Limitations to squeezing in a parametric amplifier due to pump quantum fluctuations

David D. Crouch* and Samuel L. Braunstein†
Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, California 91125
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We perform discrete-mode calculations for a parametric amplifier with a quantum pump and discuss some of the limitations on calculations of this sort in quantum optics. We calculate corrections to the squeezing due to pump quantum fluctuations to order $\bar{N}^{-2}$, for a pump initially in a coherent state with average photon number $\bar{N}$. We find that the limit to the variance of the squeezed quadrature due to the quantum nature of the pump goes as $\bar{N}^{-1/2}$.

I. INTRODUCTION

The parametric amplifier[1,2] (PA) is a basic device in quantum optics and quantum electronics. It couples a pump field at frequency ω_p to signal modes at frequencies near $\omega=\omega_p/2$. In this paper we are mainly interested in the application of the PA for generating squeezed states,[3–5] i.e., quantum states for which one of a pair of canonically conjugate variables has its quantum noise (uncertainty) reduced below the vacuum level (zero point noise). The main purpose of this paper is to show that the ability of a PA to produce squeezed light is limited by the initial phase noise in the pump.

When the signal modes are initially in vacuum states, only the pump's phase can determine which quadrature will be squeezed. If the pump's phase fluctuates, then the quadrature chosen will have a slight admixture[6] of its conjugate quadrature—the noisy quadrature. This argument is treated more carefully in Sec. II for the case of phase noise in a classical pump. Calculations of the corrections to semiclassical order (i.e., to order $1/\bar{N}$ in the matrix elements, where $\bar{N}$ is the average photon number of the pump) have been previously performed for both the one-[7] and two-mode[8] PA.

Hillery and Zubairy[7] studied the one-mode PA with an interaction Hamiltonian

$$\hat{H}_{\text{int}}=i\frac{\hbar\kappa}{2}(\hat{a}^{\dagger 2}\hat{a}_p-\hat{a}^{2}\hat{a}_p^{\dagger}) \tag{1.1}$$

(up to a phase rotation of the variables), where $\hat{a}$ and $\hat{a}_p$ are the annihilation operators for the signal and pump modes, respectively, and κ is a coupling constant which is proportional to the second order nonlinear susceptibility $\chi^{(2)}$ of the medium in which the interaction is taking place. They used a path-integral technique[9] to obtain corrections at the semiclassical order. They did not claim to get the full semiclassical correction,[10] and the dominant terms they obtained for the fluctuation in the squeezed quadrature were

$$\langle\Delta\hat{x}_2^2\rangle\approx\frac{e^{-2u}}{4}+\frac{e^{2u}}{64\bar{N}}, \tag{1.2}$$

where $\hat{x}_2=-i(\hat{a}-\hat{a}^{\dagger})/2$ is the quadrature (analogous to the position operator) which is squeezed by the interaction in Eq. (1.1), and $u=\bar{N}^{1/2}\kappa t$ is a dimensionless time. This yields a minimum variance, and hence a limit to the squeezing, of

$$\langle\Delta\hat{x}_2^2\rangle_{\min}\approx\frac{1}{8\bar{N}^{1/2}}, \tag{1.3}$$

which is just what the argument of phase noise in the classical pump gives (see Sec. II).

Scharf and Walls[8] studied the two-mode PA whose interaction Hamiltonian is (again up to a rotation of the variables' phases)

$$\hat{H}_{\text{int}}=i\hbar\kappa(\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\hat{a}_p-\hat{a}_1\hat{a}_2\hat{a}_p^{\dagger}), \tag{1.4}$$

where $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators for the two signal modes. They used an asymptotic method developed by Scharf[11–13] to arrive at the dominant correction to the variance of the Hermitian variable

$$\hat{y}_2\equiv-\frac{i}{2}(\hat{a}_+-\hat{a}_+^{\dagger}), \tag{1.5}$$

as

$$\langle\Delta\hat{y}_2^2\rangle\approx\frac{e^{-2u}}{4}+\frac{e^{6u}}{1920\bar{N}}, \tag{1.6}$$

where $\hat{a}_+\equiv(\hat{a}_1+\hat{a}_2)/\sqrt{2}$. They concluded that the minimum variance obtainable by the two-mode PA would be

$$\langle\Delta\hat{y}_2^2\rangle_{\min}\approx\frac{1}{6(10\bar{N})^{1/4}}. \tag{1.7}$$

How can we compare these calculations? If one rewrites Eq. (1.4) in terms of the variables

$$\hat{a}_+\equiv\frac{1}{\sqrt{2}}(\hat{a}_1+\hat{a}_2), \tag{1.8a}$$

$$\hat{a}_-\equiv\frac{1}{\sqrt{2}}(\hat{a}_1-\hat{a}_2), \tag{1.8b}$$

then the interaction Hamiltonian may be written[14]

$$\hat{H}_{\text{int}}=i\frac{\hbar\kappa}{2}(\hat{a}_+^{\dagger 2}\hat{a}_p-\hat{a}_+^{2}\hat{a}_p^{\dagger})-i\frac{\hbar\kappa}{2}(\hat{a}_-^{\dagger 2}\hat{a}_p-\hat{a}_-^{2}\hat{a}_p^{\dagger}). \tag{1.9}$$

If the pump is now treated classically then the $\hat{a}_+$ and $\hat{a}_-$ modes become completely independent, each described by the one-mode PA Hamiltonian equation (1.1). Thus we might expect the same correction to the squeezing due to a quantum pump as found by Hillery and Zubairy [see Eq. (1.2)]. In fact, since the pump is allowed to be quantum mechanical, the $\hat{a}_+$ and $\hat{a}_-$ modes can interact with each other by modifying their common pump. Thus these modes cannot completely decouple. Even so, Scharf and Walls's results of Eqs. (1.6) and (1.7) are surprising; for a pump with $\bar{N}=10^9$ there is a large discrepancy

$$\frac{\langle \Delta\hat{y}_2^2\rangle_{\min}}{\langle \Delta\hat{x}_2^2\rangle_{\min}}=\frac{4}{3}\left[\frac{\bar{N}}{10}\right]^{1/4}=\frac{400}{3}\ . \tag{1.10}$$

The purpose of this paper is to resolve the apparent discrepancy between these two calculations, first noted by Caves and Crouch.[15] We will use three different methods to calculate the semiclassical corrections for the one- and the two-mode PA.

This paper is divided into seven sections. Section II justifies our use of discrete-mode calculations for a traveling-wave device which in principle should be given a full continuum treatment, and also reviews the argument for the contribution of phase noise in a classical pump. In Sec. III we discuss the symmetries of the discretized PA system and how they can be used to simplify our calculations.

The first method of calculation (Sec. IV) involves integrating the Heisenberg equations for the quadrature-phase amplitudes up to the required order. What is novel about the approach presented here is that we work out first the form of all the terms, and then we identify the dominant terms and proceed to calculate only those terms. This calculation yields the dominant terms up to $O(1/\bar{N}^2)$. By calculating these terms we can estimate when the semiclassical correction breaks down. We find that the semiclassical corrections [for instance, Eq. (1.2)] are valid so long as they are much less than 1.

The second method (Sec. V) is numerical. However, we are able to obtain analytic expressions for the full corrections to $O(1/\bar{N}^2)$. We can do this since the general form of these corrections has already been worked out in Sec. IV. This calculation uses a new algebra which can be viewed as a semiclassical approximation to the ordinary commutator algebra for annihilation and creation operators. This algebra is used in order to normal order the annihilation and creation operators of the pump.

The third method (Sec. VI) uses the positive-P distribution[16] to derive Fokker-Planck equations for one- and two-mode parametric amplifiers. Standard methods of stochastic calculus[17] are then used to derive Ito stochastic differential equations (SDE's) from the Fokker-Planck equations. An approximate solution of the SDE's is obtained by iteration, and the full semiclassical correction is then calculated analytically.

These three methods agree with each other. The latter two show that Hillery and Zubairy have in fact calculated the exact semiclassical corrections to the parametric approximation for the one-mode PA, namely,

$$\langle \Delta\hat{x}_2^2\rangle=\frac{e^{-2u}}{4}+\frac{e^{2u}}{64\bar{N}}[1+(3-8u)e^{-2u}-(1+8u-8u^2)e^{-4u}-3e^{-6u}]\ . \tag{1.11}$$

Similarly, the exact semiclassical expression for the two-mode PA is found to be

$$\langle|\Delta\hat{X}_2|^2\rangle=\frac{e^{-2u}}{4}+\frac{e^{2u}}{64\bar{N}}[1+(4-8u)e^{-2u}-(1+12u-8u^2)e^{-4u}-4e^{-6u}]\ , \tag{1.12}$$

where $X_2=-i(\hat{a}_1-\hat{a}_2^\dagger)/2$ is the quadrature-phase operator for the squeezed quadrature, and

$$\langle|\Delta\hat{X}|^2\rangle=\langle\hat{X}\hat{X}^\dagger\rangle_{\rm sym}-\langle\hat{X}\rangle\langle\hat{X}^\dagger\rangle =\tfrac{1}{2}\langle\hat{X}\hat{X}^\dagger+\hat{X}^\dagger\hat{X}\rangle-\langle\hat{X}\rangle\langle\hat{X}^\dagger\rangle\ . \tag{1.13}$$

The dominant corrections for the one- and two-mode calculations are the same, and agree with the dominant correction obtained by Caves and Crouch[15] from a *continuum* calculation.

II. DISCUSSION

The conventional approach to problems in quantum optics typically makes use of a mode expansion to describe the electromagnetic field. Using this approach, one can derive from an appropriate Hamiltonian *temporal* differential equations for the modal creation and annihilation operators, the spatial dependence being carried by the mode functions. Such an approach is suitable for cavity devices in which one has well defined standing-wave modes (the eigenmodes of the cavity), but not for a traveling-wave device in which such modes are nonexistent. One would like to derive *spatial* differential equations governing the evolution of the field operators through the medium, in analogy with classical nonlinear optics; the conventional approach is clearly unsuited to this purpose. Tucker and Walls[18] and Lane *et al.*[19] recognized these problems with the conventional approach and developed a continuum wave-packet formalism in an attempt to deal with them.

In this section we briefly describe a discrete-mode expansion of the electromagnetic field in terms of *wave-packet* modes that enable us to derive *spatial* equations of motion for PA's. We assume that the wave packets are short compared to the nonlinear medium through which they propagate, so that they "fit" inside the medium, allowing us to ignore boundary effects. Physically, the individual wave packet propagates from free space through the entrance boundary on a time scale short compared to the time it will spend inside the nonlinear medium; in this way the interaction is "turned on." This method is preferable to the technique often used in the conventional approach in which the interaction is suddenly turned on throughout all space, either at time $t=0$ or at some time in the remote past. We also present a heuristic argument for the dominant effect of pump quantum fluctuations on the variance of the squeezed quadrature in a PA.

