Supplementary Material for: Black hole evaporation rates without spacetime

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PREAMBLE TO THE HILBERT SPACE DESCRIPTION

In this section we provide background to the introductory discussion of the manuscript.

Causal separation implies a tensor product structure: Black holes are defined by their causal structure (their event horizons). The event horizon specifies what is inaccessible from observation by an external observer. In any quantum description of external observables what is inaccessible must be traced out — one necessarily has a tensor product structure between the exterior and the remainder of Hilbert space (the interior) $\mathcal{H}_{\rm ext}\otimes\mathcal{H}_{\rm int}$.

This observation is hardly new. It occurs automatically in field theoretic descriptions. Indeed, such a tensor product structure was explicitly utilized by Hawking [S1]. Further, it is exactly what is seen in Rindler spacetime where the uniformly accelerated observer has only access to signals on their side of the Rindler event horizon — tracing out the inaccessible degrees of freedom leaves a thermal state for the accelerated observer.

This use of the tensor product, to delineate what is outside and what is not (at the Hilbert space level), in no way implies that the spatial location of the event horizon cannot be fuzzy. These are quite separate matters.

The quantum mechanics of Hawking radiation: Whatever detailed field theoretic quantum gravity theory is ultimately developed, it is not unreasonable to expect that such a theory should allow for a description of black hole evaporation in terms of a microscopic (quantum mechanical) mechanism. As early as 1976, Hawking proposed pair creation as this mechanism: Here, pair creation is conceived to occur outside the event horizon, with one of the pair falling into the black hole (past the event horizon) and the other flying off as Hawking radiation. The big advantage of this mechanism is that it preserves the classical causal structure of the black hole even at the quantum level — Hawking's version of a quantum black hole is of a perfectly 'semi-permeable membrane' — anything can enter, nothing can leave; mass 'escapes' because negative energy is absorbed.

It was only very recently realized, however, that such a view is completely at odds with the possibility of complete unitary (quantum) evaporation of the black hole [S2]. Under Hawking's mechanism each pair created will be pair-wise entangled (entanglement between spin degrees of freedom; entanglement between spatial degrees of freedom; indeed entanglement across all degrees of freedom for the created pair). For each Hawking pair creation event when one partner of the entangled pair passes the boundary corresponding to the event horizon (as seen say by an infalling observer) the rank of entanglement across that event horizon will increase. Indeed, the structure of the tensor product provides a natural framework for quantifying entanglement across the event horizon.

However, if the rank of entanglement across the event horizon is increasing with each pair creation event then the Hilbert space dimensionality of the black hole interior cannot vanish [S2]. (We should note that the Hamiltonian constraint of describing an initially compact object with a finite mass implies that the black hole Hilbert space of any *dynamical* degrees of freedom must be effectively finite dimensional.) This did not pose any obvious problem for the *static* black hole spacetimes originally considered by Hawking. Rather, the difficulty is most glaring when considering non-static black holes that can shrink and can eventually vanish. Indeed, were Hawking's heuristic pair creation mechanism correct the complete unitary evaporation of a black hole would be utterly impossible [S2]. Here, we dub this catastrophic inconsistency as 'entanglement overload'.

For black holes to be able to eventually vanish, the original Hawking picture of a perfectly semi-permeable membrane must fail at the quantum level. In other words, entanglement overload very strongly points to the necessary breakdown of the classical causal structure of a black hole. This statement already points to the likely solution.

Evaporation as tunneling: The most straightforward way to evade entanglement overload is for Hilbert space within the black hole to 'leak away' — quantum mechanically we would call such a mechanism tunneling [S3]. Indeed, for over a decade now, such tunneling, out and across the event horizon, has been used as a powerful way of computing black hole evaporation rates including the effects of backreaction.

We suggest that the evaporation across event horizons operates by Hilbert space subsystems from the black hole interior moving to the exterior. The equation

$$|i\rangle_{\rm int}\to(U|i\rangle)_{BR}, \tag{1}$$

[Eq. (3) of the manuscript] provides the simplest mechanism for this to occur: Subsystems are dynamically selected (by some unitary U) and reassigned as radiation in an enlarged exterior Hilbert space.

