{p})$; i.e., it is the symmetrically ordered phase-space representation of the operator $f(\hat{x},\hat{p})$. We can write $f(\hat{x},\hat{p})$ in terms of the associated function as

$$f(\hat{x},\hat{p}) = \int dx\,dp\,F(x,p)\int \frac{d\sigma d\tau}{2\pi}\,[\hat{D}(\sigma,\tau)]^\dagger e^{-i(\sigma p-\tau x)}\,. \tag{A4}$$

Refer now to the analysis in Sec. II: comparing Eqs. (A1) and (A4) with Eqs. (2.14) and (2.15) shows that $\langle -x_1,-x_2|\psi_d\rangle/2\pi$ is the associated function for the operator $\hat{\Upsilon}_{0,0}$, and $\langle p_1,-p_2|\psi_d\rangle$ is the Fourier transform of the associated function.

If the operator function is a density operator $\hat{\rho} = f(\hat{x},\hat{p})$, then the associated function is the Wigner function $W(x,p) = F(x,p)$, and the Fourier transform of the Wigner function is the symmetrically ordered characteristic function $\Phi(\sigma,\tau) = \tilde{F}(\sigma,\tau)$. Suppose one wants to have $\hat{\Upsilon}_{0,0} \propto |\phi\rangle\langle\phi|$, where $|\phi\rangle$ is a system state vector that has the Wigner function $W(x,p)$ and the characteristic function $\Phi(\sigma,\tau)$. Then one chooses the position-basis wave function for the initial detector state to be

$$\langle x_1,x_2|\psi_d\rangle = \frac{1}{\sqrt{\mathcal{N}}}W(-x_1,-x_2) \tag{A5}$$

or, equivalently, chooses the momentum-basis wave function to be

$$\langle p_1,p_2|\psi_d\rangle = \frac{1}{2\pi\sqrt{\mathcal{N}}}\Phi(-p_2,p_1)\,, \tag{A6}$$

where $\mathcal{N}$ is a normalization constant defined by

$$\mathcal{N} \equiv \int dx\,dp\,[W(x,p)]^2 = \int \frac{d\sigma d\tau}{(2\pi)^2}\,|\Phi(\sigma,\tau)|^2\,. \tag{A7}$$

With these choices $\hat{\Upsilon}_{0,0}$ is given by

$$\hat{\Upsilon}_{0,0} = \frac{1}{2\pi\sqrt{\mathcal{N}}}|\phi\rangle\langle\phi|\,. \tag{A8}$$

The choice of initial detector pure state represented by Eqs. (A5) and (A6) is by no means the most general choice. It requires that the position-basis wave function (A5) be real; if $\langle x_1,x_2|\psi_d\rangle$ is not real, then $\hat{\Upsilon}_{0,0}$ is not Hermitian. Moreover, even if the position-basis wave function is real, it might not be proportional to a Wigner function for a system state, in which case $\hat{\Upsilon}_{0,0}$, although Hermitian, would have some negative eigenvalues, or it might be proportional to the Wigner function for a system mixed state, in which case $\hat{\Upsilon}_{0,0}$ would be proportional to that mixed state. It should also be noted that for many system states $|\phi\rangle$, the choice (A5) requires that the two detectors be correlated initially.

If $|\phi\rangle=|0\rangle$ is the harmonic-oscillator vacuum, then $W(x,p) = \pi^{-1}\exp[-(x^2+p^2)]$ and $\Phi(\sigma,\tau) = \exp[-\frac{1}{4}(\sigma^2+\tau^2)]$, which implies $\mathcal{N} = 1/2\pi$. Hence, we can make

$$\hat{\Upsilon}_{0,0} = \frac{1}{\sqrt{2\pi}}|0\rangle\langle 0| \tag{A9}$$

by choosing

$$\langle x_1,x_2|\psi\rangle = \left(\frac{2}{\pi}\right)^{1/2}\exp[-(x_1^2+x_2^2)] \tag{A10}$$

or

$$\langle p_1,p_2|\psi\rangle = \frac{1}{\sqrt{2\pi}}\exp[-\tfrac{1}{4}(p_1^2+p_2^2)]\,. \tag{A11}$$

This choice yields Q-function statistics in the measurement model of Sec. II.

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