We will give a brief outline of the derivation of the discrete wave-packet mode equations of motion for the PA; details will be given elsewhere. The discrete-mode expansions of the signal and pump magnetic field operators in a dispersionless medium are given by

$$\hat{B}^{(+)}_{0s}(z,t)=\sum_{n=-M}^{M}\sum_{k=-\infty}^{\infty}\left[\frac{2\pi n_0\hbar\omega_n}{c\sigma T_s}\right]^{1/2}\hat{a}_{nk}(z)\,e^{-i\omega_n[(t-n_0z/c)-kT_s]}f((t-n_0z/c)-kT_s)\ , \tag{2.1a}$$

$$\hat{B}^{(+)}_{0p}(z,t)=i\sum_{k=-\infty}^{\infty}\left[\frac{2\pi n_0\hbar\Omega_p}{c\sigma T_p}\right]^{1/2}\hat{b}_{k}(z)\,e^{-i\Omega_p[(t-n_0z/c)-kT_p]}f((t-n_0z/c)-kT_p)\ , \tag{2.1b}$$

where

$$[\hat{a}_{nk}(z),\hat{a}_{n'k'}(z)]=[\hat{b}_k(z),\hat{b}_{k'}(z)]=0\ , \tag{2.2a}$$

$$[\hat{a}_{nk}(z),\hat{a}^{\dagger}_{n'k'}(z)]=\delta_{nn'}\delta_{kk'}\ , \tag{2.2b}$$

$$[\hat{b}_k(z),\hat{b}^{\dagger}_{k'}(z)]=\delta_{kk'}\ , \tag{2.2c}$$

and

$$f_j(t)=\frac{\sin\pi t/T_j}{\pi t/T_j}\ ,\quad j=s,p \tag{2.3}$$

is the wave-packet envelope function. Here n_0 is the index of refraction of the dispersionless medium, $\Omega_p=2\Omega$ is the pump frequency, and σ is a cross-sectional area we use to account crudely for the transverse structure of the field. The discrete-mode expansions described by Eqs. (2.1a) and (2.1b) are obtained from a continuum description by dividing the signal and pump bandwidths into "bins" of width $\Delta\omega_s$ and $\Delta\omega_p$, respectively, with signal center frequency $\omega_n=\Omega+n\Delta\omega$ and pump center frequency $\Omega_p=2\Omega$.[20] Each signal (pump) bin corresponds to a train of wave packets (corresponding to different values of k) in the time domain, each of approximate duration $T_s=2\pi/\Delta\omega_s$ ($T_p=2\pi/\Delta\omega_p$) with envelope given by Eq. (2.3).

By substituting Eqs. (2.1a) and (2.1b) into Maxwell's equations, we obtain the *spatial* equations of motion

$$\frac{d\hat{a}_{nk}(z)}{dz}=\kappa'\sum_{k'=-\infty}^{\infty}e^{i\Omega_p(k'T_p-kT_s)}\frac{\sin[\pi(kT_s-k'T_p)/T_p]}{\pi(kT_s-k'T_p)/T_p}\hat{b}_{k'}(z)\hat{a}^{\dagger}_{-nk}(z)\ , \tag{2.4a}$$

$$\frac{d\hat{b}_{k}(z)}{dz}=-\frac{\kappa'}{2}\sum_{n=-M}^{M}\sum_{k'=-\infty}^{\infty}e^{i\Omega_p(k'T_s-kT_p)}\frac{\sin[\pi(k'T_s-kT_p)/T_p]}{\pi(k'T_s-kT_p)/T_p}\hat{a}_{nk'}(z)\hat{a}_{-nk'}(z)\ , \tag{2.4b}$$

where the coupling constant κ' is given by

$$\kappa'=\frac{4\pi\Omega\chi^{(2)}}{n_0^2c}\left[\frac{2\pi n_0\hbar\Omega_p}{c\sigma T_p}\right]^{1/2}. \tag{2.5}$$

Here we have assumed that $\Delta\omega_p \ll \Delta\omega_s$ (or $T_p \gg T_s$) to avoid coupling among energy nonconserving modes. By restricting our observations to the region of spacetime near $t-n_0z/c=0$, we can discard all wave packets (both signal and pump) with $k\neq0$, since $f_j(kT_j)=\delta_{k,0}$ With $\hat{a}_{n0}(z)\equiv\hat{a}_n(z)$ and $\hat{b}_0(z)\equiv\hat{a}_p(z)$, we find

$$\frac{d\hat{a}_n(z)}{dz}=\kappa'\hat{a}_p(z)\hat{a}^{\dagger}_{-n}(z)\ , \tag{2.6a}$$

$$\frac{d\hat{a}_p(z)}{dz}=-\frac{\kappa'}{2}\sum_{n=-M}^{M}\hat{a}_n(z)\hat{a}_{-n}(z)\ . \tag{2.6b}$$

Assuming that the wave packets are narrow compared to the scale of variation set by κ', we can replace z by ct/n_0 and obtain the *temporal* equations of motion

$$\frac{d\hat{a}_n(t)}{dt}=\kappa\hat{a}_p(t)\hat{a}^{\dagger}_{-n}(t)\ , \tag{2.7a}$$

$$\frac{d\hat{a}_p(t)}{dt}=-\frac{\kappa}{2}\sum_{n=-M}^{M}\hat{a}_n(t)\hat{a}_{-n}(t)\ , \tag{2.7b}$$

where $\kappa=c\kappa'/n_0$. Equations (2.7a) and (2.7b) are identical to the Heisenberg equations of motion that are derived from the multimode Hamiltonian

$$H=i\frac{\hbar\kappa}{2}\sum_{n=-M}^{M}[\hat{a}_p(t)\hat{a}^{\dagger}_n(t)\hat{a}^{\dagger}_{-n}(t)-\hat{a}^{\dagger}_p(t)\hat{a}_n(t)\hat{a}_{-n}(t)] \tag{2.8}$$

when the conventional approach is used.

The Hamiltonian equation (2.8) correctly describes the interaction of a discrete pump mode with $2M+1$ discrete signal modes, but it does not provide a completely accurate description of traveling-wave parametric amplification, since it ignores the interaction of the pump wave packet $k=0$ with signal wave packets other than $k=0$. Ignoring as it does interactions with these wave packets, the Hamiltonian cannot correctly describe nonlinear effects such as pump depletion; it does, however, correctly describe the effect of the initial pump quantum fluctuations on the signal modes. We will show, first by a heuristic argument and then by the results of detailed calculations using the Hamiltonian equation (2.8), that the initial pump quantum fluctuations are responsible for the dominant correction to the squeezing due to the quantum nature of the pump. We also calculate higher-order corrections. By the argument just given, the exact form of these corrections cannot be related to the physical parameters of a traveling-wave PA; these corrections *are* of physical interest, however, in showing how nonlinear effects affect the squeezing, and of mathematical interest in demonstrating the computational tools we have developed to calculate them.

The wave-packet approach gives us a new and more realistic way to deal with traveling-wave problems in quantum optics; it also leads one to realize that the conventional Hamiltonian approach can lead to misleading results when used blindly. We will, however, ignore distinctions between the conventional and the wave-packet approaches through much of this paper. The point we wish to make is that the wave-packet modes are an appropriate set of modes for describing the spatial evolution of quantized electromagnetic fields in traveling-wave devices without resorting to continuum calculations.

We will now give a heuristic argument for the effect of pump quantum fluctuations on the squeezing produced by a parametric amplifier. Our treatment of parametric amplification has thus far treated the pump quantum mechanically. Under certain circumstances, one can treat the pump classically, in what is known as the parametric approximation; in this approximation, one replaces the pump operator by a c number $\alpha_p=\bar{N}^{1/2}e^{i\phi_p}$. The interaction Hamiltonian for a one-mode PA, where we ignore all modes in Eq. (2.8) except for $n=0$, is

$$H_p=i\frac{\hbar\kappa\bar{N}^{1/2}}{2}[\hat{a}^{\dagger 2}(t)\,e^{i\phi_p}-\hat{a}^{2}(t)\,e^{-i\phi_p}]\;, \tag{2.9}$$

where $\hat{a}_0(t)\equiv\hat{a}(t)$; two resulting equations of motion are

$$\frac{d\hat{a}(t)}{dt}=\kappa\bar{N}^{1/2}e^{i\phi_p}\,\hat{a}^{\dagger}(t)\;, \tag{2.10a}$$

$$\frac{d\hat{a}^{\dagger}(t)}{dt}=\kappa\bar{N}^{1/2}e^{-i\phi_p}\,\hat{a}(t)\;. \tag{2.10b}$$

We define *two* sets of quadrature-phase amplitudes,[14]

$$\hat{x}_1(t)=\frac{1}{2}[\hat{a}(t)+\hat{a}^{\dagger}(t)]\;, \tag{2.11a}$$

$$\hat{x}_2(t)=-\frac{i}{2}[\hat{a}(t)-\hat{a}^{\dagger}(t)] \tag{2.11b}$$

and

$$\hat{x}_1'(t)=\frac{1}{2}[\hat{a}(t)e^{-i\phi_p/2}+\hat{a}^{\dagger}(t)e^{i\phi_p/2}]\;, \tag{2.12a}$$

$$\hat{x}_2'(t)=-\frac{i}{2}[\hat{a}(t)e^{-i\phi_p/2}-\hat{a}^{\dagger}(t)e^{i\phi_p/2}]\;, \tag{2.12b}$$

the two sets being identical when $\phi_p=0$. The two sets of quadrature-phase amplitudes are related by the rotation

$$\hat{x}_1(t)=\hat{x}_1'(t)\cos(\phi_p/2)-\hat{x}_2'(t)\sin(\phi_p/2)\;, \tag{2.13a}$$

$$\hat{x}_2(t)=\hat{x}_2'(t)\cos(\phi_p/2)+\hat{x}_1'(t)\sin(\phi_p/2)\;, \tag{2.13b}$$

pictured in Fig. 1 for $\phi_p/2=\Delta\phi$. By substituting Eqs. (2.12a) and (2.12b) in Eqs. (2.10a) and (2.10b), we see that the quadrature-phase amplitudes $\hat{x}_1'(t)$ and $\hat{x}_2'(t)$ decouple the equations of motion:

$$\frac{d\hat{x}_1'(t)}{dt}=\kappa\bar{N}^{1/2}\hat{x}_1'(t)\Longrightarrow\hat{x}_1'(u)=\hat{x}_1'(0)\,e^{u}\;, \tag{2.14a}$$

$$\frac{d\hat{x}_2'(t)}{dt}=-\kappa\bar{N}^{1/2}\hat{x}_2'(t)\Longrightarrow\hat{x}_2'(u)=\hat{x}_2'(0)\,e^{-u}\;, \tag{2.14b}$$

where $u=\kappa\bar{N}^{1/2}t$ is a dimensionless time. For a vacuum input, one easily finds that

$$\langle\Delta\hat{x}_1'^2(u)\rangle=\tfrac{1}{4}e^{2u}\;, \tag{2.15a}$$

$$\langle\Delta\hat{x}_2'^2(u)\rangle=\tfrac{1}{4}e^{-2u}\;; \tag{2.15b}$$

the $\hat{x}_2'$ quadrature exhibits maximum squeezing when the pump's phase is ϕ_p. The corresponding noise in the quadrature-phase amplitudes $\hat{x}_1$ and $\hat{x}_2$, from Eqs. (2.13a), (2.13b), (2.15a), and (2.15b) is described by

$$\langle\Delta\hat{x}_1^2(u)\rangle=\tfrac{1}{4}e^{2u}\cos^2(\phi_p/2)+\tfrac{1}{4}e^{-2u}\sin^2(\phi_p/2)\;, \tag{2.16a}$$

$$\langle\Delta\hat{x}_2^2(u)\rangle=\tfrac{1}{4}e^{-2u}\cos^2(\phi_p/2)+\tfrac{1}{4}e^{2u}\sin^2(\phi_p/2)\;. \tag{2.16b}$$

Suppose we allow the pump's phase to fluctuate. For the quantized pump in a coherent state $|\bar{N}^{1/2}\rangle$ with mean photon number $\bar{N}$, the phase fluctuations are characterized by

$$\langle\Delta\phi_p^2\rangle=\langle\phi_p^2\rangle=\frac{1}{4\bar{N}}\;, \tag{2.17}$$

since we may choose without loss of generality $\langle\phi_p\rangle=0$. Because $\bar{N}$ is large, $\langle\Delta\phi_p^2\rangle$ will be small; we can thus approximate $\cos^2(\phi_p/2)$ by 1, $\sin^2(\phi_p/2)$ by $\langle\phi_p^2/4\rangle=1/16\bar{N}$, and the variance of the $\hat{x}_2$ quadrature by

$$\langle\Delta\hat{x}_2^2(u)\rangle\simeq\tfrac{1}{4}e^{-2u}+\frac{1}{64\bar{N}}e^{2u}\;. \tag{2.18}$$

