

# Schrödinger cats and their power for quantum information processing

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## Abstract

We outline a toolbox comprised of passive optical elements, single photon detection and superpositions of coherent states (Schrödinger cat states). Such a toolbox is a powerful collection of primitives for quantum information processing tasks. We illustrate its use by outlining a proposal for universal quantum computation. We utilize this toolbox for quantum metrology applications, for instance weak force measurements and precise phase estimation. We show in both these cases that a sensitivity at the Heisenberg limit is achievable.

**Keywords:** quantum information processing, Schrödinger cats, quantum metrology, high precision measurements, quantum toolbox

(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

Quantum optics has played a major role in the testing of fundamental properties of quantum mechanics and more recently in implementing simple quantum information protocols [1, 2]. This has been made possible because photons are easily produced and manipulated. This is especially true as the electromagnetic environment at optical frequencies can be regarded as a vacuum and is relatively decoherence free.

One of the earliest proposals for implementing a quantum logic gate was made by Milburn [3] and was based on encoding each qubit in two optical modes, each containing exactly one photon. This was a very elegant proposal, but unfortunately required massive and reversible nonlinearities. Such reversible nonlinearities are well beyond those presently available and hence it was thought quantum optics would not provide a practical path to efficient and scalable quantum computation. Knill *et al* [4] recently challenged this orthodoxy when they showed that given appropriate single photon sources and detectors, linear optics alone could create a non-

deterministic two qubit gates. Furthermore they showed that near deterministic gates could be created from these non-deterministic gates through a technique of teleporting gates [5]. This therefore provided a route for efficient and scalable quantum computation with only single photon sources, photon counting and linear optics.

This does however raise the question whether there are other architectures based on different encoding schemes which have similar characteristics. These other architectures may have advantages in that their optical circuits are less complex. We could trade off the complexity of the circuit in the KLM scheme for more complicated initial resources, for instance continuous variable multi-photon fields. The idea of encoding quantum information on continuous variables of multi-photon fields has emerged recently [6] and a number of schemes have been proposed for realizing quantum computation in this way [7–9]. A significant drawback of these proposals is that hard nonlinear interactions are required *in-line* of the computation and make such proposals difficult to implement in practice. In contrast, a recent proposal [10, 11] details a scheme

for quantum computation where the hard nonlinear interactions are only required for the *off-line* preparation of resource states. A required resource for this scheme is superpositions of coherent states (Schrödinger cat states) which are currently experimentally challenging to realize.

In this paper we outline a toolbox of techniques and states necessary for universal quantum computation with coherent states. This toolbox can also be used for quantum metrology applications and we will examine two specific examples: the detection of weak tidal forces due to gravitational radiation [1, 12, 13] and improving the sensitivity of Ramsey fringe interferometry [14, 15]. An alternative approach for these metrology applications would be the use entangled states (either through entangled qubits, or continuous variable twin beams). These entangled states can be used for improved quantum measurements [16–18] and interferometry [19].

The paper is structured as follows: section 2 describes the components of the toolbox, while section 3 describes how to achieve a universal set of gates sufficient for quantum information processing. Finally section 4 illustrates two quantum metrology examples.

## 2. The toolbox

The base components that our toolbox will contain will be passive linear optical elements such as beam-splitters and phase shifters. The beam-splitter interaction is given by

$$B(\theta) = \exp[i\theta(ab^\dagger + a^\dagger b)]. \quad (1)$$

Here  $a$  and  $b$  are the usual boson annihilation operators for the two electromagnetic field modes at the beam-splitter. The action of the beam-splitter is such that two coherent states  $|\gamma\rangle_a$  and  $|\beta\rangle_b$  get transformed as

$$B(\theta)|\gamma\rangle_a|\beta\rangle_b = |\gamma \cos \theta + i\beta \sin \theta\rangle_a |\beta \cos \theta + i\gamma \sin \theta\rangle_b. \quad (2)$$

A phase shifter is just a delay with respect to the local oscillator and can be described by the operator  $P(\theta) = \exp[i\theta\hat{a}^\dagger\hat{a}]$  which just introduces a phase to the coherent state:  $P(\theta)|\alpha\rangle = |e^{i\theta}\alpha\rangle$ . From these basic components, we can construct other operators; for instance, the displacement operator

$$D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*a) \quad (3)$$

acting on a state  $|\phi\rangle$  can be constructed by mixing that state with a strong coherent state on a weak beam-splitter. On coherent states the displacement operators just displace the coherent state:  $D(\beta)|\alpha\rangle = \exp[(\beta\alpha^* - \beta^*\alpha)/2]|\alpha + \beta\rangle$ .