Spacetime free conjecture: This brings us to the key conjecture of the manuscript: that Eq. (1) above (all equation numbers herein refer to Supplementary Material equations unless explicitly referring back to the manuscript) accurately describes the evaporation across black hole event horizons.

Our manuscript primarily investigates the consequences of Eq. (1) applied specifically to event horizons of black holes. Now the consensus appears to be that the physics of event horizons (cosmological, black hole, or those due to acceleration) is universal. In fact, it is precisely because of this generality that one should not expect Eq. (1) to bear the signatures of the detailed physics of black holes. Rather we then go on to impose the details of that physics onto this equation.

Testing this conjecture: The manuscript is devoted to exploring the implications of Eq. (1) for the evaporation rates of black holes, thus providing a test of its predictive power. To achieve this, the key pieces of physics about black holes we rely on are the no-hair theorem and the existence of Penrose processes. We assume that any quantum representation of a black hole must have a direct correspondence to its classical counterpart where these properties hold true. Therefore, when we wish to apply the very general Hilbert space description of quantum tunneling across event horizons in Eq. (1) we need to impose conditions consistent with these classical properties of a black hole.

It is our contention that the key technical content of the manuscript [involving its Theorems 1 through 4 and leading to its Eq(14)] provides strong evidence in support of the conjecture that Eq. (1) describes the evaporation across black hole event horizons. Importantly, the generality of this equation suggests that evidence which supports the validity of Eq. (1) for black holes likely implies its more universal validity as a description of evaporation across arbitrary event horizons.

TECHNICAL PROOFS AND MINOR NOTES

Proof of Theorem 1: We observe that it is trivial to verify that any function $\Gamma(\varepsilon|M)$ of the form $e^{f(M-\varepsilon)-f(M)+h(\varepsilon)}$ satisfies $\Gamma(\varepsilon_1|M)\,\Gamma(\varepsilon_2|M-\varepsilon_1) = \Gamma(\varepsilon_2|M)\,\Gamma(\varepsilon_1|M-\varepsilon_2)$. To prove this is the general solution set $\gamma(\varepsilon, M) = \ln\Gamma(\varepsilon|M)$. Then γ satisfies an additive equation

$$\gamma(\varepsilon_1, M-\varepsilon_2)+\gamma(\varepsilon_2, M) = \gamma(\varepsilon_2, M-\varepsilon_1)+\gamma(\varepsilon_1, M). \quad (2)$$

Taking the partial derivative of this equation with-respect-to ε_2 and then setting $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = 0$ yields

$$\gamma_2(\varepsilon, M) = \gamma_1(0, M) - \gamma_1(0, M-\varepsilon), \quad (3)$$

where $\gamma_1(\varepsilon, M) \equiv \partial\gamma(\varepsilon, M)/\partial\varepsilon$ and $\gamma_2(\varepsilon, M) \equiv \partial\gamma(\varepsilon, M)/\partial M$. A general solution to this equation is given by

$$\gamma(\varepsilon, M) = \int_{M-\varepsilon}^{\infty}\gamma_1(0, M')\,\mathrm{d}M' - \int_{M}^{\infty}\gamma_1(0, M')\,\mathrm{d}M' + h(\varepsilon), \quad (4)$$

where $h(\varepsilon)$ is an arbitrary function. Now setting

$$f(M) = \int_{M}^{\infty}\gamma_1(0, M')\,\mathrm{d}M', \quad (5)$$

we have $\gamma(\varepsilon, M) = f(M-\varepsilon) - f(M) + h(\varepsilon)$. ■

Proof of Theorem 2: The case of energy and charge which are scalar observables is obvious. We have to consider angular momentum only. As is well-known angular momentum operators generate the Lie algebra $su(2)$. The finite-dimensional representations of this algebra are *completely reducible*, that is, the state space can be decomposed into a direct sum of *irreducible* representations. Moreover, since the black hole is to be in a definite angular momentum state each of the summands must have the same J^2 eigenvalue. It therefore suffices to focus on any one irreducible summand. We will freely use the standard properties of irreducible representations. Since the post-evaporation state of a black hole must be a spin-coherent state and the only orthogonal set of spin-coherent states are of the form $\{R(\theta,\phi)\,|j, j\rangle\,, R(\theta,\phi)\,|j,-j\rangle\}$ the general state of the evaporated particle and black hole is given by