The pump can be considered classical and the parametric approximation valid when the correction term is small, that is, when

$$\bar{N}\gg\tfrac{1}{16}\exp(4\bar{N}^{1/2}\kappa t)\;, \tag{2.19}$$

where we have used the definition of u. The second term of Eq. (2.18) is the dominant correction to the variance of the squeezed quadrature due to pump quantum fluctuations. Because of the quantum nature of the pump, phase fluctuations are unavoidable; Fig. 1 illustrates their effect. The solid ellipse represents squeezing with a classical pump (i.e., the parametric approximation) with a well defined phase $\phi_p=0$. When $\phi_p\neq 0$, the ellipse is rotated by an angle $\phi_p/2$, as demonstrated by Eqs. (2.13a) and (2.13b). Pump phase fluctuations cause the orientation of

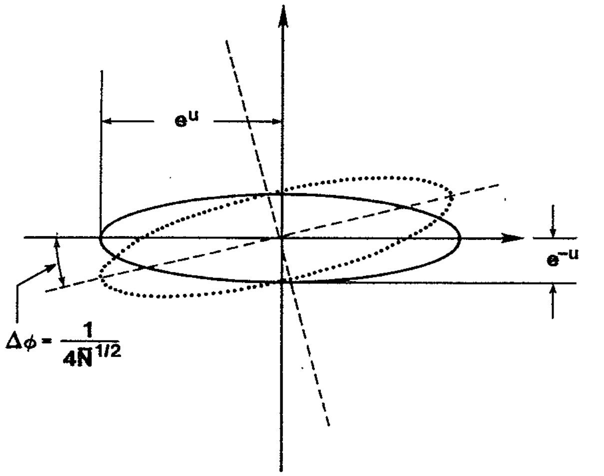

FIG. 1. The effect of pump phase fluctuations on squeezing. The ellipse with solid lines represents ideal squeezing, in which the pump has a well defined phase. Phase fluctuations in the pump cause the orientation of the ellipse to fluctuate about $\phi_p=0$, with the characteristic angle $\Delta\phi=1/(4\bar{N}^{1/2})$, feeding noise from the amplified quadrature into the squeezed quadrature.

the ellipse to fluctuate about $\phi_p=0$ with the characteristic angle $\Delta\phi=\langle\phi_p^2/4\rangle^{1/2}=1/4\bar{N}^{1/2}$, as represented by the dotted ellipse in Fig. 1, feeding noise from the amplified quadrature into the squeezed quadrature. One also sees why amplitude fluctuations are unimportant. Amplitude fluctuations merely produce fluctuations in the gain (or rate of squeezing); they do not couple noise in the amplified quadrature into the squeezed quadrature.

The above argument is for a one-mode PA, but it is easily extended to any number of modes; the same argument has been given for a continuum-mode PA,[15] yielding the same dominant correction as given by Eq. (2.18), but with a bandwidth determined by phase mismatching that allows one to relate $\bar{N}$ to the pump power. In the parametric approximation, the signal modes interact in pairs at frequencies $\Omega+n\Delta\omega$ and $\Omega-n\Delta\omega$; there are no interactions among different pairs in this approximation, and each pair can thus be considered separately. The correction that we have been discussing is due to fluctuations in the initial state of the pump, and has nothing to do with back action from the signal modes—pump depletion being one example of such back action—which would depend on the number of signal modes. The initial fluctuations act on each pair of modes in the same manner as described in Eq. (2.18) for the one-mode PA, yielding *for each pair of modes* a correction identical to that of Eq. (2.18). This correction is then independent of the number of signal modes, justifying our one-mode treatment.

The arguments given above are not rigorous; we have cited quantum mechanics as the ultimate source of pump phase fluctuations, yet we have treated their effect on squeezing classically. What we have given is a plausibility argument for and a physical picture of the dominant effect due to such fluctuations. The validity of our arguments will be confirmed by our detailed calculations showing the correction terms in Eq. (2.18) to be the dominant effect of pump quantum fluctuations on the squeezing, independent of the number of signal modes.

III. SYMMETRIES OF THE MULTIMODE PA

We shall begin our analysis by studying the symmetries of the multimode PA, with the pump in a coherent state and all signal modes initially in vacuum states. This will allow us to concentrate on the few matrix elements that are not constrained by symmetry. Since the multimode PA combines the one- and two-mode PA's as subsystems, we will be able to study the symmetries of these subsystems along with those of the larger system.

From Eqs. (2.7a) and (2.7b), the Heisenberg equations of motion in the interaction picture are

$$\frac{d\hat{a}_n}{dt}=\kappa\hat{a}_p\hat{a}^\dagger_{-n}\ , \tag{3.1}$$

$$\frac{d\hat{a}_p}{dt}=-\frac{\kappa}{2}\sum_{n=-M}^{M}\hat{a}_n\hat{a}_{-n}\ . \tag{3.2}$$

The quantities that are measured by a balanced homodyne detector are the variances of the quadrature phase amplitudes,[14]

$$\hat{X}_{(n)1}=\frac{1}{2}(\hat{a}_n+\hat{a}^\dagger_{-n})\ , \tag{3.3a}$$

$$\hat{X}_{(n)2}=-\frac{i}{2}(\hat{a}_n-\hat{a}^\dagger_{-n})\ . \tag{3.3b}$$

Inverting these definitions gives

$$\hat{a}_n=\hat{X}_{(n)1}+i\hat{X}_{(n)2}\ , \tag{3.4a}$$

$$\hat{a}^\dagger_{-n}=\hat{X}_{(n)1}-i\hat{X}_{(n)2}\ , \tag{3.4b}$$

and similar definitions for the pump yield

$$\hat{a}_p=\alpha_0+\hat{P}_1+i\hat{P}_2\ , \tag{3.5}$$

where the coherent amplitude α_0 of the pump has been written explicitly (α_0 is chosen real for convenience). It is worth noting that

$$\hat{X}_{(n)i}=\hat{X}^\dagger_{(-n)i}\ , \tag{3.6}$$

and that $\hat{P}_i$ and $\hat{X}_{(0)i}$ are both Hermitian. The equations of motion [Eqs. (2.7)] in these new variables are

$$\frac{d\hat{X}_{(n)1}}{du}=\hat{X}_{(n)1}+\frac{1}{\alpha_0}(\hat{X}_{(n)1}\hat{P}_1+\hat{X}_{(n)2}\hat{P}_2)\ , \tag{3.7a}$$

$$\frac{d\hat{X}_{(n)2}}{du}=-\hat{X}_{(n)2}+\frac{1}{\alpha_0}(\hat{X}_{(n)1}\hat{P}_2-\hat{X}_{(n)2}\hat{P}_1)\ , \tag{3.7b}$$

$$\frac{d\hat{P}_1}{du}=-\frac{1}{2\alpha_0}\sum_{n=-M}^{M}(\hat{X}_{(n)1}\hat{X}^\dagger_{(n)1}-\hat{X}_{(n)2}\hat{X}^\dagger_{(n)2})\ , \tag{3.7c}$$

$$\frac{d\hat{P}_2}{du}=-\frac{1}{2\alpha_0}\sum_{n=-M}^{M}(\hat{X}_{(n)1}\hat{X}^\dagger_{(n)2}+\hat{X}_{(n)2}\hat{X}^\dagger_{(n)1})\ , \tag{3.7d}$$

where $u\equiv\kappa\alpha_0 t$.

We are interested in symmetries of the time evolved matrix elements. Thus we want symmetries which preserve both the equations of motion and the initial state of the system. The coherent part of the pump has been subtracted out in our choice of variables [see Eq. (3.5)]; in terms of the pump quadrature-phase amplitudes $\hat{P}_1$ and $\hat{P}_2$, the effective initial state of the pump is vacuum. As any phase shift, reflection, or rotation of the quadratures leaves the vacuum invariant, we seek such transformations which leave the equations of motion [Eq. (3.7)] invariant.

The symmetries may now be classed as follows. (i) Time-reversal:

$$\hat{X}_{(n)1} \leftrightarrow \hat{X}_{(n)2} , \tag{3.8a}$$

$$\hat{P}_2 \rightarrow -\hat{P}_2 , \tag{3.8b}$$

$$u \rightarrow -u . \tag{3.8c}$$

One consequence of this symmetry is that the squeezed quadrature is simply the time-reversed amplified quadrature,

$$\langle | \Delta\hat{X}_{(n)1}(u)|^2 \rangle = \langle | \Delta\hat{X}_{(n)2}(-u)|^2 \rangle . \tag{3.9}$$

(ii) Reflection:

$$\hat{X}_{(n)1} \rightarrow -\hat{X}_{(n)1} \quad (\text{or } \hat{X}_{(n)2} \rightarrow -\hat{X}_{(n)2}) , \tag{3.10a}$$

$$\hat{P}_2 \rightarrow -\hat{P}_2 , \tag{3.10b}$$

which tells us that

$$\langle \hat{X}_{(n)1}(u) \rangle = \langle \hat{X}_{(n)2}(u) \rangle = \langle \hat{P}_2(u) \rangle \equiv 0 . \tag{3.11}$$

Hence the pump's phase does not drift, and the signal modes do not acquire a coherent piece. A similar symmetry for the pump amplitude $\hat{P}_1$ is absent because the pump may, for example, become depleted. Another consequence of this symmetry is that the conjugate quadratures remain uncorrelated, e.g.,

$$\langle \hat{X}_{(n)1}(u) \hat{X}^\dagger_{(n)2}(u) \rangle \equiv 0 . \tag{3.12}$$

(iii) Time stationary noise:[14]

$$\hat{X}_{(n)1}(u) \rightarrow e^{i\theta_n} \hat{X}_{(n)1}(u) , \quad n \neq 0 , \tag{3.13a}$$

$$\hat{X}_{(n)2}(u) \rightarrow e^{i\theta_n} \hat{X}_{(n)2}(u) ; \quad n \neq 0 . \tag{3.13b}$$

We cannot apply this symmetry to the one-mode PA (corresponding to $n=0$) since the quadratures $\hat{X}_{(0)1}$ and $\hat{X}_{(0)2}$ are Hermitian [see Eqs. (3.3)]. But we can say that matrix elements like

$$\langle \hat{X}_{(n)i}(u) \hat{X}_{(n')j}(u) \rangle$$

and

$$\langle \hat{X}_{(n)i}(u) \hat{X}_{(n')j}(u) \hat{X}^\dagger_{(m)k}(u) \rangle$$

are identically zero for $n \neq 0$. With a combination of the above symmetries we may determine all quadratic matrix elements given only

$$\langle | \Delta\hat{X}_{(n)2}|^2 \rangle = \langle \hat{X}_{(n)2} \hat{X}^\dagger_{(n)2} \rangle_{\text{sym}} . \tag{3.14}$$

At this point it is worth asking how the quantity $\langle \Delta\hat{y}_2^2 \rangle$ [Eq. (1.6)] calculated by Scharf and Walls[8] is related to the quadrature-phase amplitudes. They considered the two-mode PA and the quantity

$$\hat{y}_2 = \frac{1}{\sqrt{2}} (\hat{X}_{(1)2} + \hat{X}^\dagger_{(1)2}) \tag{3.15}$$

[Eqs. (1.5), (1.8), and (3.3)]. When the time stationary noise and reflection symmetries are used one finds

$$\langle \Delta\hat{y}_2^2 \rangle = \tfrac{1}{2} \langle \hat{X}^2_{(1)2} \rangle + \tfrac{1}{2} \langle \hat{X}^{\dagger 2}_{(1)2} \rangle + \langle \hat{X}_{(1)2} \hat{X}^\dagger_{(1)2} \rangle_{\text{sym}} - \langle \hat{y}_2 \rangle^2$$
$$= \langle | \Delta\hat{X}_{(1)2}|^2 \rangle , \tag{3.16}$$

which is just the variance of the squeezed quadrature-phase amplitude.

IV. DOMINANT TERMS FOR THE MULTIMODE PA

In this section we shall determine the dominant time behavior of the squeezed quadrature-phase amplitude due to the quantum nature of the pump. As in Sec. III we study the multimode PA, extracting the one- and two-mode results at the end.