To these passive elements we also want to add single photon counters, which can resolve the ‘quanta’ in the electromagnetic field, and homodyne detectors. While high-efficiency homodyne detection is currently achievable [20], single photon counters are extremely challenging, but there is an active research program to construct them (see for instance [21, 22]).

Finally, to this collection of elements we add the ability to generate optical ‘Schrödinger cat’ states. These are states which are coherent superpositions of coherent states  $|\alpha\rangle$  for different  $\alpha$ . In particular, we are interested in the even and odd cat states:

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{\mathcal{N}_\pm}} [|\alpha\rangle \pm |-\alpha\rangle], \quad (4)$$

where  $\mathcal{N}_\pm = 2 \pm 2e^{-2|\alpha|^2}$ . There are several proposals on how to generate these states (e.g. [23, 24]). It is easy to show that the even (odd) cat states have only even (odd) photon number terms—which is where they get their name. From this we can see that the two states are orthogonal and a single photon counter will be able to distinguish between them.

One of the most powerful features of this toolbox that is not immediately obvious is that we now have the ability to easily generate entangled states [25]. By combining a single mode cat state of the form  $|\sqrt{2}\alpha\rangle + |-\sqrt{2}\alpha\rangle$  with the vacuum state on a 50/50 beam-splitter, the output state is of the form of a Bell state in the  $\{|\alpha\rangle, |-\alpha\rangle\}$  subspace:

$$|\Psi\rangle = \frac{1}{\sqrt{\mathcal{N}}} [|\alpha\rangle|\alpha\rangle + |-\alpha\rangle|-\alpha\rangle], \quad (5)$$

where  $\mathcal{N}$  is the normalization constant. In this subspace we can also perform Bell-basis measurements by simply running the Bell state creation in reverse: we interfere the two modes at a beam splitter, then use photon counters to measure the number of photons in each output mode [9]. We can then identify the four possible results: (i) (even, 0), (ii) (odd, 0), (iii) (0, even), (iv) (0, odd), where  $(m, n)$  indicates counting  $m$  and  $n$  photons in the two modes respectively. These results correspond to each of the four Bell-cat states: (i)  $(|-\alpha, -\alpha\rangle + |\alpha, \alpha\rangle)/\sqrt{2}$ , (ii)  $(|-\alpha, -\alpha\rangle - |\alpha, \alpha\rangle)/\sqrt{2}$ , (iii)  $(|-\alpha, \alpha\rangle + |\alpha, -\alpha\rangle)/\sqrt{2}$ , or (iv)  $(|-\alpha, \alpha\rangle - |\alpha, -\alpha\rangle)/\sqrt{2}$ . Note that there is also a fifth possibility of detecting zero photons in both modes which indicates a failure of the measurement. Fortunately, this occurs with probability of only  $\sim e^{-\alpha^2}$ , and for  $\alpha$  moderately large this is insignificant.

We can go further with entanglement and generate multi-mode entangled states. If a single mode cat state  $|\alpha\rangle + |-\alpha\rangle$  is input into one mode of an  $N$  port symmetric beam-splitter with the remaining input ports empty. The output state from this beam-splitter is then the massively entangled GHZ-like state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[ \left| \frac{\alpha}{\sqrt{N}}, \frac{\alpha}{\sqrt{N}}, \dots, \frac{\alpha}{\sqrt{N}} \right\rangle + \left| -\frac{\alpha}{\sqrt{N}}, -\frac{\alpha}{\sqrt{N}}, \dots, -\frac{\alpha}{\sqrt{N}} \right\rangle \right]. \quad (6)$$

## 3. Universal quantum logic gates

The first application of this toolbox that we will review is a scheme for quantum computation with coherent states [11, 10]. Consider an encoding of logical qubits in coherent states with the logical 0 and 1 states being  $|0\rangle_L = |-\alpha\rangle$  and  $|1\rangle_L = |\alpha\rangle$  respectively. (An entirely equivalent encoding would be  $|0\rangle_L = |0\rangle$  and  $|1\rangle_L = |2\alpha\rangle$  as discussed in [11], and these two encodings are simply related by a displacement  $D(-\alpha)$ .) For convenience and without loss of generality we will choose  $\alpha$  to be real. These qubits are not strictly orthogonal, but the approximation is good for  $\alpha$  even moderately large as  $|\langle\alpha|-\alpha\rangle|^2 = e^{-4\alpha^2}$ . For  $\alpha \geq 2$  the overlap between the zero and one logic qubit states is only  $|\langle\alpha|-\alpha\rangle|^2 \leq 10^{-6}$ .