$$\begin{aligned}\Phi &= \alpha'\otimes R(\theta,\phi)\,|j,j\rangle + \beta'\otimes R(\theta,\phi)\,|j,-j\rangle \\ &= R(\theta,\phi)\otimes R(\theta,\phi)(\alpha\otimes|j,j\rangle + \beta\otimes|j,-j\rangle),\end{aligned} \quad (6)$$

where $R(\theta,\phi)\alpha = \alpha'$ and $R(\theta,\phi)\beta = \beta'$ denote (unnormalized) states of the evaporated particle. Let the operator representing J^2 on the product space be denoted J^2_{tot}. Then the condition that Φ be an eigenstate of $J^2\otimes\mathbb{1}$, $\mathbb{1}\otimes J^2$ and J^2_{tot} implies it must be an eigenstate of the operator

$$\begin{aligned}\tilde{J} &\equiv J^2_{\text{tot}} - J^2\otimes\mathbb{1} - \mathbb{1}\otimes J^2 \\ &= J_+\otimes J_- + J_-\otimes J_+ + 2J_z\otimes J_z.\end{aligned} \quad (7)$$

As $\tilde{J}$ is invariant under $R(\theta,\phi)\otimes R(\theta,\phi)$ this implies that

$$\begin{aligned}&J_+\alpha\otimes J_-\,|j,j\rangle + J_-\beta\otimes J_+\,|j,-j\rangle \\ &+2j(J_z\alpha\otimes|j,j\rangle - J_z\beta\otimes|j,-j\rangle) \\ &= x(\alpha\otimes|j,j\rangle + \beta\otimes|j,-j\rangle)\end{aligned} \quad (8)$$

where x is a real number.

First, suppose that $j > 1$. Then the vectors $|j,j\rangle$, $|j,-j\rangle$, $J_-\,|j,j\rangle$ and $J_+\,|j,j\rangle$ are mutually orthogonal and the above equation can be satisfied if and only if either $\beta = 0$ and $J_+\alpha = 0$ or $\alpha = 0$ and $J_-\beta = 0$. We conclude that in this case the only allowed forms of Φ are (up to a global rotation) $|j_p, j_p\rangle\otimes|j,j\rangle$ and $|j_p,-j_p\rangle\otimes|j,-j\rangle$ where $|j_p, j_p\rangle$ ($|j_p,-j_p\rangle$) is the highest (lowest) eigenvector in the particle's angular momentum space. Clearly these states are always J^2 eigenstates for any value of J. We call such states for Φ standard. To conserve $J_{\text{tot},\hat{n}}$ the state of the mother black hole must be $R(\theta,\phi)\,|j+j_p, \pm(j+j_p)\rangle$ respectively.

Next suppose $j = 1$. It follows from Eq. (8) that besides the standard states the state (up to a global rotation)

$$\frac{1}{\sqrt{2}}(|j_p, -1\rangle \otimes |1, 1\rangle - |j_p, 1\rangle \otimes |1, -1\rangle), \tag{9}$$

is also an eigenstate of J^2_{tot} with total angular momentum of the mother black hole $j' = j_p$. As this is a $J_{\text{tot},\hat{z}}$ eigenstate with zero eigenvalue no orientation can conserve $J_{\text{tot},\hat{n}}$ of the original black hole. We therefore rule this class of states out.