The clear way to proceed in obtaining an expansion in α_0^{-1} is to take the Heisenberg equations [Eq. (3.7)] and iterate them. It is easier first to write them as integral equations:

$$\hat{X}_{(n)1}(u) = e^{u} \hat{X}_{(n)1}(0) + \frac{e^{u}}{\alpha_0} \int_0^u du' e^{-u'} [\hat{X}_{(n)1}(u') \hat{P}_1(u') + \hat{X}_{(n)2}(u') \hat{P}_2(u')] , \tag{4.1a}$$

$$\hat{X}_{(n)2}(u) = e^{-u} \hat{X}_{(n)2}(0) + \frac{e^{-u}}{\alpha_0} \int_0^u du' e^{u'} [\hat{X}_{(n)1}(u') \hat{P}_2(u') - \hat{X}_{(n)2}(u') \hat{P}_1(u')] , \tag{4.1b}$$

$$\hat{P}_1(u) = \hat{P}_1(0) - \frac{1}{2\alpha_0} \int_0^u du' \sum_{n=-M}^{M} [\hat{X}_{(n)1}(u') \hat{X}^\dagger_{(n)1}(u') - \hat{X}_{(n)2}(u') \hat{X}^\dagger_{(n)2}(u')] , \tag{4.1c}$$

$$\hat{P}_2(u) = \hat{P}_2(0) - \frac{1}{2\alpha_0} \int_0^u du' \sum_{n=-M}^{M} [\hat{X}_{(n)1}(u') \hat{X}^\dagger_{(n)2}(u') + \hat{X}_{(n)2}(u') \hat{X}^\dagger_{(n)1}(u')] . \tag{4.1d}$$

By substituting the quadrature-phase amplitudes correct to $O(1/\alpha_0^n)$ into Eq. (4.1), we will obtain expressions for them correct to $O(1/\alpha_0^{n+1})$. Although this procedure is easy up to $O(1/\bar{N})$ [i.e., $O(1/\alpha_0^2)$], it becomes prohibitive in obtaining even the $O(1/\bar{N}^2)$ correction, this correction requiring around 1000 terms. Instead of keeping every term we shall determine which terms can yield a dominant contribution and then calculate them.

Our objective is to determine the time dependence of the various terms that appear at each order in the expansion. Thus we start by keeping only information about time dependence in the integral equations (4.1). This leaves us with the "equations"

$$X_1(u)\approx e^{u}+\frac{e^{u}}{\alpha_0}\int_0^u du' e^{-u'}[X_1(u')P_1(u')+X_2(u')P_2(u')] , \tag{4.2a}$$

$$X_2(u)\approx e^{-u}+\frac{e^{-u}}{\alpha_0}\int_0^u du' e^{u'}[X_1(u')P_2(u')+X_2(u')P_1(u')] , \tag{4.2b}$$

$$P_1(u)\approx 1+\frac{1}{\alpha_0}\int_0^u du'[X_1(u')X_1(u')+X_2(u')X_2(u')] , \tag{4.2c}$$

$$P_2(u)\approx 1+\frac{1}{\alpha_0}\int_0^u du'[X_1(u')X_2(u')+X_2(u')X_1(u')] , \tag{4.2d}$$

where we have thrown away numerical coefficients and sums over different modes. We shall even treat these equations as c-number equations. Next we define the symbol

$$\tilde{u}^{\,n}\equiv u^{n}+u^{n-1}+\cdots+u+1 \tag{4.3}$$

to represent an arbitrary polynomial of order n in the scaled time with all its coefficients suppressed. With these simplifications Eqs. (4.2) become easy to iterate. The information we are left with is the *form* of solution at each order in the expansion; for instance, the squeezed quadrature-phase amplitude has the form

$$\begin{aligned} X_2\approx{}& e^{-u}+\frac{1}{\alpha_0}(e^{u}+\tilde{u}e^{-u})+\frac{1}{\alpha_0^2}(e^{u}+\tilde{u}^{\,2}e^{-u}+e^{-3u})\\ &+\frac{1}{\alpha_0^3}(e^{3u}+\tilde{u}^{\,2}e^{u}+\tilde{u}^{\,3}e^{-u}+\tilde{u}e^{-3u})\\ &+\frac{1}{\alpha_0^4}(\tilde{u}e^{3u}+\tilde{u}^{\,3}e^{u}+\tilde{u}^{\,4}e^{-u}+\tilde{u}^{\,2}e^{-3u}+e^{-5u})+\cdots \end{aligned} \tag{4.4}$$

to $O(1/\alpha_0^4)$. Squaring this expression gives us the form of the variance:

$$\begin{aligned} \langle X_2^2\rangle\approx{}& e^{-2u}+\frac{1}{\bar{N}}(e^{2u}+\tilde{u}+\tilde{u}^{\,2}e^{-2u}+e^{-4u})\\ &+\frac{1}{\bar{N}^2}(e^{4u}+\tilde{u}^{\,2}e^{2u}+\tilde{u}^{\,3}+\tilde{u}^{\,4}e^{-2u}\\ &+\tilde{u}^{\,2}e^{-4u}+e^{-6u})+\cdots . \end{aligned} \tag{4.5}$$

The terms multiplied by an odd power of α_0^{-1} contain an odd number of quadrature operators, and thus vanish by the reflection symmetry. The dominant terms at each order in $1/\bar{N}$ are pure exponentials. Thus when we calculate the dominant terms for the squeezed quadrature, we may throw away any polynomial times an exponential if the polynomial is nontrivial.

We now use these simplifications to iterate Eq. (4.1), retaining only the dominant term at each order:

$$\hat{X}_{(n)1}(u)\approx e^{u}\hat{X}_{(n)1}(0)-\frac{e^{3u}}{8\alpha_0^2}\hat{X}_{(n)1}(0)\sum_{m=-M}^{M}\hat{X}_{(m)1}(0)\hat{X}^{\dagger}_{(m)1}(0)+O(1/\alpha_0^3) , \tag{4.6a}$$

$$\hat{X}_{(n)2}(u)\approx e^{-u}\hat{X}_{(n)2}(0)+\frac{e^{u}}{2\alpha_0}\hat{X}_{(n)1}(0)\hat{P}_2(0)-\frac{e^{3u}}{16\alpha_0^3}\hat{X}_{(n)1}(0)\sum_{m=-M}^{M}\hat{X}_{(m)1}(0)\hat{X}^{\dagger}_{(m)1}(0)\hat{P}_2(0)+O(1/\alpha_0^4) , \tag{4.6b}$$

$$\hat{P}_1(u)\approx\hat{P}_1(0)-\frac{e^{2u}}{4\alpha_0}\sum_{m=-M}^{M}\hat{X}_{(m)1}(0)\hat{X}^{\dagger}_{(m)1}(0)+O(1/\alpha_0^3) , \tag{4.6c}$$

$$\hat{P}_2(u)\approx\hat{P}_2(0)-\frac{e^{2u}}{4\alpha_0^2}\sum_{m=-M}^{M}\hat{X}_{(m)1}(0)\hat{X}^{\dagger}_{(m)1}(0)\hat{P}_2(0)+O(1/\alpha_0^3) . \tag{4.6d}$$

This gives the variance for the quadrature $\hat{X}_{(n)2}$ as

$$\begin{aligned} \langle|\Delta\hat{X}_{(n)2}(u)|^2\rangle\approx{}& e^{-2u}\langle\hat{X}_{(n)2}(0)\hat{X}^{\dagger}_{(n)2}(0)\rangle_{\rm sym}+\frac{e^{2u}}{4\bar{N}}\langle\hat{X}_{(n)1}(0)\hat{X}^{\dagger}_{(n)1}(0)\rangle_{\rm sym}\langle\hat{P}_2^2(0)\rangle\\ &-\frac{e^{4u}}{64\bar{N}^2}\Big\langle 2\hat{X}_{(n)1}(0)\sum_m\hat{X}_{(m)1}(0)\hat{X}^{\dagger}_{(m)1}(0)\hat{X}^{\dagger}_{(n)1}(0)+\hat{X}^{\dagger}_{(n)1}(0)\hat{X}_{(n)1}(0)\sum_m\hat{X}_{(n)1}(0)\hat{X}^{\dagger}_{(n)1}(0)\\ &+\sum_m\hat{X}_{(n)1}(0)\hat{X}^{\dagger}_{(n)1}(0)\hat{X}^{\dagger}_{(n)1}(0)\hat{X}_{(n)1}(0)\Big\rangle\langle\hat{P}_2^2(0)\rangle+O(1/\bar{N}^3) . \end{aligned} \tag{4.7}$$

Restricting this calculation to one mode ($M=0$; $n=0$ only) gives

$$\langle \Delta \hat{x}_2^2(u) \rangle \approx \frac{e^{-2u}}{4} + \frac{e^{2u}}{64\bar{N}} - \frac{3e^{4u}}{1024\bar{N}^2} , \quad (4.8)$$

and for two modes ($n=\pm 1$ only)

$$\langle |\Delta \hat{X}_2(u)|^2 \rangle \approx \frac{e^{-2u}}{4} + \frac{e^{2u}}{64\bar{N}} - \frac{4e^{4u}}{1024\bar{N}^2} . \quad (4.9)$$

When $M \geq 0$ the full M-mode calculation gives

$$\langle |\Delta \hat{X}_{(n)2}(u)|^2 \rangle \approx \frac{e^{-2u}}{4} + \frac{e^{2u}}{64\bar{N}} - \frac{(3+4M)e^{4u}}{1024\bar{N}^2} , \quad (4.10)$$

which confirms our argument that the dominant semiclassical correction is independent of the number of signal modes.

For the M-mode PA the dominant term at $O(1/\bar{N}^2)$ (and higher orders) is small compared to the $O(1/\bar{N})$ term when

$$e^{2u} \ll \frac{16\bar{N}}{3+4M} . \quad (4.11)$$

This is similar to the restrictions for the one-mode PA (i.e., $M=0$) calculations done by Hillery and Zubairy,[7] except that they need the added restriction on their results that $u \lesssim 1$.

Our result shows that so long as the condition in Eq. (4.11) holds, the semiclassical approximation is sufficient to determine the limit to the squeezing, and thus the squeezing is limited by the pump phase noise. If the condition of Eq. (4.11) does not hold, the semiclassical approximation breaks down, and only a full quantum treatment to all orders can be relied upon.

V. SEMINUMERICAL METHOD

The central theme of this paper is to determine the behavior of the squeezed quadrature's variance $\langle |\Delta \hat{X}_2|^2 \rangle$ as a function of the scaled time u. Looking at Eq. (4.5), we can see that we have already derived the form of this variance to $O(1/\bar{N}^2)$, and in Eqs. (4.8) to (4.10) we have also calculated the coefficients of the dominant terms to this order. It is nonetheless worth calculating the coefficients for the subdominant terms appearing in Eq. (4.5), since this allows us to check directly whether or not these terms have coefficients large enough to overcome their smaller relative growth at times when the phase noise begins to dominate. Further, we shall show that it is relatively simple to derive the exact expression for $\langle |\Delta \hat{X}_2|^2 \rangle$ to $O(1/\bar{N}^2)$ (or higher) using a new algebra and some numerical assistance. We hope that exposition of this new technique will be an adequate motivation for presenting the exact form of $\langle |\Delta \hat{X}_2|^2 \rangle$ to $O(1/\bar{N}^2)$ for the one- and the two-mode PA. In Sec. II we showed how to discretize a continuum mode calculation. Clearly the fineness of this discretization should not appear in any physical quantities relevant to the continuum system. We find, as we argued in Sec. II, that all but the dominant correction at $O(1/\bar{N})$ do depend in detail on the discretization prescription. If ever these terms become important, then the discrete-mode equations no longer model accurately the continuum system.

Let us see how we might proceed in finding the unknown coefficients in Eq. (4.5). To $O(\bar{N}^0)$ there is 1 coefficient, to $O(1/\bar{N}^1)$ there are 7, and to $O(1/\bar{N}^2)$ there are 17. We can calculate the first terms in a power-series expansion of $\langle |\Delta \hat{X}_2|^2 \rangle$ in the scaled time u; similarly we can do a power-series expansion of the form given in Eq. (4.5) with a set of unknown coefficients. By equating the coefficients of u and $\bar{N}$ in these expansions, we can obtain sufficient simultaneous equations to solve for the unknown coefficients of Eq. (4.5).