It is well known that one set of universal gates for qubits is comprised of arbitrary single qubit rotations together with an entangling gate. The single qubit rotations for our qubits can be built from four basic single qubit gates. The first two gates are the bit and sign flip operations and are given as follows.

- A bit-flip: the logical value of a qubit can be flipped by delaying it with respect to the local oscillator by half a cycle. Thus the ‘bit-flip’ gate  $X$  is given by  $X = P(\pi)$ . For example,  $X(\mu|-\alpha\rangle + \nu|\alpha\rangle) = \mu|\alpha\rangle + \nu|-\alpha\rangle$ .
- A sign-flip: the sign flip gate  $Z$  can be achieved using a teleportation protocol and the maximally entangled resource (5). Consider that we wish to sign flip the qubit  $\mu|-\alpha\rangle + \nu|\alpha\rangle$ . A Bell state measurement is performed between one half of the resource (5) and the qubit of interest. Depending on which of the four possible outcomes are found the other half of the Bell state is projected into one of the following four states with equal probability: (i)  $\mu|-\alpha\rangle + \nu|\alpha\rangle$ , (ii)  $\mu|-\alpha\rangle - \nu|\alpha\rangle$ , (iii)  $\mu|\alpha\rangle + \nu|-\alpha\rangle$ , and (iv)  $\mu|\alpha\rangle - \nu|-\alpha\rangle$ . The bit flips in results three and four can be corrected using the  $X$  gate above. After  $X$  correction the gate has two possible outcomes: either the identity has been applied, in which case we repeat the process, or else the required transformation:

$$Z(\mu|-\alpha\rangle + \nu|\alpha\rangle) = \mu|-\alpha\rangle - \nu|\alpha\rangle. \quad (7)$$

The teleportation trick used in the  $Z$  gate is incredibly useful and can be used to ‘clean up’ qubits that move slightly away from the  $\{|-\alpha\rangle, |\alpha\rangle\}$  subspace [10]. The remaining two operations are arbitrary rotations about the  $Z$  and  $X$  axis and like the sign flip operation  $Z$  they also use a teleportation protocol to achieve the gate. These operations are given by the following.

- An arbitrary rotation  $\phi$  about the  $Z$  axis, schematically depicted in figure 1 can be implemented by first displacing our arbitrary input qubit  $\mu|-\alpha\rangle + \nu|\alpha\rangle$  by a small amount  $\beta = \alpha\theta$  in the imaginary direction. This results in the state

$$\mu e^{-i\theta\alpha^2} |-\alpha(1-i\theta)\rangle + \nu e^{i\theta\alpha^2} |\alpha(1+i\theta)\rangle \quad (8)$$

which is a small distance outside the computational space. The teleportation then projects us back into the qubit space resulting in the state

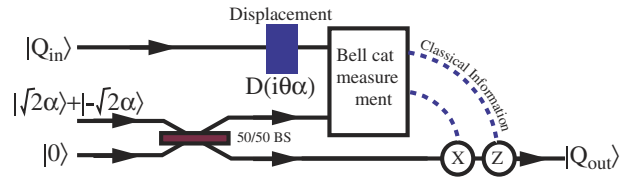
$$e^{-\theta^2\alpha^2/2} (e^{-i2\theta\alpha^2} \mu|-\alpha\rangle + e^{i2\theta\alpha^2} \nu|\alpha\rangle). \quad (9)$$

This is a rotation around  $Z$  by  $4\theta\alpha^2$ . This gate is near deterministic for a sufficiently small values of  $\theta^2\alpha^2$ . Repeated application of this gate can build up a finite rotation with high probability.