Next suppose $j = \frac{1}{2}$. In addition to the standard states there are other possibilities. The product space decomposes into two irreducible representations corresponding to total angular momentum $j' = j_p \pm \frac{1}{2}$. They are generated respectively by highest weight vectors (up to a global rotation)

$$|j_p, j_p\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle\,, \tag{10}$$

for $x = j_p$ corresponding to $j' = j_p + \frac{1}{2}$ and

$$\frac{1}{\sqrt{2j_p + 1}}\big(\sqrt{2j_p}\,|j_p, j_p\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle - |j_p, j_p - 1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle\big), \tag{11}$$

for $x = -1 - j_p$ corresponding to $j' = j_p - \frac{1}{2}$. Starting with either of these we can generate the other $J_{\hat{z}}$ eigenvectors (in this globally rotated basis) by successive applications of the J_- operator. However, considering conservation of $J_{\text{tot},\hat{n}}$ disallows any of these extra eigenvectors. Therefore, when quantized along the $\hat{n}$ axis the mother black hole had the state $|j_p + \frac{1}{2}, j_p + \frac{1}{2}\rangle$ and $|j_p - \frac{1}{2}, j_p - \frac{1}{2}\rangle$ respectively (with the exception of the case $j_p = \frac{1}{2}$ for the latter mother black hole state with $j' = 0$ where the orientation of the quantization axis is arbitrary).

This leaves only $j = 0$ which is trivial. It is now easy to check that in every case allowed by global conservation laws the statement of the theorem holds true. ■

Proof of Theorem 3: Let $\gamma(\vec{x}, \vec{X}) = \ln \Gamma(\vec{x}|\vec{X})$. Then γ satisfies

$$\gamma(\vec{x}, \vec{X}) + \gamma(\vec{x}', \vec{X} - \vec{x}) = \gamma(\vec{x}', \vec{X}) + \gamma(\vec{x}, \vec{X} - \vec{x}') \tag{12}$$

Taking the partial derivative with-respect-to x'_i and then setting $x'_i = 0$ yields

$$\frac{\partial \gamma(\vec{x}, \vec{X})}{\partial X_i} = \frac{\partial \gamma}{\partial x_i}\Big|_{(\vec{0}, \vec{X})} - \frac{\partial \gamma}{\partial x_i}\Big|_{(\vec{0}, \vec{X} - \vec{x})}. \tag{13}$$

As in Theorem 1, the solution to the above partial differential equation for $i = n$ can be inferred from Eq. (3) above by treating all variables except the last "conjugate" pair (x_n, X_n) as constants, so

$$\gamma(\vec{x}, \vec{X}) = f_n(\vec{X} - \vec{x}) - f_n(\vec{X}) + h_n(\vec{x}, \hat{X}) \tag{14}$$

where $\hat{X} = \{X_1, \ldots, X_{n-1}\}$ without dependence on X_n; the function $h_n(\vec{x}, \hat{X})$ is otherwise arbitrary. Now, substituting this into the functional equation (12) for γ and noting that

$$f_n(\vec{X} - \vec{x}) - f_n(\vec{X}) \tag{15}$$

is a already a solution of it, we see that h_n satisfies the equations

$$\frac{\partial h_n(\vec{x}, \hat{X})}{\partial X_i} = \frac{\partial h_n}{\partial x_i}\Big|_{(\vec{0}, \hat{X})} - \frac{\partial h_n}{\partial x_i}\Big|_{(\vec{0}, \hat{X} - \hat{x})} \tag{16}$$

Now consider the function $\partial h_n(\vec{x}, \hat{X})/\partial x_n$ we have

$$\begin{aligned} \frac{\partial}{\partial X_i}\Big(\frac{\partial h_n(\vec{x}, \hat{X})}{\partial x_n}\Big) &= \frac{\partial}{\partial x_n}\Big(\frac{\partial h_n(\vec{x}, \hat{X})}{\partial X_i}\Big) \\ = \frac{\partial}{\partial x_n}\Big(\frac{\partial h_n}{\partial x_i}\Big|_{(\vec{0}, \hat{X})} &- \frac{\partial h_n}{\partial x_i}\Big|_{(\vec{0}, \hat{X} - \hat{x})}\Big) \equiv 0, \end{aligned} \tag{17}$$

since $\hat{x}$ has no dependence on x_n. Hence the function $\partial h_n(\vec{x}, \hat{X})/\partial x_n$ can have no dependence on any X_i. Consequently the function $h_n(\vec{x}, \hat{X})$ must have the form

$$h_n(\vec{x}, \hat{X}) = u_n(\vec{x}) + \gamma_{n-1}(\hat{x}, \hat{X}). \tag{18}$$