Since at $O(1/\bar{N}^2)$ we must find 17 unknown coefficients, we will need an expansion of $\langle |\Delta \hat{X}_2|^2 \rangle = \langle |\hat{X}_2|^2 \rangle$ to $O(u^{16})$. To avoid repetition we shall describe the calculation only for the one-mode PA with interaction Hamiltonian $\hat{H}_{\text{int}} = i\kappa(a^{\dagger 2}b - a^2 b^\dagger)/2$; the variance of the evolving quadrature-phase amplitude may be written

$$\langle 0;\alpha_0 | \hat{x}_2^2(u) | 0;\alpha_0 \rangle = \langle 0;\alpha_0 | \exp(iu\hat{H}_{\text{int}}/\kappa\alpha_0) \hat{x}_2^2 \exp(-iu\hat{H}_{\text{int}}/\kappa\alpha_0) | 0;\alpha_0 \rangle$$
$$= \left\langle 0;\alpha_0 \left| \hat{x}_2^2 + iu \left[\frac{\hat{H}_{\text{int}}}{\kappa\alpha_0}, \hat{x}_2^2 \right] - \frac{u^2}{2} \left[\frac{\hat{H}_{\text{int}}}{\kappa\alpha_0}, \left[\frac{\hat{H}_{\text{int}}}{\kappa\alpha_0}, \hat{x}_2^2 \right] \right] + \cdots \right| 0;\alpha_0 \right\rangle , \quad (5.1)$$

where the scaled time is $u = \kappa\alpha_0 t$, and $|0;\alpha_0\rangle = |0\rangle \otimes |\alpha_0\rangle$ is the initial state (vacuum for the signal mode and coherent state for the pump mode). The signal and pump mode annihilation operators are a and b, respectively. Since the signal mode is in vacuum its annihilation and creation operators may be treated directly as ladder operators on a number state. For the pump mode we need to normal order its annihilation and creation operators.

Our first simplification comes from only needing the next to semiclassical approximation, i.e., $O(1/\bar{N}^2)$, for the calculation. Thus when we are normal ordering the pump mode operators we may throw away all terms generated by more than two applications of the commutation relation $[b, b^\dagger] = 1$. This can be done automatically by using a new algebra which we now present. For simplicity we start with a description of this algebra good to semiclassical order [i.e., $O(1/\bar{N})$].

Let us start with more general considerations: many calculations in quantum optics require the expectation value of a product of creation and annihilation operators in a coherent state, e.g.,

$$\langle \beta | b^{\dagger 3} b^2 b^\dagger b^3 b^\dagger b | \beta \rangle . \quad (5.2)$$

If we are interested only in calculating this to semiclassical order, then we need only keep the terms up to $O(1/|\beta|^2)$ times the dominant ("classical") term. The dominant term is given by replacing the operators b and $b^\dagger$ with the c numbers β and β^*, respectively. For the semiclassical correction we need only perform the first of a series of commutation relations — never performing more than one for each term [since we are only interested in terms to order $\hbar$ or equivalently $O(1/|\beta|^2)$ times the classical piece]. To do this we make use of the commutation relation

$$[b,b^\dagger]=1 \tag{5.3}$$

and the notation

$$B_{n,m}^{\sigma}\equiv b^{\dagger n}b^{m}+\sigma\, b^{\dagger n-1}b^{m-1} ; \tag{5.4}$$

the creation and annihilation operators are written as

$$B_{1,0}^{0}=b^\dagger , \tag{5.5a}$$

$$B_{0,1}^{0}=b . \tag{5.5b}$$

To semiclassical order in the amplitude $|\beta|$ the following relation allows us to normal order any expansion:

$$B_{n,m}^{\sigma}B_{n',m'}^{\sigma'}=B_{n+n',m+m'}^{\sigma+\sigma'+mn'} . \tag{5.6}$$

Clearly $m\times n'$ is just the number of times the commutation relation is required in order to pass m annihilation operators from $B_{n,m}^{\sigma}$ past the n' creation operators in $B_{n',m'}^{\sigma'}$. As an example, the evaluation of the matrix element in Eq. (5.2) can be worked out to semiclassical order in the coherent state of complex amplitude β:

$$\begin{aligned}\langle\beta|\,b^{\dagger 3}b^2b^\dagger b^3b^\dagger b\,|\beta\rangle&=\langle\beta|\,B_{3,2}^{0}B_{1,3}^{0}B_{1,1}^{0}\,|\beta\rangle\\&=\langle\beta|\,B_{5,6}^{7}\,|\beta\rangle\\&=\beta^{*5}\beta^{6}+7\beta^{*4}\beta^{5}+O(|\beta|^7) .\end{aligned} \tag{5.7}$$

For calculations at next to semiclassical order, we simply extend the algebra to be good to $O(1/\bar{N}^2)$. In this case we modify the notation so

$$B_{n,m}^{\sigma;\tau}\equiv b^{\dagger n}b^{m}+\sigma\, b^{\dagger n-1}b^{m-1}+\tau\, b^{\dagger n-2}b^{m-2} , \tag{5.8}$$

$$B_{1,0}^{0;0}=b^\dagger , \tag{5.9a}$$

$$B_{0,1}^{0;0}=b . \tag{5.9b}$$

After some calculation we find

$$B_{n,m}^{\sigma;\tau}B_{n',m'}^{\sigma';\tau'}=B_{n+n',m+m'}^{\sigma+\sigma'+mn';\tau+\tau'+\sigma\sigma'+\sigma(m-1)n'+\sigma'm(n'-1)+(m,n')} , \tag{5.10}$$

where

$$(m,n)=m(m-1)n(n-1)/2 . \tag{5.11}$$

This extended algebra allows us, for example, to determine the $O(|\beta|^7)$ term in Eq. (5.7) to be $8\beta^{*3}\beta^4$.

In calculating $\langle\hat{x}_2^2\rangle$, rather than proceeding precisely as suggested by Eq. (5.1), we used this "B algebra" to obtain instead the time evolved state of the system

$$\begin{aligned}\exp[u(a^{\dagger 2}b-a^2b^\dagger)/2]\,|0;\alpha_0\rangle&=|0;\alpha_0\rangle+\frac{\sqrt{2}u}{2}B_{0,1}^{0;0}\,|2;\alpha_0\rangle\\&\quad+\frac{u^2}{8}(\sqrt{4!}B_{0,2}^{0;0}\,|4;\alpha_0\rangle-2B_{1,1}^{0;0}\,|0;\alpha_0\rangle)+\cdots .\end{aligned} \tag{5.12}$$

Here $|n;\alpha_0\rangle=|n\rangle\otimes|\alpha_0\rangle$ is the outer product of a number state for the signal with the pump's initial coherent state. Equation (5.12) shows us the first terms in a power-series expansion in u, although we actually retained the first 17 terms. To get $\langle\hat{x}_2^2\rangle$ we need only wedge the operator $\hat{x}_2^2$ between pairs of the time evolved states given by Eq. (5.12).

Even after these simplifications the calculation would still be very tedious to do by hand, so we calculated Eq. (5.12) on a computer. The evolved state was represented by a multidimensional array, and numbers were calculated with limited accuracy. This allowed us to generate the power-series expansion of $\langle\hat{x}_2^2\rangle$ and hence the coefficients of Eq. (4.5) up to the accuracy used in the computations.

The final step comes in estimating the accuracy necessary to reproduce the rational coefficients that should appear in Eq. (4.5). That they are indeed rational can be seen by looking at how they would arise if we were to iterate the Heisenberg equations [Eq. (4.1)] in full. This also allows us to estimate an upper bound on the numerators and denominators for each fraction. In the worst case, to $O(1/\bar{N})$, the numerator could come from all of the 16 terms that appear at that order, and the denominator from the factors of 2 in the definition of the quadrature phase amplitudes and from factors due to the time integrations. An unambiguous calculation of this worst case rational number requires only about four significant figures in the final answer. Similarly at $O(1/\bar{N}^2)$ we need to keep ten significant figures in the final coefficients.

For the one-mode PA we find at next to semiclassical order that

$$\begin{aligned}\langle\Delta\hat{x}_2^2\rangle=\frac{e^{-2u}}{4}+\frac{e^{2u}}{64\bar{N}}[1+(3-8u)e^{-2u}-(1+8u-8u^2)e^{-4u}-3e^{-6u}]\\-\frac{1}{1024\bar{N}^2}[3e^{4u}-(19/4-24u+32u^2)e^{2u}+60-112u-(80+58u-48u^2-128u^3/3+32u^4)e^{-2u}\\+(33+48u+96u^2)e^{-4u}-45e^{-6u}/4] .\end{aligned} \tag{5.13}$$

For the two-mode PA the same technique yields a next to semiclassical result

$$\langle |\Delta\hat{X}_2|^2\rangle = \frac{e^{-2u}}{4} + \frac{e^{2u}}{64\bar{N}}[1+(4-8u)e^{-2u}-(1+12u-8u^2)e^{-4u}-4e^{-6u}]$$
$$-\frac{1}{1024\bar{N}^2}[4e^{4u}-(5-28u+40u^2)e^{2u}+96-160u-(105+112u-48u^2-224u^3/3+32u^4)e^{-2u}$$
$$+(28+32u+128u^2)e^{-4u}-18e^{-6u}] . \quad (5.14)$$

VI. STOCHASTIC DIFFERENTIAL EQUATIONS

A. The one-mode PA

The dynamic evolution of a one-mode PA is described by von Neumann's equation in the interaction picture

$$\frac{\partial\hat{\rho}_I(t)}{\partial t} = \frac{i}{\hbar}[\hat{\rho}_I(t),\hat{H}_{\text{int}}(t)] , \quad (6.1)$$

where the interaction Hamiltonian $\hat{H}_{\text{int}}(t)$ is given by Eq. (1.1). All operators are now in the interaction picture. We will assume that initially the signal mode is in the vacuum state, and the pump is in a coherent state of real amplitude α_0:

$$\hat{\rho}_I(0) = |0;\alpha_0\rangle\langle 0;\alpha_0| . \quad (6.2)$$

To solve Eq. (6.1), it is convenient to project the density operator $\hat{\rho}_I(t)$ onto a suitable set of basis states. The positive-P representation[16] is an off-diagonal representation obtained from an expansion on a coherent state basis:

$$\hat{\rho}_I(t) = \int\int\int\int P(\alpha,\alpha_p,\beta,\beta_p,t)\,\hat{\Lambda}(\alpha,\alpha_p,\beta,\beta_p) \times d^2\alpha\, d^2\alpha_p\, d^2\beta\, d^2\beta_p , \quad (6.3)$$

where the operator $\hat{\Lambda}$ is given by

$$\hat{\Lambda}(\alpha,\alpha_p,\beta,\beta_p) \equiv \frac{|\alpha;\alpha_p\rangle\langle\beta^*;\beta_p^*|}{\langle\beta^*;\beta_p^*|\alpha;\alpha_p\rangle} = e^{-(\alpha\beta+\alpha_p\beta_p)}e^{\alpha\hat{a}^\dagger+\alpha_p\hat{a}_p^\dagger} \times |0;0\rangle\langle 0;0|\,e^{\beta\hat{a}+\beta_p\hat{a}_p} . \quad (6.4)$$

By substituting Eqs. (1.1), (6.3), and (6.4) into Eq. (6.1) and integrating by parts, we find the Fokker-Planck equation

$$\frac{\partial P}{\partial\tau} = \left[-\alpha_p\beta\frac{\partial}{\partial\alpha} - \alpha\beta_p\frac{\partial}{\partial\beta} + \frac{\alpha^2}{2}\frac{\partial}{\partial\alpha_p} + \frac{\beta^2}{2}\frac{\partial}{\partial\beta_p} + \frac{\alpha_p}{2}\frac{\partial^2}{\partial\alpha^2} + \frac{\beta_p}{2}\frac{\partial^2}{\partial\beta^2}\right]P , \quad (6.5)$$

where $\tau = \kappa t$ and $P \equiv P(\alpha,\alpha_p,\beta,\beta_p,\tau)$.