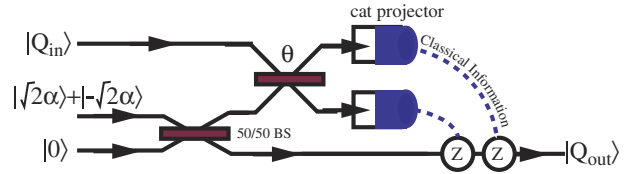
- The fourth gate to consider is a rotation of  $\pi/2$  about the  $X$  axis. The gate is shown schematically in figure 2. For an arbitrary input state  $\mu|-\alpha\rangle + \nu|\alpha\rangle$ , the interaction  $C_a C_b U_{BS}$  produces the output state (after correcting with  $X$  and  $Z$ )

$$e^{-\theta^2\alpha^2/4} [(e^{i2\theta\alpha^2} \mu + e^{-i2\theta\alpha^2} \nu)|-\alpha\rangle + (e^{-i2\theta\alpha^2} \mu + e^{i2\theta\alpha^2} \nu)|\alpha\rangle], \quad (10)$$

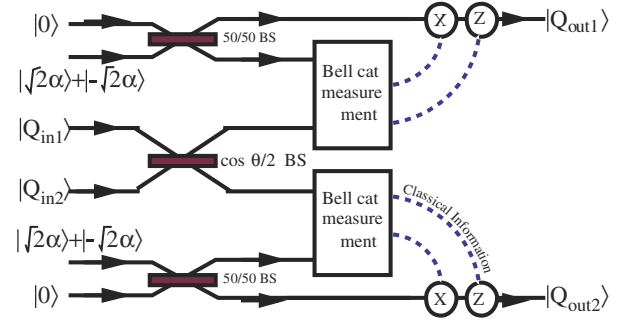
where  $C_a$  and  $C_b$  represent cat state projections onto either the even or odd parity cat (i.e. photon counting and conditioning on even or odd numbers of photons). By choosing  $4\theta\alpha^2 = \pi/2$  the gate will implement a  $\pi/2$  rotation around the  $X$  axis.



**Figure 1.** Schematics for implementing a rotation around  $Z$ . We begin by first shifting our qubit a small distance out of the computational basis and then using teleportation to project back into the qubit space.



**Figure 2.** Schematics for implementing a rotation of  $\pi/2$  about the  $X$  axis.



**Figure 3.** Schematics of implementing an entangling gate. For a sufficiently small value of  $\theta^2\alpha^2$  this gate is near deterministic. Repeated application of this gate can build up to a gate locally equivalent to a CNOT gate, with high probability.

By combining these gates it is possible to achieve an arbitrary single qubit rotation. If we can supplement these gates with a single two-qubit entangling operation between the qubits, then we have a universal set.

- We can implement an entangling gate in a similar way to the single qubit  $Z$  rotation. A schematic circuit for the gate is depicted in figure 3. If both our qubits are first mixed on a beam-splitter and are then projected back into the qubit space of  $\{|\pm\alpha\rangle\}$  using teleportation, we find for an arbitrary input state  $\nu|-\alpha\rangle_a |-\alpha\rangle_b + \mu|\alpha\rangle_a |-\alpha\rangle_b + \tau|-\alpha\rangle_a |\alpha\rangle_b + \gamma|\alpha\rangle_a |\alpha\rangle_b$  that the resultant state is

$$e^{i2\theta\alpha^2} \nu|-\alpha\rangle_a |-\alpha\rangle_b + e^{-i2\theta\alpha^2} \mu|\alpha\rangle_a |-\alpha\rangle_b + e^{-i2\theta\alpha^2} \tau|-\alpha\rangle_a |\alpha\rangle_b + e^{i2\theta\alpha^2} \gamma|\alpha\rangle_a |\alpha\rangle_b, \quad (11)$$

where, as before, we have assumed orthogonality of the qubit basis state and  $\theta^2\alpha^2 \ll 1$ . If we choose  $4\theta\alpha^2 = \pi/2$  then this gate will implement a CNOT up to single-qubit rotations [10].

This then completes a universal set of gates. In [10] details are given on using further nested teleportation to make these gates deterministic without requiring  $\theta^2\alpha^2 \ll 1$ . These gates can be used for both quantum computation and communication.

## 4. Quantum metrology

In this section we illustrate the utility of the Schrödinger cat states for two metrology applications—the detection of weak forces, and high precision phase measurements.