The function γ_{n-1} satisfies the functional equation (12) with $n - 1$ pairs of conjugate variables. Hence

$$\gamma(\vec{x}, \vec{X}) = f_n(\vec{X} - \vec{x}) - f_n(\vec{X}) + u_n(\vec{x}) + \gamma_{n-1}(\hat{x}, \hat{X}). \tag{19}$$

Using this argument recursively and absorbing the different functions together, we conclude that

$$\gamma(\vec{x}, \vec{X}) = f(\vec{X} - \vec{x}) - f(\vec{X}) + h(\vec{x}). \tag{20}$$

■

Note: From Theorem 4, permutation symmetry yields

$$\Gamma(\vec{x}|\vec{X}) = e^{f(\vec{X} - \vec{x}) - f(\vec{X}) + h(\vec{x})}. \tag{21}$$

For infinitessimal $\vec{x}$ backreaction should be negligible and we should recover the Hawking thermal spectrum, i.e.,

$$\Gamma(\vec{x}|\vec{X}) \simeq e^{-\vec{\nabla} f(\vec{X}) \cdot \vec{x} + h(\vec{0})} \equiv N e^{-\vec{\nabla} \mathcal{S}(\vec{X}) \cdot \vec{x}}, \quad \forall \vec{X}. \tag{22}$$

Here $\mathcal{S}(\vec{X})$ is the thermodynamic entropy of the black hole, N is a normalization constant and without loss of generality we have absorbed any linear part of h into f. Solving $\vec{\nabla} f(\vec{X}) = \vec{\nabla} \mathcal{S}(\vec{X})$ yields $f(\vec{X}) = \mathcal{S}(\vec{X})$ since $f(\vec{0})$ may be chosen arbitrarily. Note that the reasoning provided in the manuscript does *not* rely on this argument *nor* on consistency with the Hawking thermal spectrum.

Proof of Theorem 4: Let $\vec{X} \in \Sigma^o - K$, then by definition there is some component X_i of $\vec{X}$ such that $\partial \mathcal{I}/\partial X_i \neq 0$ at $\vec{X}$. Without loss of generality we may take $i = n$. Then there is some neighborhood O of $\vec{X}$

such that $\partial\mathcal{I}/\partial X_n \neq 0$ at every point in O. Consider the continuously differentiable map $F : O \to O$

$$F(\vec{X}) = (X_1, \ldots, X_{n-1}, \mathcal{I}(\vec{X})). \tag{23}$$

The Jacobian of F is simply $|\partial\mathcal{I}/\partial X_n|$ and does not vanish anywhere in O. From the inverse function theorem then there is a neighborhood $\tilde{O} \subset O$ such that F is *invertible* in $\tilde{O}$. Thus any $\vec{X} \in \tilde{O}$ can be written in the new coordinate system as $\vec{X} = (X_1, \ldots, X_{n-1}, \mathcal{I}(\vec{X}))$. Let θ be the corresponding function that represents Θ in the new coordinates. Then for $\vec{X}_1, \vec{X}_2 \in \tilde{O}$ and $\vec{X}'_1, \vec{X}'_2 \in \tilde{O}'$ the hypothesis in Eq. (10) [of the manuscript] is equivalent to $\theta(X_{1,1}, \ldots, X_{1,n-1}, \mathcal{I}(\vec{X}_1), X'_{1,1}, \ldots, X'_{1,n-1}, \mathcal{I}(\vec{X}'_1)) = \theta(X_{2,1}, \ldots, X_{2,n-1}, \mathcal{I}(\vec{X}_2), X'_{2,1}, \ldots, X'_{2,n-1}, \mathcal{I}(\vec{X}'_2))$ But this is precisely the statement that θ is independent of the first $n-1$ coordinates in each argument. Hence $\Theta(\vec{X}, \vec{X}') = \theta(\mathcal{I}(\vec{X}), \mathcal{I}(\vec{X}'))$ in $\tilde{O} \times \tilde{O}'$. This must be true for every pair of points in $\Sigma^o - K$. Note that although for another pair of points say $\vec{Y}, \vec{Y}' \in \Sigma^o - K$ the new θ_Y may be a different function, θ and θ_Y must match in any common domain since Θ is globally defined. Hence there is a continuously differentiable function θ such that the assertion of the theorem holds for any pair of arguments in $\Sigma^o - K$. Since the latter is a dense subset of Σ, θ can be uniquely extended to the whole of $\Sigma \times \Sigma$ by continuity. ∎