In its present form, Eq. (6.5) is a complex eight-dimensional Fokker-Planck equation. The analyticity of $\hat{\Lambda} \equiv \hat{\Lambda}(\alpha,\alpha_p,\beta,\beta_p)$, however, allows us some freedom of choice in interpreting the derivatives.[21] By properly interpreting the derivatives in Eq. (6.5), we obtain a real Fokker-Planck equation with positive-semidefinite diffusion. Using the standard methods of stochastic calculus,[17] this eight-dimensional Fokker-Planck equation yields a set of eight real, first-order Ito stochastic differential equations (SDE's). When written in complex notation, the resulting SDE's are

$$d\alpha = \alpha_p\beta\, d\tau + \sqrt{\alpha_p}\, dW_1 , \quad (6.6a)$$

$$d\beta = \alpha\beta_p\, d\tau + \sqrt{\beta_p}\, dW_2 , \quad (6.6b)$$

$$d\alpha_p = -\tfrac{1}{2}\alpha^2\, d\tau , \quad (6.6c)$$

$$d\beta_p = -\tfrac{1}{2}\beta^2\, d\tau , \quad (6.6d)$$

where, in the Ito calculus,

$$dW_1^2 = dW_2^2 = d\tau . \quad (6.6e)$$

The Wiener increments dW_1 and dW_2 are real and independent.

Although we cannot solve Eqs. (6.6) analytically, an approximate solution is possible for a nearly classical pump. We assume that the stochastic pump mode variables α_p and β_p consist of a mean amplitude α_0 (chosen to be real) plus fluctuations $\Delta\alpha$ and $\Delta\beta$:

$$\alpha_p \equiv \alpha_0 + \Delta\alpha , \quad (6.7a)$$

$$\beta_p \equiv \alpha_0 + \Delta\beta . \quad (6.7b)$$

We define new variables x_1, p_1, x_2, and p_2 by

$$x_1 = \tfrac{1}{2}(\alpha+\beta) , \quad x_2 = -\frac{i}{2}(\alpha-\beta) , \quad (6.8a)$$

$$p_1 = \tfrac{1}{2}(\Delta\alpha+\Delta\beta) , \quad p_2 = -\frac{i}{2}(\Delta\alpha-\Delta\beta) . \quad (6.8b)$$

It is convenient to change variables once again. We define the variables z_1 and z_2 by

$$z_1 = x_1 e^{-u} , \quad (6.9a)$$

$$z_2 = x_2 e^{u} , \quad (6.9b)$$

where $u = \alpha_0\tau$. The resulting SDE's are

$$dz_1=\frac{1}{\alpha_0}(z_1p_1+z_2p_2e^{-2u})\,du+\frac{1}{2}e^{-u}\left[\left[1+\frac{p_1+ip_2}{\alpha_0}\right]^{1/2}dV_1+\left[1+\frac{p_1-ip_2}{\alpha_0}\right]^{1/2}dV_2\right], \quad (6.10a)$$

$$dz_2=\frac{1}{\alpha_0}(z_1p_2e^{2u}-z_2p_1)\,du-\frac{i}{2}e^{u}\left[\left[1+\frac{p_1+ip_2}{\alpha_0}\right]^{1/2}dV_1-\left[1+\frac{p_1-ip_2}{\alpha_0}\right]^{1/2}dV_2\right], \quad (6.10b)$$

$$dp_1=\frac{1}{2\alpha_0}(z_2^2e^{-2u}-z_1^2e^{2u})\,du, \quad (6.10c)$$

$$dp_2=-\frac{1}{\alpha_0}z_1z_2\,du, \quad (6.10d)$$

where $dV_1=\sqrt{\alpha_0}\,dW_1$ and $dV_2=\sqrt{\alpha_0}\,dW_2$.

We use an iterative procedure to obtain an approximate solution of Eqs. (6.10). The square roots are expanded in a Taylor series:

$$\left[1+\frac{p_1\pm ip_2}{\alpha_0}\right]^{1/2}=1+\frac{1}{\alpha_0}\frac{p_1\pm ip_2}{2}-\frac{1}{\alpha_0^2}\frac{(p_1\pm ip_2)^2}{8}+\cdots. \quad (6.11)$$

Substituting Eq. (6.11) into Eqs. (6.10) and integrating formally, we find

$$z_1(u)=z_1(0)+\frac{1}{\alpha_0}\int_0^u[z_1(x)p_1(x)+z_2(x)p_2(x)e^{-2x}]\,dx+\frac{1}{\sqrt{2}}\int_0^u e^{-x}\left[dV+\frac{1}{\alpha_0}\left[\frac{1}{2}p_1(x)dV+\frac{i}{2}p_2(x)dW\right]+\cdots\right], \quad (6.12a)$$

$$z_2(u)=z_2(0)+\frac{1}{\alpha_0}\int_0^u[z_1(x)p_2(x)e^{2x}-z_2(x)p_1(x)]\,dx-\frac{i}{\sqrt{2}}\int_0^u e^{x}\left[dW+\frac{1}{\alpha_0}\left[\frac{1}{2}p_1(x)dW+\frac{i}{2}p_2(x)dV\right]+\cdots\right], \quad (6.12b)$$

$$p_1(u)=p_1(0)+\frac{1}{2\alpha_0}\int_0^u[z_2^2(x)e^{-2x}-z_1^2(x)e^{2x}]\,dx, \quad (6.12c)$$

$$p_2(u)=p_2(0)-\frac{1}{\alpha_0}\int_0^u z_1(x)z_2(x)\,dx. \quad (6.12d)$$

Here we have defined two new independent Wiener increments

$$dV(x)=\frac{dV_1(x)+dV_2(x)}{\sqrt{2}}, \quad (6.13a)$$

$$dW(x)=\frac{dV_1(x)-dV_2(x)}{\sqrt{2}}. \quad (6.13b)$$

The new Wiener increments defined in Eqs. (6.13) correspond to a rotation of the old Wiener increments, dV_1 and dV_2, and hence retain the same correlation matrix.[17]

We can ignore the initial values $z_1(0)=x_1(0)$, $z_2(0)=x_2(0)$, $p_1(0)$ and $p_2(0)$ in subsequent calculations because all moments involving these quantities are zero. To see this, we observe that the P function gives normally ordered averages for all moments α^n and β^n, and all normally ordered averages are initially zero for the case studied here. By extension, all moments involving the initial values $x_1(0)$, $x_2(0)$, $p_1(0)$, and $p_2(0)$ are zero.

The formal solution [Eqs. (6.12)] yields an approximate solution, valid for short interaction times and large pump amplitude, when the stochastic variables are expanded in a perturbation series in the reciprocal of the pump amplitude:

$$\theta=\sum_{n=0}^{\infty}\alpha_0^{-n}\,\theta^{(n)}. \quad (6.14)$$

By substituting the expansion Eq. (6.14) into the formal solution Eqs. (6.12) and equating equal powers of α_0^{-n}, we obtain an approximate solution to the set of SDE's: (i) to zeroth order,

$$z_1^{(0)}(u)=\frac{1}{\sqrt{2}}\int_0^u e^{-x}dV, \quad (6.15a)$$

$$z_2^{(0)}(u)=-\frac{i}{\sqrt{2}}\int_0^u e^{x}dW, \quad (6.15b)$$

$$p_1^{(0)}(u)=p_2^{(0)}(u)=0; \quad (6.15c)$$

(ii) to first order,

$$z_1^{(1)}(u)=z_2^{(1)}(u)=0, \quad (6.16a)$$

$$p_1^{(1)}(u)=\tfrac{1}{2}\int_0^u[z_2^{(0)2}(x)e^{-2x}-z_1^{(0)2}(x)e^{2x}]\,dx, \quad (6.16b)$$

$$p_2^{(1)}(u)=-\int_0^u z_1^{(0)}(x)z_2^{(0)}(x)\,dx; \quad (6.16c)$$

and (iii) to second order,

$$z_1^{(2)}(u)=\int_0^u [z_1^{(0)}(x)p_1^{(1)}(x)+z_2^{(0)}(x)p_2^{(1)}(x)e^{-2x}]\,dx+\frac{1}{2\sqrt{2}}\int_0^u e^{-x}[p_1^{(1)}(x)\,dV+ip_2^{(1)}(x)\,dW]\ , \tag{6.17a}$$

$$z_2^{(2)}(u)=\int_0^u [z_1^{(0)}(x)p_2^{(1)}(x)e^{2x}-z_2^{(0)}(x)p_1^{(1)}(x)]\,dx-\frac{i}{2\sqrt{2}}\int_0^u e^{x}[p_1^{(1)}(x)\,dW+ip_2^{(1)}(x)\,dV]\ , \tag{6.17b}$$

$$p_1^{(2)}(u)=p_2^{(2)}(u)=0\ . \tag{6.17c}$$

The SDE's corresponding to the zeroth-order solutions $z_1^{(0)}(u)$ and $z_2^{(0)}(u)$ are

$$dz_1^{(0)}=\frac{1}{\sqrt{2}}e^{-u}\,dV\ , \tag{6.18a}$$

$$dz_2^{(0)}=-\frac{i}{\sqrt{2}}e^{u}\,dW\ . \tag{6.18b}$$

We define the new variables

$$A_1(u)=z_1^{(0)}(u)\,e^{u}\ , \tag{6.19a}$$

$$A_2(u)=iz_2^{(0)}(u)\,e^{-u}\ , \tag{6.19b}$$

with the resulting SDE's

$$dA_1=A_1\,du+\frac{1}{\sqrt{2}}\,dV\ , \tag{6.20a}$$

$$dA_2=-A_2\,du+\frac{1}{\sqrt{2}}\,dW\ . \tag{6.20b}$$

Equations (6.20) describe two independent Ornstein-Uhlenbeck processes, each with zero mean. Thus $z_1^{(0)}(u)$ and $z_2^{(0)}(u)$, apart from the exponential factors e^{u} and e^{-u}, respectively, are themselves Ornstein-Uhlenbeck processes. They are Gaussian variables; all higher-order moments can be expressed in terms of second-order moments. With this in mind, we can formulate a pair of rules to guide us through the remaining calculations: (i) a rule for quadratic moments,

$$\langle z_1^{(0)}(u)z_1^{(0)}(w)\rangle_{\text{av}}=\tfrac{1}{4}(1-e^{-2w})\quad u\geq w\ , \tag{6.21a}$$

$$\langle z_2^{(0)}(u)z_2^{(0)}(w)\rangle_{\text{av}}=-\tfrac{1}{4}(e^{2w}-1)\quad u\geq w\ , \tag{6.21b}$$

$$\langle z_1^{(0)}(u)z_2^{(0)}(w)\rangle_{\text{av}}=0\ , \tag{6.21c}$$

and (ii) a rule for quartic moments,

$$\begin{aligned}\langle z_1^{(0)}(u)z_1^{(0)}(v)z_1^{(0)}(w)z_1^{(0)}(z)\rangle_{\text{av}}&=\langle z_1^{(0)}(u)z_1^{(0)}(v)\rangle_{\text{av}}\langle z_1^{(0)}(w)z_1^{(0)}(z)\rangle_{\text{av}}\\&\quad+\langle z_1^{(0)}(u)z_1^{(0)}(w)\rangle_{\text{av}}\langle z_1^{(0)}(v)z_1^{(0)}(z)\rangle_{\text{av}}\\&\quad+\langle z_1^{(0)}(u)z_1^{(0)}(z)\rangle_{\text{av}}\langle z_1^{(0)}(v)z_1^{(0)}(w)\rangle_{\text{av}}\ ,\end{aligned} \tag{6.22a}$$

$$\begin{aligned}\langle z_2^{(0)}(u)z_2^{(0)}(v)z_2^{(0)}(w)z_2^{(0)}(z)\rangle_{\text{av}}&=\langle z_2^{(0)}(u)z_2^{(0)}(v)\rangle_{\text{av}}\langle z_2^{(0)}(w)z_2^{(0)}(z)\rangle_{\text{av}}\\&\quad+\langle z_2^{(0)}(u)z_2^{(0)}(w)\rangle_{\text{av}}\langle z_2^{(0)}(v)z_2^{(0)}(z)\rangle_{\text{av}}\\&\quad+\langle z_2^{(0)}(u)z_2^{(0)}(z)\rangle_{\text{av}}\langle z_2^{(0)}(v)z_2^{(0)}(w)\rangle_{\text{av}}\ ,\end{aligned} \tag{6.22b}$$

where $\langle\ \rangle_{\text{av}}$ denotes an average in the positive-P representation.