### 4.1. The detection of weak forces

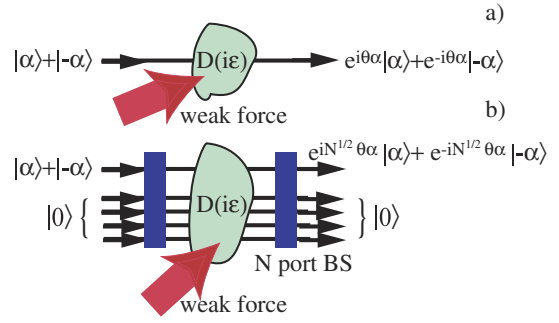
Before we begin our discussion of the application of Schrödinger cats states to weak force detection [13], it is essential to establish the best classical limit. It is well known that when a classical force given by  $F(t)$  acts for a fixed time on a simple harmonic oscillator, it displaces the complex amplitude of this oscillator in phase space. The resulting amplitude and phase of the displacement are determined by the time dependence of the force [26]. If the oscillator begins in a coherent state  $|\alpha_0\rangle$  (with  $\alpha_0$  real) then a displacement  $D(i\epsilon)$  (chosen to be orthogonal to the coherent amplitude of the initial state) causes the coherent state to evolve to  $e^{i\epsilon\alpha_0}|\alpha_0 + i\epsilon\rangle$ . The maximum signal to noise ratio is then  $\text{SNR} = S/\sqrt{V} = 2\epsilon$ . This must be greater than unity for the displacement to be resolved and hence this establishes the standard quantum limit (SQL) [1] of  $\epsilon_{\text{SQL}} \geq 1/2$ .

It is also well known that this limit may be overcome if the oscillator is prepared in a non-classical state. However, what is the sensitivity achieved by (4), and does this reach the ultimate (Heisenberg) limit? When a weak classical force acts on the even photon number cat state  $|\alpha\rangle + |-\alpha\rangle$  with  $\alpha$  real (see figure 4(a)) it is displaced to

$$|\phi\rangle_{\text{out}} \approx \frac{1}{\sqrt{2}} (e^{i2\epsilon\alpha}|\alpha\rangle + e^{-i2\epsilon\alpha}|-\alpha\rangle). \quad (12)$$

Our problem is thus reduced to finding the optimal readout to be able to distinguish (12) from  $|\alpha\rangle + |-\alpha\rangle$ . The theory of optimal parameter estimation [27] indicates that the limit on the precision with which the parameter  $\epsilon\alpha$  can be determined is  $(\delta\theta)^2 \geq 1/\text{var}(\hat{\sigma}_x)_{\text{in}}$ , where  $\text{var}(\hat{\sigma}_x)_{\text{in}}$  is the variance in the generator of the rotation in the input state  $|\alpha\rangle + |-\alpha\rangle$ . In this case the variance is simply unity. It thus follows that the minimum detectable force is  $\epsilon \geq 1/2\sqrt{\bar{n}}$ , where  $\bar{n}$  is the mean photon number given by  $\bar{n} = |\alpha|^2$ . It is straightforward to show this ‘measurement’ is the Heisenberg limit for a displacement measurement. An interesting question is what type of measurement is required to achieve this limit. In effect we need to be able to distinguish the even parity cat state from the odd parity cat state. Currently this is experimentally challenging. However, by performing a Hadamard operation (one of the single qubit gates discussed previously), the even and odd Schrödinger cats are transformed to the coherent states  $|\alpha\rangle$  from  $|-\alpha\rangle$  which can be easily distinguished via a standard homodyne measurement.

If the weak force acts over a reasonable spatial range it would be possible to have a number of spatial modes of light being affected. Could this help us exceed the limit above, even if we constrained the total mean photon number of the entire multimode system? We depict in figure 4(b) a schematic for the setup of a proposed experiment. Using a single mode cat state and an  $N$  port symmetric beam-splitter we can generate the state (6), which has a total mean photon number of  $n_{\text{tot}} = |\alpha|^2$ . We now assume that the weak force acts simultaneously on all



**Figure 4.** Schematic diagram of the action of a weak force causing a displacement  $D(i\epsilon)$  on a Schrödinger cats state  $|\alpha\rangle + |-\alpha\rangle$ . In (a) a single mode case is illustrated while in (b) an  $N$  mode situation is considered.