Note: The irreducible mass of a black hole with no-hair triple $\vec{X} = (M, Q, J)$ in General Relativity is

$$\mathcal{I} = \frac{1}{2}\left(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - a^2}\right)^{1/2}, \tag{24}$$

where $a \equiv J/M$. It is straightforward to check that this function satisfies the condition in Theorem 4 that $\{\vec{X} : |\vec{\nabla}\mathcal{I}(\vec{X})| = 0\}$ is nowhere dense.

Note: It has been noted in the literature [S4, S5, S6] that Eq. (1) [of the manuscript] for the Schwarzschild case naively satisfies the relation [S7]

$$\Gamma(\varepsilon_1|M)\Gamma(\varepsilon_2|M - \varepsilon_1) = \Gamma(\varepsilon_1 + \varepsilon_2|M). \tag{25}$$

by symmetry of $\varepsilon_1 + \varepsilon_2$ it is trivial to use Theorem 1 from our manuscript to write down the general solution to Eq. (25) as

$$\Gamma(\varepsilon|M) \equiv e^{f(M-\varepsilon)-f(M)}, \tag{26}$$

for some function f.

Note: Consider one form of the well-known Cauchy functional equation [S8]

$$G(a)G(b) = G(a + b). \tag{27}$$

Its unique solution is the exponential family of functions.

Naively, Eq. (25) is apparently a natural generalization to the Cauchy equation (27) when incorporating conservation laws. However, as already noted [S7] its interpretation is problematic. By contrast, the functional equation

$$\Gamma(\vec{x}|\vec{X})\,\Gamma(\vec{x}'|\vec{X} - \vec{x}) \;=\; \Gamma(\vec{x}'|\vec{X})\,\Gamma(\vec{x}|\vec{X} - \vec{x}'), \tag{28}$$

[Eq. (7) of the manuscript] provides a truly non-trivial generalization to the Cauchy functional equation in the presence of conservation laws. Its interpretation is clear as a permutation symmetry (see manuscript) and further it includes Eq. (25) as a special case.

[S1] S. W. Hawking, Phys. Rev. D **14**, 2460 (1976).

[S2] H. Nikolić, Int. J. Mod. Phys. D **14**, 2257 (2005). S. D. Mathur, Class. Quantum Grav. **26**, 224001 (2009).

[S3] S. L. Braunstein and K. Życzkowski, arXiv:0907.1190.

[S4] B. C. Zhang, et al., Phys. Lett. B **675**, 98 (2009).

[S5] M. K. Parikh, arXiv:hep-th/0402166.

[S6] M. A. Arzano, J. M. Medved and E. C. Vagenas, JHEP **09**, 037 (2005).

[S7] In fact, this is possible only if the normalization is excluded or absorbed into the measure. More worrying, perhaps, is the interpretation of this relation. Why for example does it not include a sum over all ways of partitioning the total energy $\varepsilon_1 + \varepsilon_2$ into two (or more) subcomponents?

[S8] J. Aczél, Lectures on Functional Equations and their Applications (Academic Press, New York, 1966).