The squeezing in the signal mode is easily calculated by the repeated application of (i) and (ii). The signal-mode quadrature-phase amplitudes are defined by Eqs. (2.11), and the pump mode quadrature-phase amplitudes are defined by Eq. (3.5), or more explicitly by

$$\hat{P}_1=\tfrac{1}{2}(\hat{a}_p+\hat{a}_p^{\dagger})\ ,\quad \hat{P}_2=-\frac{i}{2}(\hat{a}_p-\hat{a}_p^{\dagger})\ . \tag{6.23}$$

The expectation values of the signal-mode quadrature-phase amplitudes $\hat{x}_1$ and $\hat{x}_2$ are zero when the signal is initially vacuum, as shown by Eq. (3.11). We then find that the uncertainties in $\hat{x}_1$ and $\hat{x}_2$ are

$$\langle\Delta\hat{x}_1^2\rangle=\tfrac{1}{4}+\langle x_1^2(u)\rangle_{\text{av}}=\tfrac{1}{4}+\langle z_1^2(u)\rangle_{\text{av}}e^{2u}\ , \tag{6.24a}$$

$$\langle\Delta\hat{x}_2^2\rangle=\tfrac{1}{4}+\langle x_2^2(u)\rangle_{\text{av}}=\tfrac{1}{4}+\langle z_2^2(u)\rangle_{\text{av}}e^{-2u}\ . \tag{6.24b}$$

We see from Eqs (6.24) that x_1 and x_2 are the c-number equivalents of the quadrature-phase amplitudes $\hat{x}_1$ and $\hat{x}_2$, respectively. To second order in α_0^{-1}, the uncertainties are

$$\langle\Delta\hat{x}_1^2\rangle=\tfrac{1}{4}+\langle z_1^{(0)2}(u)\rangle_{\text{av}}e^{2u}+\frac{1}{\alpha_0^2}\langle 2z_1^{(0)}(u)z_1^{(2)}(u)\rangle_{\text{av}}e^{2u}\ , \tag{6.25a}$$

$$\langle\Delta\hat{x}_2^2\rangle=\tfrac{1}{4}+\langle z_2^{(0)2}(u)\rangle_{\text{av}}e^{-2u}+\frac{1}{\alpha_0^2}\langle 2z_2^{(0)}(u)z_2^{(2)}(u)\rangle_{\text{av}}e^{-2u}\ . \tag{6.25b}$$

Application of (i) yields the ideal squeezing:

$$\langle\Delta\hat{x}_1^2\rangle_{\text{ideal}}=\tfrac{1}{4}+\langle z_1^{(0)2}(u)\rangle_{\text{av}}e^{2u}=\tfrac{1}{4}e^{2u}\ , \tag{6.26a}$$

$$\langle\Delta\hat{x}_2^2\rangle_{\text{ideal}}=\tfrac{1}{4}+\langle z_2^{(0)2}(u)\rangle_{\text{av}}e^{-2u}=\tfrac{1}{4}e^{-2u}\ . \tag{6.26b}$$

Repeated applications of (i) and (ii) yield the quadrature variances correct to semiclassical order $\alpha_0^{-2}=\bar{N}^{-1}$:

$$\langle\Delta\hat{x}_1^2\rangle=\tfrac{1}{4}e^{2u}+\frac{1}{8\bar{N}}[u^2e^{2u}+u(e^{2u}+1)-(3\sinh^2u+2)\sinh u\,e^{u}-\sinh^2u]\ , \tag{6.27a}$$

$$\langle\Delta\hat{x}_2^2\rangle=\tfrac{1}{4}e^{-2u}+\frac{1}{8\bar{N}}[u^2e^{-2u}-u(e^{-2u}+1)+(3\sinh^2u+2)\sinh u\,e^{-u}-\sinh^2u]\ , \tag{6.27b}$$

which are exactly the results obtained by Hillery and Zubairy.[7] Equation (6.27a) also agrees to $O(\bar{N}^{-1})$ with the result calculated via the seminumerical method [Eq. (5.13)]. The variance of the squeezed quadrature, including the dominant correction for at best moderate squeezing only, is

$$\langle \Delta \hat{x}_2^2 \rangle = \tfrac{1}{4} e^{-2u} + \frac{1}{64\bar{N}} e^{2u} , \tag{6.28}$$

which agrees with Eq. (2.18), validating our heuristic picture of the effects of pump fluctuations on squeezing.

B. The two-mode PA

The analysis of the two-mode PA is similar to that of the one-mode PA. The equation of motion is again von Neumann's equation in the interaction representation, Eq. (6.1), with the two-mode interaction Hamiltonian $\hat{H}_{\rm int}(t)$ given by Eq. (1.4). Initially, we assume the signal modes are in vacuum states, and the pump mode is in a coherent state:

$$\hat{\rho}(0) = \hat{\rho}_I(0) = |0;0;\alpha_0\rangle\langle 0;0;\alpha_0| . \tag{6.29}$$

By substituting the two-mode versions of Eqs. (6.3) and (6.4) into Eq. (6.1) and integrating by parts, we find the Fokker-Planck equation for the two-mode PA:

$$\frac{\partial P}{\partial \tau} = \left[-\beta_2\alpha_p \frac{\partial}{\partial \alpha_1} - \alpha_2\beta_p \frac{\partial}{\partial \beta_1} - \beta_1\alpha_p \frac{\partial}{\partial \alpha_2} - \alpha_1\beta_p \frac{\partial}{\partial \beta_2} + \alpha_1\alpha_2 \frac{\partial}{\partial \alpha_p} + \beta_1\beta_2 \frac{\partial}{\partial \beta_p} + \alpha_p \frac{\partial^2}{\partial\alpha_1\partial\alpha_2} + \beta_p \frac{\partial^2}{\partial\beta_1\partial\beta_2} \right] P , \tag{6.30}$$

where $\tau = \kappa t$ and $P \equiv P(\alpha_1,\alpha_2,\alpha_p,\beta_1,\beta_2,\beta_p,\tau)$.

Proceeding as in the one-mode case, we can derive a set of Ito SDE's from the Fokker-Planck equation, Eq. (6.30):

$$d\alpha_1 = \beta_2\alpha_p\, d\tau + \sqrt{\alpha_p}\, dW_1 , \tag{6.31a}$$

$$d\alpha_2 = \beta_1\alpha_p\, d\tau + \sqrt{\alpha_p}\, dW_1^* , \tag{6.31b}$$

$$d\beta_1 = \alpha_2\beta_p\, d\tau + \sqrt{\beta_p}\, dW_2 , \tag{6.31c}$$

$$d\beta_2 = \alpha_1\beta_p\, d\tau + \sqrt{\beta_p}\, dW_2^* , \tag{6.31d}$$

$$d\alpha_p = -\alpha_1\alpha_2\, d\tau , \tag{6.31e}$$

$$d\beta_p = -\beta_1\beta_2\, d\tau . \tag{6.31f}$$

The *complex* Wiener increments dW_1 and dW_2 are defined by

$$dW_1 = \frac{dW_{1x} + i dW_{1y}}{\sqrt{2}} , \tag{6.32a}$$

$$dW_2 = \frac{dW_{2x} + i dW_{2y}}{\sqrt{2}} , \tag{6.32b}$$

where dW_{1x}, dW_{1y}, dW_{2x}, and dW_{2y} are independent, real Wiener increments, and, in the Ito calculus,

$$dW_{1s}^2 = dW_{2s}^2 = d\tau , \quad s = x,y . \tag{6.32c}$$

The complex pump amplitudes α_p and β_p are again assumed to consist of a large, real mean value α_0 plus small fluctuations, as in Eqs. (6.7). We define the new variables X_1, Y_1, X_2, Y_2, P_1, and P_2 by

$$X_1 = \tfrac{1}{2}(\alpha_1 + \beta_2) , \quad X_2 = -\frac{i}{2}(\alpha_1 - \beta_2) , \tag{6.33a}$$

$$Y_1 = \tfrac{1}{2}(\alpha_2 + \beta_1) , \quad Y_2 = -\frac{i}{2}(\alpha_2 - \beta_1) , \tag{6.33b}$$

$$P_1 = \tfrac{1}{2}(\Delta\alpha + \Delta\beta) , \quad P_2 = -\frac{i}{2}(\Delta\alpha - \Delta\beta) . \tag{6.33c}$$

It is convenient to change variables one more time:

$$U_1 = X_1 e^{-u} , \quad U_2 = X_2 e^{u} , \tag{6.34a}$$

$$Z_1 = Y_1 e^{-u} , \quad Z_2 = Y_2 e^{u} , \tag{6.34b}$$

where $u = \alpha_0\tau$. The resulting SDE's are

$$dU_1 = \frac{1}{\alpha_0}(U_1P_1 + U_2P_2e^{-2u})\, du + \frac{1}{2}e^{-u}\left[\left[1 + \frac{P_1 + iP_2}{\alpha_0}\right]^{1/2} dV_1 + \left[1 + \frac{P_1 - iP_2}{\alpha_0}\right]^{1/2} dV_2^* \right] , \tag{6.35a}$$

$$dZ_1 = \frac{1}{\alpha_0}(Z_1P_1 + Z_2P_2e^{-2u})\, du + \frac{1}{2}e^{-u}\left[\left[1 + \frac{P_1 + iP_2}{\alpha_0}\right]^{1/2} dV_1^* + \left[1 + \frac{P_1 - iP_2}{\alpha_0}\right]^{1/2} dV_2 \right] , \tag{6.35b}$$

$$dU_2 = \frac{1}{\alpha_0}(U_1P_2e^{2u} - U_2P_1)\, du - \frac{i}{2}e^{u}\left[\left[1 + \frac{P_1 + iP_2}{\alpha_0}\right]^{1/2} dV_1 - \left[1 + \frac{P_1 - iP_2}{\alpha_0}\right]^{1/2} dV_2^* \right] , \tag{6.35c}$$

$$dZ_2 = \frac{1}{\alpha_0}(Z_1P_2e^{2u} - Z_2P_1)\, du - \frac{i}{2}e^{u}\left[\left[1 + \frac{P_1 + iP_2}{\alpha_0}\right]^{1/2} dV_1^* - \left[1 + \frac{P_1 - iP_2}{\alpha_0}\right]^{1/2} dV_2 \right] , \tag{6.35d}$$

$$dP_1 = \frac{1}{\alpha_0}(U_2Z_2e^{-2u} - U_1Z_1e^{2u})\, du , \tag{6.35e}$$

$$dP_2 = -\frac{1}{\alpha_0}(U_1Z_2 + U_2Z_1)\, du , \tag{6.35f}$$

where $dV_1=\sqrt{\alpha_0}\,dW_1$ and $dV_2=\sqrt{\alpha_0}\,dW_2$.