modes of this  $N$  party entangled state, displacing them each by an amount  $D(i\epsilon)$  (for  $\epsilon \ll 1$ ). The resulting state after the action of the force is

$$|\psi(\theta)\rangle = \frac{1}{\sqrt{2}} \left[ e^{i2\sqrt{N}\epsilon\alpha} \left| \frac{\alpha}{\sqrt{N}}, \dots, \frac{\alpha}{\sqrt{N}} \right\rangle + e^{-i2\sqrt{N}\epsilon\alpha} \left| -\frac{\alpha}{\sqrt{N}}, \dots, -\frac{\alpha}{\sqrt{N}} \right\rangle \right], \quad (13)$$

where we have neglected the small displacement that occurs to the coherent state. The theory of optimal parameter estimation indicates that the limit on the precision with which the displacement parameter  $\epsilon$  be estimated is bounded by

$$\epsilon_{\min} = \frac{1}{\sqrt{N[1+4n_{\text{tot}}]}} \sim \frac{1}{2\sqrt{Nn_{\text{tot}}}} \quad (14)$$

for  $n_{\text{tot}} \gg 1$ . If, however, we had used  $N$  independent cat states each with a photon number  $n_{\text{tot}}/N$  then  $\epsilon_{\min}$  for the entire system would have scaled as  $\epsilon_{\min} \sim 1/\sqrt{4n_{\text{tot}}}$ , which is the same result we obtained for the single mode case. For large  $n_{\text{tot}}$ , the preferred regime to work in, we find that the  $N$  mode entangled situation gives an extra  $\sqrt{N}$  improvement over the single mode cat situation for the same total mean photon number. Now how do we interpret such results? The effect that we are seeing is due to the weak force acting equally on all  $N$  modes and the state between the  $N$  port beam-splitters being highly entangled. Does this result in a violation of the Heisenberg limit of  $1/\sqrt{n_{\text{tot}}}$  which we previously mentioned? The answer is no. A careful analysis using parameter estimation of this multimode situation indicates that our result is at the Heisenberg limit. For displacement measurements the Heisenberg limit does depend on the number of modes.

These results indicate that subject to the spatial bandwidth of the weak classical force it seems optimal for a cat state with fixed mean photon  $n_{\text{tot}}$  to be split and entangled over as many modes as feasible. This in the absence of loss gives the best sensitivity. Such techniques are likely to work for other non-classical continuous variable states.

In the situation just discussed, maximum sensitivity required the classical force to displace the cat states in a direction orthogonal to the phase of the superposed coherent amplitudes. In general there is no way to arrange this beforehand, as the phase of the displacement depends on an unknown time dependence of the classical force. However,



a simple generalization of the previous cat states can be used to relax this constraint. If we consider that the oscillator is initially prepared in the state

$$|\psi\rangle_{\text{in}} = |\alpha\rangle + |i\alpha\rangle + |-i\alpha\rangle + |-\alpha\rangle \quad (15)$$

with  $\alpha$  real, then under the action of a weak force characterized by a complex amplitude displacement  $\beta$ , the output state is

$$|\psi\rangle_{\text{out}} = e^{2i\theta}|\alpha\rangle + e^{2i\phi}|i\alpha\rangle + e^{-2i\phi}|-i\alpha\rangle + e^{-2i\theta}|-\alpha\rangle, \quad (16)$$

where  $\theta = \alpha \text{Im}(\beta)$  and  $\phi = \alpha \text{Re}(\beta)$ . The state now carries information on both the real and imaginary components of the displacement due to the force which may be extracted by measuring the projection operator onto the initial state.