We can obtain an approximate solution to Eqs. (6.35) just as we did in the one-mode case; we formally integrate Eqs. (6.35), expand the square roots, and substitute the expansion Eq. (6.14). We have found the approximate solution up to second order in α_0^{-1}: (i) to zeroth order,

$$U_1^{(0)}(u)=\tfrac{1}{2}\int_0^u e^{-x}(dS_1+idS_2)\ , \tag{6.36a}$$

$$Z_1^{(0)}(u)=U_1^{(0)*}(u)\ , \tag{6.36b}$$

$$U_2^{(0)}(u)=\tfrac{1}{2}\int_0^u e^{x}(dS_3-idS_4)\ , \tag{6.36c}$$

$$Z_2^{(0)}(u)=-U_2^{(0)*}(u)\ , \tag{6.36d}$$

$$P_1^{(0)}(u)=P_2^{(0)}(u)=0\ ; \tag{6.36e}$$

(ii) to first order,

$$U_1^{(1)}(u)=Z_1^{(1)}(u)=0\ , \tag{6.37a}$$

$$U_2^{(1)}(u)=Z_2^{(1)}(u)=0\ , \tag{6.37b}$$

$$P_1^{(1)}(u)=\int_0^u [U_2^{(0)}(x)Z_2^{(0)}(x)e^{-2x}-U_1^{(0)}(x)Z_1^{(0)}(x)e^{2x}]\,dx\ , \tag{6.37c}$$

$$P_2^{(1)}(u)=-\int_0^u [U_1^{(0)}(x)Z_2^{(0)}(x)+U_2^{(0)}(x)Z_1^{(0)}(x)]\,dx\ ; \tag{6.37d}$$

and (iii) to second order,

$$U_1^{(2)}(u)=\int_0^u [U_1^{(0)}(x)P_1^{(1)}(x)+U_2^{(0)}(x)P_2^{(1)}(x)e^{-2x}]\,dx+\tfrac{1}{4}\int_0^u e^{-x}[P_1^{(1)}(x)(dS_1+idS_2)-P_2^{(1)}(x)(dS_3-idS_4)]\ , \tag{6.38a}$$

$$Z_1^{(2)}(u)=\int_0^u [Z_1^{(0)}(x)P_1^{(1)}(x)+Z_2^{(0)}P_2^{(1)}(x)e^{-2x}]\,dx+\tfrac{1}{4}\int_0^u e^{-x}[P_1^{(1)}(x)(dS_1-idS_2)+P_2^{(1)}(x)(dS_3+idS_4)]\ , \tag{6.38b}$$

$$U_2^{(2)}(u)=\int_0^u [U_1^{(0)}(x)P_2^{(1)}(x)e^{2x}-U_2^{(0)}(x)P_1^{(1)}(x)]\,dx+\tfrac{1}{4}\int_0^u e^{x}[P_1^{(1)}(x)(dS_3-idS_4)+P_2^{(1)}(x)(dS_1+idS_2)]\ , \tag{6.38c}$$

$$Z_2^{(2)}(u)=\int_0^u [Z_1^{(0)}(x)P_2^{(1)}(x)e^{2x}-Z_2^{(0)}(x)P_1^{(1)}(x)]\,dx-\tfrac{1}{4}\int_0^u e^{x}[P_1^{(1)}(x)(dS_3+idS_4)-P_2^{(1)}(x)(dS_1-idS_2)]\ , \tag{6.38d}$$

$$P_1^{(2)}(u)=P_2^{(2)}(u)=0\ . \tag{6.38e}$$

Notice that in Eqs. (6.36) and Eqs. (6.38) we have replaced the complex noise increments by the real Wiener increments dS_1, dS_2, dS_3, and dS_4, where, using Eqs. (6.32),

$$dS_1=\frac{dW_{1x}+dW_{2x}}{\sqrt{2}}\ ,\quad dS_2=\frac{dW_{1y}-dW_{2y}}{\sqrt{2}}\ , \tag{6.39a}$$

$$dS_3=\frac{dW_{1y}+dW_{2y}}{\sqrt{2}}\ ,\quad dS_4=\frac{dW_{1x}-dW_{2x}}{\sqrt{2}}\ . \tag{6.39b}$$

Also note that, as in the one-mode case, we have dropped all contributions arising from the initial conditions.

By comparing Eqs. (6.36) with Eqs. (6.15), we see that the zeroth-order solutions $U_1^{(0)}(u)$, $Z_1^{(0)}(u)$, $U_2^{(0)}(u)$, and $Z_2^{(0)}(u)$ have real and imaginary parts that are Gaussian variables. Let

$$U_1^{(0)}(u)=Z_1^{(0)*}(u)=Q_1(u)+iQ_2(u)\ , \tag{6.40a}$$

$$U_2^{(0)}(u)=-Z_2^{(0)*}(u)=Q_3(u)+iQ_4(u)\ , \tag{6.40b}$$

where $Q_1(u)$, $Q_2(u)$, $Q_3(u)$, and $Q_4(u)$ are *independent* Gaussian variables with zero mean. We can use Eqs. (6.40) to generalize the one-mode rules [Eqs. (6.21) and Eqs. (6.22)] to the two-mode case: (i) a rule for quadratic moments,

$$\langle Q_1(u)Q_1(w)\rangle_{\rm av}=\langle Q_2(u)Q_2(w)\rangle_{\rm av}=\tfrac{1}{8}(1-e^{-2w})\ ,\quad u\ge w\ , \tag{6.41a}$$

$$\langle Q_3(u)Q_3(w)\rangle_{\rm av}=\langle Q_4(u)Q_4(w)\rangle_{\rm av}=\tfrac{1}{8}(e^{2w}-1)\ ,\quad u\ge w\ , \tag{6.41b}$$

$$\langle Q_i(u)Q_j(u)\rangle_{\rm av}=0\ ,\quad i\ne j \tag{6.41c}$$

and (ii) a rule for quartic moments,

$$\langle Q_1(u)Q_1(v)Q_1(w)Q_1(z)\rangle_{\rm av}=\langle Q_1(u)Q_1(v)\rangle_{\rm av}\langle Q_1(w)Q_1(z)\rangle_{\rm av}+\langle Q_1(u)Q_1(w)\rangle_{\rm av}\langle Q_1(v)Q_1(z)\rangle_{\rm av}+\langle Q_1(u)Q_1(z)\rangle_{\rm av}\langle Q_1(v)Q_1(w)\rangle_{\rm av}\ , \tag{6.42a}$$

$$\langle Q_2(u)Q_2(v)Q_2(w)Q_2(z)\rangle_{\rm av}=\langle Q_1(u)Q_1(v)Q_1(w)Q_1(z)\rangle_{\rm av}\ , \tag{6.42b}$$

$$\langle Q_3(u)Q_3(v)Q_3(w)Q_3(z)\rangle_{\rm av}=\langle Q_3(u)Q_3(v)\rangle_{\rm av}\langle Q_3(w)Q_3(z)\rangle_{\rm av}+\langle Q_3(u)Q_3(w)\rangle_{\rm av}\langle Q_3(v)Q_3(z)\rangle_{\rm av}+\langle Q_3(u)Q_3(z)\rangle_{\rm av}\langle Q_3(v)Q_3(w)\rangle_{\rm av}\ , \tag{6.42c}$$

$$\langle Q_4(u)Q_4(v)Q_4(w)Q_4(z)\rangle_{\rm av}=\langle Q_3(u)Q_3(v)Q_3(w)Q_3(z)\rangle_{\rm av}\ . \tag{6.42d}$$

We can calculate the two-mode squeezing by repeated application of (i) and (ii). The two-mode quadrature-phase amplitudes are defined by

$$\hat{X}_1=\frac{1}{2}(\hat{a}_1+\hat{a}_2^{\dagger})\ ,\quad \hat{X}_2=-\frac{i}{2}(\hat{a}_1-\hat{a}_2^{\dagger})\ , \tag{6.43a}$$

$$\hat{P}_1=\frac{1}{2}(\hat{a}_p+\hat{a}_p^{\dagger})\ ,\quad \hat{P}_2=-\frac{i}{2}(\hat{a}_p-\hat{a}_p^{\dagger})\ . \tag{6.43b}$$

The two-mode signal quadrature-phase amplitudes are *not* Hermitian operators. In terms of the stochastic c numbers, the mean-square uncertainties in the two-mode quadrature-phase amplitudes are given by

$$\langle |\Delta\hat{X}_1|^2\rangle = \tfrac{1}{4} + \langle X_1 Y_1\rangle_{av} - \langle X_1\rangle_{av}\langle Y_1\rangle_{av} , \quad (6.44a)$$

$$\langle |\Delta\hat{X}_2|^2\rangle = \tfrac{1}{4} + \langle X_2 Y_2\rangle_{av} - \langle X_2\rangle_{av}\langle Y_2\rangle_{av} . \quad (6.44b)$$

From Eq. (1.13) and Eqs. (6.44) we see that X_1 and Y_1 are the c-number equivalents of the operators $\hat{X}_1$ and its Hermitian conjugate $\hat{X}_1^\dagger$, respectively, and X_2 and Y_2 are the c-number equivalents of the operators $\hat{X}_2$ and its Hermitian conjugate $\hat{X}_2^\dagger$. Substituting Eqs. (6.14), (6.36), (6.37), and (6.38) into Eqs. (6.44), we have to second order in α_0^{-1}

$$\langle |\Delta\hat{X}_1|^2\rangle = \tfrac{1}{4} + \langle U_1^{(0)}(u)Z_1^{(0)}(u)\rangle_{av} e^{2u} + \langle U_1^{(0)}(u)Z_1^{(2)}(u) + U_1^{(2)}(u)Z_1^{(0)}(u)\rangle_{av} e^{2u} , \quad (6.45a)$$

$$\langle |\Delta\hat{X}_2|^2\rangle = \tfrac{1}{4} + \langle U_2^{(0)}(u)Z_2^{(0)}(u)\rangle_{av} e^{-2u} + \frac{1}{\alpha_0^2}\langle U_2^{(0)}(u)Z_2^{(2)}(u) + U_2^{(2)}(u)Z_2^{(0)}(u)\rangle_{av} e^{-2u} , \quad (6.45b)$$

since the expectation values of X_1, Y_1, X_2, and Y_2 are zero.

Repeated application of the two-mode rules (i) and (ii) yield the mean-squared uncertainties correct to semiclassical order $\alpha_0^{-2} = \bar{N}^{-1}$:

$$\langle |\Delta\hat{X}_1|^2\rangle = \tfrac{1}{4}e^{2u} + \frac{1}{8\bar{N}}[u^2e^{2u} + u(\tfrac{3}{2}e^{2u}+1) - (4\sinh^2 u + \tfrac{5}{2})\sinh u\, e^{u} - \tfrac{3}{2}\sinh^2 u] , \quad (6.46a)$$

$$\langle |\Delta\hat{X}_2|^2\rangle = \tfrac{1}{4}e^{-2u} + \frac{1}{8\bar{N}}[u^2e^{-2u} - u(\tfrac{3}{2}e^{-2u}+1) + (4\sinh^2 u + \tfrac{5}{2})\sinh u\, e^{-u} - \tfrac{3}{2}\sinh^2 u] , \quad (6.46b)$$

which is slightly different from the one-mode result, Eqs. (6.27), and agrees to $O(\bar{N}^{-1})$ with the result calculated via the seminumerical method [Eq. (5.14)]. When the dominant correction only is kept, the uncertainty in the squeezed quadrature is

$$\langle |\Delta\hat{X}_2|^2\rangle \cong \tfrac{1}{4}e^{-2u} + \frac{1}{64\bar{N}}e^{2u} , \quad (6.47)$$

which is the same as the dominant correction found for the one-mode case, Eq.(6.28), in contrast to the result obtained by Scharf and Walls.[8]

VII. CONCLUSION

We have calculated, for the one- and the two-mode PA, the explicit corrections for squeezing to order $1/\bar{N}^2$, due to a quantum pump in a coherent state with an average photon number $\bar{N}$. We found that the pump's phase noise is responsible for the dominant contribution to the limitations on squeezing for any number of signal modes. We also briefly discuss when traveling-wave calculations can be treated by Hamiltonian methods in the most direct way. This was done by discretizing the continuum problem. Finally, the limitation to squeezing in the discrete-mode calculations we performed was shown to be insensitive to the details of our discretization process.

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*Present address: Joint Institute for Laboratory Astrophysics, University of Colorado, Boulder, CO 80309.

†Present address: Optical Sciences Center, University of Arizona, Tucson, AZ 85721.

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