#### 4.2. High precision phase measurements

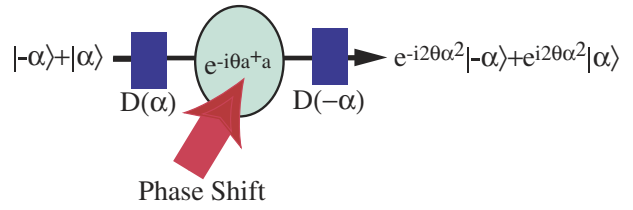
The second metrological example we are going to investigate is the estimation of phase. The classic situation to consider is Ramsey fringe interferometry, which was first introduced by Bollinger *et al* [14] in the mid-1990s. In Ramsey fringe interferometry the objective is to detect the relative phase difference between two superposed qubit basis states  $|0\rangle$  and  $|1\rangle$ . This phase difference problem reduces to a quantum parameter estimation situation in which a unitary transformation  $U(\theta) = \exp[i\theta\hat{Z}]$  (with  $\hat{Z} = |1\rangle\langle 1| - |0\rangle\langle 0|$ ) induces a relative phase in the specified basis. For example, an initial state of the form  $c_0|0\rangle + c_1|1\rangle$  evolves to  $c_0e^{-i\theta}|0\rangle + c_1e^{i\theta}|1\rangle$  under the above unitary operation. When can we distinguish these two states? Is there an optimal choice of initial state? The theory of quantum parameter estimation [27] indicates for this situation that we should choose the initial state as  $|\psi\rangle_i = (|0\rangle + |1\rangle)/\sqrt{2}$  and that the optimal measurement is a projective measurement in the basis  $|\pm\rangle = |0\rangle \pm |1\rangle$ . The probability of obtaining the result  $+$  is  $P(+|\theta) = \cos^2\theta$ . For  $N$  repetitions of this measurement the uncertainty in the inferred parameter  $\theta$  is  $\delta\theta = 1/\sqrt{N}$ . This is known as the standard quantum limit. It was noted by Bollinger *et al* [14] that a more effective way to use the  $N$  two-level systems is to first prepare them in the maximally entangled state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_2 \cdots |0\rangle_N + |1\rangle_1|1\rangle_2 \cdots |1\rangle_N), \quad (17)$$

and then subject the entire state to the unitary transformation  $U(\theta) = \prod_{i=1}^N \exp(-i\theta\hat{Z}_i)$ . After the unitary transformation the state (17) evolves to

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\exp(-iN\theta)|0\rangle_1|0\rangle_2 \cdots |0\rangle_N + \exp(iN\theta)|1\rangle_1|1\rangle_2 \cdots |1\rangle_N). \quad (18)$$

The uncertainty in the estimation of the parameter  $\theta$  then achieves the Heisenberg lower limit of  $\delta\theta = 1/N$ . This would seem to indicate, as in the weak force case, that entanglement is a critical requirement to achieve the improved sensitivity. Let us examine this point a little further for the phase estimation situation. The Hilbert space of  $N$  two-level systems is a tensor product space of dimension  $2^N$ . The entangled state given in equation (17), however, resides in a much smaller  $N + 1$  dimensional irreducible subspace of permutation symmetric



**Figure 5.** Schematics of a quantum circuit illustrating how a phase shift can be seen on an input state of the form.

states [28]. We may use an  $SU(2)$  representation to write the entangled state  $|0\rangle_1|0\rangle_2 \cdots |0\rangle_N + |1\rangle_1|1\rangle_2 \cdots |1\rangle_N$  in the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|-N/2\rangle_{N/2} + |N/2\rangle_{N/2}). \quad (19)$$

This is just an  $SU(2)$  ‘cat state’ for  $N$  two-level atoms. Hence a single  $N$  level atom can achieve the same phase sensitivity as a maximally entangled GHZ state since it can be written in the form  $|-N/2\rangle_{N/2} + |N/2\rangle_{N/2}$ . This would also seem to indicate that a superposition of coherent states (a cat state) can provide the same phase resolution. In figure 5 a schematic diagram is shown for the cat state yielding the Heisenberg limited phase resolution. Such phase shifts could be used to resolve precisely very small length intervals, a *quantum ruler* [29] in effect. As  $\alpha$  increases, a number of high visibility, narrowly spaced fringes emerge, which could enable very short length intervals to be accurately measured. As an example, suppose our laser wavelength is  $10 \mu\text{m}$ . In a standard interferometer this would enable length intervals of  $5 \mu\text{m}$  to be stepped off. However, using the cat-state technique with an  $\alpha$  of 10 leads to the fringe separation being reduced to  $1 \mu\text{m}$ .

The preceding discussion shows that entanglement is not necessary to achieve a Heisenberg limited phase measurement. What entanglement allows, however, is the creation of an effective cat state without the need to resort to creating a superposition between the ground state and a highly excited one.

## 5. Concluding remarks

In the paper we have presented a toolbox of techniques that make use of superpositions of coherent states. Using this toolbox we have presented a quantum computation scheme based on encoding qubits as coherent states, and their superposition. The optical networks required are conceptually simple and require only linear interactions, homodyne measurements and photon counting. We have concentrated on the simplest implementation, which unfortunately requires large  $\alpha$ . However, with a modest increase in complexity the non-deterministic operation of the gates at low  $\alpha$  can form the basis of a scalable system [10]. We have also shown how the same toolbox can be used to achieve extremely sensitive force detection and phase measurements.

An open and very interesting question is whether the toolbox of techniques and states we have described can be transferred to other systems where we can formulate coherent states, for example SQUIDs. In those systems it may be possible to augment or simplify the toolbox making the quantum information applications more accessible.

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## References

- [1] See for instance, Walls D F and Milburn G J 1994 *Quantum Optics* (Berlin: Springer)
- [2] Nielsen M and Chuang I 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
- [3] Milburn G J 1989 *Phys. Rev. Lett.* **62** 2124
- [4] Knill E, Laflamme L and Milburn G J 2001 *Nature* **409** 46
- [5] Gottesman D, Kitaev A and Preskill J 2001 *Phys. Rev. A* **64** 012310
- [6] Furasawa A, Sørensen J L, Braunstein S L, Fuchs C A, Kimble H J and Polzik E S 1998 *Science* **282** 706
- [7] Lloyd S and Braunstein S L 1999 *Phys. Rev. Lett.* **82** 1784
- [8] Bartlett S D, de Guise H and Sanders B C 2002 *Phys. Rev. A* **65** 052316
- [9] Jeong H and Kim M S 2002 *Phys. Rev. A* **65** 042305
- [10] Ralph T C, Gilchrist A, Milburn G J, Munro W J and Glancy S 2003 *Phys. Rev. A* **68** 042319
- [11] Ralph T C, Munro W J and Milburn G J 2001 Quantum computation with coherent states, linear interactions and superposed resources *Hewlett Packard Labs Tech Report* 2001-266
- [12] Caves C M, Thorne K S, Drever R W P, Sandberg V D and Zimmermann M 1980 *Rev. Mod. Phys.* **52** 341
- [13] Munro W J, Nemoto K, Milburn G J and Braunstein S L 2002 *Phys. Rev. A* **66** 023819
- [14] Bollinger J J, Itano W M, Wineland D J and Heinzen D J 1996 *Phys. Rev. A* **54** R4649
- [15] Huelga S F, Macchiavello C, Pellizzari T, Ekert A K, Plenio M B and Cirac J I 1997 *Phys. Rev. Lett.* **79** 3865
- [16] D'Ariano M G, Lo Presti P and Paris M G A 2001 *Phys. Rev. Lett.* **87** 270404
- [17] Kolobov M I and Fabre C 2000 *Phys. Rev. Lett.* **85** 3789
- [18] Saleh B E A, Jost B M, Fei H-B and Teich M C 1998 *Phys. Rev. Lett.* **80** 3483
- [19] D'Ariano M G, Paris M G A and Perinotti P 2002 *Phys. Rev. A* **65** 062106
- [20] Polzik E S, Carry J and Kimble H J 1992 *Phys. Rev. Lett.* **68** 3020
- [21] Kim J, Takeuchi S, Yamamoto Y and Hogue H H 1999 *Appl. Phys. Lett.* **74** 902
- [22] Takeuchi S, Kim J, Yamamoto Y and Hogue H H 1999 *Appl. Phys. Lett.* **74** 1063
- [23] Dakna M, Anhut T, Opatrny T, Knöll L and Welsch D-G 1997 *Phys. Rev. A* **55** 3184
- [24] Song S, Caves C M and Yurke B 1990 *Phys. Rev. A* **41** 5261
- [25] van Enk S J and Hirota O 2001 *Phys. Rev. A* **64** 022313
- [26] Braginsky V and Khalili F Ya 1992 *Quantum Measurement* (Cambridge: Cambridge University Press)
- [27] Braunstein S L, Caves C M and Milburn G J 1996 *Ann. Phys.* **247** 135
- [28] Dicke R H 1954 *Phys. Rev.* **93** 99
- [29] Ralph T C 2002 *Phys. Rev. A* **65** 042313